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# Fourier transform on compact Hausdorff groups

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**Abstract.** This article deals with the generalization of the abstract Fourier analysis on the compact Hausdorff group. In this paper, the generalized Fourier transform $F$ is defined as $F(\psi)(\alpha) = \int \psi(h) M_\alpha \left(h^{-1}\right) d\mu(h)$ for all $\psi \in L^2(G) \bigcap L^1(G)$, where $M_\alpha$ is a continuous unitary representation $M_\alpha : G \to UC\left(C^{n(\alpha)}\right)$ of the group $G$ in $C^{n(\alpha)}$, and its properties are studied. Also, we define the symplectic Fourier transform and the generalized Wigner function $W_A(\psi, \varphi)$ and establish the Moyal equality for the Wigner function.

We show that the homomorphism $\pi : G \to U(L^2(G/K, H_1))$ induced by $\Lambda : G \times (G/K) \to U(H_1)$ by $(\pi(\psi))(g,h) = \left(\Lambda\left(h^{-1}, g\right)\right)^{-1}\left(\psi\left(h^{-1}g\right)\right)$, $g \in G/K$, $h \in G$, $\psi \in L^2(G/K, H_1)$ is a unitary representation of the group $G$, assuming the mapping $h \mapsto (\pi(\psi))(g,h)$ is continuous as morphism $G \to U(L^2(G/K, H_1))$.

We study the unitary representation $\tilde{\pi} : G \to H$ induced by the unitary representation $V : K \to U(H_1)$ given by $\tilde{\pi}_g(\psi)(t) = \psi\left(g^{-1}t\right)$ for all $t \in G/K$.

## 1. Introduction

Let $G$ be a compact communicative group equipped with a Haar measure $\mu$ and let $\hat{G}$ be a Pontrjagin dual group consisting of the characters of $G$. A character of the group $G$ is a continuous homomorphism from $G$ to the first unitary group $U(1)$.

The Fourier transform $F$ of the function $\psi \in L^2(G) \bigcap L^1(G)$ is defined by

$$F(\psi)(\chi) = \int \psi(g) \overline{\chi(g)}\, d\mu(g) \tag{1}$$

for all $\chi \in \hat{G}$.

The inverse Fourier transform $F^{-1}$ can be expressed by a similar formula

$$F^{-1}(\psi)(\chi) = \int \psi(g) \chi(g)\, d\mu(g) \tag{2}$$

for all $\chi \in \hat{G}$.

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Mappings $F$ and $F^{-1}$ are connected so that $F(\psi)\left(\chi^{-1}\right) = F^{-1}(\psi)(\chi)$ and

$$\begin{aligned} &F(\psi)\left(\chi^{-1}\right) = \int \psi\left(g^{-1}\right)\overline{\chi(g)}\, d\mu(g) = \\ &= \int \overline{\overline{\psi(g^{-1})}\chi(g)}\, d\mu(g). \end{aligned} \tag{3}$$

**Example.** Let us consider a special case when the main group $G = R^n$ is an additive group. The representation of $R^n$ in a Hilbert space $H = L^2(R^n)$ of functions $\psi$ on $R^n$ is a shift $\tau$ given by $\tau(y,\ \psi) = \psi(\cdot - y)$. All mappings $\tau(y)\ :\ R^n \to L^2(R^n)$ constitute a semigroup. Assume $\tau(y)$ is bounded on $H = L^2(R^n)$ then representation $\tau$ is called the regular representation on $H = L^2(R^n)$.

The Fourier transform $F(\psi)$ of $\psi \in L^2(R^n) \bigcap L^1(R^n)$ is defined by

$$F(\psi)(\lambda) = \hat{\psi}(\lambda) = \int_{R^n} \exp(-i\lambda \cdot x)\, \psi(g)\ dx \tag{4}$$

for all $\lambda \in R^n$. Since the mapping $\exp(-i\lambda\cdot)\ :\ R^n \to S^1$ is continuous with respect to the compact convergence topology, homomorphism $\exp(-i\lambda\cdot)\ :\ R^n \to S^1$ can be rewritten as factorized as follows $\exp(-i\lambda\cdot)\ :\ G = R^n \xrightarrow{\lambda\cdot} R \xrightarrow{\exp(-i\lambda\cdot)} S^1 = U(1)$. The system $\{\exp(-i\lambda\cdot)\}$ constitutes an orthogonal basis in $H = L^2(R^n)$.

