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Inverse coefficient problem for quasilinear pseudo-parabolic equation by Fourier method

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Abstract. In this research, we consider a coefficient problem of an inverse problem of a quasilinear pseudo-parabolic equation with nonlocal boundary condition. We prove the existence, uniqueness and continuously dependence upon the data of the solution by iteration method. Also we consider numerical solution for this inverse problem by using linearization and finite difference method.

1. Introduction

Fourier series, which have an important place in various branches of mathematics, physics, engineering and applied sciences, are the series that are widely used especially in real world problems involving periodic oscillations. Fourier series, which are encountered in differential equation problems related to applications in many fields such as electromagnetic theory, heat transfer and conduction, physical phenomena involving oscillation, quantum theory, acoustics, magnetic and electronics, can be derived with the help of trigonometric functions. Another advantage of these series is that they allow trigonometric presentation for functions that do not need to be arbitrarily differentiable (see [7]).

Many methods are known for solving differential equations. However, sometimes it can be a difficult process to detect arbitrary functions in the differential equation that satisfy the given boundary conditions. In fact, it is impossible to solve the general solution of partial differential equations except in special cases. For these reasons, various methods have been developed for the solution of boundary value problems. One of the well-known methods is the Fourier method, which is based on the separation of variables.

The subject of inverse problems was first discussed in the 19th and 20th centuries and shed light on the solution of many problems in heat transfer, diffusion problems, nuclear physics problems and seismology. Although parabolic equations fall within the scope of inverse problems, they are a type of problem that can be obtained by utilizing the solution to be obtained under certain conditions (see [2], [3], [4], [5], [6] and [8]). Also, for some further results the readers can consider the paper [9]-[13] and [14]-[17]. We will deal with the following problem in this study.

Consider the equation

$$a(t)u_t - u_{xx} - \varepsilon u_{xxt} = r(t)f(x,t,u), \ (x,t) \in \ D, \tag{1}$$

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with the initial condition

$$u(x,0) = \varphi(x), \ \ x \in [0,\pi], \tag{2}$$

the periodic boundary condition

$$u(0,t) = u(1,t), \ u_x(1,t) = 0, \ t \in [0,T], \tag{3}$$

and the overdetermination data

$$E(t) = \int_0^1 u(x,t)dx, 0 \le t \le T, \tag{4}$$

where

$$D := \{0 < x < 1, \ 0 < t < T\}.$$

The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,1]$ and $\bar{D} \times (-\infty,\infty)$, respectively.

In Section 2, the existence and uniqueness of the solution of inverse problem (1)-(4) will be proved by using the Fourier method and iteration method. In Section 3, the continuous dependence upon the data of the inverse problem will be demonsrated. In Section 4, the numerical procedure for the solution of the inverse problem will be given. In Section 5, some examples will be presented to verify the main findings.

2. Existence and Uniqueness of the Solution of the Inverse Problem

Consider the following system of functions on the interval $[0,1]$:

$$X_0(x) = 2, \ X_{2k-1}(x) = 4\cos(2\pi kx), \ X_{2k}(x) = 4(1-x)\sin(2\pi kx), \ k = 1,2,...,$$

$$Y_0(x) = x, \ Y_{2k-1}(x) = x\cos(2\pi kx), \ Y_{2k}(x) = \sin(2\pi kx), \ k = 1,2,....$$

The systems of these functions arise in [5] for the solution of a nonlocal boundary value problem in heat conduction.

It is easy to verify that the system of function $X_k(x)$ and $Y_k(x)$, $k = 0,1,2,...$ are bi-orthonormal on $[0,1]$. They are also Riesz bases in $L_2[0,1]$ [1] .

$$u(x,t) = \sum_{k=1}^{\infty} u_k x_k.$$

Definition 2.1. *The problem of finding the pair $\{r(t), u(x,t)\}$ in (1)-(4) is called an inverse problem.*

The main result on existence and uniqueness of the solution of the inverse problem (1)-(4) is presented as follows:

We have the following assumptions on the data of the problem (1)-(4).

