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Subgradient-like extragradient algorithms for systems of variational inequalities with constraints

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Abstract. In this paper, we introduce a modified viscosity subgradient-like extragradient implicit rule with line-search process for finding a solution of a general system of variational inequalities (GSVI) with a variational inequality (VIP) and a fixed-point (FPP) constraints in Hilbert spaces. The suggested algorithms are based on the subgradient extragradient method with line-search process, hybrid Mann implicit iteration method, and composite viscosity approximation method. Under suitable restrictions, we demonstrate the strong convergence of the suggested algorithm to a solution of the GSVI with the VIP and FPP constraints, which is a unique solution of a certain hierarchical variational inequality.

1. Introduction

Let P_C be the metric (or nearest point) projection from H onto C where $\emptyset \neq C \subset H$ with C being closed and convex in a real Hilbert space H. We denote by the $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and induced norm in H, respectively. Given a nonlinear operator $T : C \to H$. Let the Fix(T) and $\mathbf{R}$ indicate the fixed-point set of T and the real-number set, respectively. Let the $\to$ and $\rightharpoonup$ represent the strong convergence and weak convergence in H, respectively. An operator $T : C \to C$ is referred as being asymptotically nonexpansive if $\exists\{\theta_\iota\}_{\iota=1}^\infty \subset [0, +\infty)$ s.t. $\lim_{\iota\to\infty} \theta_\iota = 0$ and

$$\|T^\iota u - T^\iota v\| \leq (1 + \theta_\iota)\|u - v\|, \quad \forall \iota \geq 1,\ u, v \in C. \tag{1}$$

In particular, whenever $\theta_\iota = 0\ \forall\iota \geq 1$, T is known as being nonexpansive. Given a self-mapping A on H. The classical variational inequality problem (VIP) is the one of finding $u \in C$ s.t. $\langle Au, v - u\rangle \geq 0\ \ \forall v \in C$. The solution set of the VIP is written as VI(C, A). To the best of our knowledge, one of the most effective approaches for solving the VIP is the extragradient one put forward by Korpelevich [18] in 1976, i.e., for any initial point $u_0 \in C$, let $\{u_\iota\}$ be the sequence constructed below

$$\begin{cases} v_\iota = P_C(u_\iota - \ell A u_\iota), \\ u_{\iota+1} = P_C(u_\iota - \ell A v_\iota), \quad \forall \iota \geq 0, \end{cases} \tag{2}$$

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where $\ell \in (0, \frac{1}{L})$ and L is Lipschitz constant of A. Whenever $\mathrm{VI}(C, A) \neq \emptyset$, the sequence $\{u_\iota\}$ converges weakly to a point in $\mathrm{VI}(C, A)$. Fixed point problems and variational inequalities have been studied extensively, see e.g., [1–5, 7–9, 13–17, 19–25, 27–36] and references therein.

Suppose that $B_1, B_2 : C \to H$ are two nonlinear operators. Consider the following problem of finding $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 B_1 v^* + u^* - v^*, v - u^* \rangle \geq 0, & \forall v \in C, \\ \langle \rho_2 B_2 u^* + v^* - u^*, v - v^* \rangle \geq 0, & \forall v \in C, \end{cases} \tag{3}$$

with constants $\rho_1, \rho_2 > 0$. The problem (3) is called a general system of variational inequalities (GSVI). Note that GSVI (3) can be transformed into the fixed-point problem below.

Lemma 1.1 ([5]). *For given $u^*, v^* \in C$, (u^*, v^*) is a solution of GSVI (1.3) if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of the mapping $G := P_C(I - \rho_1 B_1)P_C(I - \rho_2 B_2)$, and $y^* = P_C(I - \rho_2 B_2)x^*$.*

Suppose that the mappings B_1, B_2 are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : C \to C$ be a contraction with coefficient $\delta \in [0, 1)$ and $F : C \to H$ be κ-Lipschitzian and η-strongly monotone with constants $\kappa, \eta > 0$ such that $\delta < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0, 1]$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_\iota\}$ such that $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \neq \emptyset$, where $\mathrm{Fix}(G)$ is the fixed-point set of the mapping $G := P_C(I - \rho_1 B_1)P_C(I - \rho_2 B_2)$ for $\rho_1 \in (0, 2\alpha)$ and $\rho_2 \in (0, 2\beta)$. Recently, Cai, Shehu and Iyiola [2] proposed the modified viscosity implicit rule for finding an element of Ω, that is, for any initial $x_1 \in C$, let $\{x_\iota\}$ be the sequence constructed below

$$\begin{cases} u_\iota = \beta_\iota x_\iota + (1 - \beta_\iota) y_\iota, \\ v_\iota = P_C(u_\iota - \rho_2 B_2 u_\iota), \\ y_\iota = P_C(v_\iota - \rho_1 B_1 v_\iota), \\ x_{\iota+1} = P_C[\alpha_\iota f(x_\iota) + (I - \alpha_\iota \rho F) T^\iota y_\iota], & \forall \iota \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_\iota\}$ and $\{\beta_\iota\}$ are sequences in $(0, 1]$ such that

(i) $\sum_{\iota=1}^\infty |\alpha_{\iota+1} - \alpha_\iota| < \infty$ and $\sum_{\iota=1}^\infty \alpha_\iota = \infty$;

(ii) $\lim_{l\to\infty} \alpha_\iota = 0$ and $\lim_{\iota\to\infty} \frac{\theta_\iota}{\alpha_\iota} = 0$;

(iii) $0 < \varepsilon \leq \beta_\iota \leq 1$ and $\sum_{\iota=1}^\infty |\beta_{\iota+1} - \beta_\iota| < \infty$;

(iv) $\sum_{\iota=1}^\infty \|T^{\iota+1} y_\iota - T^\iota y_\iota\| < \infty$.

It was proved in [2] that the sequence $\{x_\iota\}$ converges strongly to an element $x^* \in \Omega$, which is a unique solution of the hierarchical variational inequality (HVI): $\langle (\rho F - f)x^*, p - x^* \rangle \geq 0$, $\forall p \in \Omega$. In 2019, Thong and Hieu [24] proposed the inertial subgradient extragradient method with line-search process for solving the monotone VIP with Lipschitz continuous mapping A and the fixed-point problem (FPP) of a quasi-nonexpansive mapping T with a demiclosedness property. Assume that $\Omega := \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \neq \emptyset$. Given $\{\alpha_\iota\} \subset [0, 1]$ and $\{\beta_\iota\} \subset (0, 1)$. For any initial $x_0, x_1 \in H$, the sequence $\{x_\iota\}$ is constructed below.

Algorithm 1.2 ([24]). *Initialization: Given $\gamma > 0$, $\ell \in (0, 1)$, $\mu \in (0, 1)$. Iterative Steps: Compute $x_{\iota+1}$ below:*

Step 1. Set $w_\iota = x_\iota + \alpha_\iota(x_\iota - x_{\iota-1})$ and calculate $v_\iota = P_C(w_\iota - \tau_\iota A w_\iota)$, where τ_ι is chosen to be the largest $\tau \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ satisfying $\tau\|Aw_\iota - Av_\iota\| \leq \mu\|w_\iota - v_\iota\|$.

Step 2. Calculate $z_\iota = P_{C_\iota}(w_\iota - \tau_\iota A v_\iota)$ with $C_\iota := \{v \in H : \langle w_\iota - \tau_\iota A w_\iota - v_\iota, v - v_\iota \rangle \leq 0\}$.

Step 3. Calculate $x_{\iota+1} = (1 - \beta_\iota) w_\iota + \beta_\iota S z_\iota$. If $w_\iota = z_\iota = x_{\iota+1}$ then $w_\iota \in \Omega$.

Again set $\iota := \iota + 1$ and go to Step 1.

It was proven in [24] that under suitable conditions, $\{x_\iota\}$ converges weakly to an element of Ω. Very recently, Reich et al. [22] suggested the modified projection-type method for solving the pseudomonotone VIP with uniform continuity mapping A. Given a sequence $\{\alpha_\iota\} \subset (0, 1)$ and a contraction $f : C \to C$ with constant $\delta \in [0, 1)$. For any initial $x_1 \in C$, the sequence $\{x_\iota\}$ is constructed below.

