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On the generalized Ostrowski type inequalities for co-ordinated convex functions

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Abstract. The purpose of this article is to establish some generalized Ostrowski type inequalities and integral inequalities in the coordinate plane for convex functions of 2 variables. For this, we will specify a generalized identity, and with the help of this integral identity, we will examine the Ostrowski, trapezoid, and midpoint type integral inequalities, including Riemann integral and Riemann-Liouville fractional integral. In this way, we aim to contribute to the generalization of integral inequalities, an important topic in mathematical analysis.

1. Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [21] is the following classical integral inequality associated with the differentiable mappings:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\left\|f'\right\|_\infty=\sup\limits_{t\in(a,b)}\left|f'(t)\right|<+\infty$. Then, we have the following inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

The Ostrowski integral inequality is a powerful tool in mathematical analysis that has been extensively studied in the literature. It has numerous applications in various fields of mathematics, including calculus, functional analysis, and numerical analysis. In addition to the Riemann and Riemann fractional integrals, the inequality has also been extended to include other integral operators such as the Hadamard fractional integral and the Erdélyi-Kober fractional integral. Moreover, recent studies have also explored

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the Ostrowski integral inequality in the context of multi-dimensional integrals and its generalizations to higher order derivatives. These developments have opened up new avenues for research and have further enriched our understanding of this important mathematical inequality, see ([3]-[9], [12], [22], [23], [29]).

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, midpoint, Simpson or Jensen inequalities ect.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequalities [18]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

One of the best-known inequalities about the integral mean is the Hermite Hadamard inequalities. These inequalities give an estimate on both sides of the mean value of a convex function and also make the convex function integrable. It should also be noted that some classical mean inequalities can be derived from Hadamard's inequality under the utility of peculiar convex functions f. Inequalities for these convex functions also play an important role in analysis and in the fields of pure and applied mathematics. The absolute value of the second part of these inequalities is known in the literature as the trapezoidal inequality and was given by Dragomir and Agarwal in 1998 [11]. Then, the absolute value of the first part of the inequalities introduced by Kırmancı is known as the midpoint inequality in [14].

The purpose of this article is to create some generalized integral inequalities in the coordinate plane for convex functions of 2 variables. For this, we will specify a generalized identity, and with the help of this integral identity, we will examine Ostrwoski-type inequalities involving Riemann integral and Riemann-Liouville fractional integral. We will also consider some integral inequalities on the left and right sides of the Hermite-Hadamard type inequalities. This study aims to contribute to the generalization of integral inequalities, which is an important topic in mathematical analysis.

2. Preliminaries

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \le tf(x,y) + (1-t)f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [10]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ \le\ & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,w) + (1-t)(1-s)f(y,w). \end{aligned} \tag{2.1}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [10]). Dragomir first proved Hermite-Hadamard inequalities for co-ordinated convex mappings in [10]. The midpoint and trapezoid type inequalities for co-ordinated convex functions were established in the papers [17] and [26], respectively. Moreover, Sarikaya obtained Hermite-Hadamard inequalities for functions with two variables by utilizing Riemann-Liouville fractional integrals in [27]. Whereas Sarikaya gave the corresponding fractional trapezoid inequalities for co-ordinated convex functions in [27], Tunç et al. presented fractional midpoint type inequalities for co-ordinated convex functions in [30]. In the literature, there are numerous papers related to Ostrowski and Hermite-Hadamard inequalities for several type co-ordinated convex functions. For several recent results concerning Ostrowski and Hermite-Hadamard's inequalities for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([2], [3], [7], [8], [10], [12], [17], [19], [20], [24]).

Recently, in [10], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\le\;& \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
\le\;& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (2.2)\\
\le\;& \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
\le\;& \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

Generalized double fractional integrals are given by Sarikaya et al. in [26], as follows:

Definition 2. *Let $f \in L_1([a,b]\times[c,d])$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ of order $\alpha,\beta>0$ with $a,c\ge 0$ are defined by*

$$J^{\alpha,\beta}_{a+,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt,\;\; x>a,\; y>c,$$

$$J^{\alpha,\beta}_{a+,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\;\; x>a,\; y<d,$$

$$J^{\alpha,\beta}_{b-,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt,\;\; x<b,\; y>c,$$

and

$$J^{\alpha,\beta}_{b-,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\;\; x<b,\; y<d,$$

respectively. Here, Γ is the Gamma function,

$$J^{0,0}_{a+,c+}f(x,y)=J^{0,0}_{a+,d-}f(x,y)=J^{0,0}_{b-,c+}f(x,y)=J^{0,0}_{b-,d-}f(x,y)=f(x,y),$$

and

$$J^{1,1}_{a+,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y f(t,s)dsdt.$$

For some recent results connected with fractional integral inequalities see ([1], [4], [9], [13], [15], [16], [25]-[30]).

