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Iterative approximation of a solution of system of generalized nonlinear variational-like inclusions and a fixed point of total asymptotically nonexpansive mappings

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Abstract. This paper investigates the problem of finding a point lying in the intersection of the set of solutions of a system of generalized nonlinear variational-like inclusions and the set of fixed points of a total asymptotically nonexpansive mapping. To achieve this aim, a new iterative algorithm is suggested. Finally, the strong convergence and stability of the sequence generated by our proposed iterative algorithm to a common point of the above two sets are proved. The results presented in this paper are new, and improve and generalize many known corresponding results.

1. Introduction

The study of variational inequality theory is due primarily to Stampacchia [61] and Fichera [29] which made their works independently in 1964. Initially, one of the attractions of this theory was its applications to many questions of physical interest, but during the last decades, variational inequalities were shown to be very useful across a wide of subjects, ranging from nonsmooth mechanics, physics, and engineering to economics. For this reason, a large number of problems arising in diverse branches of pure and applied sciences can be found which lead to mathematical models expressed in terms of variational inequalities. Because of the importance and active impact of variational inequalities, the mathematical literature dedicated to this field is extensive, and the progress made in the past five decades is impressive. It is worthwhile to stress that variational inequalities cannot be used as a suitable mathematical model for dealing with many problems arising in different fields of science and engineering. In fact, to deal with such problems, it needs that variational inequalities to be generalized and extended. This fact was one of the main incentives of

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researchers to develop and generalize various classes of variational inequalities in many different directions using novel and innovative techniques. For a detailed description of these generalizations along with relevant commentaries, the reader is referred to [7, 8] and the references therein.

Among the generalizations, variational inclusion has been recognized as a very effective and powerful tool for studying a wide class of linear and nonlinear problems arising in many diverse fields of science and engineering. This is the reason why in the last twenty years, there has been an increasing interest in studying a wide class of variational inclusion problems in the setting of Hilbert and Banach spaces. This field still continues to be an active topic for research motivated by numerous applications to problems in optimization and control, economics and transportation equilibrium, engineering science, etc.

In parallel to the study of variational inequalities and variational inclusions, considerable research efforts have been devoted to the design of methods to approximate the solutions of the variational inequalities/inclusions and related optimization problems in the setting of different spaces. Among the methods that have appeared in the literature, the method based on the resolvent operator technique, which is a generalization of projection method, is of interest and importance and has been the focus of much research in the past several decades. For more related details, we refer the readers to [4, 9, 10, 25–28, 37, 47, 57, 66] and the references therein.

Browder [14] and Minty [54, 55] are known as the initiators of the study of problems with monotone operators. Of course, it should be pointed out that the term monotone operator was first coined by Kačurovskiĭ [41], who proved that the subdifferential of a convex function on a Hilbert space is monotone. Maximal monotone operators appear in several branches of applied mathematics such as optimization, partial differential equations, and variational analysis. The introduction of the concept of accretive mapping was first made independently by Browder [15] and Kato [42]. The motivation for studying m-accretive mappings, those accretive ones which satisfy the range condition, comes from the study of nonlinear semigroups, differential equations in Banach spaces, and fully nonlinear partial differential equations. Thanks to their extraordinary utility and broad applicability in science and engineering, in the last decades, much attention has been given to develop and generalize the notions of maximal monotone operators and m-accretive mappings. By the same taken, the class of P-η-accretive mappings was initially introduced by Kazmi and Khan [44] in the setting of real q-uniformly smooth Banach space, and can be viewed as a unifying framework for the classes of maximal η-monotone operators [35], η-subdifferential operators [24, 48], generalized m-accretive mappings [34], H-monotone operators [25], general H-monotone operators [67], H-accretive mappings [26] and (H, η)-monotone operators [28]. They defined the resolvent operator (P-η-proximal-point mapping) associated with a P-η-accretive mapping and derived some properties in connection with it. Unfortunately, some errors occurred in the proofs of their preliminaries results, in particular, in the proof of the theorem concerning the Lipschitz continuity of the resolvent operator associated with a P-η-accretive mapping. These errors have resolved one year later in a work due to Peng and Zhu [57], who reviewed the class of P-η-accretive mappings and presented the correct versions of relevant assertions. The authors considered a system of variational inclusions involving P-η-accretive mappings in real q-uniformly smooth Banach spaces and verified the existence and uniqueness of its solution under some appropriate conditions. With the aim of approximating this unique solution, they proposed a Mann iterative algorithm and discussed its convergence under some suitable assumptions.

On the other hand, it is well known that many problems in topology, nonlinear analysis, mathematical economics, differential equations, control theory, optimization problem and game theory give rise to fixed point problems for some suitable (uni-valued or set-valued) mappings. This is why fixed point theory became one of the most interesting area of research in the last fifty years. During the past few decades, considerable effort has been aimed to develop efficient iterative algorithms to compute approximate fixed points of a nonexpansive mapping, see, for example, [1, 22, 40, 60, 63] and the references therein. At the same time, as we know, the study of nonexpansive mappings is closely connected with concepts of monotone and accretive mappings, two classes of operators which arise naturally in the theory of differential equations. Due to the important role and many diverse applications of nonexpansive mappins in the theory of fixed points, within the past 50 years or so, many authors devoted their attention to introducing and presenting several interesting generalized nonexpansive mappings in different contexts and to study the appropriate conditions for existence of fixed points of them.