Algorithm 1.3 ([22]). *Initialization: Given* $\mu > 0$, $\ell \in (0, 1)$, $\lambda \in (0, \frac{1}{\mu})$. *Iterative Steps: Given the current iterate* x_ι, *calculate* $x_{\iota+1}$ *as follows:*

Step 1. Compute $y_\iota = P_C(x_\iota - \lambda A x_\iota)$ *and* $r_\lambda(x_\iota) := x_\iota - y_\iota$. *If* $r_\lambda(x_\iota) = 0$, *then stop;* x_ι *is a solution of* VI(C, A). *Otherwise,*

Step 2. Compute $w_\iota = x_\iota - \tau_\iota r_\lambda(x_\iota)$, *where* $\tau_\iota := \ell^{j_\iota}$ *and* j_ι *is the smallest nonnegative integer j fulfilling*

$$\langle Ax_\iota - A(x_\iota - \ell^j r_\lambda(x_\iota)), r_\lambda(x_\iota)\rangle \le \frac{\mu}{2}\|r_\lambda(x_\iota)\|^2.$$

Step 3. Compute $x_{\iota+1} = \alpha_\iota f(x_\iota) + (1 - \alpha_\iota)P_{C_\iota}(x_\iota)$, *where* $C_\iota := \{x \in C : h_\iota(x_\iota) \le 0\}$ *and* $h_\iota(x) = \langle Aw_\iota, x - x_\iota\rangle + \frac{\tau_\iota}{2\lambda}\|r_\lambda(x_\iota)\|^2$.

Again set $\iota := \iota + 1$ *and go to Step 1.*

It was proven in [22] that under mild conditions, $\{x_\iota\}$ converges strongly to an element of VI(C, A). In a real Hilbert space H, we always assume that the VIP, GSVI, HVI and FPP represent a variational inequality problem, a general system of variational inequalities, a hierarchical variational inequality and a fixed-point problem of an asymptotically nonexpansive mapping, respectively. We introduce the modified viscosity subgradient-like extragradient implicit rule for finding a solution of the GSVI with the VIP and FPP constraints. The suggested algorithms are based on the subgradient extragradient method with line-search process, hybrid Mann implicit iteration method, and composite viscosity approximation method. Under suitable restrictions, we demonstrate the strong convergence of the suggested algorithms to a solution of the GSVI with the VIP and FPP constraints, which is a unique solution of a certain HVI. In addition, an illustrated example is provided to illustrate the feasibility and applicability of our proposed rule.

The structure of this article is organized below: In Section 2, we present some concepts and basic tools for further use. Section 3 treats the convergence analysis of the suggested algorithms. In the end, Section 4 applies our main results to solve the GSVI, VIP and FPP in an illustrated example. Our results improve and extend the corresponding results announced by some others, e.g., Cai, Shehu and Iyiola [2], Thong and Hieu [24], and Reich et al. [22].

2. Preliminaries

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and assume that $\emptyset \ne C \subset H$ with C being convex and closed. For a given sequence $\{z_k\} \subset H$ we use $z_k \to z^*$ (resp., $z_k \rightharpoonup z^*$) indicate the strong (resp., weak) convergence of $\{z_k\}$ to z^*. An operator $\Psi : C \to H$ is referred to as being

(a) L-Lipschitzian (or L-Lipschitz continuous) if $\exists L > 0$ s.t. $\|\Psi u - \Psi v\| \le L\|u - v\|$, $\forall u, v \in C$;

(b) pseudomonotone if $\langle \Psi u, v - u\rangle \ge 0 \Rightarrow \langle \Psi v, v - u\rangle \ge 0$, $\forall u, v \in C$;

(c) α-strongly monotone if $\exists \alpha > 0$ s.t. $\langle \Psi u - \Psi v, u - v\rangle \ge \alpha\|u - v\|^2$, $\forall u, v \in C$;

(d) β-inverse-strongly monotone if $\exists \beta > 0$ s.t. $\langle \Psi u - \Psi v, u - v\rangle \ge \beta\|\Psi u - \Psi v\|^2$, $\forall u, v \in C$;

(e) sequentially weakly continuous if $\forall \{v_k\} \subset C$, the relation holds: $v_k \rightharpoonup v \Rightarrow \Psi v_k \rightharpoonup \Psi v$.

It is clear that each monotone mapping is pseudomonotone but the converse is not true. It is known that $\forall v \in H$, $\exists|$ (nearest point) $P_C v \in C$ s.t. $\|v - P_C v\| \le \|v - w\|$ $\forall w \in C$. P_C is referred to as a nearest point (or metric) projection of H onto C. Recall that the following conclusions hold (see [10]):

(a) $\langle v - w, P_C v - P_C w\rangle \ge \|P_C v - P_C w\|^2$, $\forall v, w \in H$;

(b) $w = P_C v \Leftrightarrow \langle v - w, u - w\rangle \le 0$, $\forall v \in H, u \in C$;

(c) $\|v - w\|^2 \ge \|v - P_C v\|^2 + \|w - P_C v\|^2$, $\forall v \in H, w \in C$;

(d) $\|v - w\|^2 = \|v\|^2 - \|w\|^2 - 2\langle v - w, w\rangle$, $\forall v, w \in H$;

(e) $\|sv + (1-s)w\|^2 = s\|v\|^2 + (1-s)\|w\|^2 - s(1-s)\|v-w\|^2, \ \forall v, w \in H, s \in [0,1]$.

The following lemmas will be used for demonstrating our main results in the sequel.

Lemma 2.1. *Let the mapping $B : C \to H$ be γ-inverse-strongly monotone. Then, for a given $\lambda \geq 0$,*

$$\|(I - \lambda B)u - (I - \lambda B)v\|^2 \leq \|u - v\|^2 - \lambda(2\gamma - \lambda)\|Bu - Bv\|^2.$$

In particular, if $0 \leq \lambda \leq 2\gamma$, then $I - \lambda B$ is nonexpansive.

Using Lemma 2.1, we immediately derive the following lemma.

Lemma 2.2. *Let the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : C \to C$ be defined as $G := P_C(I - \rho_1 B_1)P_C(I - \rho_2 B_2)$. If $0 \leq \rho_1 \leq 2\alpha$ and $0 \leq \rho_2 \leq 2\beta$, then $G : C \to C$ is nonexpansive.*

Lemma 2.3 ([5]). *Let $A : C \to H$ be pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, v - u\rangle \geq 0 \ \forall v \in C$, if and only if $\langle Av, v - u\rangle \geq 0, \ \forall v \in C$.*

Lemma 2.4 ([26]). *Let $\{a_l\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{l+1} \leq (1 - \lambda_l)a_l + \lambda_l\gamma_l, \ \forall l \geq 1$, where $\{\lambda_l\}$ and $\{\gamma_l\}$ are sequences of real numbers such that (i) $\{\lambda_l\} \subset [0,1]$ and $\sum_{l=1}^{\infty} \lambda_l = \infty$, and (ii) $\limsup_{l\to\infty} \gamma_l \leq 0$ or $\sum_{l=1}^{\infty} |\lambda_l\gamma_l| < \infty$. Then $\lim_{l\to\infty} a_l = 0$.*

Later on, we will make use of the following lemmas to demonstrate our main results.

Lemma 2.5 ([12]). *Let H_1 and H_2 be two real Hilbert spaces. Suppose that $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

Lemma 2.6 ([11]). *Let h be a real-valued function on H and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then $\mathrm{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \ \forall x \in C$, where $\mathrm{dist}(x, K)$ denotes the distance of x to K.*

Lemma 2.7 ([6]). *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{u_k\}$ is a sequence in C such that $u_k \rightharpoonup u \in C$ and $(I - T)u_k \to 0$, then $(I - T)u = 0$, where I is the identity mapping of X.*

The following lemmas are very crucial to the convergence analysis of the proposed algorithms.

Lemma 2.8 ([20]). *Let $\{\Gamma_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{m_k}\}$ of $\{\Gamma_m\}$ which satisfies $\Gamma_{m_k} < \Gamma_{m_k+1}$ for each integer $k \geq 1$. Define the sequence $\{\phi(m)\}_{m \geq m_0}$ of integers below:*

$$\phi(m) = \max\{k \leq m : \Gamma_k < \Gamma_{k+1}\},$$

where integer $m_0 \geq 1$ such that $\{k \leq m_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then the following hold:

(i) $\phi(m_0) \leq \phi(m_0 + 1) \leq \cdots$ *and* $\phi(m) \to \infty$;

(ii) $\Gamma_{\phi(m)} \leq \Gamma_{\phi(m)+1}$ *and* $\Gamma_m \leq \Gamma_{\phi(m)+1}, \ \forall m \geq m_0$.

Lemma 2.9 ([26]). *Let the number $\lambda \in (0,1]$ and the mapping $T : C \to C$ be nonexpansive. Let the mapping $T^\lambda : C \to H$ be defined as $T^\lambda u := (I - \lambda\rho F)Tu, \ \forall u \in C$ with $F : C \to H$ being κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \rho < \frac{2\eta}{\kappa^2}$, i.e., $\|T^\lambda u - T^\lambda v\| \leq (1 - \lambda\tau)\|u - v\|, \ \forall u, v \in C$, where $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0,1]$.*

3. Main Results

In this section, let the feasible set C be a nonempty closed convex subset of a real Hilbert space H, and assume always that the following conditions hold.