The main part of the paper is devoted to the generalization of the Fourier transform and the Fourier-Stieltjes calculus, and developing the basic apparatus of a new approach to problems of quantum physics, so we propose a new type of the Wigner function and establish the Moyal identity for it. The Wigner function $W(\psi,\ \varphi)$ allows us to define the wavepacket transform $W_\varphi(\psi)$ with the window $\varphi$ by $W_\varphi(\psi) = (2\pi)^{\frac{n}{2}}\ W(\psi,\ \varphi)$ where the function $\psi \in S(R^n)$ is going backward and the window $\varphi \in S(R^n)$ moves forward at the same speed.

For a function $\psi \in L^2(R^n \oplus R^n) \bigcap L^1(R^n \oplus R^n)$, the classical symplectic Fourier transform $F_\sigma$ is given by $F_\sigma(\psi)(\lambda) = F\psi(J\lambda)$ where $J$ is the standard symplectic matrix $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ and $I$ is an identity matrix. We propose to generalize the symplectic Fourier transform as a function defined on $\hat{G}_\sigma$ by an integral $F_\sigma(\psi)(\chi_\sigma) = \int \psi(h)\,\overline{\chi_\sigma(h)}d(\mu \otimes \mu)(h)$ where $\hat{G}_\sigma$ is a set of all continuous homomorphisms from $G \oplus G$ to $USp(2)$, so $\hat{G}_\sigma$ constitutes a group with the operation of pointwise multiplication and the uniform convergence topology. For the generalized Wigner function, the analog of Moyal identity can be proved so for arbitrary $\varphi \in L^2(G)$ the mapping $\psi \mapsto W_A(\psi,\ \varphi)$ is a partial isometry on a closed subspace of $L^2(G \oplus G)$ thus the wavepacket transform can be defined by $W^A_\varphi(\psi) = A_1 W_A(\psi,\ \varphi)\quad :\quad L^2(G) \to L^2(G \oplus G)$ with the window $\varphi \in L^2(G)$, this approach facilitates analysis of Bopp calculus.

## 2. The results of Peter-Weyl theorems

Let $G$ be a compact Hausdorff group equipped with a Haar measure $\mu$.

**Definition 1.** *A complete Hilbert algebra of the square-integrable functions on the group G is denoted by* $L^2(G)$.

According to the Peter-Weyl theorem, $L^2(G)$ algebra can be represented as an orthogonal sum $\oplus_{\alpha\in R}\Lambda_\alpha = L^2(G)$ of topologically simple algebras $\Lambda_\alpha$, where $\Lambda_\alpha$ equals to matrix algebra $M_{n(\alpha)}(C)$ of $(n(\alpha))^2$-dimension, where $\alpha$ is a finite-dimensional representation of the compact group $G$. Each function $\Lambda_\alpha\ :\ G \to M_{n(\alpha)}(C)$ is a continuous function on the compact group $G$.

**Definition 2**. *The set of all equivalence classes of an irreducible representation of the group G is called* $\hat{G}$.

From $\phi_\alpha = \sum_{k=1,..,n(\alpha)} e_k$, we have $\sum_{k=1,..,n(\alpha)} \psi * e_k = \psi * \phi_\alpha$ for the presentation $\psi = \sum_\alpha \psi * \phi_\alpha$. Each element $\Lambda_\alpha$ uniquely corresponds with a continuous function, so that for each finite-dimensional representation $\alpha$ there is a decomposition $\Lambda_\alpha = \oplus_{1\le k\le n(\alpha)}\Lambda_\alpha * m_k$ where $m_k$ is an irreducible idempotent, and so that $\phi_\alpha = \sum_{k=1,..,n(\alpha)} m_k$. Let $\{a_k\}_{1\le k\le n(\alpha)}$ be a Hilbert basis in $\Lambda_\alpha * m_1$ with the condition $a_k \in m_k * \Lambda_\alpha * m_1$.