(A_1) $E(t) \in C^1[0,T], r(t) \in C[0,T]$,

(A_2) $\varphi(x) \in C^2[0,1], \varphi(0) = \varphi(1), \varphi'(1) = 0, \varphi''(0) = \varphi''(1)$,

(A_3)

(1) Let the function $f(x,t,u)$ is continuous with respect to all arguments in $\bar{D} \times (-\infty,\infty)$ and satisfies the following condition

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \le b(t,x) \left| u - \tilde{u} \right| \ , n = 0,1,2,$$

where $b(x,t) \in L_2(D), \ b(x,t) \ge 0,$

(2) $f(x,t,u)\in C^4[0,1], t\epsilon[0,T],\ f(x,t,u)\big|_{x=0} = f(x,t,u)\big|_{x=1},\ f_x(x,t,u)\big|_{x=1} = 0,\ f_{xx}(x,t,u)\big|_{x=0} = f_{xx}(x,t,u)\big|_{x=1},$

(3) $\int\limits_0^1 f(x,t,u)\,dx \neq 0,\ \forall t\epsilon[0,T].$

By applying the procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3) for arbitrary $r(t)\in C[0,T]$:

$$\begin{aligned} u(x,t) &= \left[\varphi_0 + \int\limits_0^t r(\tau) f_0(\tau) d\tau\right] X_0(x) \\ &+ \sum_{k=1}^{\infty}\left[\varphi_{2k}\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int\limits_0^t r(\tau) f_{2k}(\tau) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\, d\tau\right] X_{2k}(x) \\ &+ \sum_{k=1}^{\infty}\left[\left(\varphi_{2k-1} - \frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}\varphi_{2k}\right) e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}\right] X_{2k-1}(x) \\ &+ \sum_{k=1}^{\infty}\left[\frac{1}{1+\varepsilon(2\pi k)^2}\int\limits_0^t r(\tau)\left(f_{2k-1}(\tau) - \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\,(t-\tau)\, f_{2k}(\tau)\right)\, e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\, d\tau\right] X_{2k-1}(x) \end{aligned} \tag{5}$$

where

$$\varphi_k = \int\limits_0^1 \varphi(x) Y_k(x) dx,\ \ f_k(t) = \int\limits_0^1 f(x,t,u) Y_k(x) dx, k = 0,1,2,....$$

Assume that

$$\begin{aligned} u_0(t) &= \varphi_0 + 2\int\limits_0^t\int\limits_0^1 r(\tau) f(\xi,\tau,u(\xi,\tau))\xi d\xi d\tau \\ u_{2k}(t) &= \varphi_{2k}\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int\limits_0^t\int\limits_0^1 r(\tau) f(\xi,\tau,u(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi\ d\xi d\tau \\ u_{2k-1}(t) &= \left(\varphi_{2k-1} - \frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}\varphi_{2k}\right) e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} \\ &+ \frac{1}{1+\varepsilon(2\pi k)^2}\int\limits_0^t\int\limits_0^1 r(\tau) f(\xi,\tau,u(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\ \xi\cos 2\pi k\xi d\xi d\tau \\ &- \frac{(2\pi k)^2}{(1+\varepsilon(2\pi k)^2)^2}\int\limits_0^t\int\limits_0^1 (t-\tau) r(\tau) f(\xi,\tau,u(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi\ d\xi d\tau. \end{aligned}$$

Differentiating (4), we obtain

$$\int\limits_0^1 u_t(x,t) dx = E'(t), 0 \le t \le T. \tag{6}$$

from (5) and (6), we get

$$r(t) = \frac{E'(t)}{\int_0^t f(x,t,u)dx} + \frac{\sum_{k=1}^{\infty} \frac{8\pi k}{1+\varepsilon(2\pi k)^2}\left(\varphi_{2k}\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 r(\tau) f(\xi,\tau,u(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi \, d\xi d\tau\right)}{\int_0^t f(x,t,u)dx}. \tag{7}$$

Definition 2.2. *Denote the set of continuous functions* $\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t), k = 1, ..., n\}$, *on* $[0, T]$ *satisfying the condition* $2\max_{0\le t\le T}|u_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}|u_{2k-1}(t)|\right) < \infty$, *by* $\mathbf{B}_1$. *Let* $\|u(t)\| = 2\max_{0\le t\le T}|u_0(t)| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}|u_{2k-1}(t)|\right)$, *be the norm in* $\mathbf{B}_1$.

It can be shown that $\mathbf{B}_1$ is the Banach space.

Theorem 2.3. *Let assumptions* $(A_1) - (A_3)$ *be satisfied. Then, the inverse problem (1)-(4) has a unique solution in* D.