3. Fractional Inequalities for co-ordinated convex functions

Throughout this section, we will use the following sympol

$$
\begin{aligned}
& S(H_1,H_2)=4H_1(1)H_2(1)f(x,y)-2H_1(1)H_2(0)[f(x,c)+f(x,d)] \\
& -2H_1(0)H_2(1)[f(a,y)+f(b,y)] \\
& +H_1(0)H_2(0)[f(a,c)+f(a,d)+f(b,c)+f(b,d)] \\
& -\frac{2H_2(1)}{x-a}\int_a^x H_1'\left(\frac{u_1-a}{x-a}\right)f(u_1,y)\,du_1 \\
& -\frac{2H_2(1)}{b-x}\int_x^b H_1'\left(\frac{b-u_2}{b-x}\right)f(u_2,y)\,du_2 \\
& +\frac{H_2(0)}{x-a}\int_a^x H_1'\left(\frac{u_1-a}{x-a}\right)[f(u_1,c)+f(u_1,d)]\,du_1 \\
& +\frac{H_2(0)}{b-x}\int_x^b H_1'\left(\frac{b-u_2}{b-x}\right)[f(u_2,c)+f(u_2,d)]\,du_2 \\
& -\frac{2H_1(1)}{y-c}\int_c^y H_2'\left(\frac{v_1-c}{y-c}\right)f(x,v_1)\,dv_1-\frac{2H_1(1)}{d-y}\int_y^d H_2'\left(\frac{d-v_2}{d-y}\right)f(x,v_2)\,dv_2 \\
& +\frac{H_1(0)}{y-c}\int_c^y H_2'\left(\frac{v_1-c}{y-c}\right)[f(a,v_1)+f(b,v_1)]\,dv_1 \\
& +\frac{H_1(0)}{d-y}\int_y^d H_2'\left(\frac{d-v_2}{d-y}\right)[f(a,v_2)+f(b,v_2)]\,dv_2 \\
& +\frac{1}{(x-a)(y-c)}\int_a^x\int_c^y H_1'\left(\frac{u_1-a}{x-a}\right)H_2'\left(\frac{v_1-c}{y-c}\right)f(u_1,v_1)\,dv_1du_1 \\
& +\frac{1}{(x-a)(d-y)}\int_a^x\int_y^d H_1'\left(\frac{u_1-a}{x-a}\right)H_2'\left(\frac{d-v_2}{d-y}\right)f(u_1,v_2)\,dv_2du_1 \\
& +\frac{1}{(b-x)(y-c)}\int_x^b\int_c^y H_1'\left(\frac{b-u_2}{b-x}\right)H_2'\left(\frac{v_1-c}{y-c}\right)f(u_2,v_1)\,dv_1du_2 \\
& +\frac{1}{(b-x)(d-y)}\int_x^b\int_y^d H_1'\left(\frac{b-u_2}{b-x}\right)H_2'\left(\frac{d-v_2}{d-y}\right)f(u_2,v_2)\,dv_2du_2
\end{aligned}
$$

In this part, we will give the following inequalities by using convex functions of 2-variables on the co-ordinates. In order to prove our main results, we need the following lemma.

Lemma 1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$ with $0\le a<b,\ 0\le c<d$ and $H_1,H_2:[0,1]\to R$ be two positive differentiable functions. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$
\begin{aligned}
& S(H_1,H_2) \qquad (3.1)\\
= \;& (x-a)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\,dsdt
\end{aligned}
$$

$$
\begin{aligned}
&-(x-a)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\,dsdt\\
&-(b-x)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\,dsdt\\
&+(b-x)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\,dsdt.
\end{aligned}
$$

Proof. By integration by parts, we get

$$
\begin{aligned}
W_1 &= \int_0^1\int_0^1 H_1(t)H_2(s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\,dsdt \qquad (3.2)\\
&= \int_0^1 H_1(t)\left\{H_2(s)\,\frac{1}{y-c}\frac{\partial f}{\partial t}(tx+(1-t)a,sy+(1-s)c)\Big|_0^1\right.\\
&\quad\left.-\frac{1}{y-c}\int_0^1 F_2'(s)\frac{\partial f}{\partial t}(tx+(1-t)a,sy+(1-s)c)\,ds\right\}dt\\
&= \frac{1}{y-c}\int_0^1 H_1(t)\left[H_2(1)\frac{\partial f}{\partial t}(tx+(1-t)a,y)-H_2(0)\frac{\partial f}{\partial t}(tx+(1-t)a,c)\right]dt\\
&\quad-\frac{1}{y-c}\int_0^1 H_2'(s)\left[\int_0^1 H_1(t)\frac{\partial f}{\partial t}(tx+(1-t)a,sy+(1-s)c)\,dt\right]ds\\
&= \frac{1}{y-c}\left\{\frac{H_1(t)}{x-a}\left[H_2(1)f(tx+(1-t)a,y)-H_2(0)f(tx+(1-t)a,c)\right]\Big|_0^1\right.\\
&\quad\left.-\frac{1}{x-a}\int_0^1 H_1'(s)\left[H_2(1)f(tx+(1-t)a,y)-H_2(0)f(tx+(1-t)a,c)\right]ds\right\}\\
&\quad-\frac{1}{y-c}\int_0^1 H_2'(s)\left\{\frac{H_1(t)}{x-a}f(tx+(1-t)a,sy+(1-s)c)\Big|_0^1\right.\\
&\quad\left.-\frac{1}{x-a}\int_0^1 H_1'(t)f(tx+(1-t)a,sy+(1-s)c)\,dt\right\}ds\\
&= \frac{1}{(x-a)(y-c)}\{H_1(1)[H_2(1)f(x,y)-H_2(0)f(x,c)]\\
&\quad-H_1(0)[H_2(1)f(a,y)-H_2(0)f(a,c)]\}\\
&\quad-\frac{1}{(x-a)(y-c)}\int_0^1 H_1'(t)\\
&\quad\times[H_2(1)f(tx+(1-t)a,y)-H_2(0)f(tx+(1-t)a,c)]\,dt\\
&\quad-\frac{1}{(x-a)(y-c)}\int_0^1 H_2'(s)[H_1(1)f(x,sy+(1-s)c)-H_1(0)f(a,sy+(1-s)c)]\,ds\\
&\quad+\frac{1}{(x-a)(y-c)}\int_0^1\int_0^1 H_1'(t)H_2'(s)f(tx+(1-t)a,sy+(1-s)c)\,dsdt.
\end{aligned}
$$

