



On the solution of the operator Riccati equations and invariant subspaces in the weighted Bergman space of the unit ball

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Abstract. We consider the Riccati operator equations on the weighted Bergman space $A_\alpha^2(\mathbb{B}_n)$ of the unit ball \mathbb{B}_n in \mathbb{C}^n and investigate the properties of their solutions. Our discussion uses the Berezin symbols method.

1. Introduction and background

Through the paper, \mathbb{B}_n will denote the unit ball in \mathbb{C}^n . Let ν be the normalized Lebesgue volume measure on \mathbb{B}_n . For $-1 < \alpha < +\infty$, we denote by ν_α the measure on \mathbb{B}_n defined by

$$d\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z),$$

where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant such that $\nu_\alpha(\mathbb{B}_n) = 1$. For $1 \leq p < \infty$, we write $\|\cdot\|_{\alpha,p}$ for the norm in $L^p(\mathbb{B}_n, d\nu_\alpha)$ and $\langle \cdot, \cdot \rangle_\alpha$ for the inner product on $L^2(\mathbb{B}_n, d\nu_\alpha)$. Recall that the Bergman space $A_\alpha^2(\mathbb{B}_n)$ is the space of holomorphic functions which are square-integrable with respect to measure $d\nu_\alpha$ on \mathbb{B}_n . It is known that (see, for instance, Hedenmalm, Korenblum and Zhu [17]) the reproducing kernel K_w^α and normalized reproducing kernel k_w^α of the space $A_\alpha^2(\mathbb{B}_n)$ are given by

$$K_w^\alpha(z) := \frac{1}{(1 - \langle z, w \rangle)^{n+\alpha+1}}$$

and

$$k_w^\alpha(z) := \frac{(1 - |w|^2)^{\frac{n+\alpha+1}{2}}}{(1 - \langle z, w \rangle)^{n+\alpha+1}},$$

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respectively, for $z, w \in \mathbb{B}_n$. The reproducing property means that

$$\langle f, K_w^\alpha \rangle_\alpha = f(w)$$

for every $f \in A_\alpha^2(\mathbb{B}_n)$ and all $z, w \in \mathbb{B}_n$. The orthogonal projection P_α of $L^2(\mathbb{B}_n, dv_\alpha)$ onto $A_\alpha^2(\mathbb{B}_n)$ is given by

$$(P_\alpha f)(w) := \langle f, K_w^\alpha \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \frac{1}{(1 - \langle w, z \rangle)^{n+\alpha+1}} dv_\alpha(z),$$

for $f \in L^2(\mathbb{B}_n, dv_\alpha)$ and $w \in \mathbb{B}_n$. Given $h \in L^1(\mathbb{B}_n, dv_\alpha)$, the Toeplitz operator

$$T_h : A_\alpha^2(\mathbb{B}_n) \rightarrow A_\alpha^2(\mathbb{B}_n)$$

and multiplication operator

$$M_h : A_\alpha^2(\mathbb{B}_n) \rightarrow L^2(\mathbb{B}_n, dv_\alpha)$$

are defined, respectively, by

$$(T_h f)(z) = \int_{\mathbb{B}_n} \frac{f(w)h(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} dv_\alpha(w) \tag{1}$$

and

$$M_h(f) = hf.$$

Recall that for a bounded linear operator A on Bergman space $A_\alpha^2(\mathbb{B}_n)$, its Berezin symbol (transform) is defined by (see [2–5, 9, 15, 17])

$$\tilde{A}(z) := \langle Ak_z(w), k_z(w) \rangle, \quad z, w \in \mathbb{B}_n.$$

Since by the Cauchy-Schwarz inequality

$$|\tilde{A}(z)| \leq \|Ak_z(w)\| \|k_z(w)\| \leq \|A\|$$

for all $z \in \mathbb{B}_n$, the Berezin symbol \tilde{A} is a bounded complex-valued function on \mathbb{B}_n . The present paper is motivated mostly by the papers [10, 16]. Namely, in the present article, by using the Berezin symbol technique, we study the operator Riccati equation

$$XAX + XB - CX - D = 0 \tag{2}$$

with coefficient operators A, B, C, D in $\mathcal{B}(A_\alpha^2(\mathbb{B}_n))$.

We also investigate invariant subspaces of a Toeplitz operator on the weighted Bergman space $A_\alpha^2(\mathbb{B}_n)$ in terms of Berezin symbols.