Proof. An iteration for Fourier coefficient of (5) is defined as follows:

$$\begin{aligned}
u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \int_0^t\int_0^1 r^{(N)}(\tau) f(\xi,\tau,u^{(N)}(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \xi d\xi d\tau \\
u_{2k}^{(N+1)}(t) &= u_{2k}^{(0)}(t) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 r^{(N)}(\tau) f(\xi,\tau,u^{(N)}(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi \, d\xi d\tau \\
u_{2k-1}^{(N+1)}(t) &= u_{2k-1}^{(0)}(t) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 r^{(N)}(\tau) f(\xi,\tau,u^{(N)}(\xi,\tau)) \xi \cos 2\pi k\xi e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} d\xi d\tau \\
&\quad - \frac{(2\pi k)^2}{(1+\varepsilon(2\pi k)^2)^2}\int_0^t\int_0^1 (t-\tau) r^{(N)}(\tau) f(\xi,\tau,u^{(N)}(\xi,\tau)) \sin 2\pi k\xi \, e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} d\xi d\tau
\end{aligned} \tag{8}$$

where, $N = 0, 1, 2, ...$

An iteration for (7) is defined as:

$$r^{(N)}(t) = \frac{E'(t) + \sum_{k=1}^{\infty} \frac{8\pi k}{1+\varepsilon(2\pi k)^2}\left(\varphi_{2k}\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 r(\tau) f(\xi,\tau,u^{(N)}(\xi,\tau)) e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi \, d\xi d\tau\right)}{\int_0^t f(x,t,u^{(N)})dx} \tag{9}$$

Let

$$\begin{aligned}
u_0^{(0)}(t) &= \varphi_0, u_{2k}^{(0)}(t) = \varphi_{2k} e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}, \\
u_{2k-1}^{(0)}(t) &= \left(\varphi_{2k-1} - \frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}\varphi_{2k}\right) e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}.
\end{aligned}$$

From the conditions of the theorem, we have $u^{(0)}(t) \in \mathbf{B}_1$.
Let us write $N = 0$ in (8).

$$\begin{aligned} u_0^{(1)}(t) &= u_0^{(0)}(t) + \int_0^t \int_0^1 r^{(0)}(\tau)[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)]\xi d\xi d\tau \\ &+ \int_0^t \int_0^1 r^{(0)}(\tau) f(\xi,\tau,0)\xi d\xi d\tau. \end{aligned}$$

Applying the Cauchy inequality, the Lipschitzs condition and taking the maximum of both sides of the last inequality, we have:

$$\begin{aligned} & 2\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| \\ \le\ & 2\left|\varphi_0\right| + 2\sqrt{T}\,\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{C[0,T]} + 2\sqrt{T}\left\|f(x,t,0)\right\|_{L_2(D)}. \end{aligned}$$

$$\begin{aligned} & u_{2k}^{(1)}(t) \\ =\ & u_{2k}^{(0)}(t) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 r^{(0)}(\tau)[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)]e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi d\xi d\tau \\ & + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 r^{(0)}(\tau) f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi d\xi d\tau. \end{aligned}$$

Applying the Cauchy inequality, the Hölder inequality, the Bessel inequality, the Lipschitz condition and taking maximum of both sides of the last inequality, we obtain:

$$\begin{aligned} 4\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right| \le\ & 4\sum_{k=1}^{\infty}\left|\varphi_{2k}\right| + \frac{\pi}{3\sqrt{2}}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} \\ & + \frac{\pi}{3\sqrt{2}}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}}. \end{aligned}$$

Applying the same estimations to $u_{2k-1}^{(1)}(t)$, we deduce

$$\begin{aligned} & 4\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right| \\ \le\ & 4\sum_{k=1}^{\infty}\left|\varphi_{2k-1}\right| + 4|T|\sum_{k=1}^{\infty}\left|\varphi_{2k}\right| \\ & + \left(\frac{\pi}{3\sqrt{2}} + \frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} \\ & + \left(\frac{\pi}{3\sqrt{2}} + \frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} M. \end{aligned}$$

Finally, we have the following inequality:

$$\begin{aligned}
&\left\|u^{(1)}(t)\right\|_{\mathbf{B}_1}\\
=\;& 2\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right| + \max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right|\right)\\
\le\;& \left\|\varphi\right\|\\
&+\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}}\\
&+\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}}M.
\end{aligned}$$

where $\left\|\varphi\right\| = 2\left|\varphi_0\right| + 4\sum\limits_{k=1}^{\infty}\left[(1+|T|)\left|\varphi_{2k}\right| + \left|\varphi_{2k-1}\right|\right]$. Hence $u^{(1)}(t)\in\mathbf{B}_1$. In the same way, for a general value of N, we provide

$$\begin{aligned}
&\left\|u^{(N)}(t)\right\|_{\mathbf{B}_1}\\
=\;& 2\max_{0\le t\le T}\left|u_0^{(N)}(t)\right| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{2k-1}^{(N)}(t)\right|\right)\\
\le\;& \left\|\varphi\right\|\\
&+\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(N-1)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(N-1)}(t)\right\|_{\mathbf{C[0,T]}}\\
&+\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r^{(N-1)}(t)\right\|_{\mathbf{C[0,T]}}M.
\end{aligned}$$