By similar way, by integration by parts, we also get,

$$
\begin{aligned}
W_2 &= \int_0^1 \int_0^1 H_1(t) H_2(s) \frac{\partial^2 f}{\partial t \partial s}(tx + (1-t)a, sy + (1-s)d)\, dsdt \\
&= -\frac{1}{(x-a)(d-y)} \{H_1(1)[H_2(1) f(x,y) - H_2(0) f(x,d)] \\
&\quad - H_1(0)[H_2(1) f(a,y) - H_2(0) f(a,d)]\} \\
&\quad + \frac{1}{(x-a)(d-y)} \int_0^1 H_1'(t) \\
&\quad \times [H_2(1) f(tx + (1-t)a, y) - H_2(0) f(tx + (1-t)a, d)]\, dt \\
&\quad + \frac{1}{(x-a)(d-y)} \int_0^1 H_2'(s) \\
&\quad \times [H_1(1) f(x, sy + (1-s)d) - H_1(0) f(a, sy + (1-s)d)]\, ds \\
&\quad - \frac{1}{(x-a)(d-y)} \int_0^1 \int_0^1 H_1'(t) H_2'(s) f(tx + (1-t)a, sy + (1-s)d)\, dsdt,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
W_3 &= \int_0^1 \int_0^1 H_1(t) H_2(s) \frac{\partial^2 f}{\partial t \partial s}(tx + (1-t)b, sy + (1-s)c)\, dsdt \\
&= -\frac{1}{(b-x)(y-c)} \{H_1(1)[H_2(1) f(x,y) - H_2(0) f(x,c)] \\
&\quad - H_1(0)[H_2(1) f(b,y) - H_2(0) f(b,c)]\} \\
&\quad + \frac{1}{(b-x)(y-c)} \int_0^1 H_1'(t) \\
&\quad \times [H_2(1) f(tx + (1-t)b, y) - H_2(0) f(tx + (1-t)b, c)]\, dt \\
&\quad + \frac{1}{(b-x)(y-c)} \int_0^1 H_2'(s) \\
&\quad \times [H_1(1) f(x, sy + (1-s)c) - H_1(0) f(b, sy + (1-s)c)]\, ds \\
&\quad - \frac{1}{(b-x)(y-c)} \int_0^1 \int_0^1 H_1'(t) H_2'(s) f(tx + (1-t)b, sy + (1-s)c)\, dsdt,
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
W_4 &= \int_0^1 \int_0^1 H_1(t) H_2(s) \frac{\partial^2 f}{\partial t \partial s}(tx + (1-t)b, sy + (1-s)d)\, dsdt \\
&= \frac{1}{(b-x)(d-y)} \{H_1(1)[H_2(1) f(x,y) - H_2(0) f(x,d)] \\
&\quad - H_1(0)[H_2(1) f(b,y) - H_2(0) f(b,d)]\} \\
&\quad - \frac{1}{(b-x)(d-y)} \int_0^1 H_1'(t) \\
&\quad \times [H_2(1) f(tx + (1-t)b, y) - H_2(0) f(tx + (1-t)b, d)]\, dt
\end{aligned} \tag{3.5}
$$

$$-\frac{1}{(b-x)(d-y)}\int_0^1 H_2'(s)$$
$$\times\left[H_1(1)f(x,sy+(1-s)d)-H_1(0)f(b,sy+(1-s)d)\right]ds$$
$$+\frac{1}{(b-x)(d-y)}\int_0^1\int_0^1 H_1'(t)H_2'(s)f(tx+(1-t)b,sy+(1-s)d)\,dsdt.$$

We multiply both sides of (3.2) by $(x-a)(y-c)$, (3.3) by $-(x-a)(d-y)$, (3.4) by $-(b-x)(y-c)$ and (3.5) by $(b-x)(d-y)$, then we add the resulting equalities up yielding

$$\begin{aligned}
W &= (x-a)(y-c)W_1-(x-a)(d-y)W_2-(b-x)(y-c)W_3+(b-x)(d-y)W_4\\
&= 4H_1(1)H_2(1)f(x,y)-2H_1(1)H_2(0)\left[f(x,c)+f(x,d)\right]\\
&\quad -2H_1(0)H_2(1)\left[f(a,y)+f(b,y)\right]\\
&\quad +H_1(0)H_2(0)\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\
&\quad -2H_2(1)\int_0^1 H_1'(t)f(tx+(1-t)a,y)\,dt-2H_2(1)\int_0^1 H_1'(t)f(tx+(1-t)b,y)\,dt\\
&\quad +H_2(0)\int_0^1 H_1'(t)\left[f(tx+(1-t)a,c)+f(tx+(1-t)a,d)\right]dt\\
&\quad +H_2(0)\int_0^1 H_1'(t)\left[f(tx+(1-t)b,c)+f(tx+(1-t)b,d)\right]dt\\
&\quad -2H_1(1)\int_0^1 H_2'(s)f(x,sy+(1-s)c)\,ds-2H_1(1)\int_0^1 H_2'(s)f(x,sy+(1-s)d)\,ds\\
&\quad +H_1(0)\int_0^1 H_2'(s)\left[f(a,sy+(1-s)c)+f(b,sy+(1-s)c)\right]ds\\
&\quad +H_1(0)\int_0^1 H_2'(s)\left[f(a,sy+(1-s)d)+f(b,sy+(1-s)d)\right]ds\\
&\quad +\int_0^1\int_0^1 H_1'(t)H_2'(s)f(tx+(1-t)a,sy+(1-s)c)\,dsdt\\
&\quad +\int_0^1\int_0^1 H_1'(t)H_2'(s)f(tx+(1-t)a,sy+(1-s)d)\,dsdt\\
&\quad +\int_0^1\int_0^1 H_1'(t)H_2'(s)f(tx+(1-t)b,sy+(1-s)c)\,dsdt\\
&\quad +\int_0^1\int_0^1 H_1'(t)H_2'(s)f(tx+(1-t)b,sy+(1-s)d)\,dsdt.
\end{aligned}$$