$T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$, and $A : H \to H$ is pseudomonotone and uniformly continuous on C, s.t. $\|Az\| \le \liminf_{n\to\infty} \|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup z$.

$B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$ where $G := P_C(I - \rho_1 B_1)P_C(I - \rho_2 B_2)$ for $\rho_1 \in (0, 2\alpha)$ and $\rho_2 \in (0, 2\beta)$.

$f : C \to H$ is a contraction with constant $\delta \in [0, 1)$, and $F : C \to H$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$.

$\{\sigma_n\} \subset [0, 1]$ and $\{\alpha_n\} \subset (0, 1]$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$ and $0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < 1$.

Algorithm 3.1. *Initialization: Given $\mu > 0$, $s \in (0, 1)$, $\ell \in (0, 1)$, $\lambda \in (0, \frac{1}{\mu})$. Let $x_1 \in C$ be arbitrary. Iterative Steps: Given the current iterate x_n, calculate x_{n+1} below:*

Step 1. Calculate $w_n = (1 - \sigma_n)x_n + \sigma_n[su_n + (1 - s)T^n x_n]$ with

$$\begin{cases} v_n = P_C(w_n - \rho_2 B_2 w_n), \\ u_n = P_C(v_n - \rho_1 B_1 v_n). \end{cases}$$

Step 2. Calculate $y_n = P_C(w_n - \lambda A w_n)$ and $r_\lambda(w_n) := w_n - y_n$.

Step 3. Calculate $t_n = w_n - \tau_n r_\lambda(w_n)$, where $\tau_n := \ell^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$\langle Aw_n - A(w_n - \ell^j r_\lambda(w_n)), w_n - y_n\rangle \le \frac{\mu}{2}\|r_\lambda(w_n)\|^2. \tag{5}$$

Step 4. Compute $z_n = P_{C_n}(w_n)$ and $x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \rho F)T^n z_n]$, where $C_n := \{u \in C : h_n(u) \le 0\}$ and

$$h_n(u) = \langle At_n, u - w_n\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2. \tag{6}$$

Again put $n := n + 1$ and return to Step 1.

Lemma 3.2. *The Armijo-type search approach* (5) *is well defined, and the relation holds: $\lambda^{-1}\|r_\lambda(w_n)\|^2 \le \langle r_\lambda(w_n), Aw_n\rangle$.*

Proof. Noticing that $\ell \in (0, 1)$ and the uniformly continuity of A on C, one has $\lim_{j\to\infty}\langle Aw_n - A(w_n - \ell^j r_\lambda(w_n)), r_\lambda(w_n)\rangle = 0$. In the case of $r_\lambda(w_n) = 0$, one has $j_n = 0$. Otherwise, from $r_\lambda(w_n) \neq 0$, it follows that $\exists$ (integer) $j_n \ge 0$ fulfilling (5). Using firm nonexpansivity of P_C, one gets $\langle u - P_C v, u - v\rangle \ge \|u - P_C v\|^2$, $\forall u \in C, v \in H$. Putting $u = w_n$ and $v = w_n - \lambda A w_n$, one has $\lambda\langle w_n - P_C(w_n - \lambda A w_n), Aw_n\rangle \ge \|w_n - P_C(w_n - \lambda A w_n)\|^2$, that hence arrives at $\langle r_\lambda(w_n), Aw_n\rangle \ge \lambda^{-1}\|r_\lambda(w_n)\|^2$. □

Lemma 3.3. *Let h_n be the function formulated in* (6)*. Then, $h_n(\varrho) \le 0$, $\forall \varrho \in \Omega$. In addition, when $r_\lambda(w_n) \neq 0$, one has $h_n(w_n) > 0$.*

Proof. The latter assertion of Lemma 3.3 is clear. Next, let us show the former assertion. As a matter of fact, take a $\varrho \in \Omega$ arbitrarily. By Lemma 2.3 one has $\langle At_n, t_n - \varrho\rangle \ge 0$. Hence one gets

$$\begin{aligned} h_n(\varrho) &= \langle At_n, \varrho - w_n\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2 \\ &= \langle At_n, t_n - w_n\rangle + \langle At_n, \varrho - t_n\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2 \\ &\le -\tau_n\langle At_n, r_\lambda(w_n)\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2. \end{aligned} \tag{7}$$

Moreover, from (5) one has $\langle Aw_n - At_n, r_\lambda(w_n)\rangle \le \frac{\mu}{2}\|r_\lambda(w_n)\|^2$. Thus, by Lemma 3.2 one gets

$$\langle At_n, r_\lambda(w_n)\rangle \ge -\frac{\mu}{2}\|r_\lambda(w_n)\|^2 + \langle r_\lambda(w_n), Aw_n\rangle \ge (-\frac{\mu}{2} + \frac{1}{\lambda})\|r_\lambda(w_n)\|^2. \tag{8}$$

This together with (7), leads to

$$h_n(\varrho) \le -\frac{\tau_n}{2}(\frac{1}{\lambda} - \mu)\|r_\lambda(w_n)\|^2. \tag{9}$$

So, we obtain the desired result. □

Lemma 3.4. *Let $\{w_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ be the bounded sequences constructed in Algorithm 3.1. Assume that $x_n - x_{n+1} \to 0$, $x_n - Gw_n \to 0$, $w_n - y_n \to 0$, $x_n - z_n \to 0$ and $x_n - T^n x_n \to 0$. If $T^n x_n - T^{n+1}x_n \to 0$ and $\exists\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z \in C$, then $z \in \Omega$.*

Proof. From Algorithm 3.1, we get $w_n - x_n = \sigma_n[s(u_n - x_n) + (1-s)(T^n x_n - x_n)]$, $\forall n \ge 1$, and hence $\|w_n - x_n\| \le \sigma_n[s\|u_n - x_n\| + (1-s)\|T^n x_n - x_n\|]$. Utilizing the assumptions $\liminf_{n\to\infty} \sigma_n > 0$, $u_n - x_n \to 0$ and $x_n - T^n x_n \to 0$, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{10}$$

Putting $q_n := \alpha_n f(x_n) + (I - \alpha_n \rho F)T^n z_n$, by Algorithm 3.1 we obtain that $x_{n+1} = P_C q_n$ and $q_n - T^n z_n = \alpha_n f(x_n) - \alpha_n \rho F T^n z_n$, which immediately yields

$$\begin{aligned}
\|x_n - T^n z_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n z_n\| \\
&\le \|x_n - x_{n+1}\| + \|q_n - T^n z_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n)\| + \alpha_n\|\rho F T^n z_n\|.
\end{aligned}$$

Since $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$ and $\{x_n\}, \{z_n\}$ are bounded, we obtain

$$\lim_{n\to\infty} \|x_n - T^n z_n\| = 0.$$

We claim that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. In fact, Using the asymptotical nonexpansivity of T, one deduces that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}z_n\| + \|T^{n+1}z_n - Tx_n\| \\
&\le \|x_n - T^n z_n\| + (1+\theta_n)\|z_n - x_n\| + \|T^n x_n - T^{n+1}x_n\| + (1+\theta_{n+1})\|x_n - z_n\| + (1+\theta_1)\|T^n z_n - x_n\| \\
&= (2+\theta_1)\|x_n - T^n z_n\| + (2 + \theta_n + \theta_{n+1})\|z_n - x_n\| + \|T^n x_n - T^{n+1}x_n\|.
\end{aligned}$$

Since $x_n - z_n \to 0$, $x_n - T^n z_n \to 0$ and $T^n x_n - T^{n+1}x_n \to 0$, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{11}$$

Also, let us show that $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$. In fact, by Lemma 2.2 we know that $G : C \to C$ is nonexpansive for $\rho_1 \in (0, 2\alpha)$ and $\rho_2 \in (0, 2\beta)$. Again from Algorithm 3.1, we have $u_n = Gw_n$. Since

$$\|Gx_n - x_n\| \le \|Gx_n - Gw_n\| + \|Gw_n - x_n\| \le \|x_n - w_n\| + \|u_n - x_n\|,$$

Noticing that $u_n - x_n \to 0$ and $x_n - w_n \to 0$ (due to (10)), we obtain

$$\lim_{n\to\infty} \|Gx_n - x_n\| = 0. \tag{12}$$

Next, let us show $z \in \mathrm{VI}(C, A)$. Indeed, noticing $x_n - w_n \to 0$ and $x_{n_k} \rightharpoonup z$, we know that $w_{n_k} \rightharpoonup z$. Since C is convex and closed, from $\{w_n\} \subset C$ and $w_{n_k} \rightharpoonup z$ we get $z \in C$. In what follows, we consider two cases. If $Az = 0$, then it is clear that $z \in \mathrm{VI}(C, A)$ because $\langle Az, y - z\rangle \ge 0$, $\forall y \in C$. Assume that $Az \ne 0$. Then it follows from $w_n - x_n \to 0$ and $x_{n_k} \rightharpoonup z$ that $w_{n_k} \rightharpoonup z$ as $k \to \infty$. Utilizing the assumption on A, instead of the sequentially weak continuity of A, we get $0 < \|Az\| \le \liminf_{k\to\infty} \|Aw_{n_k}\|$. So, we might assume that

$\|Aw_{n_k}\| \neq 0\ \forall k \geq 1$. On the other hand, from $y_n = P_C(w_n - \lambda A w_n)$, we have $\langle w_n - \lambda A w_n - y_n, x - y_n \rangle \leq 0$, $\forall x \in C$, and hence

$$\frac{1}{\lambda}\langle w_n - y_n, x - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \leq \langle Aw_n, x - w_n\rangle, \quad \forall x \in C. \tag{13}$$

According to the uniform continuity of A on C, one knows that $\{Aw_n\}$ is bounded (due to Lemma 2.5). Note that $\{y_n\}$ is bounded as well. Thus, from (13) we get $\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0$, $\forall x \in C$.