**Definition 3**. *For every finite-dimensional representation $\alpha$, we define a matrix $M_\alpha(g)$ of $n(\alpha)\times n(\alpha)$-dimension with coefficients*

$$a_{ij}(g) = (n(\alpha))^{-1}\left(a_i(g) * \overline{a_j(g^{-1})}\right) \tag{5}$$

*for $1 \le i \le n(\alpha)$ and $1 \le j \le n(\alpha)$.*

From definition 3 we have $a_{ii} = m_i$.

**Definition 4**. *The Fourier transform $F(\psi)$ of the function $\psi \in L^1(G)$ is a mapping defined by*

$$F(\psi)(\alpha) = \int \psi(h) M_\alpha\left(h^{-1}\right) d\mu(h), \tag{6}$$

*where $M_\alpha$ is a continuous unitary representation $M_\alpha : G \to UC\left(C^{n(\alpha)}\right)$ of the group $G$ in $C^{n(\alpha)}$.*

We denote the set $\bigcap_\alpha M_{n(\alpha)}(C)$ by $\Theta\left(\hat{G}\right)$.

**Theorem** *(first theorem) 1. Let $G$ be a compact group then the mapping $F : L^2(G) \to L^2\left(\hat{G}\right)$ defined by*

$$F(\psi)(\alpha) = \int \psi(g) M_\alpha\left(g^{-1}\right) d\mu(g) \tag{7}$$

*is an isometric isomorphism.*

*For each element $\psi \in L^2(G)$, we have a representation*

$$\psi = \sum_\alpha n(\alpha) \sum_{i,k=1,\dots,n(\alpha)} \langle\langle F(\psi)(\alpha)(e_i(\alpha)), (e_k(\alpha))\rangle\rangle \phi_{ik}(\alpha), \tag{8}$$

*where $\{e_i(\alpha)\}_{i=1,\dots,n(\alpha)}$ is an orthonormal basis in $C^{n(\alpha)}$ and coordinate functions $\phi_{ik}$ are defined as*

$$\phi_{ik}(\alpha)(g) = \langle M_\alpha(g) e_i(\alpha), e_k(\alpha)\rangle \tag{9}$$

*for all $g \in G$ and $i,\ k = 1, \dots, n(\alpha)$.*

**Theorem** *(second theorem) 2. Let $G$ be a compact group then the inverse Fourier transform $F^{-1} : L^2\left(\hat{G}\right) \to L^2(G)$ is defined by*

$$\psi(g) = \sum_\alpha n(\alpha)\, tr(F(\psi)(\alpha) M_\alpha(g)) \tag{10}$$

*for any Fourier transform $F(\psi) \in L^2\left(\hat{G}\right)$ of $\psi \in L^2(G)$ and the series converges in $L^2$.*

## 3. The structure of $L^2$ - algebra

Let $G$ be a compact group then $L^2(G)$ is a separable complete Hilbert algebra. Let $\ell$ be a closed left ideal of $L^2(G)$ and let $\psi,\ \varphi \in \ell$ then there exist a sequence $\{e_n\}$ of irreducible self-adjoint idempotents $e_n$ of $\ell$ such that $\psi = \sum_n \psi e_n$ and $\langle\psi,\ \varphi\rangle = \langle\sum_n \psi e_n,\ \sum_n \varphi e_n\rangle$.

We remind matrix coefficients of $G$ are mappings $g \mapsto \phi^*\left(M_\alpha(g)\phi\right)$ for all $\phi^*,\ \phi \in C^{n(\alpha)}$.

Theorem (orthogonality of matrix coefficients). Let $\alpha$ be an irreducible representation of the compact group $G$ in the separable Hilbert space $H$. Then for all given $\psi_1,\ \varphi_1,\ \psi_2,\ \varphi_2 \in H$, there is a strictly positive constant $d$ such that

$$\int_G \langle\alpha(g)\psi_1,\ \varphi_1\rangle \overline{\langle\alpha(g)\psi_2,\ \varphi_2\rangle}\, d\mu(g) = \frac{1}{d}\langle\psi_1,\ \psi_2\rangle\langle\varphi_2,\ \varphi_1\rangle. \tag{11}$$

The PeterWeyl theorem allows us to elucidate the structure of $L^2(G)$ algebra as follows.