Note that Berezin symbol of operators plays important role in operator theory (see [5, 9, 17]). In particular, it is important in the characterization of compact operators (see Nordgren and Rosenthal [15]), including compact Toeplitz operators (see [1, 17], and references therein). The method of Berezin symbols is motivated by its connection with quantum physics, see Berezin [2, 3].

2. Riccati equation on the set of Toeplitz operators

For $f \in L^1(\mathbb{B}_n, dv_\alpha)$, we define the Berezin transform (symbol) of f to be the function \widetilde{f} , i.e.,

$$\widetilde{f}(z) := \int_{\mathbb{B}_n} f(w) |k_z^\alpha(w)|^2 dv_\alpha(w). \tag{3}$$

If f is bounded, then \widetilde{f} is a bounded function on \mathbb{B}_n .

According to the definition of Berezin transform, the mean oscillation of f in the weighted Bergman metric is the function $MO(f)(z)$ defined on \mathbb{B}_n by (see, [6])

$$MO(f)(z) := \left| \widetilde{f}^2(z) - \left| \widetilde{f}(z) \right|^2 \right|.$$

For $z \in \mathbb{B}_n$, let φ_z be the automorphism of \mathbb{B}_n such that $\varphi_z(0) = z$ and $\varphi_z = \varphi_z^{-1}$. Thus, we have the change-of-variable formula

$$\int_{\mathbb{B}_n} h(\varphi_z(w)) |k_z^\alpha(w)|^2 dv_\alpha(w) = \int_{\mathbb{B}_n} h(w) dv_\alpha(w)$$

for every $h \in L^1(\mathbb{B}_n, dv_\alpha)$.

For $f \in L^1(\mathbb{B}_n, dv_\alpha)$, the Berezin transform of Toeplitz operator T_f coincides the Berezin transform of the function f , that is $\widetilde{T}_f(z) = \widetilde{f}(z)$ for all $z \in \mathbb{B}_n$. For more properties and facts of the Berezin transform, see, for example, Engliš [5], Nordgren and Rosenthal [15] and Zhu [17].

Recall that $BMO_\partial := BMO_\partial(\mathbb{B}_n)$ is defined to be the space of functions f such that

$$\|f\|_{BMO_\partial} := \sup_{z \in \mathbb{B}_n} MO(f)(z) < +\infty.$$

Let $VMO_\partial := VMO_\partial(\mathbb{B}_n)$ be the subspace of BMO_∂ consisting of functions f with

$$\lim_{z \rightarrow \partial \mathbb{B}_n} \left(\left| \widetilde{f}^2(z) - \left| \widetilde{f}(z) \right|^2 \right| \right) = 0.$$

Also, we denote by $BA := BA(\mathbb{B}_n)$ (see [14]) the space of functions f with

$$\|f\|_{BA} := \sup_{z \in \mathbb{B}_n} \left[\left| \widetilde{f}^2(z) \right| \right]^{1/2},$$

and finally, we define $VA := VA(\mathbb{B}_n)$ (see [14]) as the subspace of BA consisting of functions f satisfying

$$\lim_{z \rightarrow \partial \mathbb{B}_n} \left| \widetilde{f}^2(z) \right| = 0.$$

It is easy to see that if $f \in VA$ then $f \in VMO_\partial$. For each $f \in VA$, it can be obtained by Hölder inequality that

$$\left| \widetilde{f}(z) \right| = \int_{\mathbb{B}_n} |f(w)| |k_z(w)|^2 dv_\alpha(w) \leq \left[\left| \widetilde{f}^2(z) \right| \right]^{1/2} < +\infty.$$

Recall also that $H^\infty = H^\infty(\mathbb{B}_n)$ is the space of holomorphic functions f on \mathbb{B}_n such that

$$\|f\|_{H^\infty} := \sup_{z \in \mathbb{B}_n} |f(z)| < +\infty.$$

For a function $\varphi \in BMO_\partial(\mathbb{B}_n)$, we define the Toeplitz operator T_φ with symbol φ on $A_\alpha^2(\mathbb{B}_n)$ by

$$T_\varphi f = P_\alpha(\varphi f) = \int_{\mathbb{B}_n} \frac{f(w)\varphi(w)}{(1-\langle z,w \rangle)^{n+\alpha+1}} dv_\alpha(w) \text{ (see (1))}$$

for all $f \in A_\alpha^2(\mathbb{B}_n)$. For more information about Toeplitz operators, the reader can find, for example, in [1, 5, 7, 12, 13, 17].

