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The strong Dunford-Pettis relatively compact property of order p

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Abstract. We introduce and study Banach lattices with the strong Dunford-Pettis relatively compact property of order p $(1 \le p < \infty)$; that is, spaces in which every weakly p-compact and almost Dunford-Pettis set is relatively compact. We also introduce the notion of the weak Dunford-Pettis property of order p and then characterize this property in terms of sequences. In particular, in terms of disjoint weakly compact operators into c_0, an operator characterization of those Banach lattices with the weak Dunford-Pettis property of order p is given. Moreover, some results about Banach lattices with the positive Dunford-Pettis relatively compact property of order p are presented.

1. Introduction and preliminaries

Throughout this paper X will denote a Banach space, E will denote a Banach lattice, $E^+ = \{x \in E : x \ge 0\}$ refers to the positive cone of E and $Sol(A)$ (the solid hull of A) is the set $Sol(A) = \{y \in E : |y| \le |x|, \text{ for some } x \in A\}$. If A is a subset of X and for each weak* null (weakly null) sequence (x_n^*) in X^*,

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0,$$

then we say that A is limited (Dunford-Pettis). Banach spaces whose limited (Dunford-Pettis) sets are relatively compact are called Gelfand-Phillips (Dunford-Pettis relatively compact property or $\mathrm{DP}_{rc}\mathrm{P}$) spaces. A dual Banach space with the weak Radon-Nikodym property has the $\mathrm{DP}_{rc}\mathrm{P}$. A Banach space X has the Dunford-Pettis property if each weakly compact operator on X is Dunford-Pettis (i.e. it carries weakly null sequences to norm null ones) [8, 10, 11]

A bounded subset A of E is said to be an almost limited (almost Dunford-Pettis) set, if every disjoint weak* null (disjoint weakly null) sequence (x_n^*) in E^* converges uniformly to zero on A [4, 7]. According to the definition of these sets, the stronger version of the $\mathrm{DP}_{rc}\mathrm{P}$ is considered and introduced the class of Banach lattices with the strong $\mathrm{DP}_{rc}\mathrm{P}$ which is shared by those Banach lattices whose almost Dunford-Pettis subsets are relatively compact. In fact discrete KB-spaces are exactly the same as the Banach lattices with the strong $\mathrm{DP}_{rc}\mathrm{P}$ [2].

A Banach lattice E has the Schur (positive Schur) property if each weakly null (positive weakly null)

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sequence in E is norm null [19, 21]. Also E has the weak Dunford-Pettis property if each weakly compact operator from E into each Banach space is an almost Dunford-Pettis operator (i.e. it carries disjoint weakly null sequences to norm null ones) [20].

We remember some definitions and terminologies. A sequence (x_n) of X is called weakly p-summable, if for each $x^* \in X^*$, $(x^*(x_n)) \in \ell_p$. Also $(x_n) \subset X$ is called weakly p-convergent to $x \in X$ if $(x_n - x) \in \ell_p^w(X)$, where $\ell_p^w(X)$ is the space of weakly p-summable sequences of X and the weakly ∞-convergent sequences are the weakly convergent sequences. A bounded subset $A \subset X$ is called relatively weakly p-compact, if each sequence in A has a weakly p-convergent subsequence. If B_X is a weakly p-compact set, then X is called weakly p-compact [6].

Castillo mentioned the p-convergent operators for the first time. In fact, p-convergent operators are precisely those operators which transformed weakly p-compact subsets into relatively compact subsets. Equivalently, an operator $T : E \to X$ is p-convergent if for every sequence $(x_n) \in \ell_p^w(E)$, $\|Tx_n\| \to 0$. Later disjoint p-convergent were introduced. An operator T is called disjoint p-convergent if for every disjoint weakly p-summable sequence (x_n), $\|Tx_n\| \to 0$. A Banach space X has the Dunford-Pettis property of order p (p-Dunford-Pettis property) if each weakly compact operator T from X into each Banach space Y is p-convergent [6, 23].

The notion of the Schur property of order p (i.e. p-Schur property) as a generalization of the Schur property is introduced and then some examples are presented. Following Zeekoei, X the p-Schur property if every weakly p-summable sequence in X is norm null. A Banach lattice E has the p-positive Schur property if every $(x_n) \in \ell_p^w(E)_+$ is norm null [23].

A Banach space X has the p-DP_{rc}P if each weakly p-compact and Dunford-Pettis subset of X is relatively compact; or equivalently, every Dunford-Pettis sequence $(x_n) \in \ell_p^w(X)$ is norm null [14].

In the first part of this paper, we introduce the concept of the strong p-DP_{rc}P for Banach lattices and consider the connection between this concept with other well-known properties. In particular, we provide Banach lattices in which two properties p-Schur and p-positive Schur coincide. Also, the p-weak Dunford-Pettis property is introduced and using disjoint p-convergent operators some characterizations are obtained. As an application, we provide the conditions under which two properties p-weak Dunford-Pettis and p-Dunford-Pettis will be the same. Finally, Banach lattices with the positive p-DP_{rc}P are studied and some of the results derived from this property are given. In all cases we assume that $1 \leq p < \infty$, unless otherwise stated. We refer the reader to references [1, 17] for the theory of operators and Banach lattices.