**Theorem. (*First*) 3.** *Let G be a compact Hausdorff group then $L^2(G)$ is a complete Hausdorff-Hilbert algebra, which can be decomposed into a countable or finite Hilbert sum $L^2(G) = \oplus_{\alpha\in R}\Lambda_\alpha$ of topologically simple orthogonal algebras $\Lambda_\alpha$ under conditions $\Lambda_{\alpha_1}\Lambda_{\alpha_2} = \{0\}$ for all $\alpha_1 \neq \alpha_2$. Each simple algebra $\Lambda_\alpha$ can be decomposed as a finite sum $\Lambda_\alpha = \oplus_j \ell_j$ of minimal left ideals such that there does not exist a pair of isomorphic ideals $\ell_j$. Since G is a compact group, there exists an isomorphism of $\Lambda_\alpha$ to finite-dimensional matrix algebra $M_{n(\alpha)}$.*

(***Second***) 4. *Let $U : G \to U_R(H)$ be a unitary representation of a group G in the separable Hilbert space H. Then Hilbert space H can be presented as a direct sum of finite irreducible representations each of the representations is equivalent to the matrix $\overline{M_{n(\alpha)}}$.*

Proof. The first part follows from the density in Hilbert space $L^2(G)$ of the set of matrix coefficients of the compact group $G$ and the theorem of orthogonality of matrix coefficients. Under the density, we mean that for every fixed $\psi \in L^2(G)$ and for any $\varepsilon > 0$ there exists a matrix coefficient $\tilde{\psi}$ such that $\left\|\psi - \tilde{\psi}\right\| < \varepsilon$.

To show the validity of the second part of the theorem, we employ the first part of the theorem so that for any $\varphi \in C(G)$ and $\varepsilon > 0$ there exists matrix coefficient $\tilde{\psi}$ such that

$$\left\|\int_G \left(\varphi(g) - \tilde{\psi}(g)\right)\alpha(g) f \, d\mu(g)\right\| < \varepsilon \left\|f\right\| \tag{12}$$

for all $f \in H$.

Let $\tilde{\tilde{\psi}}(g) = \phi^*\left(\hat{\alpha}(g)\phi\right)$ be a matrix coefficient of the same dimensional dual representation $\hat{\alpha}$ on n $E$. We define a nonzero mapping $E^* \mapsto H$ by

$$\left(\phi \mapsto \int_G \phi^*\left(\hat{\alpha}\left(g^{-1}\right)\phi\right)\alpha(g) f \, d\mu(g)\right) \in Hom^G(E^*, H). \tag{13}$$

The image $\left(\phi \mapsto \int_G \phi^*\left(\hat{\alpha}\left(g^{-1}\right)\phi\right)\alpha(g) f \, d\mu(g)\right)(E^*)$ is a nonempty finite-dimensional subspace of $H$. We partially order a set $\Xi$ of finite-dimensional irreducible invariant subsets by the inclusion. Employing the choice axion, we have that there exists a maximal $\theta_{\max}$ element of the partially ordered set $\Xi$. Assuming the span of $\theta_{\max}$ does not coincide with Hilbert space $H$ then the complement of the span of $\theta_{\max}$ contains at least one irreducible subspace so $\theta_{\max}$ can not be maximal since their union is larger than $\theta_{\max}$, thus we obtain that the span of $\theta_{\max}$ does coincides with the Hilbert space $H$.

By the second part of the last theorem, we have obtained that *let $U : G \to U_R\left(L^2(G)\right)$ a unitary representation of a compact group G in $L^2(G)$. Then $L^2(G)$ decomposed into a direct sum of finite irreducible representations each of the representations is equivalent to the matrix $\overline{M_{n(\alpha)}}$.*

## 4. Induce representation of a locally compact group

Let $G$ be a locally compact separable group and let $K$ be a closed subgroup of $G$. The $G/K$ is a metrizable space with a positive Borel measure $\mu$ on $G/K$. Our goal is to construct a unitary representation $\pi : G \to U(H)$ and the Hilbert space $H$ under the assumption that the unitary representation $V : K \to U(H_1)$ is given and $H_1$ is a separable Hilbert space.