For $\varphi \in H^\infty$, T_φ is just the operator of multiplication by φ on the space $A_\alpha^2(\mathbb{B}_n)$, i.e., $T_\varphi f = \varphi f$, $f \in A_\alpha^2(\mathbb{B}_n)$. Then we have that

$$\begin{aligned} \langle T_\varphi^* k_z^\alpha, f \rangle &= \langle k_z^\alpha, T_\varphi f \rangle = \langle k_z^\alpha, \varphi f \rangle \\ &= \overline{\langle \varphi f, k_z^\alpha \rangle} = \overline{(\varphi f)(z)} = \overline{\varphi(z)k_z^\alpha}, \end{aligned}$$

hence, $T_\varphi^* k_z^\alpha = \overline{\varphi(z)}k_z^\alpha$ for all $z \in \mathbb{B}_n$.

In this section, we study Riccati operator equation (2) on the set of Toeplitz operators $T_\varphi \in \mathcal{B}(A_\alpha^2(\mathbb{B}_n))$ in terms of Berezin symbols, which firstly studied by Karavayev [10] in the Hardy Hilbert space $H^2(\mathbb{D})$ over the unit disc \mathbb{D} of \mathbb{C} . In the sequel, this approach attracted attention of several authors in [7, 8, 11, 12, 14]. The following extension of Engliš’s result, [5], is essential in what follows, so, we start with the proof of the following auxiliary lemma.

Lemma 2.1. *Let $\varphi \in VMO_\partial(\mathbb{B}_n)$. Then*

$$\lim_{z \rightarrow \partial \mathbb{B}_n} \|T_\varphi k_z^\alpha - \widetilde{\varphi}(z)k_z^\alpha\|_{A_\alpha^2(\mathbb{B}_n)} = 0.$$

Proof. Indeed, it is easy to see that

$$\begin{aligned} \|T_\varphi k_z^\alpha - \widetilde{\varphi}(z)k_z^\alpha\|_{A_\alpha^2(\mathbb{B}_n)}^2 &= \langle T_\varphi k_z^\alpha - \widetilde{\varphi}(z)k_z^\alpha, T_\varphi k_z^\alpha - \widetilde{\varphi}(z)k_z^\alpha \rangle \\ &= \|T_\varphi k_z^\alpha\|^2 - \overline{\widetilde{\varphi}(z)} \langle T_\varphi k_z^\alpha, k_z^\alpha \rangle - \widetilde{\varphi}(z) \langle k_z^\alpha, T_\varphi k_z^\alpha \rangle + |\widetilde{\varphi}(z)|^2. \end{aligned}$$

Since $\widetilde{T}_\varphi(z) = \widetilde{\varphi}(z)$, we have

$$\|T_\varphi k_z^\alpha - \widetilde{\varphi}(z)k_z^\alpha\|_{A_\alpha^2(\mathbb{B}_n)}^2 = \|T_\varphi k_z^\alpha\|^2 - |\widetilde{\varphi}(z)|^2.$$

On the other hand,

$$\|T_\varphi k_z^\alpha\|^2 = \|P_\alpha(\varphi k_z^\alpha)\|^2 \leq \|\varphi k_z^\alpha\|_{L^2(\mathbb{B}_n, dV_\alpha)}^2 = |\widetilde{\varphi}|^2(z).$$

Then

$$\|T_\varphi k_z^\alpha - \widetilde{\varphi}(z)k_z^\alpha\|_{A_\alpha^2(\mathbb{B}_n)}^2 \leq |\widetilde{\varphi}|^2(z) - |\widetilde{\varphi}(z)|^2.$$

For $\varphi \in VMO_\partial$, we have

$$|\widetilde{\varphi}|^2(z) - |\widetilde{\varphi}(z)|^2 \rightarrow 0, \quad z \rightarrow \partial \mathbb{B}_n \text{ non-tangentially,}$$

which proves the lemma. \square

In next theorem, we consider the case that Toeplitz operator is a solution of the Riccati equation (2) on the space $A_\alpha^2(\mathbb{B}_n)$.