It can be claimed that the class of asymptotically nonexpansive mappings which its study was initiated by Goebel and Kirk [30] in 1972, is one of the most important and well known generalizations of nonexpansive mappings. Later, Sahu [58] succeeded to introduce the class of nearly asymptotically nonexpansive mappings as an extension of the class of asymptotically nonexpansive mappings. Another successfully attempt was made one year later by Alber et al. [3], who introduced a more general class than the class of nearly asymptotically nonexpansive mappings the so-called total asymptotically nonexpansive mappings. The efforts in this direction have been continued and several other interesting classes of generalized nonexpansive mappings have been appeared in the literature. For more details and further information about these generalizations, we refer the reader to [3, 9, 12, 18, 20, 30, 39, 56, 58] and the references contained therein. Besides, the existence of a deep and close relation between the variational inequality (inclusion) problems and the fixed point problems motivated many researchers in recent years to investigate the problem of approximating the solution of a variational inequality (inclusion) problem which is also fixed point of a given operator. For details, an interested reader is referred to [5, 6, 9, 11–13, 16, 17, 21, 38, 59, 62, 64, 69] and the references therein.

The remainder of this paper is organised as follows. In Sect. 2, we resume basic definitions and facts about P-η-accretive mappings and provide some concrete interesting examples relating to them. In Sect. 3, a system of generalized nonlinear variational-like inclusions (for short, SGNVLI) involving P-η-accretive mappings is considered and under some suitable conditions, the existence and uniqueness of the solution of the SGNVLI is proved. In Sect. 4, our main attention is paid to the investigation of the problem of finding a common element of the solutions set of the SGNVLI and the fixed points set of a given total asymptotically nonexpansive mapping. With the aim of approximating such a point, a new iterative algorithm is suggested. In the end, Sect. 4 is closed with a theorem in which the strong convergence and stability of the sequence of generated by our proposed iterative algorithm to a common point of the two sets mentioned above is demonstrated.

2. Basic Definitions and Properties

In what follows, unless otherwise stated, we always let E be a real Banach space with a norm $\|.\|$, E^* be the topological dual space of E, $\langle .,. \rangle$ be the dual pair between E and E^*, and 2^E denote the family of all the nonempty subsets of E. As usual, x^* will stand for the weak star topology in E^* and the value of a functional $x^* \in E^*$ at $x \in E$ will denote by either $\langle x, x^* \rangle$ or $x^*(x)$, as is convenient. For the sake of simplicity, the norms of E and E^* will denote by the symbol $\|.\|$. For a given multi-valued mapping $M : E \to 2^E$,

(i) the set Range(M) given by the formula

$$\text{Range}(M) := \{y \in E : \exists x \in E : (x, y) \in M\} = \bigcup_{x\in E} M(x)$$

is called the range of M;

(ii) the set Graph(M) defined by

$$\text{Graph}(M) := \{(x, u) \in E \times E : u \in M(x)\},$$

is called the graph of M.

Recall that a normed space E is said to be *strictly convex* if the unit sphere in E is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all distinct unit vectors x and y in E. It is called *smooth* if for every unit vector x in E there exists a unique $x^* \in E^*$ such that $\|x\| = \langle x, x^* \rangle = 1$. It is a well known truth that E is smooth if E^* is strictly convex, and that E is strictly convex if E^* is smooth.

Definition 2.1. *A normed space E is said to be uniformly convex if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if x and y are unit vectors in E with $\|x - y\| \geq 2\varepsilon$, then the average $(x + y)/2$ has norm at most $1 - \delta$.*

The function $\delta_E : [0, 2] \to [0, 1]$ given by the formula

$$\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\}$$

is called the *modulus of convexity* of E.

It is significant to emphasize that in the definition of $\delta_E(\varepsilon)$ we can as well take the infimum over all vectors $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$.

The function δ_E is continuous and increasing on the interval $[0, 2]$ and $\delta_E(0) = 0$. Obviously, thanks to the definition of the function δ_E, a normed space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

Definition 2.2. *A normed space E is said to be uniformly smooth if, for all $\varepsilon > 0$ there is a $\tau > 0$ such that if x and y are unit vectors in E with $\|x - y\| \leq 2\tau$, then the average $(x + y)/2$ has norm at least $1 - \varepsilon\tau$.*

The function $\rho_E : [0, +\infty) \to [0, +\infty)$ given by

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in E, \|x\| = \|y\| = 1\}$$

is called the modulus of smoothness of E. It is to be noted that the function ρ_E is convex, continuous and increasing on the interval $[0, +\infty)$ and $\rho_E(0) = 0$. In addition, $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$. In the light of the definition of the function ρ_E, a normed space E is uniformly smooth if and only if $\lim\limits_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

It is also remarkable that in the definition of $\rho_E(\tau)$, we can as well take the supremum over all vectors $x, y \in E$ with $\|x\|, \|y\| \leq 1$. Note, in particular, that any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space E is uniformly convex (resp., uniformly smooth) if and only if E^* is uniformly smooth (resp., uniformly convex). The spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see [23, 32, 49]. At the same time, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, can be found in [23, 32, 49].

For an arbitrary but fixed real number $q > 1$, the multi-valued mapping $J_q : E \to 2^{E^*}$ given by

$$J_q(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

is called the *generalized duality mapping* of E. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$, for all $x \neq 0$. Note that J_q is single-valued if E is uniformly smooth or equivalently E^* is strictly convex. If E is a Hilbert space, then J_2 becomes the identity mapping on E.

For a real constant $q > 1$, a Banach space E is called q-uniformly smooth if there exists a constant $C > 0$ such that $\rho_E(t) \leq Ct^q$ for all $t \in \mathbb{R}^+$. It is well known that (see e.g. [68]) L_q (or l_q) is q-uniformly smooth for $1 < q \leq 2$ and is 2-uniformly smooth if $q \geq 2$.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [68] proved the following result.