Since $u^{(N-1)}(t)\in\mathbf{B}_1$ and from the conditions of the theorem, we have $u^{(N)}(t)\in\mathbf{B}_1$,

$$\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k=1,2,...\}\in\mathbf{B}_1.$$

$$\begin{aligned}
&\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\\
\le\;& \frac{\left|E'(t)\right|}{M} + 4\sqrt{3}\pi\,\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}}\\
&+4\sqrt{3}\pi\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}}.
\end{aligned}$$

In the same way, for a general value of N, we have

$$\begin{aligned}
&\left\|r^{(N)}(t)\right\|_{\mathbf{C[0,T]}}\\
\le\;& \frac{\left|E'(t)\right|}{M} + 4\sqrt{3}\pi\,\|b(x,t)\|_{L_2(D)}\left\|u^{(N-1)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(N-1)}(t)\right\|_{\mathbf{C[0,T]}}\\
&+4\sqrt{3}\pi\left\|r^{(N-1)}(t)\right\|_{\mathbf{C[0,T]}}.
\end{aligned}$$

Now, we prove that the iterations $u^{(N+1)}(t)$ and $r^{(N+1)}(t)$ converge in $\mathbf{B}_1$ and $\mathbf{C[0,T]}$, respectively, as

$N \to \infty$.

$$
\begin{aligned}
& u^{(1)}(t) - u^{(0)}(t) \\
= \; & 2\left(u_0^{(1)}(t) - u_0^{(0)}(t)\right) + 4\sum_{k=1}^{\infty}[(u_{2k}^{(1)}(t) - u_{2k}^{(0)}(t)) + (u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t))] \\
= \; & 2\left(\int_0^t \int_0^1 r^{(0)}(\tau)\left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]\xi d\xi d\tau\right) \\
& +2\int_0^t \int_0^1 r^{(0)}(\tau) f(\xi,\tau,0)\xi d\xi d\tau \\
& +4\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 r^{(0)}(\tau)\left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi d\xi d\tau \\
& +4\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 r^{(0)}(\tau) f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi d\xi d\tau \\
& +4\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 r^{(0)}(\tau)\left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\xi\cos 2\pi k\xi d\xi d\tau \\
& +4\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 r^{(0)}(\tau) f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\xi\cos 2\pi k\xi d\xi d\tau \\
& -4\sum_{k=1}^{\infty}\frac{(2\pi k)^2}{(1+\varepsilon(2\pi k)^2)^2}\int_0^t \int_0^1 (t-\tau)\, r^{(0)}(\tau)\left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi d\xi d\tau \\
& -4\sum_{k=1}^{\infty}\frac{(2\pi k)^2}{(1+\varepsilon(2\pi k)^2)^2}\int_0^t \int_0^1 (t-\tau)\, r^{(0)}(\tau) f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi d\xi d\tau
\end{aligned}
$$

Applying Cauchy inequality, Hölder inequality, Lipschitzs condition and Bessel inequality to the last equation, we obtain:

$$
\begin{aligned}
& \left\|u^{(1)}(t) - u^{(0)}(t)\right\|_{B_1} \\
\leq \; & \left(2\sqrt{T} + \frac{\pi}{3\sqrt{2}} + \frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} \\
& +\left(2\sqrt{T} + \frac{2}{\sqrt{3}} + 4\sqrt{2}T\right)\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} M.
\end{aligned}
$$

$$
\begin{aligned}
A \; = \; & \left(2\sqrt{T} + \frac{\pi}{3\sqrt{2}} + \frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} \\
& +\left(2\sqrt{T} + \frac{\pi}{3\sqrt{2}} + \frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r^{(0)}(t)\right\|_{\mathbf{C[0,T]}} M.
\end{aligned}
$$

Applying the same estimations, we get

$$\begin{aligned}&\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1}\\ \leq\ &\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u^{(1)}-u^{(0)}\right\|_{B_1}\\ &+\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r^{(1)}(t)-r^{(0)}(t)\right\|_{\mathbf{C[0,T]}}M.\end{aligned}$$

$$\begin{aligned}&\left\|r^{(1)}-r^{(0)}\right\|_{\mathbf{C[0,T]}}\\ \leq\ &\frac{4\sqrt{3}}{M(1-4\sqrt{3})}\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u^{(1)}-u^{(0)}\right\|_{\mathbf{B}_1}.\end{aligned}$$

$$\begin{aligned}&\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{\mathbf{B}_1}\\ \leq\ &\left(\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left(1+\frac{4\sqrt{3}}{M(1-4\sqrt{3})}\right)\right)\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}A\end{aligned}$$

where

$$B=\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left(1+\frac{4\sqrt{3}}{M(1-4\sqrt{3})}\right)$$