2. Strong p-DP_{rc}P and p-weak DP property

Recently, Banach spaces with the p-DP_{rc}P are studied and some results are obtained. Motivated by the notion of the p-DP_{rc}P, we define the so-called strong p-DP_{rc}P. We are then able to describe a relationship between strong p-DP_{rc}P and p-DP_{rc}P.

Definition 2.1. *A Banach lattice E has the strong p-$\mathrm{DP}_{rc}P$ if each almost Dunford-Pettis weakly p-compact subset of E is relatively compact.*

It is easily seen that E has the strong p-DP_{rc}P if and only if each almost Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)$ is norm null. If $1 \leq p < q$ and E has the strong q-DP_{rc}P, then E has the strong p-DP_{rc}P.

Theorem 2.2. *Each Banach lattice with the strong p-$\mathrm{DP}_{rc}P$ is a KB-space, but the converse is false.*

Proof. If a Banach lattice has the strong p-DP_{rc}P, then it contains no copy of c_0 (since c_0 does not have the strong p-DP_{rc}P) and so it is a KB-space. For the converse, the space $L^1[0,1]$ is an example of a KB-space without the (strong) p-DP_{rc}P. In fact, Rademacher sequences (r_n) in are weakly 2-summable (see [18, Proposition 3.6]) and Dunford-Pettis in $L^1[0,1]$ (by the Dunford-Pettis property), but $\|r_n\| = 1$ for all n. Thus, $L^1[0,1]$ does not have the (strong) p-DP_{rc}P for all $2 \leq p \leq \infty$. □

Note that, the space $L^1[0,1]$ is a non-discrete Banach lattice with the 1-Schur property. We need to consider some preliminary results for Banach spaces with the 1-Schur property. If X is weakly sequentially

complete, then it has no copy of c_0 (and so it has the the 1-Schur property). The converse holds, if X has an unconditional basis [16, Theorem 1.c.13].
In fact, James space J is a Banach space without any copy of c_0; that is J has the 1-Schur property, while it is not weakly sequentially complete. But the situation is different in Banach lattices. A Banach lattice E does not contain a copy of c_0 if and only if E has the 1-Schur property. The converse of Theorem 2.2 holds when E is discrete.

Proposition 2.3. *If E is a discrete Banach lattice, then the following are equivalent:*

(a) E has the strong $DP_{rc}P$*,*

(b) E has the strong p*-*$DP_{rc}P$*,*

(c) E has the p*-*$DP_{rc}P$*,*

(d) E has the $DP_{rc}P$*.*

Proof. Only $(d) \Rightarrow (a)$ needs a proof. To this end, it is enough to note that each Banach lattice with the $DP_{rc}P$ is a KB-space and each discrete KB-space has the strong $DP_{rc}P$ [2, Theorem 2.3]. □

Each Banach lattice with the strong p-$DP_{rc}P$ has the p-$DP_{rc}P$, but the converse is false.

Example 2.4. *The Lorentz sequence space* $d_{\omega,1}$ *has the* $DP_{rc}P$ *and so it has the* p*-*$DP_{rc}P$*. But* $d_{\omega,1}$ *cannot have the strong* p*-*$DP_{rc}P$*, for* $1 < p < \infty$*. In fact,* $d_{\omega,1}$ *is a Banach lattice with the weak Dunford-Pettis property and without the p-Schur property, for* $1 < p < \infty$ *([20, Example 1, p. 231]).*

The following properties are easily verified:

Proposition 2.5. *(a) If a Banach space X contains a complemented copy of* ℓ_1*, then* X^* *cannot have the* p*-*$DP_{rc}P$*.*

(b) If a Banach lattice E contains a sublattice isomorphic to ℓ_1*, then* E^* *cannot have the* p*-*$DP_{rc}P$*.*

Proof. (a). If X has a complemented copy of ℓ_1, then by [8, Theorem 10 in Chapter V] X^* has an isomorphic copy of ℓ_∞ and so it cannot have the p-$DP_{rc}P$.
(b). If E contains a sublattice isomorphic to ℓ_1, then by [17, Proposition 2.3.12] E^* contains a sublattice isomorphic to ℓ_∞ and so it cannot have the strong p-$DP_{rc}P$. □

We give some sufficient conditions under which two properties strong ∞-$DP_{rc}P$ and strong $DP_{rc}P$ are the same. At first recall that the concept of (L)-weak* sequentially continuous lattice operations. A sequence (f_n) in E^* is called an L-sequence if for each weakly null sequence (x_n) in E, $f_n(x_n) \to 0$. Also, E^* has (L)-weak* sequentially continuous lattice operations if for every L-sequence (f_n) of E^* satisfying $f_n \xrightarrow{w^*} 0$, $|f_n| \xrightarrow{w^*} 0$ and in this case each disjoint weakly compact set (that is, set whose disjoint sequences in the solid hull is weakly null) is weakly conditionally compact. Note that almost Dunford-Pettis sets are disjoint weakly compact [22].