Theorem 2.2. Suppose that $B = T_\varphi^*$, $C = T_\psi$ such that φ, ψ are nonconstant functions on H^∞ , and let A and D be bounded linear operators on $A_\alpha^2(\mathbb{B}_n)$. If $h \in VA(\mathbb{B}_n)$ and the corresponding Toeplitz operator T_h is a solution of the operator Riccati equation (2), then

$$\lim_{z \rightarrow \partial \mathbb{B}_n} (\widetilde{A}(z)\widetilde{h}^2(z) + (\overline{\varphi(z)} - \psi(z))\widetilde{h}(z) - \widetilde{D}(z)) = 0. \tag{4}$$

Proof. In fact, since T_h is a solution of equation (2), then

$$T_h A T_h + T_h T_\varphi^* - T_\psi T_h - D = 0.$$

Then, $[T_h A T_h + T_h T_\varphi^* - T_\psi T_h - D]^\sim(z) = 0$ for all $z \in \mathbb{B}_n$, that is

$$\langle (T_h A T_h + T_h T_\varphi^* - T_\psi T_h - D)k_z^\alpha, k_z^\alpha \rangle = 0, \forall z \in \mathbb{B}_n.$$

Therefore

$$\begin{aligned} 0 &= \langle T_h A T_h k_z^\alpha, k_z^\alpha \rangle + \langle T_h T_\varphi^* k_z^\alpha, k_z^\alpha \rangle - \langle T_\psi T_h k_z^\alpha, k_z^\alpha \rangle - \langle D k_z^\alpha, k_z^\alpha \rangle \\ &= \langle A T_h k_z^\alpha, T_h^* k_z^\alpha - \widetilde{T}_h^*(\lambda) k_z^\alpha \rangle + \widetilde{T}_h(\lambda) \langle A T_h k_z^\alpha, k_z^\alpha \rangle \\ &\quad + \langle T_h (T_h^* k_z^\alpha - \widetilde{T}_\varphi^*(\lambda) k_z^\alpha), k_z^\alpha \rangle + \widetilde{T}_\varphi^*(z) \widetilde{T}_h(z) \\ &\quad - \langle T_\psi (T_h k_z^\alpha - \widetilde{T}_h(\lambda) k_z^\alpha), k_z^\alpha \rangle - \widetilde{T}_\psi(\lambda) \widetilde{T}_h(z) - \widetilde{D}(z) \\ &= \langle A T_h k_z^\alpha, T_h^* k_z^\alpha - \widetilde{T}_h^*(\lambda) k_z^\alpha \rangle \\ &\quad + \widetilde{T}_h(\lambda) \left[\langle A (T_h k_z^\alpha - \widetilde{T}_h(\lambda) k_z^\alpha), k_z^\alpha \rangle + \widetilde{T}_h(\lambda) \widetilde{A}(z) \right] \\ &\quad + \langle T_\varphi^* k_z^\alpha - \widetilde{T}_\varphi^*(\lambda) k_z^\alpha, T_h^* k_z^\alpha \rangle + \widetilde{T}_\varphi^*(\lambda) \widetilde{T}_h(z) - \langle T_h k_z^\alpha - \widetilde{T}_h(\lambda) k_z^\alpha, T_\psi^* k_z^\alpha \rangle \\ &\quad - \widetilde{T}_\psi(\lambda) \widetilde{T}_h(z) - \widetilde{D}(z) \\ &= [\widetilde{A}(z)\widetilde{h}^2(z) + (\overline{\varphi(z)} - \psi(z))\widetilde{h}(z) - \widetilde{D}(z)] + \langle A T_h k_z^\alpha, T_h^* k_z^\alpha - \widetilde{h}(z) k_z^\alpha \rangle \\ &\quad + \widetilde{h}(z) \langle T_h k_z^\alpha - \widetilde{h}(z) k_z^\alpha, A^* k_z^\alpha \rangle + \langle \overline{\varphi(z)} k_z^\alpha - \overline{\psi(z)} k_z^\alpha, T_h^* k_z^\alpha \rangle, \end{aligned}$$

which by Lemma 2.1 yields

$$\begin{aligned} \left| \widetilde{A}(z)\widetilde{h}^2(z) + (\overline{\varphi(z)} - \psi(z))\widetilde{h}(z) - \widetilde{D}(z) \right| &\leq \|A T_h\| \left\| T_h^* k_z^\alpha - \widetilde{h}(z) k_z^\alpha \right\| \\ &\quad + \|T_h\| \|A\| \left\| T_h k_z^\alpha - \widetilde{h}(z) k_z^\alpha \right\| \rightarrow 0 \text{ as } z \rightarrow \partial \mathbb{B}_n. \end{aligned}$$