Thus, using the change of the variable $u_1=tx+(1-t)a$, $u_2=tx+(1-t)b$, $v_1=sy+(1-s)c$, and $v_2=sy+(1-s)d$ for $(t,s)\in[0,1]\times[0,1]$, we can write

$$\begin{aligned}
W &= (x-a)(y-c)W_1-(x-a)(d-y)W_2-(b-x)(y-c)W_3+(b-x)(d-y)W_4\\
&= 4H_1(1)H_2(1)f(x,y)-2H_1(1)H_2(0)\left[f(x,c)+f(x,d)\right]
\end{aligned}$$

$$-2H_1(0)H_2(1)[f(a,y)+f(b,y)]$$

$$+H_1(0)H_2(0)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]$$

$$-\frac{2H_2(1)}{x-a}\int_a^x H_1'\left(\frac{u_1-a}{x-a}\right)f(u_1,y)\,du_1-\frac{2H_2(1)}{b-x}\int_x^b H_1'\left(\frac{b-u_2}{b-x}\right)f(u_2,y)\,du_2$$

$$+\frac{H_2(0)}{x-a}\int_a^x H_1'\left(\frac{u_1-a}{x-a}\right)[f(u_1,c)+f(u_1,d)]\,du_1$$

$$+\frac{H_2(0)}{b-x}\int_x^b H_1'\left(\frac{b-u_2}{b-x}\right)[f(u_2,c)+f(u_2,d)]\,du_2$$

$$-\frac{2H_1(1)}{y-c}\int_c^y H_2'\left(\frac{v_1-c}{y-c}\right)f(x,v_1)\,dv_1-\frac{2H_1(1)}{d-y}\int_y^d H_2'\left(\frac{d-v_2}{d-y}\right)f(x,v_2)\,dv_2$$

$$+\frac{H_1(0)}{y-c}\int_c^y H_2'\left(\frac{v_1-c}{y-c}\right)[f(a,v_1)+f(b,v_1)]\,dv_1$$

$$+\frac{H_1(0)}{d-y}\int_y^d H_2'\left(\frac{d-v_2}{d-y}\right)[f(a,v_2)+f(b,v_2)]\,dv_2$$

$$+\frac{1}{(x-a)(y-c)}\int_a^x\int_c^y H_1'\left(\frac{u_1-a}{x-a}\right)H_2'\left(\frac{v_1-c}{y-c}\right)f(u_1,v_1)\,dv_1du_1$$

$$+\frac{1}{(x-a)(d-y)}\int_a^x\int_y^d H_1'\left(\frac{u_1-a}{x-a}\right)H_2'\left(\frac{d-v_2}{d-y}\right)f(u_1,v_2)\,dv_2du_1$$

$$+\frac{1}{(b-x)(y-c)}\int_x^b\int_c^y H_1'\left(\frac{b-u_2}{b-x}\right)H_2'\left(\frac{v_1-c}{y-c}\right)f(u_2,v_1)\,dv_1du_2$$

$$+\frac{1}{(b-x)(d-y)}\int_x^b\int_y^d H_1'\left(\frac{b-u_2}{b-x}\right)H_2'\left(\frac{d-v_2}{d-y}\right)f(u_2,v_2)\,dv_2du_2.$$

which completes the proof of (3.1). □

Corollary 1. *With the assumptations in Lemma 1,*
i) we choose $H_1(t)=t, H_2(s)=s$ *on* $[0,1]$, *then the equality (3.1) becomes the equality*

$$f(x,y)-\frac{1}{2(x-a)}\int_a^x f(u_1,y)\,du_1-\frac{1}{2(b-x)}\int_x^b f(u_2,y)\,du_2$$

$$-\frac{1}{2(y-c)}\int_c^y f(x,v_1)\,dv_1-\frac{1}{2(d-y)}\int_y^d f(x,v_2)\,dv_2$$

$$+\frac{1}{4(x-a)(y-c)}\int_a^x\int_c^y f(u_1,v_1)\,dv_1du_1$$

$$+\frac{1}{4(x-a)(d-y)}\int_a^x\int_y^d f(u_1,v_2)\,dv_2du_1$$

$$+\frac{1}{4(b-x)(y-c)}\int_x^b\int_c^y f(u_2,v_1)\,dv_1du_2$$

$$+\frac{1}{4(b-x)(d-y)}\int_x^b\int_y^d f(u_2,v_2)\,dv_2du_2$$

$$= \frac{(x-a)(y-c)}{4}\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)c)\,dsdt$$
$$-\frac{(x-a)(d-y)}{4}\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)d)\,dsdt$$
$$-\frac{(b-x)(y-c)}{4}\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)c)\,dsdt$$
$$+\frac{(b-x)(d-y)}{4}\int_0^1\int_0^1 ts\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)d)\,dsdt.$$