Theorem 2.6. *If* E^* *has* (L)*-weak** *sequentially continuous lattice operations, then strong* ∞*-*$DP_{rc}P$ *and the strong* $DP_{rc}P$ *are equivalent.*

Proof. Clearly strong $DP_{rc}P$ implies that the strong ∞-$DP_{rc}P$. For the converse suppose that E has the strong ∞-$DP_{rc}P$. Then each almost Dunford-Pettis weakly null sequence in E is norm null. Let $A \subset E$ be an almost DP set. Since E^* has (L)-weak* sequentially continuous lattice operations, then A is weakly conditionally compact. On the other hand, the difference set $A - A$ is almost DP. Thus we can prove that A is relatively compact and so E has the strong $DP_{rc}P$. □

Remark 2.7. *Continuous linear image of an almost Dunford-Pettis set under a positive linear projection is almost Dunford-Pettis. Also, if Y is a complemented subspace of X and* $A \subseteq Y$ *is a Dunford-Pettis set in X, then A is a Dunford-Pettis set in Y.*

Note that a Banach space X has the Schur (resp. p-Schur) property if and only if each closed linear separable subspace of X has the Schur (resp. p-Schur) property. From the above remark we observe the following interesting result:

Theorem 2.8. *For each $1 \leq p \leq \infty$, the following are equivalent:*

(a) E has the strong p-$DP_{rc}P$,

(b) every closed separable sublattice of E is contained in a complemented sublattice Z of E with the strong p-$DP_{rc}P$,

(c) E is the direct sum of two spaces with the strong p-$DP_{rc}P$.

Proof. $(a) \Rightarrow (b)$. It is obvious. In fact, strong p-$\mathrm{DP}_{rc}\mathrm{P}$ is inherited by closed sublattices.
$(b) \Rightarrow (a)$. Suppose that A is a weakly p-compact almost Dunford-Pettis subset of E and that $(x_n) \subseteq A$. Then there is a subsequence (x_{n_k}) of (x_n) that is almost Dunford-Pettis weakly p-convergent to some $x \in A$. Consider the closed linear span of (x_n). By hypothesis, it is contained in a complemented sublattice Z of E with the strong p-$\mathrm{DP}_{rc}\mathrm{P}$. So, (x_n) is almost Dunford-Pettis in Z. Since Z has the strong p-$\mathrm{DP}_{rc}\mathrm{P}$, so (x_{n_k}) is norm convergent to x.
$(a) \Rightarrow (c)$. Consider $E = E \oplus \{0\}$.
$(c) \Rightarrow (a)$. Let $E = Y \oplus Z$ such that Y and Z have the strong p-$\mathrm{DP}_{rc}\mathrm{P}$. Consider the positive linear projections $P_Y : E \to Y$ and $P_Z : E \to Z$. Assume that A is a weakly p-compact almost Dunford-Pettis subset of E. Then $P_Y(A)$ and $P_Z(A)$ are weakly p-compact and they are almost Dunford-Pettis. For each $(x_n) \subseteq A$ there is $y_n \in P_Y(A)$ and $z_n \in P_Z(A)$ such that $x_n = y_n + z_n$. Since $P_Y(A)$ and $P_Z(A)$ have the strong p-$\mathrm{DP}_{rc}\mathrm{P}$, so the sequences (y_n) and (z_n) have convergent subsequences (y_{n_k}) and (z_{n_k}); that is, there are $y \in P_Y(A)$ and $z \in P_Z(A)$ such that $y_{n_k} \to y$ and $z_{n_k} \to z$. So $x_{n_k} \to y + z \in A$, since A is weakly p-compact. Hence A is compact and so E has the strong p-$\mathrm{DP}_{rc}\mathrm{P}$. □

Now, we introduce and study the notion of p-weak Dunford-Pettis property which plays an important role in the present paper and provide some characterizations of Banach lattices with this property.

Definition 2.9. *A Banach lattice E has the p-weak Dunford-Pettis property if each weakly compact operator T from E into each Banach space Y is disjoint p-convergent.*

Clearly, the weak Dunford-Pettis property implies the p-weak Dunford-Pettis property, but the converse is false. In fact, every discrete Banach lattice with the p-Schur and without the Schur property has the p-weak Dunford-Pettis property, but it does not have the weak Dunford-Pettis property. It is important to note that E has the Schur property if and only if E has the weak Dunford-Pettis property and strong $\mathrm{DP}_{rc}\mathrm{P}$. In the following proposition a similar characterization of the p-Schur property is discussed:

Proposition 2.10. *For each Banach lattice E, the following assertions are equivalent:*

(a) E has p-Dunford-Pettis property and p-$DP_{rc}P$,

(b) E has p-weak Dunford-Pettis property and strong p-$DP_{rc}P$,

(c) E has the p-Schur property.

We have the following example of a space with the p-weak Dunford-Pettis property and without the weak Dunford-Pettis property.