Lemma 2.3. *Let E be a real uniformly smooth Banach space. For a real constant $q > 1$, E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q \|y\|^q.$$

In order to proceed our next step, we also recall the following basic important concepts and some known results which will be used in this work.

Definition 2.4. *Let E be a real q-uniformly smooth Banach space and let $P : E \to E$ and $\eta : E \times E \to E$ be the mappings. Then P is said to be*

(i) *η-accretive if*

$$\langle P(x) - P(y), J_q(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in E;$$

(ii) *strictly η-accretive if P is η-accretive and equality holds if and only if $x = y$;*

(iii) *γ-strongly η-accretive (or strongly η-accretive with a constant $\gamma > 0$) if there exists a constant $\gamma > 0$ such that*

$$\langle P(x) - P(y), J_q(\eta(x,y))\rangle \geq \gamma\|x-y\|^q, \quad \forall x,y \in E;$$

(iv) *μ-Lipschitz continuous if there exists a constant $\mu > 0$ such that*

$$\|P(x) - P(y)\| \leq \mu\|x-y\|, \quad \forall x,y \in E.$$

It should be pointed out that if $\eta(x,y) = x - y$, for all $x,y \in E$, then parts (i) to (iii) of Definition 2.4 reduce to the definitions of accretivity, strict accretivity and strong accretivity of the mapping P, respectively.

Definition 2.5 ([26, 57]). *Let E be a real q-uniformly smooth Banach space, $P : E \to E$ be a single-valued mapping and $M : E \to 2^E$ be a multi-valued mapping. Then M is said to be*

(i) *accretive if*

$$\langle u - v, J_q(x-y)\rangle \geq 0, \quad \forall (x,u),(y,v) \in \mathrm{Graph}(M);$$

(ii) *m-accretive if M is accretive and $(I + \lambda M)(E) = E$ holds for every real constant $\lambda > 0$, where I denotes the identity mapping on E;*

(iii) *P-accretive if M is accretive and $(P + \lambda M)(E) = E$ holds for every $\lambda > 0$.*

In 2001, Huang and Fang [34] were the first to introduce and study the notion of generalized m-accretive (also referred to as m-η-accretive and also η-m-accretive [19]) mappings as a generalization of m-accretive mappings as follows.

Definition 2.6 ([19, 34]). *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping and $M : E \to 2^E$ be a multi-valued mapping. M is said to be*

(i) *η-accretive if*

$$\langle u - v, J_q(\eta(x,y))\rangle \geq 0, \quad \forall (x,u),(y,v) \in \mathrm{Graph}(M);$$

(ii) *generalized m-accretive if M is η-accretive and $(I + \lambda M)(E) = E$ holds for every real constant $\lambda > 0$.*

It is worthwhile to stress that M is a generalized m-accretive mapping if and only if M is η-accretive and there is no other η-accretive mapping whose graph contains strictly Graph(M). The generalized m-accretivity is to be understood in terms of inclusion of graphs. If $M : E \to 2^E$ is a generalized m-accretive mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the η-accretivity. In fact, the extended mapping is no longer η-accretive. In other words, for every pair $(x,u) \in E \times E\backslash\mathrm{Graph}(M)$ there exists $(y,v) \in \mathrm{Graph}(M)$ such that $\langle u - v, J_q(\eta(x,y))\rangle < 0$. In view of the arguments mentioned above, a necessary and sufficient condition for a multi-valued mapping $M : E \to 2^E$ to be generalized m-accretive is that the property

$$\langle u - v, J_q(\eta(x,y))\rangle \geq 0, \quad \forall (y,v) \in \mathrm{Graph}(M)$$

is equivalent to $(x,u) \in \mathrm{Graph}(M)$. The above characterization of generalized m-accretive mappings provides a useful and manageable way for recognizing that an element u belongs to $M(x)$.

The introduction of the notion of P-η-accretive (which also referred to as (H,η)-accretive) mappings was first made by Peng and Zhu [57], and Kazmi and Khan [44] which provides a unifying framework for H-accretive (P-accretive) mappings, (H,η)-monotone operators [28], H-monotone operators [25], generalized m-accretive mappings, m-accretive mappings, maximal η-monotone operators [34], and maximal monotone operators.

Definition 2.7 ([44, 57]). *Let E be a real q-uniformly smooth Banach space, $P : E \to E$ and $\eta : E \times E \to E$ be single-valued mappings and $M : E \to 2^E$ be a multi-valued mapping. M is said to be P-η-accretive if M is η-accretive and $(P + \lambda M)(E) = E$ holds for every $\lambda > 0$.*

The following new example illustrates that for given mappings $\eta : E \times E \to E$ and $P : E \to E$, a P-η-accretive mapping may be neither P-accretive nor generalized m-accretive.