ii) we choose $H_1(t)=t^\alpha\,(\alpha>0)$, $H_2(s)=s^\beta\,(\beta>0)$ *on* $[0,1]$, *then the equality (3.1) becomes the fractional integral equality*

$$f(x,y)-\frac{\Gamma(\alpha+1)}{2(x-a)^\alpha}J^\alpha_{x-}f(a,y)-\frac{\Gamma(\alpha+1)}{2(b-x)^\alpha}J^\alpha_{x+}f(b,y)$$
$$-\frac{\Gamma(\beta+1)}{2(y-c)^\beta}J^\beta_{y-}f(x,c)-\frac{\Gamma(\beta+1)}{2(d-y)^\beta}J^\beta_{y+}f(x,d)$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x-a)^\alpha(y-c)^\beta}J^{\alpha,\beta}_{x-,y-}f(a,c)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x-a)^\alpha(d-y)^\beta}J^{\alpha,\beta}_{x-,y+}f(a,d)$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-x)^\alpha(y-c)^\beta}J^{\alpha,\beta}_{x+,y-}f(b,c)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-x)^\alpha(d-y)^\beta}J^{\alpha,\beta}_{x+,y+}f(b,d)$$
$$= \frac{(x-a)(y-c)}{4}\int_0^1\int_0^1 t^\alpha s^\beta\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)c)\,dsdt$$
$$-\frac{(x-a)(d-y)}{4}\int_0^1\int_0^1 t^\alpha s^\beta\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)d)\,dsdt$$
$$-\frac{(b-x)(y-c)}{4}\int_0^1\int_0^1 t^\alpha s^\beta\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)c)\,dsdt$$
$$+\frac{(b-x)(d-y)}{4}\int_0^1\int_0^1 t^\alpha s^\beta\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b, sy+(1-s)d)\,dsdt$$

iii) we choose $H_1(t)=(1-t)^\alpha\,(\alpha>0)$, $H_2(s)=(1-s)^\beta\,(\beta>0)$ *on* $[0,1]$, *then the equality (3.1) becomes the fractional integral equality*

$$\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$
$$+\frac{\Gamma(\alpha+1)}{2(x-a)^\alpha}J^\alpha_{a+}[f(x,c)+f(x,d)]-\frac{\Gamma(\alpha+1)}{2(b-x)^\alpha}J^\alpha_{b-}[f(x,c)+f(x,d)]$$
$$+\frac{\Gamma(\beta+1)}{2(y-c)^\beta}J^\beta_{c+}[f(a,y)+f(b,y)]+\frac{\Gamma(\beta+1)}{2(d-y)^\beta}J^\beta_{d-}[f(a,y)+f(b,y)]$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x-a)^\alpha(y-c)^\beta}J^{\alpha,\beta}_{a+,c+}f(x,y)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x-a)^\alpha(d-y)^\beta}J^{\alpha,\beta}_{a+,d-}f(x,y)$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-x)^\alpha(y-c)^\beta}J^{\alpha,\beta}_{b-,c+}f(x,y)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-x)^\alpha(d-y)^\beta}J^{\alpha,\beta}_{b-,d-}f(x,y)$$
$$= \frac{(x-a)(y-c)}{4}\int_0^1\int_0^1 (1-t)^\alpha(1-s)^\beta\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)c)\,dsdt$$
$$-\frac{(x-a)(d-y)}{4}\int_0^1\int_0^1 (1-t)^\alpha(1-s)^\beta\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a, sy+(1-s)d)\,dsdt$$

$$-\frac{(b-x)(y-c)}{4}\int_0^1\int_0^1 (1-t)^{\alpha}(1-s)^{\beta}\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\,dsdt$$
$$+\frac{(b-x)(d-y)}{4}\int_0^1\int_0^1 (1-t)^{\alpha}(1-s)^{\beta}\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\,dsdt$$

vi) we choose $H_1(t)=\ln(1+t)$, $H_2(s)=\ln(1+s)$ *on* $[0,1]$, *then the equality (3.1) becomes the equality*

$$\begin{aligned}
&\ln 2 f(x,y)-\frac{1}{2}\int_a^x \frac{f(u_1,y)}{x+u_1-2a}du_1-\frac{1}{2(b-x)}\int_x^b \frac{f(u_2,y)}{2b-x-u_2}du_2\\
&-\frac{1}{2}\int_c^y \frac{f(x,v_1)}{y+v_1-2c}dv_1-\frac{1}{2(d-y)}\int_y^d \frac{f(x,v_2)}{2d-y-v_2}dv_2\\
&+\frac{1}{4\ln 2}\int_a^x\int_c^y \frac{f(u_1,v_1)}{(x+u_1-2a)(y+v_1-2c)}dv_1du_1\\
&+\frac{1}{4\ln 2}\int_a^x\int_y^d \frac{f(u_1,v_2)}{(x+u_1-2a)(2d-y-v_2)}dv_2du_1\\
&+\frac{1}{4\ln 2}\int_x^b\int_c^y \frac{f(u_2,v_1)}{(2b-x-u_2)(y+v_1-2c)}dv_1du_2\\
&+\frac{1}{4\ln 2}\int_x^b\int_y^d \frac{f(u_2,v_2)}{(2b-x-u_2)(2d-y-v_2)}dv_2du_2\\
=\;&\frac{(x-a)(y-c)}{4\ln 2}\int_0^1\int_0^1 \ln(1+t)\ln(1+s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\,dsdt\\
&-\frac{(x-a)(d-y)}{4\ln 2}\int_0^1\int_0^1 \ln(1+t)\ln(1+s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\,dsdt\\
&-\frac{(b-x)(y-c)}{4\ln 2}\int_0^1\int_0^1 \ln(1+t)\ln(1+s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\,dsdt\\
&+\frac{(b-x)(d-y)}{4\ln 2}\int_0^1\int_0^1 \ln(1+t)\ln(1+s)\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\,dsdt
\end{aligned}$$