Example 2.11. *All the spaces ℓ_q have the p-Schur and so p-weak Dunford-Pettis property, for $1 < p < \infty$ and $1 < q < p'$. On the other hand, these spaces have the strong $DP_{rc}P$ property and so non of them can have the weak Dunford-Pettis property.*

Each Banach lattice with the p-Dunford-Pettis property has the p-weak Dunford-Pettis property, but the converse is false.

Example 2.12. *Lorentz sequence space $d_{\omega,1}$ has the weak Dunford-Pettis (and so p-weak Dunford-Pettis) property. But it does not have p-Dunford-Pettis property. In fact, $d_{\omega,1}$ is a Banach lattice with the $DP_{rc}P$ and without the p-Schur property, for $1 < p < \infty$.*

Using Proposition 1 of [20], we obtain a similar characterization for the p-weak Dunford-Pettis property in a Banach lattice in terms of disjoint weakly p-summable sequences.

Proposition 2.13. *If E is a Banach lattice. Then the following are equivalent:*

(a) E has the p-weak Dunford-Pettis property,

(b) every weakly compact operator $T : E \to c_0$ is disjoint p-convergent,

(c) for every disjoint sequence $(x_n) \in \ell_p^w(E)_+$ and every weakly null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$.

The following proposition characterizes some Banach lattices with the p-weak Dunford-Pettis property. In Chapter 16 of [9] one finds a discussion of type and cotype in Banach lattices which is of particular importance in the present paper. The fact that every disjoint sequence in the solid hull of a weakly p-compact set of a Banach lattice E with non-trivial type is weakly p-summable, plays an important role in the proofs of many results concerning p-weak Dunford-Pettis property.

Proposition 2.14. *For a Banach lattice E with the type q (with $1 < q \leq 2$) and $p \geq q'$, the following are equivalent:*

(a) E has the p-weak Dunford-Pettis property,

(b) for every disjoint sequence $(x_n) \in \ell_p^w(E)$ and every disjoint weakly null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$,

(c) the solid hull of a weakly p-compact set is almost Dunford-Pettis,

(d) each weakly p-compact set is almost Dunford-Pettis,

(e) for every sequence $(x_n) \in \ell_p^w(E)$ and every disjoint weakly null sequence (x_n^*) in E_+^*, $x_n^*(x_n) \to 0$.

Proof. The reader should note that our arguments are similar to arguments in the proof of Theorem 2.7 of [4]. Only $(b) \Rightarrow (c)$ needs to be proved.
Let W be a weakly p-compact set in E. By [23, Lemma 4.2.1] each disjoint sequence in $B := Sol(W)$ is weakly p-summable. So by hypothesis for every disjoint weakly null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$. Hence B is almost Dunford-Pettis. □

The following example shows that the p-weak Dunford-Pettis property is not preserved under linear homeomorphisms, unlike the p-Dunford-Pettis property. Indeed, each complemented sublattice (closed ideal) of a Banach lattice with the weak Dunford-Pettis property have this property too [20, Proposition 3] and the same will be true for the p-weak Dunford-Pettis property.

Example 2.15. *The Orlicz space $L^{\phi^*}(0,1)$, where $\phi(r) = (e-1)^{-1}(e^{r^2} - 1)$ has the weak Dunford-Pettis property (and so it has the p-weak Dunford-Pettis property) and also it has a complemented copy of ℓ_2 spanned by the Rademacher functions. Hence $L^{\phi^*}(0,1)$ is isomorphic to $L^{\phi^*}(0,1) \oplus \ell_2$ [21] . Since ℓ_2 does not have the p-weak Dunford-Pettis property for all $p \geq 2$, then $L^{\phi^*}(0,1) \oplus \ell_2$ does not have the p-weak Dunford-Pettis property for all $p \geq 2$.*

We now formulate a characterization of the p-Schur property.

Theorem 2.16. *For a Banach lattice E with the type q (with $1 < q \leq 2$) and $p \geq q'$, the following are equivalent:*

(a) E has the p-Schur property,

(b) E has the strong p-$DP_{rc}P$ and p-positive Schur property.

Proof. $(a) \Rightarrow (b)$. It is obvious.
$(b) \Rightarrow (a)$. Let A be a weakly p-compact set in E. Since E has the p-positive Schur property, by [23, Proposition 3.1.21] each operator on E is disjoint p-convergent. Then, E has the p-weak Dunford-Pettis property and Proposition 2.14 implies that A is almost Dunford-Pettis and so relatively compact, by the strong p-$DP_{rc}P$ of E. Thus E has the p-Schur property. □

3. p-weak Dunford-Pettis property and some classes of operators

In this section we provide some results on disjoint weakly compact operators and their applications in the study of geometrical properties of Banach lattices. In terms of disjoint weakly compact operators into c_0, we give an operator characterization of those Banach lattices with the p-weak Dunford-Pettis property. Every disjoint weakly compact operator $T : E \to c_0$ can be uniquely determined by a disjoint weakly null sequence $(x_n^*) \subset E^*$ such that $Tx = (\langle x, x_n^* \rangle)$, for all $x \in E$. Similar [13, Proposition 2.3], we can provide a characterization of almost Dunford-Pettis sets:

Proposition 3.1. *A subset $A \subset E$ is almost Dunford-Pettis if and only if for each disjoint weakly compact operator $T : E \to c_0$, $T(A)$ is relatively compact.*

J. X. Chen et al. [7] characterized the weak DP* property (i.e. each relatively weakly compact set is almost limited) of a Banach lattice: a σ-Dedekind complete Banach lattice E has the weak DP* property if and only if each continuous operator $T : E \to c_0$ is almost Dunford-Pettis. Comparing this with Proposition 2.14 in the present paper we naturally posed the following theorem.

