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Algebraic Andô dilation

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Abstract. We solve the Andô dilation problem for linear maps on a vector space asked by Krishna and Johnson in *[Oper. Matrices, 2022]*. More precisely, we show that any commuting linear maps on vector spaces can be dilated to commuting injective linear maps.

1. Introduction

As is well known, one of the fundamental questions in Hilbert space operator theory is the following: How to understand a bounded linear operator? The first and the easiest class of operators are unitary operators which are completely understood using spectral theory (see [4]). Hence we try to understand any bounded linear operator using unitary. In 1950, Halmos noticed that any contraction can be placed in the first entry of a 2 by 2 operator matrix which is unitary. Since any bounded linear operator can be converted to contraction dividing by its norm, Halmos result assures that many properties of any bounded linear operator can be obtained using unitaries. It is very interesting to note that Chapter 23 of legendary book [6] by Halmos is dedicated to unitary dilations of operators on Hilbert space. This chapter has examples of unitary dilations of operators including the zero operator. Other interesting examples are in Chapter 3 of the book [19]. In 1953, Sz.-Nagy showed that [17] (see the Appendix of the book [14] for English translation of this paper) Halmos unitary dilation can be extended which works for all powers of given contraction. In 1955, Schaffer gave an explicit construction of unitary dilation of contraction derived by Sz.-Nagy [15].

After a decade of work of Sz.-Nagy [17, 18], Andô [1] made a breakthrough result in the dilation theory of contractions on a Hilbert space which states as follows.

Theorem 1.1. *[1] (**Andô Dilation**) Let $\mathcal{H}$ be a Hilbert space and $T, S : \mathcal{H} \to \mathcal{H}$ be commuting contractions. Then there exists a Hilbert space $\mathcal{K}$ which contains $\mathcal{H}$ isometrically and a pair of commuting unitaries $U, V : \mathcal{K} \to \mathcal{K}$ such that*

$$T^nS^m = P_{\mathcal{H}}U^nV^m|_{\mathcal{H}}, \quad \forall n, m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\},$$

where $P_{\mathcal{H}} : \mathcal{K} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{H}$.

After the work of Andô, Parrott [13] showed that it is not possible to improve Theorem 1.1 for more than two commuting contractions. Later, Andô dilation is derived for commuting contractions on Banach spaces [16]. In 2021, in the paper [9], while continuing the work of Bhat, De and Rakshit [2] on dilations of linear maps on vector spaces, Krishna and Johnson [9] asked the following problem.

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Question 1.2. *[9] **Whether there is an Andô dilation for linear maps on vector spaces? More precisely, whether commuting linear maps on vector spaces can be dilated to commuting bijective linear maps?***

We note here that the notion of magic contractions have been introduced and Sz.-Nagy dilation along with p-adic von-Neumann inequality and p-adic von Neumann mean ergodic theorem are derived in [8]. In the same paper [8], Number Theory connections with Dilation Theory is given using Quadratic Reciprocity Law. On the other extreme, Sz.-Nagy dilation for self-adjoint morphisms on indefinite inner product modules over *-rings of characteristic 2 is derived in [7].

In this paper, we solve Question 1.2 partially by showing that we can go upto injective linear maps.

2. Algebraic Andô Dilation

We first give a different proof Theorem 2.1 than given in [2] which helps us to give a proof of algebraic version of Andô dilation.

Theorem 2.1. *[2] (Algebraic Sz.-Nagy Dilation or Bhat-De-Rakshit Dilation) Let $\mathcal{V}$ be a vector space and $T : \mathcal{V} \to \mathcal{V}$ be a linear map. Then there is a vector space $\mathcal{W}$ containing $\mathcal{V}$ through a natural coordinate injective map and an injective linear map $U : \mathcal{W} \to \mathcal{W}$ such that*

$$\text{(Dilation equation)} \quad T^n = P_{\mathcal{V}} U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{Z}_+,$$

where $P_{\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ is a coordinate projection (idempotent) onto $\mathcal{V}$.