Let $\left\{\phi_k\right\}$ be a Hilbert basis of $H_1$ so that an arbitrary function $\psi : G/K \to H_1$ can be presented as a convergent sequence $\sum_k \psi_k\phi_k = \psi$, where $\psi_k : G/K \to C$ so that we take

$$\left\|\psi(g)\right\|^2_{H_1} = \sum_k \left|\psi_k(g)\right|^2. \tag{14}$$

The Egoroff theorem yields that the $\mu$-measurability of each function $\psi_k : G/K \to C$ of the sequence $\{\psi_k\}$ implies the $\mu$-measurability of the function $\psi : G/K \to H_1$. For the arbitrary basis $\left\{\phi_k\right\}$ of a Hilbert basis of $H_1$, we denote $L^2(G/K, H_1)$ the space of all $\mu$-measurable functions $G/K \to H_1$ so that we have the following equalities

$$\int_{G/K} \left\|\psi(g)\right\|^2_{H_1} d\mu(g) = \sum_k \int_{G/K} \left|\psi_k(g)\right|^2 d\mu(g) = \sum_k \left\|\psi_k\right\|^2_{L^2}.$$

The inner product in $L^2(G/K, H_1)$ is given by

$$\int_{G/K} \langle \psi(g), \varphi(g) \rangle d\mu(g) = \sum_k \int_{G/K} \psi_k(g) \overline{\varphi_k(g)} d\mu(g)$$

for any pair $\psi, \varphi \in L^2(G/K, H_1)$ which is presented as $\psi = \sum_k \psi_k \phi_k$ and $\varphi = \sum_k \varphi_k \phi_k$. Now, we can consider a quotient space of $L^2(G/K, H_1)$ as a space of all classes of equivalent functions of $L^2(G/K, H_1)$, this quotient space will be again denoted by $L^2(G/K, H_1)$.

**Theorem.** *Let G be a locally compact separable group and K be a closed subgroup of G. Let $\mu$ be a positive Borel measure $\mu$ on $G/K$. Then the space $L^2(G/K, H_1)$ of all equivalence classes of all $\mu$-measurable functions $G/K \to H_1$ is a separable Hilbert space under the assumption that $H_1$ is a separable Hilbert space.*

Proof. Assume the sequence $\left\{\psi_j = \sum_k \psi_{j,k}\phi_k\right\} \subset L^2(G/K, H_1)$ satisfies the Cauchy condition in $L^2(G/K, H_1)$, for any $\varepsilon > 0$, there exists some $j_0$ such that the inequality

$$\begin{aligned} &\int_{G/K} \left\|\psi_i(g) - \psi_j(g)\right\|_{H_1}^2 d\mu(g) = \\ &= \sum_k \int_{G/K} \left|\psi_{i,k} - \psi_{j,k}\right|^2 d\mu(g) \leq \varepsilon \end{aligned}$$

holds for all $i, j > j_0$. Thus, the sequence $\{\psi_{i,k}\}_{i\geq 1} \subset L^2(H_1, C)$ satisfies the Cauchy condition. So, for any $\varepsilon > 0$, there exists an element $\gamma_k \in L^2(H_1, C)$ and some $k_0$ such that we have

$$\sum_{k=1,\ldots,k_0} \left\|\gamma_k - \psi_{j,k}\right\|_{L^2}^2 \leq \varepsilon$$

and

$$\sum_{k=1,\ldots,k_0} \left\|\gamma_k\right\|_{L^2}^2 \leq \sum_{k=1,\ldots,k_0} \left\|\gamma_k - \psi_{j,k}\right\|_{L^2}^2 + \sum_{k=1,\ldots,k_0} \left\|\psi_{j,k}\right\|_{L^2}^2 \leq \varepsilon + \left\|\psi_j\right\|_{L^2}^2$$

so $\sum_{k=1,\ldots} \left\|\gamma_k\right\|_{L^2}^2 = \left\|\gamma\right\|^2 < \infty$, the inequality

$$\sum_{k=1,\ldots} \left\|\gamma_k - \psi_{j,k}\right\|_{L^2}^2 \leq \varepsilon$$

holds for all $j > j_0$, thus, we have

$$\lim_{j\to\infty} \psi_j = \gamma,$$

the limit is understood in a topology of $L^2(G/K, H_1)$. The set of functions $\psi = \sum_{k=1,\ldots,k_0} \psi_k \phi_k$ that can be presented as a finite linear combination of $\mu$-measurable $\psi_k(g) = \left\langle \psi(g), \phi_k \right\rangle$ and elements of the basis $\left\{\phi_k\right\}$ is dense in $L^2(G/K, H_1)$ with the natural norm.