Theorem 3.2. *Let E be a Banach lattice with the type q (with $1 < q \leq 2$) and let $p \geq q'$, then the following are equivalent:*

- *(a) E has the p-weak Dunford-Pettis property,*
- *(b) every disjoint weakly compact operator $T : E \to c_0$ is p-convergent,*
- *(c) every disjoint weakly compact operator $T : E \to c_0$ is disjoint p-convergent.*

Proof. $(a) \Rightarrow (b)$. Let A be a weakly p-compact set in E. Then by the p-weak Dunford-Pettis property of E, A is almost Dunford-Pettis. From Proposition 3.1 for every disjoint weakly compact operator $T : E \to c_0$, $T(A)$ is a relatively compact set in c_0; that is, T is p-convergent.
$(b) \Rightarrow (c)$. It is clear.
$(c) \Rightarrow (a)$. We have to show that for each disjoint sequence $(x_n) \in \ell_p^w(E)$ and each disjoint weakly null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$. Consider the disjoint weakly compact operator $T : E \to c_0$ defined by $Tx = (\langle x, x_n^* \rangle)$, for all $x \in E$. According to (c), T is a disjoint p-convergent operator. Therefore, $\|Tx_n\| \to 0$, and hence $x_n^*(x_n) \to 0$, as desired. □

In order to study the connection between p-weak Dunford-Pettis property of E and E^*, we need the following lemma. In fact, the relationship between disjoint p-convergent and Dunford-Pettis operators is described.

Lemma 3.3. *Let E be a Banach lattice with the type q (with $1 < q \leq 2$) and $p \geq q'$ and F be discrete with order continuous norm. Then every positive operator $T : E \to F$ is disjoint p-convergent if and only if T is Dunford-Pettis.*

Proof. Let W be a relatively weakly compact subset of E and $T : E \to F$ be a positive disjoint p-convergent operator. It is enough to show that $T(W)$ is relatively compact. From [23, Lemma 4.2.1] every disjoint sequence (x_n) in $A := Sol(W)$ is weakly p-summable and so $\|Tx_n\| \to 0$. By a consequence of two theorems 13.3 and 13.5 of [1], $T(A)$ is an almost order bounded set in F. Since F is discrete with order continuous norm, then $T(A)$ is relatively compact. Hence, T is a Dunford-Pettis operator. □

Theorem 3.4. *Let E be a Banach lattice with the type q (with $1 < q \leq 2$) and $p \geq q'$. If E^* has the p-weak Dunford-Pettis property, then E has the p-weak Dunford-Pettis property.*

Proof. Let $(x_n) \in \ell_p^w(E)_+$ be a disjoint sequence and (x_n^*) be a weakly null sequence in E^*. Consider the disjoint positive weakly compact operator $T : E^* \to c_0$ defined by $Tf = (\langle f, x_n \rangle)$, for all $f \in E^*$. According to the p-weak Dunford-Pettis property of E^*, T is a disjoint p-convergent operator. Therefore by Lemma 3.3, T is a Dunford-Pettis operator. Hence $\|Tx_n^*\| \to 0$ and so $x_n^*(x_n) \to 0$. □

From [20, Theorem 4], E^* has the positive Schur property if and only if E has the weak Dunford-Pettis property and E contains no complemented copy of ℓ_1. For the p-positive Schur and p-weak Dunford-Pettis properties, this theorem is also expressed. At first we formulate the following lemma which describe the coincidences of disjoint p-convergent and compact operators.

Lemma 3.5. *Let E be a Banach lattice with the type q (with $1 < q \leq 2$) and $p \geq q'$ such that E^* be a KB-space and F be a discrete Banach lattice with order continuous norm. Then each positive operator $T : E \to F$ is disjoint p-convergent if and only if it is compact.*

Proof. Suppose that $T : E \to F$ is a positive disjoint p-convergent operator. Since F is discrete with order continuous norm, then by Lemma 3.3 T is a Dunford-Pettis operator. On the other hand E^* is a KB-space and so by [17, Theorem 2.4.14] each disjoint sequence (x_n) in B_E is weakly null. Then by [1, Theorem 13.5] an operator T carries unit ball of E into almost order bounded subsets of F which are relatively compact. Hence T is compact. □

Using Lemma 3.5, we obtain a characterization for the p-positive Schur property as follows:

Theorem 3.6. *Let E be a Banach lattice with the type q (with $1 < q \leq 2$) and $p \geq q'$. Then these are equivalent:*

(a) E^* has the p-positive Schur property.