Next, we start to state the first theorem containing the Ostrowski type ineqaulity for fractional integrals.

Theorem 3. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* Δ *in* $\mathbb{R}^2$ *with* $0\le a<b,\ 0\le c<d$ *and* $H_1,H_2:[0,1]\to R$ *be two positive differentiable functions. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then one has the inequality:*

$$\begin{aligned}
&|S(H_1,H_2)| \qquad\qquad (3.6)\\
\le\;&(x-a)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|\right.\\
&\left.+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right\}dsdt\\
&+(x-a)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|\right.\\
&\left.+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}dsdt\\
&+(b-x)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|\right.
\end{aligned}$$

$$+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\Big\}dsdt$$

$$+(b-x)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\Big\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|$$

$$+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\Big\}dsdt.$$

Proof. Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is convex function on the co-ordinates on Δ, from Lemma 1, we have

$$\begin{aligned}
&|S(H_1,H_2)|\\
\leq\ & (x-a)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)c)\right|dsdt\\
&+(x-a)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)a,sy+(1-s)d)\right|dsdt\\
&+(b-x)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)c)\right|dsdt\\
&+(b-x)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\left|\frac{\partial^2 f}{\partial t\partial s}(tx+(1-t)b,sy+(1-s)d)\right|dsdt\\
\leq\ & (x-a)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\Big\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|\\
&+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\Big\}dsdt\\
&+(x-a)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\Big\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|\\
&+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\Big\}dsdt\\
&+(b-x)(y-c)\int_0^1\int_0^1 H_1(t)H_2(s)\Big\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|\\
&+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\Big\}dsdt\\
&+(b-x)(d-y)\int_0^1\int_0^1 H_1(t)H_2(s)\Big\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|\\
&+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\Big\}dsdt.
\end{aligned}$$

By adapting the integral in above inequality, we have the inequality (3.6). □

Corollary 2. *With the assumptations in Theorem 3,*
i) we choose $H_1(t)=t, H_2(s)=s$ *on* $[0,1]$*, then the inequality (3.6) becomes the inequality*

$$\Big|f(x,y)-\frac{1}{2(x-a)}\int_a^x f(u_1,y)\,du_1-\frac{1}{2(b-x)}\int_x^b f(u_2,y)\,du_2 \tag{3.7}$$

$$-\frac{1}{2(y-c)}\int_c^y f(x,v_1)\,dv_1-\frac{1}{2(d-y)}\int_y^d f(x,v_2)\,dv_2$$

$$
\begin{aligned}
& +\frac{1}{4(x-a)(y-c)} \int_a^x \int_c^y f(u_1, v_1)\, dv_1 du_1 \\
& +\frac{1}{4(x-a)(d-y)} \int_a^x \int_y^d f(u_1, v_2)\, dv_2 du_1 \\
& +\frac{1}{4(b-x)(y-c)} \int_x^b \int_c^y f(u_2, v_1)\, dv_1 du_2 \\
& +\frac{1}{4(b-x)(d-y)} \int_x^b \int_y^d f(u_2, v_2)\, dv_2 du_2 \Bigg| \\
\leq\ & \frac{(b-a)(d-c)}{9}\left|\frac{\partial^2 f}{\partial t \partial s}(x, y)\right| \\
& +\frac{(b-a)}{18}\left[(y-c)\left|\frac{\partial^2 f}{\partial t \partial s}(x, c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t \partial s}(x, d)\right|\right] \\
& +\frac{(d-c)}{18}\left[(x-a)\left|\frac{\partial^2 f}{\partial t \partial s}(a, y)\right|+(b-x)\left|\frac{\partial^2 f}{\partial t \partial s}(b, y)\right|\right] \\
& +\frac{(x-a)}{36}\left[(y-c)\left|\frac{\partial^2 f}{\partial t \partial s}(a, c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t \partial s}(a, d)\right|\right] \\
& +\frac{(b-x)}{36}\left[(y-c)\left|\frac{\partial^2 f}{\partial t \partial s}(b, c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t \partial s}(b, d)\right|\right].
\end{aligned}
$$

ii) we choose $H_1(t) = t^{\alpha}\,(\alpha > 0)$, $H_2(s) = s^{\beta}\,(\beta > 0)$ *on* $[0,1]$, *then the inequality (3.6) becomes the fractional integral inequality*