Example 2.8. Let $\phi : \mathbb{Z} \to (0, +\infty)$ and consider the complex linear space $l^2_\phi(\mathbb{Z})$, the weighted $l^2(\mathbb{Z})$ space, consisting of all bi-infinite complex sequences $\{z_n\}_{n=-\infty}^{\infty}$ such that $\sum\limits_{n=-\infty}^{\infty} |z_n|^2\phi(n) < \infty$. It is known that

$$l^2_\phi(\mathbb{Z}) = \{z = \{z_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |z_n|^2\phi(n) < \infty, z_n \in \mathbb{C}\}$$

with respect to the inner product $\langle ., . \rangle : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to \mathbb{C}$ defined by

$$\langle z, w\rangle = \sum_{n=-\infty}^{\infty} z_n\overline{w_n}\phi(n), \quad \forall z = \{z_n\}_{n=-\infty}^{\infty}, w = \{w_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}),$$

is a Hilbert space. The above inner product induces a norm on $l^2_\phi(\mathbb{Z})$ as follows:

$$\|z\|_{l^2_\phi(\mathbb{Z})} = \sqrt{\langle z, z\rangle} = (\sum_{n=-\infty}^{\infty} |z_n|^2\phi(n))^{\frac{1}{2}}, \quad \forall z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}).$$

Thereby, $(l^2_\phi(\mathbb{Z}), \|.\|_{l^2_\phi(\mathbb{Z})})$ is a 2-uniformly smooth Banach space.

Any element $z = \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z})$ can be written as

$$\begin{aligned}
z = &\sum_{s\in\{\pm1,\pm3,\dots\}} \Big[(\dots, 0, 0, \dots, 0, x_{2s-1} + iy_{2s-1}, 0, x_{2s+1} + iy_{2s+1}, 0, 0, \dots) \\
&+ (\dots, 0, 0, \dots, 0, x_{2s} + iy_{2s}, 0, x_{2s+2} + iy_{2s+2}, 0, 0, \dots)\Big] \\
= &\sum_{s\in\{\pm1,\pm3,\dots\}} \Big[\frac{y_{2s-1} + y_{2s+1} - i(x_{2s-1} + x_{2s+1})}{2}(\dots, 0, 0, \dots, 0, i_{2s-1}, 0, i_{2s+1}, 0, 0, \dots) \\
&+ \frac{y_{2s-1} - y_{2s+1} - i(x_{2s-1} - x_{2s+1})}{2}(\dots, 0, 0, \dots, 0, i_{2s-1}, 0, -i_{2s+1}, 0, 0, \dots) \\
&+ \frac{y_{2s} + y_{2s+2} - i(x_{2s} + x_{2s+2})}{2}(\dots, 0, 0, \dots, 0, i_{2s}, 0, i_{2s+2}, 0, 0, \dots) \\
&+ \frac{y_{2s} - y_{2s+2} - i(x_{2s} - x_{2s+2})}{2}(\dots, 0, 0, \dots, 0, i_{2s}, 0, -i_{2s+2}, 0, 0, \dots)\Big] \\
= &\sum_{s\in\{\pm1,\pm3,\dots\}} \Big[\frac{y_{2s-1} + y_{2s+1} - i(x_{2s-1} + x_{2s+1})}{2}\delta_{2s-1,2s+1} + \frac{y_{2s-1} - y_{2s+1} - i(x_{2s-1} - x_{2s+1})}{2}\delta'_{2s-1,2s+1} \\
&+ \frac{y_{2s} + y_{2s+2} - i(x_{2s} + x_{2s+2})}{2}\delta_{2k,2k+2} + \frac{y_{2s} - y_{2s+2} - i(x_{2s} - x_{2s+2})}{2}\delta'_{2s,2s+2}\Big],
\end{aligned}$$

where for each $s \in \{\pm1, \pm3, \dots\}$, $\delta_{2s-1,2s+1} = (\dots, 0, 0, \dots, 0, i_{2s-1}, 0, i_{2s+1}, 0, 0, \dots)$, with i in the $(2s-1)$th and $(2s+1)$th positions and 0's elsewhere, $\delta'_{2s-1,2s+1} = (\dots, 0, 0, \dots, 0,\ i_{2s-1}, 0,\ -i_{2s+1}, 0, 0, \dots)$, i and $-i$ at the $(2s-1)$th and $(2s+1)$th coordinates, respectively, and all other coordinates are zero, $\delta_{2s,2s+2} = (\dots, 0, 0, \dots, 0, i_{2s}, 0, i_{2s+2}, 0, 0, \dots)$, i at the $(2s)$th and $(2s+2)$th coordinates, and all other coordinates are zero, and $\delta'_{2s,2s+2} = (\dots, 0, 0, \dots, 0, i_{2s}, 0, -i_{2s+2}, 0, 0, \dots)$, with i and $-i$ at the $(2s)$th and $(2s+2)$th places, respectively, and 0's everywhere else. Thus, the set

$$\mathfrak{B} = \{\delta_{2s-1,2s+1}, \delta'_{2s-1,2s+1}, \delta_{2s,2s+2}, \delta'_{2s,2s+2} : s = \pm1, \pm3, \dots\}$$

spans the Banach space $l^2_\phi(\mathbb{Z})$. It can be easily checked that the set $\mathfrak{B}$ is linearly independent and so it is a basis for $l^2_\phi(\mathbb{Z})$. For each $s \in \{\pm 1, \pm 3, \dots\}$, let use now take

$$\varrho_{2s-1,2s+1} = (\dots, 0, 0, \dots, 0, \frac{1}{\sqrt{2\phi(2s-1)}} i_{2s-1}, 0, \frac{1}{\sqrt{2\phi(2s+1)}} i_{2s+1}, 0, 0, \dots)$$
$$\varrho'_{2s-1,2s+1} = (\dots, 0, 0, \dots, 0, \frac{1}{\sqrt{2\phi(2s-1)}} i_{2s-1}, 0, -\frac{1}{\sqrt{2\phi(2s+1)}} i_{2s+1}, 0, 0, \dots)$$
$$\varrho_{2s,2s+2} = (\dots, 0, 0, \dots, 0, \frac{1}{\sqrt{2\phi(2s)}} i_{2s}, 0, \frac{1}{\sqrt{2\phi(2s+2)}} i_{2s+2}, 0, 0, \dots)$$

and

$$\varrho'_{2s,2s+2} = (\dots, 0, 0, \dots, 0, \frac{1}{\sqrt{2\phi(2s)}} i_{2s}, 0, -\frac{1}{\sqrt{2\phi(2s+2)}} i_{2s+2}, 0, 0, \dots).$$