(b) E has the p-weak Dunford-Pettis property and E^* is a KB-space.

Proof. $(a) \Rightarrow (b)$. If E^* has the p-positive Schur property, then the identity operator on E^* is disjoint p-convergent and so E^* has the p-weak Dunford-Pettis property. Then by Theorem 3.4, E has the p-weak Dunford-Pettis property. Also E^* has order continuous norm which is equivalent E^* is a KB-space.
$(b) \Rightarrow (a)$. We have to show that every sequence $(x_n^*) \in \ell_p^w(E^*)_+$, $\|x_n^*\| \to 0$. Consider the weakly compact positive operator $T : E \to c_0$ defined by $Tx = (\langle x, x_n^* \rangle)$, for all $x \in E$. According to Proposition 2.13, T is disjoint p-convergent. By Lemma 3.5, T is a compact operator and so $\|x_n^*\| \to 0$, which implies that E^* has the p-positive Schur property. □

The following condition on the underlying Banach lattices ensures that disjoint p-convergent operators are compact.

Theorem 3.7. *Let E be an AM-space with unit and F be a discrete Banach lattice with order continuous norm. Then each operator $T : E \to F$ is disjoint p-convergent if and only if it is compact.*

Proof. Since the closed unit ball B_E is order bounded, then each disjoint sequence (x_n) in B_E is order bounded and so it is weakly p-summable. Moreover $T : E \to F$ is disjoint p-convergent and then $\|T(x_n)\| \to 0$. Hence by [1, Theorem 13.5], $T(B_E)$ is an almost order bounded in F. Since F is discrete with order continuous norm, then $T(B_E)$ is relatively compact and so T is a compact operator. □

The following theorem leads to an improvement of proposition 1 of [20]. Recall that E has the weak Dunford-Pettis property if and only if every weakly compact operator $T : E \to c_0$ is almost Dunford-Pettis. In the following theorem, we show that the same result can be obtained with disjoint weakly compact operators.

Theorem 3.8. *If E is a Banach lattice, then the following are equivalent:*

(a) E has the weak Dunford-Pettis property,

(b) every disjoint weakly compact operator $T : E \to c_0$ is Dunford-Pettis,

(c) every disjoint weakly compact operator $T : E \to c_0$ is almost Dunford-Pettis.

Proof. $(a) \Rightarrow (b)$. Let A be a relatively weakly compact set in E. Then by the weak Dunford-Pettis property of E, A is an almost DP set [4, Theorem 2.7]. From Proposition 3.1, for every disjoint weakly compact operator $T : E \to c_0$, $T(A)$ is a relatively compact set in c_0 and so T is Dunford-Pettis.
$(b) \Rightarrow (c)$. It is clear.
$(c) \Rightarrow (a)$. From [5, Corollary 2.6], assume by way of contradiction that there is a weakly null sequence (x_n) in E^+ and a disjoint weakly null sequence (x_n^*) in E^* such that $|\langle x_n^*, x_n\rangle| > \epsilon$, for all n and some $\epsilon > 0$. Consider the operator $T : E \to c_0$ defined by

$$Tx = (\langle x, x_n^*\rangle) \quad , \quad x \in E.$$

It is clear that T is a disjoint weakly compact operator. But T is not almost Dunford-Pettis, since (x_n) is a positive weakly null sequence and $\|Tx_n\| \geq x_n^*(x_n) \geq \epsilon$, for all n [3, Theorem 2.2]. □

An operator $T : E \to X$ is order weakly compact, if $T[-x, x]$ is relatively weakly compact, for all $x \in E^+$. Alternatively, T is order weakly compact if and only if $\|T(x_n)\| \to 0$ for each order bounded disjoint sequence in E [1, Theorem 18.6].

Theorem 3.9. *Each disjoint p-convergent operator $T : E \to X$ is order weakly compact.*

Proof. Let (x_n) be an order bounded disjoint sequence in a Banach lattice E. Then (x_n) is weakly null (see [1, p. 186]). Actually, it has been shown that, this sequence is weakly 1-summable. So by the known fact $\ell_p^w(X) \subset \ell_q^w(X)$ for all $1 \leq p \leq q$ and $\frac{1}{p} + \frac{1}{q} = 1$; the sequence (x_n) is weakly p-summable. Since $T : E \to X$ is a disjoint p-convergent operator, then $\|T(x_n)\| \to 0$ and so T is order weakly compact. □

The converse of Theorem 3.9 is false. In fact, identity operator on c_0 is order weakly compact, but it is not disjoint p-convergent.