This implies the desired assertion (4), which proves the theorem. \square

Corollary 2.3. Suppose that the limits $\widetilde{A}(\xi) := \lim_{z \rightarrow \partial \mathbb{B}_n} \widetilde{A}(z)$ and $\widetilde{D}(\xi) := \lim_{z \rightarrow \partial \mathbb{B}_n} \widetilde{D}(z)$ exist for almost all points $\xi \in \partial \mathbb{B}_n$, and verify

$$(\overline{\varphi(\xi)} - \psi(\xi))^2 + 4\widetilde{A}(\xi)\widetilde{D}(\xi) = 0 \tag{5}$$

for almost all points $\xi \in \partial \mathbb{B}_n$. Let $h \in VA(\mathbb{B}_n)$. If T_h is a solution of Riccati equation (2), then

$$\lim_{z \rightarrow \xi \in \partial \mathbb{B}_n} \widetilde{h}^2(z) = -\frac{\widetilde{D}(\xi)}{\widetilde{A}(\xi)}$$

for almost all $\xi \in \partial \mathbb{B}_n$.

Proof. Indeed, since T_h satisfies equation (2), we have obtain by Theorem 2.2 that

$$\lim_{z \rightarrow \partial \mathbb{B}_n} (\widetilde{A}(z) \widetilde{h}^2(z) + (\overline{\varphi(z)} - \psi(z)) \widetilde{h}(z) - \widetilde{D}(z)) = 0.$$

Further,

$$\begin{aligned} & \lim_{z \rightarrow \partial \mathbb{B}_n} \widetilde{A}(z) \left[\widetilde{h}^2(z) + \frac{\overline{\varphi(z)} - \psi(z)}{\widetilde{A}(z)} \widetilde{h}(z) - \frac{\widetilde{A}(z) \widetilde{D}(z)}{\widetilde{A}^2(z)} \right] \\ &= \lim_{z \rightarrow \partial \mathbb{B}_n} \widetilde{A}(z) \left[\widetilde{h}^2(z) + \frac{\overline{\varphi(z)} - \psi(z)}{\widetilde{A}(z)} \widetilde{h}(z) + \frac{(\overline{\varphi(z)} - \psi(z))^2}{4\widetilde{A}^2(z)} \right] \\ &= \lim_{z \rightarrow \partial \mathbb{B}_n} \widetilde{A}(z) \left(\widetilde{h}(z) + \frac{\overline{\varphi(z)} - \psi(z)}{2\widetilde{A}(z)} \right)^2 = 0. \end{aligned}$$

In particular,

$$\lim_{z \rightarrow \xi \in \partial \mathbb{B}_n} \widetilde{h}(z) = \frac{\psi(\xi) - \overline{\varphi(\xi)}}{2\widetilde{A}(\xi)},$$

hence we have that

$$\lim_{z \rightarrow \xi \in \partial \mathbb{B}_n} \widetilde{h}^2(z) = -\frac{\widetilde{D}(\xi)}{\widetilde{A}(\xi)}$$

for almost all $\xi \in \partial \mathbb{B}_n$, as desired. \square

3. Invariant subspaces of compact Toeplitz operators on $A_\alpha^2(\mathbb{B}_n)$

Note that the solvability of the Riccati operator equation in concrete operator classes is an important problem of operator theory in the Hilbert space H . For example, the existence of a nontrivial solution of equation (2) for fixed $A \in \mathcal{B}(H)$, $B = D = 0$ and $C = A$ on the set \mathcal{P}_h of all orthogonal projections $P \in \mathcal{B}(H)$ is equivalent to the positive solution of the famous Invariant Subspace Problem in the infinite dimensional separable complex Hilbert space, since $TE \subset E$ if and only if

$$(I - P_E)TP_E = 0, \tag{6}$$

where $P_E : H \rightarrow E$ is an orthogonal projection onto the closed subspace $E \subset H$. In this section, we discuss the structure of invariant subspaces of Toeplitz operator T_f on the weighted Bergman space $A_\alpha^2(\mathbb{B}_n)$ in terms of Berezin symbols.