Evidently, the set

$$\{\varrho_{2s-1,2s+1}, \varrho'_{2s-1,2s+1}, \varrho_{2s,2s+2}, \varrho'_{2s,2s+2} : s = \pm 1, \pm 3, \dots\}$$

is linearly independent and

$$\|\varrho_{2s-1,2s+1}\|_{l^2_{\phi(\mathbb{Z})}} = \|\varrho'_{2s-1,2s+1}\|_{l^2_{\phi(\mathbb{Z})}} = \|\varrho_{2s,2s+2}\|_{l^2_{\phi(\mathbb{Z})}} = \|\varrho'_{2s,2s+2}\|_{l^2_{\phi(\mathbb{Z})}} = 1,$$

that is, it is orthonormal. Let the mappings $M : l^2_\phi(\mathbb{Z}) \to 2^{l^2_\phi(\mathbb{Z})}$, $\eta : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$ and $P : l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$ be defined, respectively, as

$$M(z) = \begin{cases} \Theta, & z = \varrho_{2t,2t+2}, \\ -z + \left\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\right\}_{n=-\infty}^{\infty}, & z \neq \varrho_{2t,2t+2}, \end{cases}$$

$$\eta(z, w) = \begin{cases} \alpha(w - z), & z, w \neq \varrho_{2t,2t+2}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and $P(z) = \beta z + \gamma\left\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\right\}_{n=-\infty}^{\infty}$, for all $z, w \in l^2_\phi(\mathbb{Z})$, where

$$\Theta = \{\varrho_{2s-1,2s+1} - \varrho_{2t,2t+2}, \varrho'_{2s-1,2s+1} - \varrho_{2t,2t+2}, \varrho_{2s,2s+2} - \varrho_{2t,2t+2}, \varrho'_{2s,2s+2} - \varrho_{2t,2t+2} : s = \pm 1, \pm 3, \dots\},$$

$\alpha, \beta, \gamma \in \mathbb{R}$ are arbitrary constants such that $\beta < 0 < \alpha$, $t \in \{\pm 1, \pm 3, \dots\}$ is chosen arbitrarily but fixed, and $\mathbf{0}$ is the zero vector of the space $l^2_\phi(\mathbb{Z})$. Taking into account that $\sum\limits_{n=-\infty}^{\infty} n^4 e^{-n^2} = 2\sum\limits_{n=1}^{\infty} n^4 e^{-n^2}$ and $\sum\limits_{n=1}^{\infty} n^4 e^{-n^2}$ is convergent, it follows that $\sum\limits_{n=-\infty}^{\infty} n^4 e^{-n^2} < \infty$, and so $\left\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\right\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z})$. Then, for all $z, w \in l^2_\phi(\mathbb{Z})$, $z \neq w \neq \varrho_{2t,2t+2}$, yields

$$\begin{aligned}
\langle M(z) - M(w), J_2(z - w)\rangle &= \langle M(z) - M(w), z - w\rangle \\
&= \langle -z + \left\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\right\}_{n=-\infty}^{\infty} \\
&\quad + w - \left\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\right\}_{n=-\infty}^{\infty}, z - w\rangle \\
&= \langle w - z, z - w\rangle = -\|z - w\|^2_{l^2_\phi(\mathbb{Z})} \\
&= -\left(\sum_{n=-\infty}^{\infty} |z_n - w_n|^2 \phi(n)\right)^{\frac{1}{2}} < 0,
\end{aligned}$$

which means that M is not accretive and so it is not a P-accretive mapping. For any given $z, w \in l^2_\phi(\mathbb{Z})$, $z \neq w \neq \varrho_{2t,2t+2}$, we have

$$\begin{aligned}\langle M(z) - M(w), J_2(\eta(z,w))\rangle &= \langle M(z) - M(w), \eta(z,w)\rangle \\ &= \langle -z + \Big\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\Big\}_{n=-\infty}^{\infty} \\ &\quad + w - \Big\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\Big\}_{n=-\infty}^{\infty}, \alpha(w-z)\rangle \\ &= \alpha\|z-w\|^2_{l^2_\phi(\mathbb{Z})} = \alpha(\sum_{n=-\infty}^{\infty}|z_n - w_n|^2\phi(n))^{\frac{1}{2}} > 0.\end{aligned}$$

For each of the cases when $z \neq w = \varrho_{2t,2t+2}$, $w \neq z = \varrho_{2t,2t+2}$ and $z = w = \varrho_{2t,2t+2}$, due to the fact that $\eta(z,w) = \mathbf{0}$, we conclude that

$$\langle u - v, J_2(\eta(z,w))\rangle = \langle u - v, \eta(z,w)\rangle = 0, \quad \forall (x,u),(y,v) \in \mathrm{Graph}(M).$$