Proof. Our construction is motivated from the construction of Sz.-Nagy dilation of a contraction on a Hilbert space given in Chapter 1 of [18]. Given a vector space $\mathcal{V}$, let $I_{\mathcal{V}}$ be the identity operator on $\mathcal{V}$ and $\oplus_{n=0}^{\infty}\mathcal{V}$ be the vector space defined by

$$\oplus_{n=0}^{\infty}\mathcal{V} := \{(x_n)_{n=0}^{\infty}, x_n \in \mathcal{V}, \forall n \in \mathbb{Z}_+, x_n \neq 0 \text{ only for finitely many } n's\}.$$

Let $T : \mathcal{V} \to \mathcal{V}$ be a linear map. Define $\mathcal{W} := \oplus_{n=0}^{\infty}\mathcal{V}$ and

$$I : \mathcal{V} \ni x \mapsto (x, 0, \dots) \in \mathcal{W},$$
$$U : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto (Tx_0, (I_{\mathcal{V}} - T)x_0, x_1, x_2, \dots) \in \mathcal{W},$$
$$P : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}.$$

Then clearly the dilation equation is satisfied. The proof is complete if we show that U is injective. Let $(x_n)_{n=0}^{\infty} \in \mathcal{W}$ be such that $U(x_n)_{n=0}^{\infty} = 0$. Then

$$(Tx_0, (I_{\mathcal{V}} - T)x_0, x_1, x_2, \dots) = (0, 0, 0, \dots).$$

We then have $x_1 = x_2 = \cdots = 0$ and $Tx_0 = (I_{\mathcal{V}} - T)x_0 = 0$. Rewriting

$$x_0 = Tx_0 = 0.$$

Thus $(\mathcal{W}, U)$ is an injective linear dilation of T. □

Following is the most important result of this paper which we call algebraic Andô dilation.

Theorem 2.2. *(**Algebraic Andô Dilation**) Let $\mathcal{V}$ be a vector space and $T, S : \mathcal{V} \to \mathcal{V}$ be commuting linear maps. Then there is a vector space $\mathcal{W}$ containing $\mathcal{V}$ through a natural coordinate injective map and injective linear maps $U, V : \mathcal{W} \to \mathcal{W}$ such that*

$$\text{(Bivariate Dilation equation)} \quad T^nS^m = P_{\mathcal{V}} U^n V^m|_{\mathcal{V}}, \quad \forall n, m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\},$$

where $P_{\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ is a coordinate projection (idempotent) onto $\mathcal{V}$.

Proof. Our arguments are motivated from original argument for Andô dilation for commuting contractions on Hilbert spaces by Andô [1]. Define $\mathcal{W} := \oplus_{n=0}^{\infty}\mathcal{V}$ and

$$\begin{aligned}
&W_1 : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto (Tx_0, (I_{\mathcal{V}} - T)x_0, 0, x_1, x_2, \dots) \in \mathcal{W},\\
&W_2 : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto (Sx_0, (I_{\mathcal{V}} - S)x_0, 0, x_1, x_2, \dots) \in \mathcal{W},\\
&P : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}.
\end{aligned}$$

Let $x \in \mathcal{V}$ be such that $(I_{\mathcal{V}} - T)Sx = 0 = (I_{\mathcal{V}} - S)x$. Then

$$(I_{\mathcal{V}} - S)Tx = Tx - STx = Tx - TSx = T(I_{\mathcal{V}} - S)x = T0 = 0$$

and

$$(I_{\mathcal{V}} - T)x = x - Tx = x - T(Sx) = Sx - TSx = (I_{\mathcal{V}} - T)Sx = 0.$$

This obervation says that the map

$$v : \{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\} \to \{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$$

defined by

$$v((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0)) := ((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0)$$

is a well-defined injective linear map. Clearly v is surjective. We now claim that v can be extended as a bijective linear map (which we again denote by v) from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. We get two cases.
Case (i): $\dim(\mathcal{V}) < \infty$.
Let $\mathcal{Y}$ be any vector space complement of $\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ and $\mathcal{Z}$ be any vector space complement of $\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. From the dimension formula for vector spaces, we then get

$$\begin{aligned}
\dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) &= \dim(\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}) + \dim(\mathcal{Y})\\
&= \dim(\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}) + \dim(\mathcal{Z}).
\end{aligned}$$