$$
\begin{aligned}
& \Bigg| f(x, y)-\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}} J_{x-}^{\alpha} f(a, y)-\frac{\Gamma(\alpha+1)}{2(b-x)^{\alpha}} J_{x+}^{\alpha} f(b, y) \\
& -\frac{\Gamma(\beta+1)}{2(y-c)^{\beta}} J_{y-}^{\beta} f(x, c)-\frac{\Gamma(\beta+1)}{2(d-y)^{\beta}} J_{y+}^{\beta} f(x, d) \\
& +\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(x-a)^{\alpha}(y-c)^{\beta}} J_{x-, y-}^{\alpha, \beta} f(a, c)+\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(x-a)^{\alpha}(d-y)^{\beta}} J_{x-, y+}^{\alpha, \beta} f(a, d) \\
& +\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(b-x)^{\alpha}(y-c)^{\beta}} J_{x+, y-}^{\alpha, \beta} f(b, c)+\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{4(b-x)^{\alpha}(d-y)^{\beta}} J_{x+, y+}^{\alpha, \beta} f(b, d) \Bigg| \\
\leq\ & \frac{(b-a)(d-c)}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t \partial s}(x, y)\right| \\
& +\frac{(b-a)}{(\alpha+2)(\beta+1)(\beta+2)}\left[(y-c)\left|\frac{\partial^2 f}{\partial t \partial s}(x, c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t \partial s}(x, d)\right|\right] \\
& +\frac{(d-c)}{(\alpha+1)(\alpha+2)(\beta+2)}\left[(x-a)\left|\frac{\partial^2 f}{\partial t \partial s}(a, y)\right|+(b-x)\left|\frac{\partial^2 f}{\partial t \partial s}(b, y)\right|\right] \\
& +\frac{(x-a)}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left[(y-c)\left|\frac{\partial^2 f}{\partial t \partial s}(a, c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t \partial s}(a, d)\right|\right] \\
& +\frac{(b-x)}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left[(y-c)\left|\frac{\partial^2 f}{\partial t \partial s}(b, c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t \partial s}(b, d)\right|\right].
\end{aligned} \tag{3.8}
$$

iii) we choose $H_1(t) = (1-t)^{\alpha}\,(\alpha > 0)$, $H_2(s) = (1-s)^{\beta}\,(\beta > 0)$ *on* $[0,1]$, *then the inequality (3.6) becomes the* *fractional integral inequality*

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. && (3.9)\\
&+\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}J_{a+}^{\alpha}\left[f(x,c)+f(x,d)\right]-\frac{\Gamma(\alpha+1)}{2(b-x)^{\alpha}}J_{b-}^{\alpha}\left[f(x,c)+f(x,d)\right]\\
&+\frac{\Gamma(\beta+1)}{2(y-c)^{\beta}}J_{c+}^{\beta}\left[f(a,y)+f(b,y)\right]-\frac{\Gamma(\beta+1)}{2(d-y)^{\beta}}J_{d-}^{\beta}\left[f(a,y)+f(b,y)\right]\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x-a)^{\alpha}(y-c)^{\beta}}J_{a+,c+}^{\alpha,\beta}f(x,y)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(x-a)^{\alpha}(d-y)^{\beta}}J_{a+,d-}^{\alpha,\beta}f(x,y)\\
&\left.+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-x)^{\alpha}(y-c)^{\beta}}J_{b-,c+}^{\alpha,\beta}f(x,y)+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-x)^{\alpha}(d-y)^{\beta}}J_{b-,d-}^{\alpha,\beta}f(x,y)\right|\\
\leq\ &\frac{(b-a)(d-c)}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|\\
&+\frac{(b-a)}{(\alpha+1)(\alpha+2)(\beta+2)}\left[(y-c)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|\right]\\
&+\frac{(d-c)}{(\alpha+2)(\beta+1)(\beta+2)}\left[(x-a)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|+(b-x)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|\right]\\
&+\frac{(x-a)}{(\alpha+2)(\beta+2)}\left[(y-c)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right]\\
&+\frac{(b-x)}{(\alpha+2)(\beta+2)}\left[(y-c)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].
\end{aligned}
$$

vi) we choose $H_1(t)=\ln(1+t)$, $H_2(s)=\ln(1+s)$ *on* $[0,1]$, *then the inequality (3.6) becomes the inequality*

$$
\begin{aligned}
&\left|\ln 2 f(x,y)-\frac{1}{2}\int_a^x\frac{f(u_1,y)}{x+u_1-2a}du_1-\frac{1}{2(b-x)}\int_x^b\frac{f(u_2,y)}{2b-x-u_2}du_2\right. && (3.10)\\
&-\frac{1}{2}\int_c^y\frac{f(x,v_1)}{y+v_1-2c}dv_1-\frac{1}{2(d-y)}\int_y^d\frac{f(x,v_2)}{2d-y-v_2}dv_2\\
&+\frac{1}{4\ln 2}\int_a^x\int_c^y\frac{f(u_1,v_1)}{(x+u_1-2a)(y+v_1-2c)}dv_1du_1\\
&+\frac{1}{4\ln 2}\int_a^x\int_y^d\frac{f(u_1,v_2)}{(x+u_1-2a)(2d-y-v_2)}dv_2du_1\\
&+\frac{1}{4\ln 2}\int_x^b\int_c^y\frac{f(u_2,v_1)}{(2b-x-u_2)(y+v_1-2c)}dv_1du_2\\
&\left.+\frac{1}{4\ln 2}\int_x^b\int_y^d\frac{f(u_2,v_2)}{(2b-x-u_2)(2d-y-v_2)}dv_2du_2\right|\\
\leq\ &\frac{(b-a)(d-c)}{16}\left|\frac{\partial^2 f}{\partial t\partial s}(x,y)\right|\\
&+(b-a)\frac{1}{4}\left(\ln 2-\frac{1}{4}\right)\left[(y-c)\left|\frac{\partial^2 f}{\partial t\partial s}(x,c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t\partial s}(x,d)\right|\right]\\
&+(d-c)\frac{1}{4}\left(\ln 2-\frac{1}{4}\right)\left[(x-a)\left|\frac{\partial^2 f}{\partial t\partial s}(a,y)\right|+(b-x)\left|\frac{\partial^2 f}{\partial t\partial s}(b,y)\right|\right]
\end{aligned}
$$

$$+(x-a)\left(\ln 2-\frac{1}{4}\right)^2\left[(d-y)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+(y-c)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right]$$
$$+(b-x)\left(\ln 2-\frac{1}{4}\right)^2\left[(y-c)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(d-y)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].$$