To show that $z \in \mathrm{VI}(C, A)$, we now choose a sequence $\{\varsigma_k\} \subset (0, 1)$ satisfying $\varsigma_k \downarrow 0$ as $k \to \infty$. For each $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\langle Aw_{n_j}, x - w_{n_j}\rangle + \varsigma_k \geq 0, \quad \forall j \geq m_k. \tag{14}$$

Since $\{\varsigma_k\}$ is decreasing, it can be readily seen that $\{m_k\}$ is increasing. Noticing that $Aw_{m_k} \neq 0\ \forall k \geq 1$ (due to $\{Aw_{m_k}\} \subset \{Aw_{n_k}\}$), we set $\varrho_{m_k} = \frac{Aw_{m_k}}{\|Aw_{m_k}\|^2}$, we get $\langle Aw_{m_k}, \varrho_{m_k}\rangle = 1$, $\forall k \geq 1$. So, from (14) we get $\langle Aw_{m_k}, x + \varsigma_k\varrho_{m_k} - w_{m_k}\rangle \geq 0\ \forall k \geq 1$. Again from the pseudomonotonicity of A we have $\langle A(x + \varsigma_k\varrho_{m_k}), x + \varsigma_k\varrho_{m_k} - w_{m_k}\rangle \geq 0$, $\forall k \geq 1$. This immediately leads to

$$\langle Ax, x - w_{m_k}\rangle \geq \langle Ax - A(x + \varsigma_k\varrho_{m_k}), x + \varsigma_k\varrho_{m_k} - w_{m_k}\rangle - \varsigma_k\langle Ax, \varrho_{m_k}\rangle. \tag{15}$$

We claim that $\lim_{k\to\infty}\varsigma_k\varrho_{m_k} = 0$. In fact, from $x_{n_k} \rightharpoonup z \in C$ and $w_n - x_n \to 0$, we obtain $w_{n_k} \rightharpoonup z$. Note that $\{w_{m_k}\} \subset \{w_{n_k}\}$ and $\varsigma_k \downarrow 0$ as $k \to \infty$. So it follows that $0 \leq \limsup_{k\to\infty}\|\varsigma_k\varrho_{m_k}\| = \limsup_{k\to\infty}\frac{\varsigma_k}{\|Aw_{m_k}\|} \leq \frac{\limsup_{k\to\infty}\varsigma_k}{\liminf_{k\to\infty}\|Aw_{n_k}\|} = 0$. Hence we get $\varsigma_k\varrho_{m_k} \to 0$ as $k \to \infty$. Thus, letting $k \to \infty$, we deduce that the right-hand side of (15) tends to zero by the uniform continuity of A, the boundedness of $\{w_{m_k}\}, \{\varrho_{m_k}\}$ and the limit $\lim_{k\to\infty}\varsigma_k\varrho_{m_k} = 0$. Therefore, we get $\langle Ax, x - z\rangle = \liminf_{k\to\infty}\langle Ax, x - w_{m_k}\rangle \geq 0$, $\forall x \in C$. By Lemma 2.3 we have $z \in \mathrm{VI}(C, A)$.

Next we show that $z \in \Omega$. In fact, note that (11) guarantees $x_{n_k} - Tx_{n_k} \to 0$. From Lemma 2.7 it follows that $I - T$ is demiclosed at zero. Thus, from $x_{n_k} \rightharpoonup z$ we get $(I - T)z = 0$, i.e., $z \in \mathrm{Fix}(T)$. Moreover, let us show that $z \in \mathrm{Fix}(G)$. As a matter of fact, by Lemma 2.7 we know that $I - G$ is demiclosed at zero. Hence, from (12) and $x_{n_k} \rightharpoonup z$ we have $(I - G)z = 0$, i.e., $z \in \mathrm{Fix}(G)$. Consequently, $z \in \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \Omega$. This completes the proof. □

Lemma 3.5. *Let $\{w_n\}$ be the sequence constructed in Algorithm 3.1. Then,*

$$\lim_{n\to\infty}\tau_n\|r_\lambda(w_n)\|^2 = 0 \ \Rightarrow\ \lim_{n\to\infty}\|r_\lambda(w_n)\| = 0. \tag{16}$$

Proof. Assume that $\limsup_{n\to\infty}\|r_\lambda(w_n)\| = a > 0$. Then, $\exists\{n_\iota\} \subset \{n\}$ s.t. $\lim_{\iota\to\infty}\|r_\lambda(w_{n_\iota})\| = a > 0$. Note that $\lim_{\iota\to\infty}\tau_{n_\iota}\|r_\lambda(w_{n_\iota})\|^2 = 0$. First, if $\liminf_{\iota\to\infty}\tau_{n_\iota} > 0$, we might assume that $\exists\nu > 0$ s.t. $\tau_{n_\iota} \geq \nu > 0$, $\forall\iota \geq 1$. So it follows that

$$\|r_\lambda(w_{n_\iota})\|^2 = \frac{\tau_{n_\iota}}{\tau_{n_\iota}}\|r_\lambda(w_{n_\iota})\|^2 \leq \frac{\tau_{n_\iota}}{\nu}\|r_\lambda(w_{n_\iota})\|^2 = \frac{\tau_{n_\iota}}{\nu}\|r_\lambda(w_{n_\iota})\|^2, \tag{17}$$

which immediately leads to $0 < a^2 = \lim_{\iota\to\infty}\|r_\lambda(w_{n_\iota})\|^2 \leq \lim_{\iota\to\infty}\{\frac{1}{\nu}\cdot\tau_{n_\iota}\|r_\lambda(w_{n_\iota})\|^2\} = 0$. So, we attain a contradiction.

If $\liminf_{\iota\to\infty}\tau_{n_\iota} = 0$, there exists a subsequence of $\{\tau_{n_\iota}\}$, still denoted by $\{\tau_{n_\iota}\}$, s.t. $\lim_{\iota\to\infty}\tau_{n_\iota} = 0$. We now set

$$\varrho_{n_\iota} := \frac{1}{\ell}\tau_{n_\iota}y_{n_\iota} + (1 - \frac{1}{\ell}\tau_{n_\iota})w_{n_\iota} = w_{n_\iota} - \frac{1}{\ell}\tau_{n_\iota}(w_{n_\iota} - y_{n_\iota}).$$

Then, from $\lim_{\iota\to\infty}\tau_{n_\iota}\|r_\lambda(w_{n_\iota})\|^2 = 0$ we infer that

$$\lim_{\iota\to\infty}\|\varrho_{n_\iota} - w_{n_\iota}\|^2 = \lim_{\iota\to\infty}\frac{1}{\ell^2}\tau_{n_\iota}\cdot\tau_{n_\iota}\|r_\lambda(w_{n_\iota})\|^2 = 0. \tag{18}$$

Using the step size rule (5), one obtains $\langle Aw_{n_\iota} - A\varrho_{n_\iota}, r_\lambda(w_{n_\iota})\rangle > \frac{\mu}{2}\|r_\lambda(w_{n_\iota})\|^2$. Since A is uniformly continuous on bounded subsets of C, (18) ensures that

$$\lim_{\iota\to\infty} \|Aw_{n_\iota} - A\varrho_{n_\iota}\| = 0, \tag{19}$$

which hence yields $\lim_{\iota\to\infty}\|r_\lambda(w_{n_\iota})\| = 0$. So, we attain a contradiction. Therefore, $r_\lambda(w_n) \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 3.6. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.1. Then $x_n \to x^* \in \Omega$ provided $T^n x_n - T^{n+1}x_n \to 0$, where $x^* \in \Omega$ is the unique solution to the HVI: $\langle(\rho F - f)x^*, p - x^*\rangle \geq 0,\ \forall p \in \Omega$.*