By making use of the same estimations,

$$\begin{aligned}&\left\|r^{(2)}-r^{(1)}\right\|_{\mathbf{C[0,T]}}\\ \leq\ &\frac{4\sqrt{3}}{M(1-4\sqrt{3})}\left\|r^{(2)}(t)\right\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u^{(2)}-u^{(1)}\right\|_{\mathbf{B}_1}.\end{aligned}$$

$$\begin{aligned}\left\|u^{(3)}(t)-u^{(2)}(t)\right\|_{\mathbf{B}_1}\ \leq\ &\frac{A}{\sqrt{2}}B^2x\\ &\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\left\|r^{(2)}(t)\right\|_{\mathbf{C[0,T]}}\|b(x,t)\|^2_{L_2(D_T)}.\end{aligned}$$

For N :

$$\left\|r^{(N+1)}-r^{(N)}\right\|_{\mathbf{C[0,T]}}\leq\frac{4\sqrt{3}}{M(1-4\sqrt{3})}\left\|r^{(N+1)}(t)\right\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u^{(N+1)}-u^{(N)}\right\|_{\mathbf{B}_1}.$$

$$\begin{aligned}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{\mathbf{B}_1}\ \leq\ &\frac{A}{\sqrt{N!}}\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\left\|r^{(2)}(t)\right\|_{\mathbf{C[0,T]}}\cdots\left\|r^{(N)}(t)\right\|_{\mathbf{C[0,T]}}x\\ &B^N\|b(x,t)\|^N_{L_2(D_T)}.\end{aligned}\tag{10}$$

By the Weierstrass M test, we deduce the last series is uniformly convergent to an element of B_1. It is easy to see that if $u^{(N+1)}\to u^{(N)}$, $N\to\infty$, then $r^{(N+1)}\to r^{(N)}$, $N\to\infty$.
Therefore $u^{(N+1)}(t)$ and $r^{(N+1)}(t)$ converge in $\mathbf{B}_1$ and $\mathbf{C[0,T]}$, respectively.
Now, let us show that there exists u and r such that

$$\lim_{N\to\infty}u^{(N+1)}(t)=u(t),\ \lim_{N\to\infty}r^{(N+1)}(t)=r(t).$$

Applying the Cauchy inequality, the Hölder Inequality, the Lipschitzs condition and the Bessel inequality to $\left|u-u^{(N+1)}\right|$ and $\left|r-r^{(N)}\right|$

$$\begin{aligned} & \left\|r-r^{(N+1)}\right\|_{\mathbf{C[0,T]}} \\ \leq\ & \frac{4\sqrt{3}}{M(1-4\sqrt{3})}\|r(t)\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u-u^{(N+1)}\right\|_{\mathbf{B}_1} \\ & \frac{4\sqrt{3}}{M(1-4\sqrt{3})}\|r(t)\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u^{(N+1)}-u^{(N)}\right\|_{\mathbf{B}_1}. \end{aligned} \tag{11}$$

$$\begin{aligned} \left\|u-u^{(N+1)}\right\|_{\mathbf{B}_1} \leq\ & B\|r(t)\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u-u^{(N+1)}\right\|_{\mathbf{B}_1} \\ & +\|r(t)\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\left\|u^{(N+1)}-u^{(N)}\right\|_{\mathbf{B}_1}. \end{aligned}$$

Applying the Gronwall inequality to the last inequality, using inequality (9) and taking maximum of both sides, we have

$$\begin{aligned} & \left\|u(t)-u^{(N+1)}(t)\right\|_{B_1}^2 \\ \leq\ & 2\frac{A^2}{N!}B^{2(N+1)} \\ & \times\left\{\|b(x,t)\|_{L_2(D)}^{N+1}\|r(t)\|_{\mathbf{C[0,T]}}\left\|r^{(1)}(t)\right\|_{\mathbf{C[0,T]}}\left\|r^{(2)}(t)\right\|_{\mathbf{C[0,T]}}\cdots\left\|r^{(N)}(t)\right\|_{\mathbf{C[0,T]}}\right\}^2 \\ & \times\exp 2B^2\|b(x,t)\|_{L_2(D)}^2\|r(t)\|_{\mathbf{C[0,T]}}^2. \end{aligned} \tag{12}$$

Then $N\to\infty$ we obtain $u^{(N+1)}\to u$, $r^{(N+1)}\to r$.