Hence, M is an η-accretive mapping. Taking into account that for any $\varrho_{2t,2t+2} \neq z \in l^2_\phi(\mathbb{Z})$,

$$\|(I+M)(z)\|^2_{l^2_\phi(\mathbb{Z})} = \|\Big\{\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}} + i\frac{n^2}{2\sqrt{e^{n^2}\phi(n)}}\Big\}_{n=-\infty}^{\infty}\|^2_{l^2_\phi(\mathbb{Z})} = \sum_{n=-\infty}^{\infty}\frac{n^4 e^{-n^2}}{2} > 0$$

and

$$(I+M)(\varrho_{2t,2t+2}) = \{\varrho_{2s-1,2s+1}, \varrho'_{2s-1,2s+1}, \varrho_{2s,2s+2}, \varrho'_{2s,2s+2} : s = \pm 1, \pm 3, \ldots\},$$

where I is the identity mapping on $E = l^2_\phi(\mathbb{Z})$, it follows that $\mathbf{0} \notin (I+M)(l^2_\phi(\mathbb{Z}))$. Therefore, $I + M$ is not surjective which ensures that M is not a generalized m-accretive (m-η-accretive) mapping. For any $\lambda > 0$ and $z \in l^2_\phi(\mathbb{Z})$, taking $w = \frac{1}{\beta-\lambda}z + \Big\{\frac{(\gamma+\lambda)n^2}{2(\lambda-\beta)\sqrt{e^{n^2}\phi(n)}} + i\frac{(\gamma+\lambda)n^2}{2(\lambda-\beta)\sqrt{e^{n^2}\phi(n)}}\Big\}_{n=-\infty}^{\infty}$ ($\lambda \neq \beta$, because $\beta < 0$), we obtain

$$(P+\lambda M)(w) = (P+\lambda M)\Big(\frac{1}{\beta-\lambda}z + \Big\{\frac{(\gamma+\lambda)n^2}{2(\lambda-\beta)\sqrt{e^{n^2}\phi(n)}} + i\frac{(\gamma+\lambda)n^2}{2(\lambda-\beta)\sqrt{e^{n^2}\phi(n)}}\Big\}_{n=-\infty}^{\infty}\Big) = z.$$

Hence, for every $\lambda > 0$, the mapping $P + \lambda M$ is surjective and so M is a P-η-accretive mapping.

For given mappings $\eta : E \times E \to E$ and $P : E \to E$, a generalized m-accretive mapping need not be P-η-accretive. This fact is shown in the following example.

Example 2.9. Let n be an arbitrary but fixed natural number, and $E = \mathbb{R}^n$ be a real Hilbert space together with the standard linear product on $E = \mathbb{R}^n$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, \quad \forall x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n.$$

The inner product defined above induces a norm on $E = \mathbb{R}^n$ as follows:

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n}|x_i|^2}, \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Taking into account that every finite dimensional normed space is a Banach space, it follows that $(\mathbb{R}^n, \|.\|)$ is a Hilbert space and so it is a 2-uniformly smooth Banach space.

Suppose that the mappings $M : E \to E$, $\eta : E \times E \to E$ and $P : E \to E$ are defined, respectively, as

$$M(x) = M((x_i)_{i=1}^n) = \Big(\frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i x_i}\Big)_{i=1}^n,$$
$$\eta(x, y) = \eta((x_i)_{i=1}^n, (y_i)_{i=1}^n) = (\theta_i \cos^{p_i} n_i x_i \cos^{p_i} t_i y_i (\sin^{q_i} m_i y_i - \sin^{q_i} m_i x_i)\Big)_{i=1}^n$$

and $P(x) = P((x_i)_{i=1}^n) = (\varsigma_i + \xi_i \sin^{k_i} l_i x_i\Big)_{i=1}^n$, for all $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in E$, where for $i = 1, 2, \ldots, n$, ς_i are arbitrary real constants, $\alpha_i, \beta_i, \gamma_i, \theta_i, \xi_i, m_i, n_i, t_i, l_i$ are arbitrary positive real constants and p_i, q_i and k_i are arbitrary but fixed even natural numbers. Then, for any $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in E$, we have

$$\begin{aligned}
\langle M(x) - M(y), J_2(\eta(x, y))\rangle &= \langle M(x) - M(y), \eta(x, y)\rangle \\
&= \langle \Big(\frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i x_i}\Big)_{i=1}^n - \Big(\frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i y_i}\Big)_{i=1}^n, \\
&\quad \Big(\theta_i \cos^{p_i} n_i x_i \cos^{p_i} t_i y_i (\sin^{q_i} m_i y_i - \sin^{q_i} m_i x_i)\Big)_{i=1}^n \rangle \\
&= \sum_{i=1}^n \frac{\alpha_i \gamma_i \theta_i \cos^{p_i} n_i x_i \cos^{p_i} t_i y_i (\sin^{q_i} m_i y_i - \sin^{q_i} m_i x_i)^2}{(\beta_i + \gamma_i \sin^{q_i} m_i x_i)(\beta_i + \gamma_i \sin^{q_i} m_i y_i)} \geq 0,
\end{aligned}$$

which implies that M is an η-accretive mapping. Let us now define the functions $f_i : \mathbb{R} \to \mathbb{R}$, for $i = 1, 2, \ldots, n$, as $f_i(\omega) = \varsigma_i + \xi_i \sin^{k_i} l_i \omega + \frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i \omega}$ for all $\omega \in \mathbb{R}$. Then, for any $x = (x_i)_{i=1}^n \in E = \mathbb{R}^n$, we have

$$(P + M)(x) = (P + M)((x_i)_{i=1}^n) = \Big(\varsigma_i + \xi_i \sin^{k_i} l_i x_i + \frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i x_i}\Big)_{i=1}^n = \Big(f_i(x_i)\Big)_{i=1}^n.$$