Proof. Since $0 < \liminf_{n\to\infty}\sigma_n \leq \limsup_{n\to\infty}\sigma_n < 1$ and $\lim_{n\to\infty}\frac{\theta_n(2+\theta_n)}{\alpha_n} = 0$, we may assume, without loss of generality, that $\{\sigma_n\} \subset [a, b] \subset (0, 1)$ and $\theta_n(2 + \theta_n) \leq \frac{\alpha_n(\tau-\delta)}{2}$, $\forall n \geq 1$. We claim that $P_\Omega(I - \rho F + f) : C \to C$ is a contraction. In fact, for all $u, v \in C$, by Lemma 2.9, one has

$$\|P_\Omega(I - \rho F + f)(u) - P_\Omega(I - \rho F + f)(v)\| \leq [1 - (\tau - \delta)]\|u - v\|,$$

which implies that $P_\Omega(I - \rho F + f)$ is a contraction. Banach's contraction mapping principle guarantees that $P_\Omega(I - \rho F + f)$ has a unique fixed point. Say $x^* \in C$, i.e., $x^* = P_\Omega(I - \rho F + f)(x^*)$. Thus, there exists a unique solution $x^* \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ of the HVI

$$\langle(\rho F - f)x^*, p - x^*\rangle \geq 0, \quad \forall p \in \Omega. \tag{20}$$

Next we show the conclusion of the theorem. To the aim, we divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, for $x^* \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$. Then $Tx^* = x^*$ and $Gx^* = x^*$, $\forall n \geq 1$. We claim that the following inequality holds:

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \mathrm{dist}^2(w_n, C_n), \quad \forall n \geq 1. \tag{21}$$

In fact, one has

$$\begin{aligned}\|z_n - x^*\|^2 = \|P_{C_n}w_n - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|P_{C_n}w_n - w_n\|^2 \\ &= \|w_n - x^*\|^2 - \mathrm{dist}^2(w_n, C_n),\end{aligned}$$

which immediately yields

$$\|z_n - x^*\| \leq \|w_n - x^*\|, \quad \forall n \geq 1. \tag{22}$$

From the formulation of w_n, we get

$$\begin{aligned}\|w_n - x^*\| &\leq (1 - \sigma_n)\|x_n - x^*\| + \sigma_n[s\|Gw_n - x^*\| + (1 - s)\|T^n x_n - x^*\|] \\ &\leq (1 - \sigma_n)\|x_n - x^*\| + \sigma_n[s\|w_n - x^*\| + (1 - s)(1 + \theta_n)\|x_n - x^*\|] \\ &= [1 - \sigma_n + \sigma_n(1 - s)(1 + \theta_n)]\|x_n - x^*\| + s\sigma_n\|w_n - x^*\|,\end{aligned}$$

which hence arrives at

$$\begin{aligned}\|w_n - x^*\| &\leq \frac{1 - \sigma_n + \sigma_n(1 - s)(1 + \theta_n)}{1 - s\sigma_n}\|x_n - x^*\| \\ &= [1 + \frac{\sigma_n(1 - s)\theta_n}{1 - s\sigma_n}]\|x_n - x^*\| \leq (1 + \theta_n)\|x_n - x^*\|.\end{aligned}$$

This together with (22), yields

$$\|z_n - x^*\| \leq \|w_n - x^*\| \leq (1 + \theta_n)\|x_n - x^*\|, \quad \forall n \geq 1. \tag{23}$$

Thus, using (23), from Lemma 2.9 we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \alpha_n\delta\|x_n-x^*\| + (1-\alpha_n\tau)(1+\theta_n)\|z_n-x^*\| + \alpha_n\|(f-\rho F)x^*\| \\ &\le \alpha_n\delta\|x_n-x^*\| + (1-\alpha_n\tau)(1+\theta_n)^2\|x_n-x^*\| + \alpha_n\|(f-\rho F)x^*\| \\ &\le [\alpha_n\delta + (1-\alpha_n\tau) + \theta_n(2+\theta_n)]\|x_n-x^*\| + \alpha_n\|(f-\rho F)x^*\| \\ &\le [1-\alpha_n(\tau-\delta) + \frac{\alpha_n(\tau-\delta)}{2}]\|x_n-x^*\| + \alpha_n\|(f-\rho F)x^*\| \\ &= [1-\frac{\alpha_n(\tau-\delta)}{2}]\|x_n-x^*\| + \alpha_n\|(f-\rho F)x^*\| \\ &= [1-\frac{\alpha_n(\tau-\delta)}{2}]\|x_n-x^*\| + \frac{\alpha_n(\tau-\delta)}{2}\cdot\frac{2\|(f-\rho F)x^*\|}{\tau-\delta} \\ &\le \max\{\|x_n-x^*\|, \frac{2\|(f-\rho F)x^*\|}{\tau-\delta}\}.\end{aligned}$$

By induction, we obtain $\|x_n-x^*\| \le \max\{\|x_1-x^*\|, \frac{2\|(f-\rho F)x^*\|}{\tau-\delta}\}$ $\forall n \ge 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{At_n\}$, $\{Gw_n\}$ and $\{T^nz_n\}$.

Step 2. We show that

$$\begin{aligned}&[1-\alpha_n\tau+\theta_n]\{\sigma_n^2 s(1-s)\|u_n-T^nx_n\|^2 + \sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2 + (1-s)\|x_n-T^nx_n\|^2] \\ &\quad + \|z_n-w_n\|^2\} + \|q_n-x_{n+1}\|^2 \le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \alpha_nM_1\end{aligned}$$

for some $M_1 > 0$. In fact, by Lemma 2.9 and the convexity of the function $g(t) = t^2$, $\forall t \in \mathbf{R}$ we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|q_n-x^*\|^2 - \|q_n-x_{n+1}\|^2 \\ &= \|\alpha_n(f(x_n)-f(x^*)) + (I-\alpha_n\rho F)T^nz_n - (I-\alpha_n\rho F)x^* + \alpha_n(f-\rho F)x^*\|^2 - \|q_n-x_{n+1}\|^2 \\ &\le \{\alpha_n\|f(x_n)-f(x^*)\| + \|(I-\alpha_n\rho F)T^nz_n - (I-\alpha_n\rho F)x^*\|\}^2 + 2\alpha_n\langle (f-\rho F)x^*, q_n-x^*\rangle \\ &\quad - \|q_n-x_{n+1}\|^2 \\ &\le \{\alpha_n\delta\|x_n-x^*\| + [(1-\alpha_n\tau)+\theta_n]\|z_n-p\|\}^2 + 2\alpha_n\langle (f-\rho F)x^*, q_n-x^*\rangle - \|q_n-x_{n+1}\|^2 \\ &\le \alpha_n\delta\|x_n-x^*\|^2 + [(1-\alpha_n\tau)+\theta_n]\|z_n-x^*\|^2 + 2\alpha_n\langle (f-\rho F)x^*, q_n-x^*\rangle - \|q_n-x_{n+1}\|^2.\end{aligned} \tag{24}$$

On the other hand, using (23) and Algorithm 3.1 one has

$$\begin{aligned}\|z_n-x^*\|^2 &\le \|w_n-x^*\|^2 - \|z_n-w_n\|^2 \\ &\le (1-\sigma_n)\|x_n-x^*\|^2 + \sigma_n[s(1+\theta_n)^2\|x_n-x^*\|^2 + (1-s)(1+\theta_n)^2\|x_n-x^*\|^2 \\ &\quad - s(1-s)\|u_n-T^nx_n\|^2] - \sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2 + (1-s)\|x_n-T^nx_n\|^2 \\ &\quad - s(1-s)\|u_n-T^nx_n\|^2] - \|z_n-w_n\|^2 \\ &\le (1+\theta_n)^2\|x_n-x^*\|^2 - \sigma_n s(1-s)\|u_n-T^nx_n\|^2 \\ &\quad - \sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2 + (1-s)\|x_n-T^nx_n\|^2 \\ &\quad - s(1-s)\|u_n-T^nx_n\|^2] - \|z_n-w_n\|^2 \\ &= (1+\theta_n)^2\|x_n-x^*\|^2 - \sigma_n^2 s(1-s)\|u_n-T^nx_n\|^2 - \|z_n-w_n\|^2 \\ &\quad - \sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2 + (1-s)\|x_n-T^nx_n\|^2] \\ &= \|x_n-x^*\|^2 + \theta_n(2+\theta_n)\|x_n-x^*\|^2 - \sigma_n^2 s(1-s)\|u_n-T^nx_n\|^2 \\ &\quad - \sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2 + (1-s)\|x_n-T^nx_n\|^2] - \|z_n-w_n\|^2.\end{aligned} \tag{25}$$