Remark 1. *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Corollary 2, then*
i) the inequality (3.7) reduce to the following inequality

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)}\int_a^b f\left(u,\frac{c+d}{2}\right)du-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},v\right)dv\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu\right|$$
$$=\frac{(b-a)(d-c)}{9}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$+\frac{(b-a)(d-c)}{36}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\right]$$
$$+\frac{(b-a)(d-c)}{144}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right],$$

ii) the inequality (3.8) reduce to the following inequality

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f\left(a,\frac{c+d}{2}\right)+J^\alpha_{\frac{a+b}{2}+}f\left(b,\frac{c+d}{2}\right)\right]\right.$$
$$-\frac{2^{\beta-1}\Gamma(\beta+1)}{(d-c)^\beta}\left[J^\beta_{\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right)+J^\beta_{\frac{c+d}{2}+}f\left(\frac{a+b}{2},d\right)\right]$$
$$+\frac{2^{(\alpha+\beta-2)}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}-}f(a,c)+J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}+}f(a,d)\right.$$
$$\left.\left.+J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}-}f(b,c)+J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}+}f(b,d)\right]\right|$$
$$\leq\frac{(b-a)(d-c)}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$+\frac{(b-a)(d-c)}{(\alpha+2)(\beta+1)(\beta+2)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|\right]$$
$$+\frac{(b-a)(d-c)}{(\alpha+1)(\alpha+2)(\beta+2)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\right]$$
$$+\frac{(b-a)(d-c)}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right],$$

iii) the inequality (3.9) reduce to the following inequality

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$

$$+\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J_{a+}^{\alpha}\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]-J_{b-}^{\alpha}\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]\right)$$
$$+\frac{2^{\beta-1}\Gamma(\beta+1)}{(d-c)^{\beta}}\left(J_{c+}^{\beta}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right]-J_{d-}^{\beta}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right]\right)$$
$$+\frac{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a+,c+}^{\alpha,\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{a+,d-}^{\alpha,\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.\left.+J_{b-,c+}^{\alpha,\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J_{b-,d-}^{\alpha,\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$+\frac{(b-a)(d-c)}{16(\alpha+1)(\alpha+2)(\beta+2)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|\right]$$
$$+\frac{((b-a)d-c)}{16(\alpha+2)(\beta+1)(\beta+2)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\right]$$
$$+\frac{(b-a)(d-c)}{4(\alpha+2)(\beta+2)}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right],$$

iv) the inequality (3.10) reduce to the following inequality

$$\left|\ln 2 f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{2}\int_a^{\frac{a+b}{2}}\frac{f\left(u,\frac{c+d}{2}\right)}{\frac{b-3a}{2}+u}du-\frac{1}{(b-a)}\int_{\frac{a+b}{2}}^{b}\frac{f\left(u,\frac{c+d}{2}\right)}{\frac{3b-a}{2}-u}du\right.$$
$$-\frac{1}{2}\int_c^{\frac{c+d}{2}}\frac{f\left(\frac{a+b}{2},v\right)}{\frac{d-3c}{2}+v}dv-\frac{1}{(d-c)}\int_{\frac{c+d}{2}}^{d}\frac{f\left(\frac{a+b}{2},v\right)}{\frac{3d-c}{2}-v}dv$$
$$+\frac{1}{4\ln 2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\frac{f(u,v)}{\left(\frac{b-3a}{2}+u\right)\left(\frac{d-3c}{2}+v\right)}dvdu$$
$$+\frac{1}{4\ln 2}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\frac{f(u,v)}{\left(\frac{b-3a}{2}+u\right)\left(\frac{3d-c}{2}-v\right)}dvdu$$
$$+\frac{1}{4\ln 2}\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}\frac{f(u,v)}{\left(\frac{3b-a}{2}-u\right)\left(\frac{d-3c}{2}+v\right)}dvdu$$
$$\left.+\frac{1}{4\ln 2}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\frac{f(u,v)}{\left(\frac{3b-a}{2}-u\right)\left(\frac{3d-c}{2}-v\right)}dvdu\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$+(b-a)(d-c)\frac{1}{16}\left(\ln 2-\frac{1}{4}\right)\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\right]$$

$$+ (b-a)(d-c)\frac{1}{4}\left(\ln 2-\frac{1}{4}\right)^{2}\left[\left|\frac{\partial^{2} f}{\partial t\partial s}(a,d)\right| + (d-c)\left|\frac{\partial^{2} f}{\partial t\partial s}(a,c)\right| + \left|\frac{\partial^{2} f}{\partial t\partial s}(b,c)\right| + \left|\frac{\partial^{2} f}{\partial t\partial s}(b,d)\right|\right].$$

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