In the light of the fact that for each $i \in \{1, 2, \ldots, n\}$,

$$f_i(\omega) = \varsigma_i + \xi_i \sin^{k_i} l_i \omega + \frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i \omega} \geq \varsigma_i + \frac{\alpha_i}{\beta_i + \gamma_i}, \quad \forall \omega \in \mathbb{R},$$

we infer that for each $i \in \{1, 2, \ldots, n\}$, $f_i(\mathbb{R}) = [\frac{\varsigma_i(\beta_i + \gamma_i) + \alpha_i}{\beta_i + \gamma_i}, +\infty) \neq \mathbb{R}$. This fact guarantees that $(P + M)(E) \neq E$, i.e., the mapping $P + M$ is not surjective, consequently M is not a P-η-accretive mapping. Now, let λ be an arbitrary positive real constant and assume that for $i = 1, 2, \ldots, n$, the functions $g_i : \mathbb{R} \to \mathbb{R}$ are defined by $g_i(\omega) = \omega + \frac{\lambda \alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i \omega}$ for all $\omega \in \mathbb{R}$. Then, for any $x = (x_i)_{i=1}^n \in E$, we have

$$(I + \lambda M)(x) = (I + \lambda M)((x_i)_{i=1}^n) = \Big(x_i + \frac{\lambda \alpha_i}{\beta_i + \gamma_i \sin^{q_i} m_i x_i}\Big)_{i=1}^n = \Big(g_i(x_i)\Big)_{i=1}^n,$$

where I is the identity mapping on E. Since for each $i \in \{1, 2, \ldots, n\}$, $g_i(\mathbb{R}) = \mathbb{R}$, we deduce that $(I + \lambda M)(E) = E$, i.e., the mapping $I + \lambda M$ is surjective. Since $\lambda > 0$ was an arbitrary real constant, it follows that M is a generalized m-accretive mapping.

Example 2.10. Assume that $H_2(\mathbb{C})$ is the set of all Hermitian matrices with complex entries. Let us recall that a square matrix A is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e., $A^* = \overline{A^t} = A$. In the light of the definition of a Hermitian 2×2 matrix, the condition $A^* = A$ implies that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is Hermitian if and only if $a, d \in \mathbb{R}$ and $b = \bar{c}$. Hence,

$$H_2(\mathbb{C}) = \Big\{\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} | x, y, z, w \in \mathbb{R}\Big\}$$

and it is a subspace of $M_2(\mathbb{C})$, the space of all 2×2 matrices with complex entries, with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{C})$, when $M_2(\mathbb{C})$ is considered as a real vector space. Considering the scalar product on $H_2(\mathbb{C})$ as $\langle A, B\rangle := \frac{1}{2} tr(AB)$, for all $A, B \in H_2(\mathbb{C})$, it can be easily observed that $\langle .,. \rangle$ is an inner product, that is, $(H_2(\mathbb{C}), \langle .,. \rangle)$ is an inner product space. The inner product defined above induces a norm on $H_2(\mathbb{C})$ as follows:

$$\|A\| = \sqrt{\langle A,A\rangle} = \sqrt{\frac{1}{2}tr(AA)} = \Big\{\frac{1}{2}tr\Big(\begin{pmatrix} x^2+y^2+z^2 & (z+w)(x-iy) \\ (z+w)(x+iy) & x^2+y^2+w^2 \end{pmatrix}\Big)\Big\}^{\frac{1}{2}}$$
$$= \sqrt{x^2+y^2+\frac{1}{2}(z^2+w^2)}, \quad \forall A \in H_2(\mathbb{C}).$$

The finite dimensional normed space $(H_2(\mathbb{C}), \|.\|)$ is a Hilbert space and so it is a 2-uniformly smooth Banach space. Suppose that the mappings $P_1, P_2, M : H_2(\mathbb{C}) \to H_2(\mathbb{C})$ and $\eta : H_2(\mathbb{C}) \times H_2(\mathbb{C}) \to H_2(\mathbb{C})$ are defined, respectively, by

$$P_1(A) = P_1\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) = \begin{pmatrix} z^2 + \frac{1}{z^2+\alpha} - \varrho\sqrt[m]{z} & x^{2l} - iy^{2l} \\ x^{2l} + iy^{2l} & 2|w| + w^k - \theta w^n + \sigma \end{pmatrix},$$

$$P_2(A) = P_2\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) = \begin{pmatrix} |z-a| + |z-b| + z^{\delta} & x-iy \\ x+iy & w + \frac{e^w}{e^w+1} \end{pmatrix},$$

$$M(A) = M\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) = \begin{pmatrix} \varrho\sqrt[m]{z} & x^l - iy^l \\ x^l + iy^l & \theta w^n \end{pmatrix}$$

and

$$\eta(A,B) = \eta\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}, \begin{pmatrix} \widehat{z} & \widehat{x}-i\widehat{y} \\ \widehat{x}+i\widehat{y} & \widehat{w} \end{pmatrix}\Big)$$
$$= \begin{pmatrix} \beta|z\widehat{z}|(\sqrt[r]{z} - \sqrt[r]{\widehat{z}}) & x-\widehat{x}-i(y-\widehat{y}) \\ x-\widehat{x}+i(y-\widehat{y}) & \gamma e^{w\widehat{w}}(w^p - \widehat{w}^p) \end{pmatrix},$$

for all $A = \begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}, B = \begin{pmatrix} \widehat{z} & \widehat{x}-i\widehat{y} \\ \widehat{x}+i\widehat{y} & \widehat{w} \end{pmatrix} \in H_2(\mathbb{C})$, where $\alpha > 1$ and $\beta, \gamma, \varrho, \theta > 0$ and $a, b, \sigma \in \mathbb{R}$ are arbitrary constants, m, n, p, l, r, δ are arbitrary but fixed odd natural numbers and k is an arbitrary but fixed even natural number.