Substituting (25) into (24), one gets

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\delta\|x_n-x^*\|^2+[1-\alpha_n\tau+\theta_n]\|x_n-x^*\|^2+\theta_n(2+\theta_n)\|x_n-x^*\|^2-[1-\alpha_n\tau+\theta_n]\\ &\quad\times\{\sigma_n^2 s(1-s)\|u_n-T^nx_n\|^2+\sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2+(1-s)\|x_n-T^nx_n\|^2]\\ &\quad+\|z_n-w_n\|^2\}+2\alpha_n\langle(f-\rho F)x^*,q_n-x^*\rangle-\|q_n-x_{n+1}\|^2\\ &\le[1-\alpha_n(\tau-\delta)]\|x_n-x^*\|^2+2\theta_n(2+\theta_n)\|x_n-x^*\|^2-[1-\alpha_n\tau+\theta_n]\{\sigma_n^2s(1-s)\|u_n-T^nx_n\|^2\\ &\quad+\sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2+(1-s)\|x_n-T^nx_n\|^2]+\|z_n-w_n\|^2\}\\ &\quad+2\alpha_n\langle(f-\rho F)x^*,q_n-x^*\rangle-\|q_n-x_{n+1}\|^2\\ &\le\|x_n-x^*\|^2-[1-\alpha_n\tau+\theta_n]\{\sigma_n^2s(1-s)\|u_n-T^nx_n\|^2+\sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2\\ &\quad+(1-s)\|x_n-T^nx_n\|^2]+\|z_n-w_n\|^2\}+\alpha_nM_1-\|q_n-x_{n+1}\|^2,\end{aligned}$$

where $\sup_{n\ge1}2\|(f-\rho F)x^*\|\|q_n-x^*\|\le M_1$ for some $M_1>0$. This immediately implies that

$$\begin{aligned}&[1-\alpha_n\tau+\theta_n]\{\sigma_n^2s(1-s)\|u_n-T^nx_n\|^2+\sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2\\ &\quad+(1-s)\|x_n-T^nx_n\|^2]+\|z_n-w_n\|^2\}+\|q_n-x_{n+1}\|^2\\ &\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1.\end{aligned}$$

Step 3. We show that

$$[1-\alpha_n\tau+\theta_n][\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1.$$

In fact, we claim that for some $L>0$,

$$\|z_n-x^*\|^2\le\|w_n-x^*\|^2-[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2. \tag{26}$$

Since the sequence $\{At_n\}$ is bounded, there exists $L>0$ such that $\|At_n\|\le L\ \forall n\ge1$. This ensures that for all $u,v\in C_n$,

$$|h_n(u)-h_n(v)|=|\langle At_n,u-v\rangle|\le\|At_n\|\|u-v\|\le L\|u-v\|,$$

which hence implies that $h_n(\cdot)$ is L-Lipschitz continuous on C_n. By Lemmas 2.6 and 3.3, we obtain

$$\operatorname{dist}(w_n,C_n)\ge\frac{1}{L}h_n(w_n)=\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2. \tag{27}$$

Combining (21) and (27), we get $\|z_n-x^*\|^2\le\|w_n-x^*\|^2-[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2$. From (24), (23) and (26) it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\le\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\theta_n]\|z_n-x^*\|^2+2\alpha_n\langle(f-\rho F)x^*,q_n-x^*\rangle\\ &\le\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\theta_n]\{\|w_n-x^*\|^2-[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2\}+2\alpha_n\langle(f-\rho F)x^*,q_n-x^*\rangle\\ &\le\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\theta_n]\{(1+\theta_n)^2\|x_n-x^*\|^2-[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2\}+\alpha_nM_1\\ &\le[1-\alpha_n(\tau-\delta)+\theta_n]\|x_n-x^*\|^2+\theta_n(2+\theta_n)\|x_n-x^*\|^2-[(1-\alpha_n\tau)+\theta_n][\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2+\alpha_nM_1\\ &\le[1-\alpha_n(\tau-\delta)+2\theta_n(2+\theta_n)]\|x_n-x^*\|^2-[(1-\alpha_n\tau)+\theta_n][\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2+\alpha_nM_1\\ &\le\|x_n-x^*\|^2-[1-\alpha_n\tau+\theta_n][\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2+\alpha_nM_1.\end{aligned}$$

This immediately yields

$$[1-\alpha_n\tau+\theta_n][\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1.$$

Step 4. We show that

$$\|x_{n+1}-x^*\|^2 \le [1-\alpha_n(\tau-\delta)]\|x_n-x^*\|^2 + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}] + \alpha_n(\tau-\delta)[\frac{2\langle (f-\rho F)x^*, q_n-x^*\rangle}{\tau-\delta} \tag{28}$$

for some $M > 0$. In fact, from Lemma 2.9 and (23), one obtains

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|q_n-x^*\|^2\\ &\le \|\alpha_n(f(x_n)-f(x^*)) + (I-\alpha_n\rho F)T^n z_n - (I-\alpha_n\rho F)x^*\|^2 + 2\alpha_n\langle (f-\rho F)x^*, q_n-x^*\rangle\\ &\le [\alpha_n\delta + (1-\alpha_n\tau+\theta_n)]\|x_n-x^*\|^2 + \theta_n(2+\theta_n)\|x_n-x^*\|^2 + 2\alpha_n\langle (f-\rho F)x^*, q_n-x^*\rangle\\ &\le \alpha_n(\tau-\delta)[\frac{2\langle (f-\rho F)x^*, q_n-x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}] + [1-\alpha_n(\tau-\delta)]\|x_n-x^*\|^2,\end{aligned}$$

where $\sup_{n\ge1} 2(2+\theta_n)\|x_n-x^*\|^2 \le M$ for some $M > 0$.

Step 5. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (20). In fact, from (28), we have

$$\|x_{n+1}-x^*\|^2 \le \alpha_n(\tau-\delta)[\frac{2\langle (f-\rho F)x^*, q_n-x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}] + (1-\alpha_n(\tau-\delta))\|x_n-x^*\|^2. \tag{29}$$

Putting $\Gamma_n = \|x_n-x^*\|^2$, we show the convergence of $\{\Gamma_n\}$ to zero by the following two cases.

Case 1. $\exists$ (integer), $n_0 \ge 1$ s.t. $\{\Gamma_n\}$ is nonincreasing. It is clear that the limit $\lim_{n\to\infty}\Gamma_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Gamma_n-\Gamma_{n+1}) = 0$. From Step 2 and $\{\sigma_n\} \subset [a,b] \subset (0,1)$ we obtain

$$\begin{aligned}&[1-\alpha_n\tau+\theta_n]\{a^2 s(1-s)\|u_n-T^n x_n\|^2 + a(1-b)[s\|x_n-u_n\|^2\\ &\quad + (1-s)\|x_n-T^n x_n\|^2] + \|z_n-w_n\|^2\} + \|q_n-x_{n+1}\|^2\\ &\le [1-\alpha_n\tau+\theta_n]\{\sigma_n^2 s(1-s)\|u_n-T^n x_n\|^2 + \sigma_n(1-\sigma_n)[s\|x_n-u_n\|^2\\ &\quad + (1-s)\|x_n-T^n x_n\|^2] + \|z_n-w_n\|^2\} + \|q_n-x_{n+1}\|^2\\ &\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \alpha_n M_1 \le \Gamma_n - \Gamma_{n+1} + \alpha_n M_1.\end{aligned}$$

Thanks to the facts that $\theta_n \to 0$, $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, from $s \in (0,1)$ one deduces that $\lim_{n\to\infty}\|u_n - T^n x_n\| = 0$, $\lim_{n\to\infty}\|x_n-u_n\| = 0$, and

$$\lim_{n\to\infty}\|x_n-T^n x_n\| = \lim_{n\to\infty}\|z_n-w_n\| = \lim_{n\to\infty}\|q_n-x_{n+1}\| = 0. \tag{30}$$

So, it is easy to see that

$$\begin{aligned}\|w_n-x_n\| &= \sigma_n\|s(u_n-x_n) + (1-s)(T^n x_n-x_n)\|\\ &\le \|u_n-x_n\| + \|T^n x_n-x_n\| \to 0 \quad (n\to\infty),\end{aligned}$$

$$\|x_n-z_n\| \le \|x_n-w_n\| + \|w_n-z_n\| \to 0 \quad (n\to\infty),$$

and

$$\begin{aligned}\|x_n-T^n z_n\| &\le \|x_n-T^n x_n\| + \|T^n x_n-T^n z_n\|\\ &\le \|x_n-T^n x_n\| + (1+\theta_n)\|x_n-z_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