For any $p \geq 1$, VMO_α^p denotes the subspace BMO_α^p consisting of functions f such that

$$\lim_{z \rightarrow \partial \mathbb{B}_n} \left\| f \circ \varphi_z - \widetilde{f}(z) \right\|_{\alpha,p} = 0.$$

The following theorem is contained in [6].

Theorem 3.1. ([6]) Suppose $f \in VMO_\alpha^1$ and $|f| / (1 - |z|)^{4(n+\alpha+1)}$ is bounded on the unit ball \mathbb{B}_n . Then T_f is compact operator on $A_\alpha^2(\mathbb{B}_n)$ if and only if $\widetilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial \mathbb{B}_n$.

Our next result is the following.

Theorem 3.2. Suppose $f \in VMO_\alpha^1$, $|f| / (1 - |z|)^{4(n+\alpha+1)}$ is bounded on \mathbb{B}_n and $\tilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial\mathbb{B}_n$. Let $E \subset A_\alpha^2(\mathbb{B}_n)$ be a closed subspace of $A_\alpha^2(\mathbb{B}_n)$. If $T_\varphi E \subset E$, i.e., E is invariant under the Toeplitz operator T_f , then

$$\left| (T_f K_z^{\alpha,E})(z) - \tilde{f}(z) K_z^{\alpha,E}(z) \right| = o\left((1 - |z|^2)^{-(n+\alpha+1)} \right) \text{ as } z \rightarrow \partial\mathbb{B}_n, \tag{7}$$

where $K_z^{\alpha,E}$ denotes the reproducing kernel of the subspace E , which is equal $P_E K_z^\alpha$.

Proof. Suppose $T_f E \subset E$. Then it means that $(I - P_E)T_f P_E = 0$, and hence $P_E T_f P_E - T_f P_E = 0$, that is P_E is a solution of equation (6) where $T = T_f$. It is then clear from the above equality that

$$\widetilde{T_f P_E}(z) - P_E \widetilde{T_f P_E}(z) = 0$$

for all $z \in \mathbb{B}_n$. So, we have that

$$\begin{aligned} 0 &= \left\langle P_E k_z^\alpha, T_f^* k_z^\alpha \right\rangle - \left\langle T_f P_E k_z^\alpha, P_E k_z^\alpha \right\rangle \\ &= \left\langle P_E k_z^\alpha, T_f^* k_z^\alpha - \tilde{f}(z) k_z^\alpha \right\rangle + \frac{\tilde{f}(z)}{K_z^\alpha(z)} \left\langle K_z^{\alpha,E}, K_z^\alpha \right\rangle - \frac{1}{K_z^\alpha(z)} \left\langle T_f K_z^{\alpha,E}, K_z^{\alpha,E} \right\rangle. \end{aligned}$$

By considering that $T_f E \subset E$, and therefore $T_f K_z^{\alpha,E} \in E$, we have that

$$\left\langle T_f K_z^{\alpha,E}, K_z^{\alpha,E} \right\rangle = \left(T_f K_z^{\alpha,E} \right)(z)$$

for all $z \in \mathbb{B}_n$. On the other hand, we obtain from Theorem 3.1 and the Cauchy-Schwarz inequality that

$$\begin{aligned} &\lim_{z \rightarrow \xi \in \partial\mathbb{B}_n} \frac{1}{K_z^\alpha(z)} \left| \left(T_f K_z^{\alpha,E} \right)(z) - \tilde{f}(z) K_z^{\alpha,E}(z) \right| \\ &= \lim_{z \rightarrow \xi \in \partial\mathbb{B}_n} (1 - |z|^2)^{(n+\alpha+1)} \left| \left(T_f K_z^{\alpha,E} \right)(z) - \tilde{f}(z) K_z^{\alpha,E}(z) \right| \\ &= \lim_{z \rightarrow \xi \in \partial\mathbb{B}_n} \left| \left\langle P_E k_z^\alpha, T_f^* k_z^\alpha - \tilde{f}(z) k_z^\alpha \right\rangle \right| \\ &\leq \lim_{z \rightarrow \xi \in \partial\mathbb{B}_n} \|P_E k_z^\alpha\| \left\| T_f^* k_z^\alpha - \tilde{f}(z) k_z^\alpha \right\| \\ &\leq \lim_{z \rightarrow \xi \in \partial\mathbb{B}_n} \left\langle T_f^* k_z^\alpha - \tilde{f}(z) k_z^\alpha, T_f^* k_z^\alpha - \tilde{f}(z) k_z^\alpha \right\rangle^{\frac{1}{2}} \\ &= \lim_{z \rightarrow \xi \in \partial\mathbb{B}_n} \left(\left\| T_f^* k_z^\alpha \right\|^2 - \left| \tilde{f}(z) \right|^2 \right)^{1/2} \text{ (since } T k_z^\alpha - \tilde{T}(z) k_z^\alpha \perp k_z^\alpha \text{ for any } T \in B(A_\alpha^2(\mathbb{B}_n)) \text{)}. \end{aligned}$$