For the uniqueness, we assume that problem (1)-(4) has two solution pairs (r,u), (q,v). Applying the Cauchy inequality, the Hölder Inequality, the Lipshitzs condition and the Bessel inequality to $|u(t)-v(t)|$ and $\left|r(t)-q(t)\right|$, we obtain:

$$\begin{aligned} & \|u(t)-v(t)\|_{\mathbf{B}_1} \\ \leq\ & \left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\|r(t)\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\|u-v\|_{\mathbf{B}_1} \\ & +\left(2\sqrt{T}+\frac{\pi}{3\sqrt{2}}+\frac{2\pi^2\sqrt{T}}{3\sqrt{2}}\right)\left\|r(t)-q(t)\right\|M, \end{aligned} \tag{13}$$

$$\left\|r(t)-q(t)\right\| \leq \frac{4\sqrt{3}}{M(1-4\sqrt{3})}\|r(t)\|_{\mathbf{C[0,T]}}\|b(x,t)\|_{L_2(D)}\|u-v\|_{\mathbf{B}_1},$$

applying the Gronwall inequality to the last inequality, we have

$u(t)=v(t)$. Hence $r(t)=q(t)$. □

The theorem is proved.

3. Continuous Dependence of (a,u) upon the data

Theorem 3.1. *Under assumption (A1)-(A3) the solution (r,u) of the problem (1)-(4) depends continuously upon the data φ, E.*

Proof. Let $\Phi = \{\varphi,\ E,\ f\}$ and $\overline{\Phi} = \left\{\overline{\varphi},\ \overline{E},\ f\right\}$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Suppose that there exist positive constants M_i, $i = 1, 2$ such that

$$\|E\|_{C^1[0,T]} \le M_1, \left\|\overline{E}\right\|_{C^1[0,T]} \le M_1, \qquad \left\|\varphi\right\|_{C^3[0,1]} \le M_2, \left\|\overline{\varphi}\right\|_{C^3[0,1]} \le M_2.$$

Let us denote $\|\Phi\| = (\|E\|_{C^1[0,T]} + \left\|\varphi\right\|_{C^3[0,1]} + \left\|f\right\|_{C^{4,0}(\overline{D})})$. Let (r, u) and $(\overline{r}, \overline{u})$ be solutions of inverse problems (1)-(4) corresponding to the data $\Phi = \{\varphi,\ E,\ f\}$ and $\overline{\Phi} = \left\{\overline{\varphi},\ \overline{E},\ f\right\}$ respectively. According to (5), we can write

$$\begin{aligned}
&u - \overline{u} \\
= \ & 2\left(\varphi_0 - \overline{\varphi_0}\right) + 4\sum_{k=1}^{\infty} \sin 2\pi k\xi \ \left(\varphi_{2k} - \overline{\varphi_{2k}}\right) e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} \\
&+4\sum_{k=1}^{\infty} \cos 2k\xi \ \left(\varphi_{2k-1} - \overline{\varphi_{2k-1}}\right) e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} \\
&+2\left(\int_0^t \int_0^1 r(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] d\xi d\tau\right) \\
&+2\left(\int_0^t \int_0^1 (r(\tau) - \overline{r}(\tau))\, f(\xi,\tau,\overline{u}(\xi,\tau)) d\xi d\tau\right) \\
&+4\sum_{k=1}^{\infty} \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 r(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi \ d\xi d\tau \\
&+4\sum_{k=1}^{\infty} \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 (r(\tau) - \overline{r}(\tau))\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi d\xi d\tau \\
&+4\sum_{k=1}^{\infty} \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 r(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \xi \cos 2\pi k\xi \ d\xi d\tau \\
&+4\sum_{k=1}^{\infty} \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 (r(\tau) - \overline{r}(\tau))\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \xi \cos 2\pi k\xi \ d\xi d\tau \\
&-4\sum_{k=1}^{\infty} \frac{(2\pi k)^2}{(1+\varepsilon(2\pi k)^2)^2} \int_0^t \int_0^1 (t-\tau) r(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi d\xi d\tau \\
&-4\sum_{k=1}^{\infty} \frac{(2\pi k)^2}{(1+\varepsilon(2\pi k)^2)^2} \int_0^t \int_0^1 (t-\tau)\,(r(\tau) - \overline{r}(\tau))\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right] e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \sin 2\pi k\xi d\xi d\tau
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\|u(t) - \overline{u}(t)\right\|_{B_1} \ \le \ & M_4 \left\|\Phi - \overline{\Phi}\right\| \\
&+M_3\left(\int_0^t \int_0^\pi r^2(\tau) b^2(\xi,\tau) \left|u(\tau) - \overline{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}
\end{aligned} \tag{14}$$