A Banach lattice E is weak p-consistent if for each sequence $(x_n) \in \ell_p^w(E)$, we have $(|x_n|) \in \ell_p^w(E)$. In each weak p-consistent Banach lattice, two properties p-positive Schur and p-Schur are the same [23, Proposition 3.3.6]. From [23, Lemma 4.2.1] and with the same techniques used to prove [3, Theorem 2.2], we formulate the following lemma:

Lemma 3.10. *Let E be a weak p-consistent Banach lattice with the type q (with $1 < q \leq 2$), $p \geq q'$ and T be an operator from E into a Banach space Y. Then the following are equivalent:*

(a) T is a disjoint p-convergent operator,

(b) T is order weakly compact and each weakly p-compact set in E is approximately order bounded with respect to the lattice semi-norm q_T,

(c) $\|Tx_n\| \to 0$ for each sequence $(x_n) \in \ell_p^w(E)_+$,

(d) $\|Tx_n\| \to 0$ for each disjoint sequence $(x_n) \in \ell_p^w(E)_+$.

Theorem 3.11. *Let E be a weak p-consistent Banach lattice with the type q (with $1 < q \leq 2$), $p \geq q'$ and T be an operator from E into a Banach space Y. Then T is disjoint p-convergent if and only if it is a p-convergent operator.*

Proof. Let $T : E \to Y$ be a disjoint p-convergent operator. We show that T is p-convergent; that is, $\|Tx_n\| \to 0$ for each sequence $(x_n) \in \ell_p^w(E)$. Since E is a weak p-consistent Banach lattice, then $(x_n^+) \in \ell_p^w(E)$ and $(x_n^-) \in \ell_p^w(E)$. On the other hand, T is disjoint p-convergent and so by Lemma 3.10, $\|Tx_n\| \leq \|Tx_n^+\| + \|Tx_n^-\| \to 0$. Hence T is p-convergent. □

Corollary 3.12. *Let E be a weak p-consistent Banach lattice with the type q (with $1 < q \leq 2$) and $p \geq q'$. Then E has the p-weak Dunford-Pettis property if and only if E has the p-Dunford-Pettis property.*

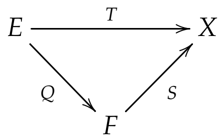
Proof. Assume that E has the p-weak Dunford-Pettis property and T is a weakly compact operator from E into a Banach space Y, then T is disjoint p-convergent. By Theorem 3.11, T is p-convergent. Hence E has the p-Dunford-Pettis property. □

It is clear that each Dunford-Pettis operator is disjoint p-convergent, but the converse is false. In fact, each operator on an AL- space (for instance, $Id_{L^1[0,1]}$) is disjoint p-convergent while it is not necessarily Dunford-Pettis. But on an AM- space with unit, these two operators are the same.

Corollary 3.13. *For each compact space K, each operator T on $C(K)$ is Dunford-Pettis if and only if it is disjoint p-convergent.*

Proof. Each disjoint p-convergent operator T is order weakly compact. Since the closed unit ball of $C(K)$ is order bounded, then T is weakly compact. Also $C(K)$ has the Dunford-Pettis property and so each weakly compact operator on $C(K)$ is Dunford-Pettis. □

Theorem 3.14. *Suppose that $T : E \to X$ admits a factorization through a Banach lattice F with the p-positive Schur property*



such that $Q : E \to F$ is a lattice homomorphism. Then T is disjoint p-convergent.

Proof. Let (x_n) be a disjoint weakly p-summable sequence in E. Since $Q : E \to F$ is a lattice homomorphism, then $(Q(x_n))$ is a disjoint weakly p-summable sequence in F and so it is norm null, by the p-positive Schur property of F. Hence $\|T(x_n)\| = \|S(Q(x_n))\| \to 0$, which implies that T is disjoint p-convergent. □

4. Positive p-DP$_{rc}$P

This section focuses on the so-called positive p-DP$_{rc}$P for Banach lattices as well as several results concerning some well-known properties of order p. The concept of positive p-DP$_{rc}$P is defined as follows:

Definition 4.1. *A Banach lattice E has the positive p-DP$_{rc}$P if each Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)_+$ is norm null.*

The positive p-DP$_{rc}$P is characterized in terms of the sequences as follows:

Theorem 4.2. *For a Banach lattice E, these are equivalent:*

(a) E has the positive p-DP$_{rc}$P,

(b) each disjoint Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)$ is norm null.

Proof. $(a) \Rightarrow (b)$. Let $(x_n) \in \ell_p^w(E)$ be a disjoint Dunford-Pettis sequence. Then from [12, 23], the sequence $(|x_n|)$ is weakly p-summable positive and Dunford-Pettis. By hypothesis (a), $\|x_n\| = \||x_n|\| \to 0$.
$(b) \Rightarrow (a)$. Suppose that $(x_n) \in \ell_p^w(E)_+$ is a Dunford-Pettis sequence, but $\|x_n\| \nrightarrow 0$. Then (x_n) is a weakly null positive sequence in E which is not norm null. By [15, Corollary 5], there is a disjoint positive subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| \nrightarrow 0$ which is a contradiction. Note that (x_{n_k}) is a disjoint Dunford-Pettis weakly p-summable sequence and by hypothesis it must be norm null. □

As a consequence of Theorem 2.13, we obtain the following characterization:

Corollary 4.3. *If E is a Banach lattice. Then the following are equivalent:*

(a) E has the p-positive Schur property,

(b) E has the p-weak Dunford-Pettis property and positive p-DP$_{rc}$P.