Since $\dim(\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}) = \dim(\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\})$,

$$\dim(\mathcal{Y}) = \dim(\mathcal{Z}).$$

Thus v can be extended bijectively and linearly from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.
Case (i): $\dim(\mathcal{V}) = \infty$.
Let $\mathcal{Y}$ be any vector space complement of $\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}$ containing the space $\{(0, x, 0, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ and $\mathcal{Z}$ be any vector space complement of $\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$ containing the space $\{(0, x, 0, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. Then

$$\dim(\mathcal{V}) = \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) \geq \dim(\mathcal{Y}) \geq \dim(\mathcal{V})$$

and

$$\dim(\mathcal{V}) = \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) \geq \dim(\mathcal{Z}) \geq \dim(\mathcal{V}).$$

Therefore $\dim(\mathcal{Y}) = \dim(\mathcal{Z})$ and hence v can be extended bijectively and linearly from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.
Define $\mathcal{V}^{(4)} := \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. We identify $\mathcal{W}$ and $\mathcal{V} \oplus (\oplus_{n=1}^{\infty}\mathcal{V}^{(4)})$ by the map

$$(x_n)_{n=0}^{\infty} \mapsto (x_0, (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots).$$

Now we define $W : \mathcal{W} \to \mathcal{W}$ by

$$W(x_n)_{n=0}^{\infty} := (x_0, v(x_1, x_2, x_3, x_4), v(x_5, x_6, x_7, x_8), \dots)$$

which becomes bijective linear map with inverse

$$W^{-1}(x_n)_{n=0}^{\infty} := (x_0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots).$$

We finally define $U := WW_1$, $V := W_2W^{-1}$ and show that $(\mathcal{W}, (U, V))$ is the required injective linear dilation of (T, S). Clearly U and V are injective. By induction, we also have the multivariate dilation equation

$$T^nS^mx = P_{\mathcal{V}}U^nV^mx, \quad \forall n, m \in \mathbb{Z}_+, \forall x \in \mathcal{V}.$$

Now we are left only with proving that U and V commute. Let $(x_n)_{n=0}^{\infty} \in \mathcal{W}$. Then

$$\begin{aligned}
UV(x_n)_{n=0}^{\infty} &= WW_1W_2W^{-1}(x_n)_{n=0}^{\infty}\\
&= WW_1W_2(x_0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots)\\
&= WW_1(Sx_0, (I_{\mathcal{V}} - S)x_0, 0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots)\\
&= W(TSx_0, (I_{\mathcal{V}} - T)Sx_0, 0, (I_{\mathcal{V}} - S)x_0, 0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots)\\
&= (TSx_0, v((I_{\mathcal{V}} - T)Sx_0, 0, (I_{\mathcal{V}} - S)x_0, 0), (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots)\\
&= (STx_0, (I_{\mathcal{V}} - S)Tx_0, 0, (I_{\mathcal{V}} - T)x_0, 0), (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots)
\end{aligned}$$

and

$$\begin{aligned}
VU(x_n)_{n=0}^{\infty} &= W_2W^{-1}WW_1(x_n)_{n=0}^{\infty} = W_2W_1(x_n)_{n=0}^{\infty}\\
&= W_2(Tx_0, (I_{\mathcal{V}} - T)x_0, 0, x_1, x_2, \dots)\\
&= (STx_0, (I_{\mathcal{V}} - S)Tx_0, 0, (I_{\mathcal{V}} - T)x_0, 0), x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \dots).
\end{aligned}$$

Therefore $VU = UV$. $\square$

Theorem 2.2 and the works presented in [3, 5, 10, 11, 15] give the following questions.