**Definition.** *Let a linear automorphism $\Lambda : G \times (G/K) \to GL(H_1)$ satisfies the conditions:*
$\Lambda(e, a) = id(H_1)$ ***for all*** $a \in G/K$
*and*

$$\Lambda(gh, a) = \Lambda(g, h \cdot a) \cdot \Lambda(h, a)$$

*for all for all $g, h \in G$ and $a \in G/K$. Then the mapping $\Lambda : G \times (G/K) \to GL(H_1)$ will be called a cocycle of the group G in a general linear group over $H_1$.*

**Theorem.** Let $V : K \to U(H_1)$ *be a unitary representation of K in $H_1$. Let $\mu$ be an outer regular, $\sigma$-inner regular, finite on compact subsets Borel measure such that*

$$\mu\left(g^{-1}E\right) = \mu(E) \tag{15}$$

*for all $g \in G$ and all $\mu$- measurable sets $E$. Let each cocycle $\Lambda : G \times (G/K) \to U(H_1)$ satisfies the following conditions: for all $s \in K$, there is $\Lambda(s,a) = U(s)$; for each $t \in G$ and $\psi \in L^2(G/K, H_1)$, the mapping $G/K \to H_1$ given by $g \mapsto \Lambda(g,t)(\psi(g))$ is $\mu$- measurable.*

*Then the homomorphism $\pi : G \to U\left(L^2(G/K,H_1)\right)$ induced by $\Lambda : G \times (G/K) \to U(H_1)$ according to*

$$(\pi(\psi))(g,h) = \left(\Lambda\left(h^{-1},g\right)\right)^{-1}\left(\psi\left(h^{-1}g\right)\right), g \in G/K, \quad h \in G, \quad \psi \in L^2(G/K,H_1)$$

*is a unitary representation of the group $G$, if the mapping $h \mapsto (\pi(\psi))(g,h)$ is continuous as $G \to U\left(L^2(G/K,H_1)\right)$.*

**Proof.** Assume $\psi \in L^2(G/K,H_1)$ and $g,h \in G$, we have

$$\left\|(\pi(\psi))(g,h)\right\|_{H_1} = \left\|\left(\Lambda\left(h,h^{-1}g\right)\right)\left(\psi\left(h^{-1}g\right)\right)\right\|_{H_1} = \left\|\psi\left(h^{-1}g\right)\right\|_{H_1}$$

so

$$\int_{G/K}\left\|\psi\left(h^{-1}g\right)\right\|^2_{H_1} d\mu(g) = \int_{G/K}\left\|\psi(g)\right\|^2_{H_1} d\mu(g),$$

thus, we obtain $\left\|(\pi(\psi))(h)\right\|_{L^2(G/K,H_1)} = \left\|\psi\right\|_{L^2(G/K,H_1)}$ for all $\psi \in L^2(G/K,H_1)$.

Thus, we have constructed the unitary representation $\pi : G \to U\left(L^2(G/K,H_1)\right)$ defined as $(\pi(\psi))(g,h) = \left(\Lambda\left(h^{-1},g\right)\right)^{-1}\left(\psi\left(h^{-1}g\right)\right)$ induced by the unitary representation $V : K \to U(H_1)$ and cocycle $\Lambda : G \times (G/K) \to U(H_1)$.

## 5. The Gerald Folland modified method

Now, we are going to construct a Hilbert space $H$ and unitary representation $\tilde{\pi} : G \to H$ induced by $V : K \to U(H_1)$ assuming that $K$ is a closed subgroup of $G$ and $\mu$ is an outer regular, $\sigma$-inner regular, finite on compact subsets Borel measure such that $\mu\left(g^{-1}E\right) = \mu(E)$ for all $g \in G$ and all $\mu$- measurable sets $E$.

Let a continuous function $\phi : G \to H_1$ be supported on a compact set. We define a function $g \mapsto \varphi_\phi(g)$ by an integral formula

$$\varphi_\phi(g) = \int_K V(h)\left(\phi(hg)\right) d\nu_K(h),$$

where $\nu_K$ is Haar's measure on the subgroup $K$.