Therefore, from (30) and Algorithm 3.1, it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|x_{n+1}-q_n\| + \|q_n-x_n\|\\ &= \|x_{n+1}-q_n\| + \|\alpha_n f(x_n) + T^n z_n - x_n - \alpha_n\rho F T^n z_n\|\\ &\le \|x_{n+1}-q_n\| + \|T^n z_n-x_n\| + \alpha_n\|f(x_n)-\rho F T^n z_n\|\\ &\le \|x_{n+1}-q_n\| + \|T^n z_n-x_n\| + \alpha_n(\|f(x_n)\| + \|\rho F T^n z_n\|)\\ &\to 0 \quad (n\to\infty).\end{aligned} \tag{31}$$

On the other hand, from Step 3 we obtain

$$[1-\alpha_n\tau+\theta_n][\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2 \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1$$
$$=\Gamma_n-\Gamma_{n+1}+\alpha_nM_1.$$

Noticing $\theta_n\to 0,\ \alpha_n\to 0$ and $\Gamma_n-\Gamma_{n+1}\to 0$, one gets

$$\lim_{n\to\infty}[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2]^2=0,$$

which together with Lemma 3.5, leads to

$$\lim_{n\to\infty}\|w_n-y_n\|=0. \tag{32}$$

From the boundedness of $\{x_n\}$, we know that $\exists$ subsequence $\{x_{n_\iota}\}\subset\{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle=\lim_{\iota\to\infty}\langle(f-\rho F)x^*,x_{n_\iota}-x^*\rangle. \tag{33}$$

Since H is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_\iota}\rightharpoonup\widetilde{x}$. Thus, from (3.29) one gets

$$\begin{aligned}\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle&=\lim_{\iota\to\infty}\langle(f-\rho F)x^*,x_{n_\iota}-x^*\rangle\\&=\langle(f-\rho F)x^*,\widetilde{x}-x^*\rangle.\end{aligned} \tag{34}$$

Since $x_n-x_{n+1}\to 0,\ Gw_n-x_n\to 0,\ w_n-y_n\to 0,\ x_n-z_n\to 0,\ x_n-T^nx_n$ and $x_{n_\iota}\rightharpoonup\widetilde{x}$, by Lemma 3.4 we infer that $\widetilde{x}\in\Omega$. Hence from (20) and (34) one has

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle=\langle(f-\rho F)x^*,\widetilde{x}-x^*\rangle\le 0, \tag{35}$$

which together with (30) and (31), arrives at

$$\begin{aligned}\limsup_{n\to\infty}\langle(f-\rho F)x^*,q_n-x^*\rangle&=\limsup_{n\to\infty}[\langle(f-\rho F)x^*,q_n-x_{n+1}+x_{n+1}-x_n\rangle+\langle(f-\rho F)x^*,x_n-x^*\rangle]\\&\le\limsup_{n\to\infty}[\|(f-\rho F)x^*\|(\|q_n-x_{n+1}\|+\|x_{n+1}-x_n\|)+\langle(f-\rho F)x^*,x_n-x^*\rangle]\le 0.\end{aligned} \tag{36}$$

Note that $\{\alpha_n(\tau-\delta)\}\subset[0,1],\ \sum_{n=1}^\infty\alpha_n(\tau-\delta)=\infty$, and

$$\limsup_{n\to\infty}[\frac{2\langle(f-\rho F)x^*,q_n-x^*\rangle}{\tau-\delta}+\frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}]\le 0.$$

Consequently, applying Lemma 2.4 to (29), one has $\lim_{n\to\infty}\|x_n-x^*\|^2=0$.

Case 2. $\exists\{\Gamma_{n_\iota}\}\subset\{\Gamma_n\}$ s.t. $\Gamma_{n_\iota}<\Gamma_{n_\iota+1}\ \forall\iota\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\phi:\mathcal{N}\to\mathcal{N}$ by

$$\phi(n):=\max\{\iota\le n:\Gamma_\iota<\Gamma_{\iota+1}\}.$$

By Lemma 2.8, we get

$$\Gamma_{\phi(n)}\le\Gamma_{\phi(n)+1}\quad\text{and}\quad\Gamma_n\le\Gamma_{\phi(n)+1}.$$

From Step 2 we have

$$\begin{aligned} &[1-\alpha_{\phi(n)}\tau+\theta_{\phi(n)}]\{a^2s(1-s)\|u_{\phi(n)}-T^{\phi(n)}x_{\phi(n)}\|^2 \\ &\quad+a(1-b)[s\|x_{\phi(n)}-u_{\phi(n)}\|^2+(1-s)\|x_{\phi(n)}-T^{\phi(n)}x_{\phi(n)}\|^2] \\ &\quad+\|z_{\phi(n)}-w_{\phi(n)}\|^2\}+\|q_{\phi(n)}-x_{\phi(n)+1}\|^2 \\ &\leq[1-\alpha_{\phi(n)}\tau+\theta_{\phi(n)}]\{\sigma_{\phi(n)}^2s(1-s)\|u_{\phi(n)}-T^{\phi(n)}x_{\phi(n)}\|^2 \\ &\quad+\sigma_{\phi(n)}(1-\sigma_{\phi(n)})[s\|x_{\phi(n)}-u_{\phi(n)}\|^2+(1-s)\|x_{\phi(n)}-T^{\phi(n)}x_{\phi(n)}\|^2] \\ &\quad+\|z_{\phi(n)}-w_{\phi(n)}\|^2\}+\|q_{\phi(n)}-x_{\phi(n)+1}\|^2 \\ &\leq\|x_{\phi(n)}-x^*\|^2-\|x_{\phi(n)+1}-x^*\|^2+\alpha_{\phi(n)}M_1 \\ &\leq\Gamma_{\phi(n)}-\Gamma_{\phi(n)+1}+\alpha_{\phi(n)}M_1, \end{aligned} \tag{37}$$

which immediately implies that $\lim_{n\to\infty}\|u_{\phi(n)}-T^{\phi(n)}x_{\phi(n)}\|=0$, $\lim_{n\to\infty}\|x_{\phi(n)}-u_{\phi(n)}\|=0$, and

$$\lim_{n\to\infty}\|x_{\phi(n)}-T^{\phi(n)}x_{\phi(n)}\|=\lim_{n\to\infty}\|z_{\phi(n)}-w_{\phi(n)}\|=\lim_{n\to\infty}\|q_{\phi(n)}-x_{\phi(n)+1}\|=0.$$

From Step 3 we get

$$\begin{aligned} [(1-\alpha_{\phi(n)}\tau)+\theta_{\phi(n)}][\frac{\tau_{\phi(n)}}{2\lambda L}\|r_\lambda(w_{\phi(n)})\|^2]^2 &\leq\|x_{\phi(n)}-x^*\|^2-\|x_{\phi(n)+1}-x^*\|^2+\alpha_{\phi(n)}M_1 \\ &=\Gamma_{\phi(n)}-\Gamma_{\phi(n)+1}+\alpha_{\phi(n)}M_1, \end{aligned}$$

which hence leads to

$$\lim_{n\to\infty}[\frac{\tau_{\phi(n)}}{2\lambda L}\|r_\lambda(w_{\phi(n)})\|^2]^2=0.$$

Utilizing the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|w_{\phi(n)}-y_{\phi(n)}\|=\lim_{n\to\infty}\|x_{\phi(n)}-z_{\phi(n)}\|=\lim_{n\to\infty}\|x_{\phi(n)+1}-x_{\phi(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,q_{\phi(n)}-x^*\rangle\leq 0. \tag{38}$$

On the other hand, from (29) we obtain

$$\alpha_{\phi(n)}(\tau-\delta)\Gamma_{\phi(n)}\leq\alpha_{\phi(n)}(\tau-\delta)[\frac{2\langle(f-\rho F)x^*,q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}],$$

which hence arrives at

$$\limsup_{n\to\infty}\Gamma_{\phi(n)}\leq\limsup_{n\to\infty}[\frac{2\langle(f-\rho F)x^*,q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}]\leq 0.$$

Thus, $\lim_{n\to\infty}\|x_{\phi(n)}-x^*\|^2=0$. Also, note that

$$\begin{aligned} \|x_{\phi(n)+1}-x^*\|^2-\|x_{\phi(n)}-x^*\|^2 &=2\langle x_{\phi(n)+1}-x_{\phi(n)},x_{\phi(n)}-x^*\rangle+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2 \\ &\leq 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2. \end{aligned} \tag{39}$$

Owing to $\Gamma_n\leq\Gamma_{\phi(n)+1}$, we get

$$\|x_n-x^*\|^2\leq\|x_{\phi(n)}-x^*\|^2+2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2\to 0(n\to\infty).$$

That is, $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

Using the similar arguments to the proof of Theorem 3.6, we can deduce the following theorem.