Question 2.3. (i) *Whether there is an explicit (matrix) construction of algebraic Andô dilation?*

(ii) *Whether there is a Halmos dilation for commuting linear maps on vector spaces?*

(iii) *Whether there is an Egerváry N-dilation for commuting linear maps on vector spaces?*

(iv) *Does Theorem 2.2 hold for more than two commuting linear maps?*

(v) *Can the dilated injective linear maps U, V in Theorem 2.2 be improved to bijective linear maps?*

Remark 2.4. *After the posting of this article on arXiv, problems (iv) and (v) in Question 2.3 have been solved by V. Muller [12].*

3. Conclusions

1. In 1950, Halmos showed that every contraction on a Hilbert space can be lifted to a unitary [5].
2. In 1953, Sz.-Nagy derived his dilation theorem [17].
3. In 1955, Schaffer gave a simple proof of Sz.-Nagy dilation result [15].
4. In 1963, Andô showed that Sz.-Nagy dilation holds for two commuting contractions [1].
5. In 1973, Stroescu derived Andô dilation for contractions on Banach spaces [16].
6. In 2021, Bhat, De and Rakshit introduced set theoretic and vector space approach to dilation theory [2]. Later, Krishna and Johnson continued this study in 2022 [9].
7. In this paper, we derived Andô dilation for linear maps on vector spaces.

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References

[1] T. Andô. On a pair of commutative contractions. *Acta Sci. Math. (Szeged)*, 24:88–90, 1963.
[2] B. V. Rajarama Bhat, Sandipan De, and Narayan Rakshit. A caricature of dilation theory. *Adv. Oper. Theory*, 6(4):Paper No. 63, 20, 2021.
[3] E. Egerváry. On the contractive linear transformations of n-dimensional vector space. *Acta Sci. Math. (Szeged)*, 15:178–182, 1954.
[4] P. R. Halmos. What does the spectral theorem say? *Amer. Math. Monthly*, 70:241–247, 1963.
[5] Paul R. Halmos. Normal dilations and extensions of operators. *Summa Brasil. Math.*, 2:125–134, 1950.
[6] Paul Richard Halmos. *A Hilbert space problem book*, volume 17 of *Encyclopedia of Mathematics and its Applications*. Springer-Verlag, New York-Berlin, second edition, 1982.
[7] K. Mahesh Krishna. Indefinite Halmos, Egervary and Sz.-Nagy dilations. *arXiv:2210.05646 v1 [math.RA] 27 September*, 2022.
[8] K. Mahesh Krishna. p-adic magic contractions, p-adic von Neumann inequality and p-adic Sz.-Nagy dilation. *arXiv:2209.12012v1 [math.NT] 24 September*, 2022.
[9] K. Mahesh Krishna and P. Sam Johnson. Dilations of linear maps on vector spaces. *Oper. Matrices*, 16(2):465–477, 2022.
[10] Eliahu Levy and Orr Moshe Shalit. Dilation theory in finite dimensions: the possible, the impossible and the unknown. *Rocky Mountain J. Math.*, 44(1):203–221, 2014.
[11] John E. McCarthy and Orr Moshe Shalit. Unitary N-dilations for tuples of commuting matrices. *Proc. Amer. Math. Soc.*, 141(2):563–571, 2013.
[12] Vladimir Muller. Liftings and dilations of commuting systems of linear mappings on vector spaces. *The Czech academy of Sciences, Preprint No. 72-2022*, 2022.
[13] Stephen Parrott. Unitary dilations for commuting contractions. *Pacific J. Math.*, 34:481–490, 1970.
[14] Frigyes Riesz and Béla Sz.-Nagy. *Functional analysis*. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1990.
[15] J. J. Schäffer. On unitary dilations of contractions. *Proc. Amer. Math. Soc.*, 6:322, 1955.
[16] Elena Stroescu. Isometric dilations of contractions on Banach spaces. *Pacific J. Math.*, 47:257–262, 1973.
[17] Béla Sz.-Nagy. Sur les contractions de l'espace de Hilbert. *Acta Sci. Math. (Szeged)*, 15:87–92, 1953.
[18] Bela Sz.-Nagy, Ciprian Foias, Hari Bercovici, and Laszlo Kerchy. *Harmonic analysis of operators on Hilbert space*. Universitext. Springer, New York, 2010.
[19] Pei Yuan Wu and Hwa-Long Gau. *Numerical ranges of Hilbert space operators*, volume 179 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2021.