The Hilbert space $H$ is defined as the completion of the set of all functions $\varphi_\phi$ in the norm naturally induced by the inner product given by

$$\langle\psi_1,\psi_2\rangle = \int_{G/K}\langle\psi_1(g),\psi_2(g)\rangle_{H_1} d\mu(gK)$$

for all functions $\psi_1$ and $\psi_2$ such that sets $P(\sup p(\psi_k))$, $\quad k = 1,2$ are compact and $\psi_k(gh) = V\left(h^{-1}\right)(\psi_k(g))$, $k = 1,2$ for all $g \in G$, $\quad h \in K$, where $P : G \to G/K$ is the quotient mapping.

*The unitary representation $\tilde{\pi} : G \to H$ induced by unitary representation $V : K \to U(H_1)$ is defined as $\tilde{\pi}_g(\psi)(t) = \psi\left(g^{-1}t\right)$ for all $t \in G/K$.*

Let $G$ be a compact separable group and let $K$ be a closed subgroup of $G$. Let us take $H_1 = C$ then to construct a unitary representation $G \mapsto L^2(G/K, C)$, we can use the Peter-Weyl theorem to consider a restriction $W : K \to U\left(C^{n(\alpha)}\right)$ of representation $M_\alpha : G \to U\left(C^{n(\alpha)}\right)$ on the subgroup $K$. By the second Peter-Weyl theorem, we can define orthogonal projection $P_{n(\alpha)} : C^{n(\alpha)} \to P_{n(\alpha)}\left(C^{n(\alpha)}\right) \subset C^{n(\alpha)}$ by

$$M_\alpha\left(\frac{1}{n(\overline{\alpha})}\chi(\overline{\alpha})\right) = \frac{1}{n(\alpha)}\int_K M_\alpha(h)\overline{\chi(\alpha)(h)}\, d\nu_K(h).$$

So, there is decomposition $C^{n(\alpha)} = \underset{\beta}{\oplus} P_\beta\left(C^\beta\right)$ where $\beta$ is representation on $K$, and the Hilbert space $L^2(G/K,\ C)$ can be presented in the form $\oplus L_\alpha$ of a Hilbert series of subspaces $L_\alpha \subset \Lambda_\alpha$ so that $L_\alpha = \underset{i=1,\ldots,d}{\oplus}\ \underset{j=1,\ldots,n(\alpha)}{\oplus} C \cdot \left(n(\alpha)\, a_i(g) * \overline{a_j(g^{-1})}\right)$ if the trivial representation $\gamma$ of the subgroup $K$ is $d = \frac{\alpha}{\gamma} \geq 1$ times in the restriction of $M_\alpha$ to $K$.

## 6. The symplectic Fourier transform and a generalization of the ambiguity function and Wigner functions

The set $Sp(2n, K)$ of all symplectic matrices over the field $K$ is called a symplectic group. The compact symplectic group $Sp(2n, C) \bigcap U(2n)$ is denoted by $USp(2n)$.

Now, let $G$ be a compact communicative group with a Haar measure $\mu$ on $G$. We define a group $\hat{G}_\sigma$ as a group of all continuous homomorphisms from $G \oplus G$ to $USp(2)$.

**Definition 5.** *The symplectic Fourier transform $F_\sigma$ of $\psi \in L^2(G \oplus G) \bigcap L^1(G \oplus G)$ is defined by*

$$F_\sigma(\psi)(\chi_\sigma) = \int_{G\times G} \psi(h)\, \overline{\chi_\sigma(h)} d(\mu \otimes \mu)(h) \tag{16}$$

***for all*** $\chi_\sigma \in \hat{G}_\sigma$.

The inverse of the symplectic Fourier transform $F_\sigma^{-1}$ is the same Fourier transform $F_\sigma$.

Now, let $A$ be a compact communicative algebra.

Let $\psi,\ \varphi \in L^2(A)$. We define the pair of functions $Am(\psi,\ \varphi)$ and $W_A(\psi,\ \varphi)$ by formulae

$$Am(\psi,\ \varphi)(\chi,\ z) = \int_A \overline{\chi(y)}\ \psi\left(y + \frac{1}{2}z\right)\overline{\varphi\left(y - \frac{1}{2}z\right)} d\mu(y) \tag{17}$$

and

$$W_A(\psi,\ \varphi)(\chi,\ z) = \int_A \overline{\chi(y)}\ \psi\left(z + \frac{1}{2}y\right)\overline{\varphi\left(z - \frac{1}{2}y\right)} d\mu(y), \tag{18}$$

these functions will be called ambiguity and Wigner functions respectively.