$$\begin{aligned}\left\|r(t)-q(t)\right\|_{\mathbf{C[0,T]}} &\leq M_5\left\|\Phi-\overline{\Phi}\right\| \\ &+M_6\left\|r(t)\right\|_{\mathbf{C[0,T]}}\left\|u-\overline{u}\right\|_{B_1}\end{aligned}$$

applying Gronwall's inequality to (14), we obtain:

$$\begin{aligned}\left\|u(t)-\overline{u}(t)\right\|_{B_1}^2 &\leq 2M_4^2\left\|\Phi-\overline{\Phi}\right\|^2 \\ &x\exp 2M_3^2\left(\int_0^t\int_0^\pi r^2(\tau)b^2(\xi,\tau)d\xi d\tau\right).\end{aligned}$$

For $\Phi\rightarrow\overline{\Phi}$ then $u\rightarrow\overline{u}$. Hence $r\rightarrow\overline{r}$. □

4. Numerical Procedure for the nonlinear problem (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\frac{\partial u^{(n)}}{\partial t} = \frac{\partial^2 u^{(n)}}{\partial x^2}+r(t)f(x,t,u^{(n-1)}),\quad (x,t)\in D \tag{15}$$

$$u^{(n)}(0,t) = u^{(n)}(1,t),\quad t\in[0,T] \tag{16}$$

$$u_x^{(n)}(1,t) = 0,\ t\in[0,T] \tag{17}$$

$$u^{(n)}(x,0) = \varphi(x)\,,\quad x\in[0,1]\,. \tag{18}$$

Let $u^{(n)}(x,t)=v(x,t)$ and $f(x,t,u^{(n-1)})=\widetilde{f}(x,t)$. Then the problem (15)-(18) can be written as a linear problem:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}+r(t)\widetilde{f}(x,t)\quad (x,t)\in D \tag{19}$$

$$v(0,t) = v(1,t),\quad t\in[0,T]\ , \tag{20}$$

$$v_x(1,t) = 0,\quad t\in[0,T]\ , \tag{21}$$

$$v(x,0) = \varphi(x),\quad x\in[0,1]\ . \tag{22}$$

In [1], the problems are linear. In this problem in order to use the similar methods in these papers, firstly we use the method of the linearization, then we use the finite difference method to solve (19)-(22) with a predictor-corrector type approach.

We subdivide the intervals $[0,1]$ and $[0,T]$ into N_x and N_t subintervals of equal lengths $h=\frac{1}{N_x}$ and $\tau=\frac{T}{N_t}$, respectively. We choose the Crank-Nicolson scheme, which is absolutely stable and has a second order accuracy in both h and τ. The Crank-Nicolson scheme for (19)-(22) is as follows:

$$\begin{aligned}\frac{1}{\tau}\left(v_i^{j+1}-v_i^j\right) &= \frac{1}{2h^2}\left[\left(v_{i-1}^j-2v_i^j+v_{i+1}^j\right)\right. \\ &\left.+\left(v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1}\right)\right] \\ &+\frac{1}{4}\left(r^j+r^{j+1}\right)\left(\widetilde{f}_i^{j+1}+\widetilde{f}_i^j\right),\end{aligned} \tag{23}$$

$$v_i^0=\phi_i, \tag{24}$$

$$v_0^j=v_{N_x}^j, \tag{25}$$

$$v_{N_x-1}^j=v_{N_x+1}^j, \tag{26}$$

where $1 \le i \le N_x$ and $0 \le j \le N_t$ are the indices for the spatial and time steps respectively, $v_i^j = v(x_i, t_j)$, $\phi_i = \varphi(x_i)$, $\widetilde{f}_i^j = \widetilde{f}(x_i, t_j)$, $x_i = ih$, $t_j = j\tau$. At the $t = 0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

Now, let us construct the predicting-correcting mechanism. First, integrating the equation (1) respect to x from 0 to 1 and using (3) and (4), we obtain

$$r(t) = \frac{E'(t) + v_x(0,t)}{\int_0^1 \widetilde{f}(x,t)dx}. \tag{27}$$

The finite difference approximation of (27) is

$$r^j = \frac{\left(\left(E^{j+1} - E^j\right)/\tau\right) + \left(v_{N_x+1}^j - v_{N_x}^j\right)/h}{(\widetilde{fin})^j},$$

where $E^j = E(t_j)$, $(\widetilde{fin})^j = \int_0^1 \widetilde{f}(x, t_j)dx$, $j = 0, 1, ..., N_t$.