Proof. $(a) \Rightarrow (b)$. It is clear.
$(b) \Rightarrow (a)$. Let $(x_n) \in \ell_p^w(E)_+$ be a disjoint sequence. We have to show that $\|x_n\| \to 0$. Since E has the p-weak Dunford-Pettis property, then by Theorem 2.13 the sequence (x_n) is Dunford-Pettis and so by the positive p-DP$_{rc}$P, $\|x_n\| \to 0$. □

If E has the strong p-DP$_{rc}$P, then E has the positive p-DP$_{rc}$P (Theorem 4.2). The converse is false, in general. In fact, $L^1[0,1]$ has the positive Schur property and so by Corollary 4.3, it has the positive p-DP$_{rc}$P. By Theorem 2.2 it is clear that $L^1[0,1]$ does not have the strong p-DP$_{rc}$P for $2 \leq p \leq \infty$.

Proposition 4.4. *Each Banach lattice E with the positive p-DP$_{rc}$P is a KB-space.*

Proof. Note that c_0 does not have the positive p-DP$_{rc}$P. In fact, c_0 is a Banach lattice with the p-weak Dunford-Pettis property and without the p-positive Schur property. So if a Banach lattice E has the positive p-DP$_{rc}$P, then E does not contain a copy of c_0 and so E is a KB-space. □

Since discrete KB-spaces have the strong DP$_{rc}$P, then we conclude in each discrete Banach lattice positive p-DP$_{rc}$P and p-DP$_{rc}$P are the same. According to the result obtained in the previous section, the following result can be generally summarized.

Corollary 4.5. *If E is a discrete Banach lattice, then the following are equivalent:*

(a) E has the strong DP$_{rc}$P,

(b) E has the strong p-DP$_{rc}$P,

(c) E has the p-DP$_{rc}$P,

(d) E has the DP$_{rc}$P,

(d) E has the positive DP$_{rc}$P,

(e) E has the positive p-DP$_{rc}$P.

Naturally, one may think we can define the so-called positive strong p-DP$_{rc}$P in a Banach lattice E to be Banach lattices on which every almost Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)_+$ is norm null. Indeed, the introduction of this notion is superfluous because we have the following theorem which gives another characterization for the positive p-DP$_{rc}$P.

Theorem 4.6. *For a Banach lattice E, these are equivalent:*

(a) each almost Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)_+$ is norm null,

(b) each disjoint almost Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)$ is norm null,

(c) each disjoint Dunford-Pettis sequence $(x_n) \in \ell_p^w(E)$ is norm null.

Proof. $(a) \Rightarrow (b)$. It follows from [23, Proposition 3.1.5] and [12, Lemma 3.7]. In fact, for each disjoint sequence $(x_n) \in \ell_p^w(E)$, we have sequence $(|x_n|) \in \ell_p^w(E)$ and also it is Dunford-Pettis.
$(b) \Rightarrow (c)$. It is clear.
$(c) \Rightarrow (a)$. Suppose that $(x_n) \in \ell_p^w(E)_+$ is an almost Dunford-Pettis sequence, but $\|x_n\| \nrightarrow 0$. Then (x_n) is a weakly null positive sequence in E which is not norm null. By [15, Corollary 5], there is a disjoint positive subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| \nrightarrow 0$ which is a contradiction. Note that (x_{n_k}) is a disjoint Dunford-Pettis (by [12, Lemma 3.7]) weakly p-summable sequence and by hypothesis it must be norm null. □

In [12, Theorem 3.15] the authors derive a characterization of the positive $\mathrm{DP}_{rc}\mathrm{P}$ of a Banach lattice. In fact, a Banach lattice E has the positive $\mathrm{DP}_{rc}\mathrm{P}$ if and only if each almost Dunford-Pettis set in E is an L-weakly compact set if and only if E is a KB-space and each almost Dunford-Pettis set in E is approximately order bounded.
According to this result, we state the following proposition in which we characterize positive p-$\mathrm{DP}_{rc}\mathrm{P}$ in discrete Banach lattices.

Theorem 4.7. *If E is a discrete Banach lattice, then the following are equivalent:*

(a) E has the positive p-$DP_{rc}P$,

(b) each almost Dunford-Pettis set in E is an L-weakly compact set,

(b) E is a KB-space.

Proof. $(a) \Rightarrow (b)$. If E has the positive p-$\mathrm{DP}_{rc}\mathrm{P}$ and E is discrete, then by Corollary 4.5 E has the positive $\mathrm{DP}_{rc}\mathrm{P}$ and so each almost Dunford-Pettis set in E is an L-weakly compact set.
$(b) \Rightarrow (c)$. It follows from [12, Theorem 3.15].
$(c) \Rightarrow (a)$. Each discrete KB-space has the strong $\mathrm{DP}_{rc}\mathrm{P}$ and so by Corollary 4.5 it has the positive p-$\mathrm{DP}_{rc}\mathrm{P}$. □

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