Since $\lim_{z \rightarrow \partial\mathbb{B}_n} \tilde{f}(z) = 0$, by Theorem 3.1, T_f is compact on $A_\alpha^2(\mathbb{B}_n)$. On the other hand, the normalized reproducing kernel k_z^α converges weakly to 0 as z tends $\partial\mathbb{B}_n$, and therefore $\left\| T_f^* k_z^\alpha \right\| \rightarrow 0$ as $z \rightarrow \partial\mathbb{B}_n$. Thus, we deduce from the latter inequality that

$$\lim_{z \rightarrow \partial\mathbb{B}_n} (1 - |z|^2)^{(n+\alpha+1)} \left| \left(T_f K_z^{\alpha,E} \right)(z) - \tilde{f}(z) K_z^{\alpha,E}(z) \right| = 0,$$

which gives (7). The proof is completed. \square

References

[1] S. Axler, D. Zheng, *The Berezin transform on the Toeplitz algebra*, Studia Math. **127** (1998), 113–136.
 [2] F. A. Berezin, *Covariant and contravariant symbols for operators*, Math. USSR-Izv. **6** (1972), 1117–1151.

- [3] F. A. Berezin, *Quantization*, Math. USSR-Izv. **8** (1974), 1109–1163.
- [4] I. Chalendar, E. Fricain, M. Gürdal, M. T. Karaev, *Compactness and Berezin symbols*, Acta Sci. Math. (Szeged) **78** (2012), 315–329.
- [5] M. Engliš, *Toeplitz operators and the Berezin transform on H^2* , Linear Algebra Appl. **223/224** (1995), 171–204.
- [6] X. Feng, K. Zhang, J. Dong, X. Liu, C. Guan, *Multiplication operator with BMO symbols and Berezin transform*, J. Funct. Spaces 2015, Article ID 754646, 4 pages <http://dx.doi.org/10.1255/2015/754646>.
- [7] M. Gürdal, F. Şöhret, *Some results for Toeplitz operators on the Bergman space*, Appl. Math. Comput. **218** (2011), 789–793.
- [8] M. Gürdal, U. Yamancı, M. Garayev, *Some results for operators on a model space*, Front. Math. China **13** (2018), 287–300.
- [9] H. Hedenmalm, B. Karenblum, K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Math., vol. 199, Springer-Verlag, 2000.
- [10] M. T. Karaev, *On the Riccati equations*, Monatsh. Math. **155** (2008), 161–166.
- [11] M. T. Karaev, M. Gürdal, M. B. Huban, *Reproducing kernels, Engliš algebras and some applications*, Studia Math. **232** (2016), 113–141.
- [12] M. T. Karaev, M. Gürdal, *On the Berezin symbols and Toeplitz Operators*, Extracta Math. **25** (2010), 83–102.
- [13] S. Kılıç, *The Berezin symbol and multipliers of functional Hilbert spaces*, Proc. Amer. Math. Soc. **123** (1995), 3687–3691.
- [14] J. Liu, H. Xie, *Some properties of solutions to the Riccati equations in connection with Bergman spaces*, Rocky Mountain J. Math. **50** (2020), 651–657.
- [15] E. Nordgren, P. Rosenthal, *Boundary values of Berezin symbols*, in: Oper. Theory Adv. Appl., Birkhäuser, **73** (1994), 362–368.
- [16] S. Saltan, *On the weak limit of compact operators on the reproducing kernel Hilbert space and related questions*, An. Şt., Univ. Ovidius Constanta **24** (2016), 253–260.
- [17] K. Zhu, *Operator Theory in Function Spaces*, Second edition, Math. Surveys and Monographs, 138, Amer. Math. Soc., Providence, 2007.