The classical ambiguity and Wigner functions are defined by integrals with respect to the Lebesgue measure

$$Amb(\psi,\ \varphi)(p,\ z) = \left(\frac{1}{2\pi}\right)^n \int_A \exp(-ip\cdot y)\ \psi\left(y + \frac{1}{2}z\right)\overline{\varphi\left(y - \frac{1}{2}z\right)} dy \tag{19}$$

and

$$W(\psi,\ \varphi)(p,\ z) = \left(\frac{1}{2\pi}\right)^n \int_A \exp(-ip\cdot y)\ \psi\left(z + \frac{1}{2}y\right)\overline{\varphi\left(z - \frac{1}{2}y\right)} dy. \tag{20}$$

By changing variables $u = y + \frac{1}{2},\quad v = y - \frac{1}{2}z$, we obtain that the classical Wigner function has an exact marginal $\langle W(\psi,\ \varphi)(\cdot,\ z)\rangle = \psi(z)\overline{\varphi(z)}$ and $\langle W(\psi,\ \varphi)(p,\ \cdot)\rangle = F(\psi(p))\overline{F(\varphi(p))}$. So, we introduce the next definition.

**Definition 6.** . *Let $\psi,\ \varphi$ be $L^2(A)$. If functions $Am(\psi,\ \varphi)$ and $W_A(\psi,\ \varphi)$ defined (17) and (18) such that $W_A(\psi,\ \varphi)$ satisfies the marginal conditions*

$$\int_A W_A(\psi,\ \varphi)(\chi,\ z)\ d\mu(\chi) = \psi(z)\overline{\varphi(z)}$$

***and***

$$\int_A W_A(\psi,\ \varphi)(\chi, y)\, d\mu(y) = F(\psi(\chi))\,\overline{F(\varphi(\chi))}$$

***then functions*** $Am(\psi, \varphi)$ ***and*** $W_A(\psi, \varphi)$ ***are called the ambiguity and Wigner functions respectively.***

The centralizer $C_A(a)$ of $a$ in $A$ is the set given by

$$C_A(a) = \{g \in A \ : \ ga = ag\}.$$

For $g, \ h \in A$ we have

$$\chi(g)\overline{\chi(h)} = \begin{cases} \left|C_A(g)\right|, & if\ g\ and\ h\ are\ conjugate \\ 0 & otherwise. \end{cases}$$

We calculate an integral

$$\int_{\hat{A}} \int_A W_A(\psi_1, \varphi_1)(\chi, z)\, \overline{W_A(\psi_2, \varphi_2)(\chi, z)}\, d\mu(z)\, d\mu(\chi) =$$
$$= \underline{\int_{\hat{A}} \int_A \int_A \int_A} \overline{\chi(y)}\, \chi(x)\ \psi_1\left(z + \tfrac{1}{2}y\right)\ \overline{\psi_2\left(z + \tfrac{1}{2}x\right)} \times$$
$$\varphi_1\left(z - \tfrac{1}{2}y\right)\varphi_2\left(z - \tfrac{1}{2}x\right) d\mu(y)\, d\mu(x)\, d\mu(z)\, d\mu(\chi) =$$
$$= |A| \left\langle \psi_1, \overline{\psi_2} \right\rangle \langle \overline{\varphi_1}\ , \varphi_2 \rangle$$

so we have obtained an analog of the Moyal identity in the form of the following theorem.

**Theorem 5.** ***The Moyal equality***

$$\langle W_A(\psi_1, \varphi_1), W_A(\psi_2, \varphi_2)\rangle_{L^2} = |A| \left\langle \psi_1, \overline{\psi_2} \right\rangle_{L^2} \langle \overline{\varphi_1}, \ \varphi_2 \rangle_{L^2}$$

***or***

$$\langle W_A(\psi_1, \varphi_1), W_A(\psi_2, \varphi_2)\rangle_{L^2} = |A|\, (\psi_1, \psi_2)_{L^2}\, \overline{\langle \varphi_1, \ \varphi_2 \rangle_{L^2}}$$

***holds for all*** $\psi_1, \varphi_1, \ \psi_2, \varphi_2 \in L^2(A)$***.***

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