Theorem 3.7. *If $T : C \to C$ is nonexpansive and $\{x_n\}$ is the sequence constructed in the modified version of Algorithm 3.1, that is, for any initial $x_1 \in C$,*

$$\begin{cases} w_n = (1-\sigma_n)x_n + \sigma_n[su_n + (1-s)Tx_n], \\ v_n = P_C(I-\rho_2 B_2)w_n, \\ u_n = P_C(I-\rho_1 B_1)v_n, \\ y_n = P_C(w_n - \lambda A w_n), \\ t_n = (1-\tau_n)w_n + \tau_n y_n, \\ z_n = P_{C_n}(w_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I-\alpha_n\rho F)Tz_n], \quad \forall n \geq 1, \end{cases} \tag{40}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.1, then $x_n \to x^* \in \Omega$, where $x^* \in \Omega$ is the unique solution to the HVI: $\langle(\rho F - f)x^*, p - x^*\rangle \geq 0, \ \forall p \in \Omega$.

Remark 3.8. *Compared with the corresponding results in Cai, Shehu and Iyiola [2], Thong and Hieu [24] and Reich et al. [22], our results improve and extend them in the following aspects.*

(i) The problem of finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G)$ in [2] is extended to develop our problem of finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ where T is asymptotically nonexpansive mapping and G is the mapping defined as in Lemma 1.1, i.e., $G := P_C(I-\rho_1 B_1)P_C(I-\rho_2 B_2)$ for $\rho_1 \in (0, 2\alpha)$ and $\rho_2 \in (0, 2\beta)$. The modified viscosity implicit rule for finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G)$ in [2] is extended to develop our modified viscosity subgradient-like extragradient implicit rule with line-search process for finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$, which is based on the subgradient extragradient method with line-search process, hybrid Mann implicit iteration method, and composite viscosity approximation method.

(ii) The problem of finding an element of $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ with quasi-nonexpansive mapping T in [24] is extended to develop our problem of finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ with asymptotically nonexpansive mapping T. The inertial subgradient extragradient method with linear-search process for finding an element of $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ in [15] is extended to develop our modified viscosity subgradient-like extragradient implicit rule with line-search process for finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$, which is based on the subgradient extragradient method with line-search process, hybrid Mann implicit iteration method, and composite viscosity approximation method.

(iii) The problem of finding an element of $\mathrm{VI}(C, A)$ with pseudomonotone uniform continuity mapping A is extended to develop our problem of finding an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ with both asymptotically nonexpansive mapping T and nonexpansive mapping G. The modified projection-type method with line-search process in [22] is extended to develop our modified viscosity subgradient-like extragradient implicit rule with line-search process, e.g., the original projection step $y_n = P_C(x_n - \lambda A x_n)$ is replaced by the hybrid Mann implicit projection step $w_n = (1-\sigma_n)x_n + \sigma_n[sGw_n + (1-s)T^n x_n]$ and $y_n = P_C(w_n - \lambda A w_n)$; meantime, the original viscosity step $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_{C_n}(x_n)$ is replaced by the composite viscosity iterative step $x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \rho F)T^n P_{C_n}(w_n)]$.

4. Examples

In this section, applying our main results we deal with the GSVI, VIP and FPP in an illustrated example. Put $\rho = 2$, $\rho_1 = \rho_2 = \frac{1}{3}$, $\mu = 1$, $s = \lambda = \ell = \frac{1}{2}$, $\sigma_n = \frac{2}{3}$ and $\alpha_n = \frac{1}{3(n+1)}$.

We first provide an example of two inverse-strongly monotone mappings $B_1, B_2 : C \to H$, Lipschitz continuous and pseudomonotone mapping A and asymptotically nonexpansive mapping T with $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$. We set $H = \mathbf{R}$ and use the $\langle a, b\rangle = ab$ and $\|\cdot\| = |\cdot|$ to denote its inner product and induced norm, respectively. Moreover, we put $C = [-2, 4]$. The starting point x_1 is arbitrarily picked in $[-2, 4]$. Let $f(x) = F(x) = \frac{1}{2}x, \ \forall x \in C$ with

$$\delta = \frac{1}{2} < \zeta = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} = 1 - \sqrt{1 - 2(2 \cdot \frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

Let $B_1x = B_2x := Bx = x - \frac{1}{2}\sin x$, $\forall x \in C$. Let $A : H \to H$ and $T : C \to C$ be defined as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$ and $Tx := \frac{3}{7}\sin x$. We now claim that B is $\frac{2}{9}$-inverse-strongly monotone. In fact, since B is $\frac{1}{2}$-strongly monotone and $\frac{3}{2}$-Lipschitz continuous, we know that B is $\frac{2}{9}$-inverse-strongly monotone with $\alpha = \beta = \frac{2}{9}$. Let us show that A is pseudomonotone and Lipschitz continuous. Indeed, for each $x, y \in H$ one has

$$\begin{aligned}\|Ax - Ay\| &\le \left|\frac{\|y\| - \|x\|}{(1 + \|y\|)(1 + \|x\|)}\right| + \left|\frac{\|\sin y\| - \|\sin x\|}{(1 + \|\sin y\|)(1 + \|\sin x\|)}\right| \\ &\le \frac{\|x - y\|}{(1 + \|x\|)(1 + \|y\|)} + \frac{\|\sin x - \sin y\|}{(1 + \|\sin x\|)(1 + \|\sin y\|)} \\ &\le \|x - y\| + \|\sin x - \sin y\| \\ &\le 2\|x - y\|.\end{aligned}$$

This implies that A is Lipschitz continuous with $L = 2$. Next, we show that A is pseudomonotone. For each $x, y \in H$, it is not hard to find that

$$\begin{aligned}&\langle Ax, y - x\rangle = \left(\frac{1}{1 + |\sin x|} - \frac{1}{1 + |x|}\right)(y - x) \ge 0 \\ \Rightarrow\ &\langle Ay, y - x\rangle = \left(\frac{1}{1 + |\sin y|} - \frac{1}{1 + |y|}\right)(y - x) \ge 0.\end{aligned}$$

Moreover, it is easy to check that T is asymptotically nonexpansive with $\theta_n = (\frac{3}{7})^n$, $\forall n \ge 1$, such that $\|T^{n+1}x_n - T^n x_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\|T^n x - T^n y\| \le \frac{3}{7}\|T^{n-1}x - T^{n-1}y\| \le \cdots \le \left(\frac{3}{7}\right)^n \|x - y\| \le (1 + \theta_n)\|x - y\|,$$

and

$$\begin{aligned}\|T^{n+1}x_n - T^n x_n\| &\le \left(\frac{3}{7}\right)^{n-1}\|T^2 x_n - T x_n\| \\ &= \left(\frac{3}{7}\right)^{n-1}\left\|\frac{3}{7}\sin(T x_n) - \frac{3}{7}\sin x_n\right\| \\ &\le 2\left(\frac{3}{7}\right)^n \to 0.\end{aligned}$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = \lim_{n\to\infty} \frac{(3/7)^n}{1/3(n + 1)} = 0.$$

Therefore, $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. In this case, noticing $G = P_C(I - \rho_1 B_1)P_C(I - \rho_2 B_2) = [P_C(I - \frac{1}{3}B)]^2$, we rewrite Algorithm 3.1 as follows:

$$\begin{cases} w_n = \frac{1}{3}x_n + \frac{2}{3}[\frac{1}{2}u_n + \frac{1}{2}T^n x_n], \\ v_n = P_C(I - \frac{1}{3}B)w_n, \\ u_n = P_C(I - \frac{1}{3}B)v_n, \\ y_n = P_C(w_n - \frac{1}{2}Aw_n), \\ t_n = (1 - \tau_n)w_n + \tau_n y_n, \\ z_n = P_{C_n}(w_n), \\ x_{n+1} = P_C[\frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})T^n z_n], \quad \forall n \ge 1, \end{cases} \tag{41}$$

where for each $n \ge 1$, C_n and τ_n are chosen as in Algorithm 3.1. Then, by Theorem 3.6, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \cap\mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$.

In particular, since $Tx := \frac{3}{7}\sin x$ is also nonexpansive, we consider the modified version of Algorithm 3.1, that is,

$$\begin{cases} w_n = \frac{1}{3}x_n + \frac{2}{3}[\frac{1}{2}u_n + \frac{1}{2}Tx_n], \\ v_n = P_C(I - \frac{1}{3}B)w_n, \\ u_n = P_C(I - \frac{1}{3}B)v_n, \\ y_n = P_C(w_n - \frac{1}{2}Aw_n), \\ t_n = (1-\tau_n)w_n + \tau_n y_n, \\ z_n = P_{C_n}(w_n), \\ x_{n+1} = P_C[\frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})Tz_n], \quad \forall n \geq 1, \end{cases} \tag{42}$$

where for each $n \geq 1$, C_n and τ_n are chosen as above. Then, by Theorem 3.7, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$.

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