For $j = 0$,

$$r^0 = \frac{-\left(\left(E^1 - E^0\right)/\tau\right) + \left(\phi_{N_x+1} - \phi_{N_x}\right)/h}{(\widetilde{fin})^0},$$

and the values of ϕ_i provide us to start our computation. We denote the values of p^j, v_i^j at the s-th iteration step $r^{j(s)}$, $v_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $r^{j+1(0)} = r^j$, $v_i^{j+1(0)} = v_i^j$, $j = 0, 1, 2, ...N_t$, $i = 1, 2, ..., N_x$. At each $(s+1)$-th iteration step we first determine $r^{j+1(s+1)}$ from the formula

$$r^{j+1(s+1)} = \frac{-\left(\left(E^{j+2} - E^{j+1}\right)/\tau\right) + \left(v_{N_x+1}^{j+1(s)} - v_{N_x}^{j+1(s)}\right)/h}{(\widetilde{fin})^{j+1}}.$$

Then from (23)-(26) we obtain

$$\begin{aligned} \frac{1}{\tau}\left(v_i^{j+1(s+1)} - v_i^{j+1(s)}\right) = {} & \frac{1}{2h^2}\left[\left(v_{i-1}^{j+1(s+1)} - 2v_i^{j+1(s+1)} + v_{i+1}^{j+1(s+1)}\right)\right. \\ & \left. + \left(v_{i-1}^{j+1(s)} - 2v_i^{j+1(s)} + v_{i+1}^{j+1(s)}\right)\right] \\ & + \frac{1}{4}\left(r^{j+1(s+1)} + r^{j+1(s)}\right)\left(\widetilde{f}_i^{j+1} + \widetilde{f}_i^j\right), \end{aligned} \tag{28}$$

$$v_0^{j+1(s)} = v_{N_x+1}^{j+1(s)}, \tag{29}$$

$$v_{N_x-1}^{j(s)} = v_{N_x+1}^{j(s)}, \tag{30}$$

The system of equations (28)-(30) can be solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $r^{j+1(s+1)}$, $v_i^{j+1(s+1)} (i = 1, 2, ..., N_x)$ as r^{j+1}, $v_i^{j+1} (i = 1, 2, ..., N_x)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

5. Numerical Example

Example 5.1. *If we consider inverse problem (1)-(4), with*

$$
\begin{aligned}
f(x,t,u) &= (1+4\cos 2x - 4\sin^2 x + 4\varepsilon^2\cos 2x - 4\varepsilon^2 \sin^2 x)u\exp(-t),\\
\varphi(x) &= \exp(\cos 2x), E(t) = \frac{\pi^2}{2}\exp(\varepsilon t),\ \ x\in[0,\pi],\ t\in[0,T],
\end{aligned}
$$

then it is easy to check the analytical solution of the problem as following:

$$\{r(t),\ u(x,t)\} = \{\exp(t), \exp\ (\varepsilon t + \cos(2x))\}.$$

Let us apply the scheme which was explained in the previous section to the step sizes $h = 0.0393$, $\tau = 0.01$.

In the case when $T = 1$, *the comparisons between the analytical solution and the numerical finite difference solution are shown in Figures 1 and 2 when* $\varepsilon = 0.1$.

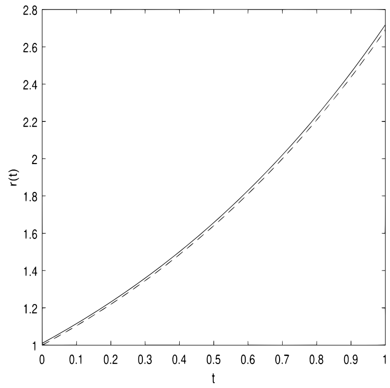

Figure 1: Exact and approximate r(t) when T=1.

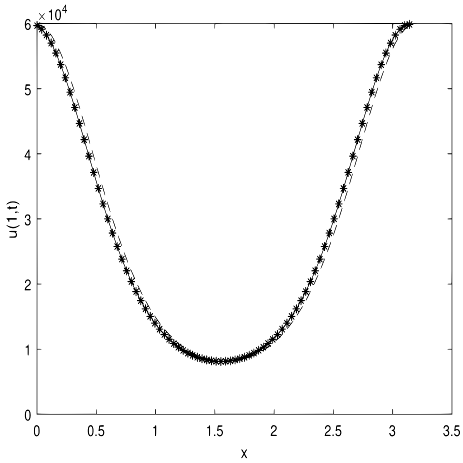

Figure 2: Exact and approximate solutions of u(x,t) at the T=1

It is clear from these results that, this method has been shown to produce stable and reasonably accurate results for these example.

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