Then, for any $A = \begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}, B = \begin{pmatrix} \widehat{z} & \widehat{x}-i\widehat{y} \\ \widehat{x}+i\widehat{y} & \widehat{w} \end{pmatrix} \in H_2(\mathbb{C})$, yields

$$\begin{aligned}
&\langle M(A)-M(B), J_2(\eta(A,B))\rangle = \langle M(A)-M(B), \eta(A,B)\rangle \\
&= \Big\langle \begin{pmatrix} \varrho(\sqrt[m]{z} - \sqrt[m]{\widehat{z}}) & x^l - \widehat{x}^l - i(y^l - \widehat{y}^l) \\ x^l - \widehat{x}^l + i(y^l - \widehat{y}^l) & \theta(w^n - \widehat{w}^n) \end{pmatrix}, \begin{pmatrix} \beta|z\widehat{z}|(\sqrt[r]{z} - \sqrt[r]{\widehat{z}}) & x-\widehat{x}-i(y-\widehat{y}) \\ x-\widehat{x}+i(y-\widehat{y}) & \gamma e^{w\widehat{w}}(w^p - \widehat{w}^p) \end{pmatrix} \Big\rangle \\
&= \frac{1}{2}tr\Big(\begin{pmatrix} \vartheta_{11}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) & \vartheta_{12}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) \\ \vartheta_{21}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) & \vartheta_{22}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) \end{pmatrix}\Big) \\
&= \frac{\varrho\beta}{2}|z\widehat{z}|(\sqrt[m]{z} - \sqrt[m]{\widehat{z}})(\sqrt[r]{z} - \sqrt[r]{\widehat{z}}) + \frac{\gamma\theta}{2}e^{w\widehat{w}}(w^n - \widehat{w}^n)(w^p - \widehat{w}^p) \\
&\quad + (x-\widehat{x})(x^l - \widehat{x}^l) + (y-\widehat{y})(y^l - \widehat{y}^l) \\
&= \frac{\varrho\beta}{2}|z\widehat{z}|(\sqrt[m]{z} - \sqrt[m]{\widehat{z}})(\sqrt[r]{z} - \sqrt[r]{\widehat{z}}) + \frac{\gamma\theta}{2}e^{w\widehat{w}}(w-\widehat{w})^2\sum_{j=1}^{n} w^{n-j}\widehat{w}^{j-1}\sum_{\nu=1}^{p} w^{p-\nu}\widehat{w}^{\delta-1} \\
&\quad + (x-\widehat{x})^2\sum_{j'=1}^{l} x^{l-j'}\widehat{x}^{j'-1} + (y-\widehat{y})^2\sum_{j''=1}^{t} y^{l-j''}\widehat{y}^{j''-1},
\end{aligned} \tag{1}$$

where

$$\begin{aligned}\vartheta_{11}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) &= \varrho\beta|z\widehat{z}|(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})+(x-\widehat{x})(x^l-\widehat{x}^l)+(y-\widehat{y})(y^l-\widehat{y}^l)\\ &\quad+i(y-\widehat{y})(x^l-\widehat{x}^l)-i(y^l-\widehat{y}^l)(x-\widehat{x}),\\ \vartheta_{12}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) &= \beta|z\widehat{z}|(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})(x-\widehat{x}-i(y-\widehat{y}))+\theta(w^n-\widehat{w}^n)(x-\widehat{x}+i(y-\widehat{y})),\\ \vartheta_{21}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) &= \varrho(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(x-\widehat{x}-i(y-\widehat{y}))+\gamma(x^l-\widehat{x}^l-i(y^l-\widehat{y}^l))e^{w\widehat{w}}(w^p-\widehat{w}^p),\\ \vartheta_{22}(x,\widehat{x},y,\widehat{y},z,\widehat{z}) &= (x-\widehat{x})(x^l-\widehat{x}^l)+(y-\widehat{y})(y^l-\widehat{y}^l)+i(y^l-\widehat{y}^l)(x-\widehat{x})\\ &\quad-i(y-\widehat{y})(x^l-\widehat{x}^l)+\gamma\theta e^{w\widehat{w}}(w^n-\widehat{w}^n)(w^p-\widehat{w}^p).\end{aligned}$$

If $z=\widehat{z}=0$ then $(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})=0$ and when $z\neq 0=\widehat{z}$, we have

$$(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})=\sqrt[m]{z}\sqrt[r]{z}=\sqrt[mr]{z^{m+r}}.$$

By arguing similarly as above, in the case where $z=0\neq\widehat{z}$, we deduce that

$$(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})=\sqrt[m]{\widehat{z}}\sqrt[r]{\widehat{z}}=\sqrt[mr]{\widehat{z}^{m+r}}.$$

Taking into account that m and r are odd natural numbers, in last both cases, it follows that

$$(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})>0.$$

For the case when $z,\widehat{z}\neq 0$, we get

$$\sqrt[m]{z}-\sqrt[m]{\widehat{z}}=\frac{z-\widehat{z}}{\sum\limits_{j=1}^{m}\sqrt[m]{z^{m-j}\widehat{z}^{j-1}}}\quad\text{and}\quad\sqrt[r]{z}-\sqrt[r]{\widehat{z}}=\frac{z-\widehat{z}}{\sum\limits_{\varsigma=1}^{r}\sqrt[r]{z^{r-\varsigma}\widehat{z}^{\varsigma-1}}}.$$

Since m and r are odd natural numbers, we have $\sum\limits_{j=1}^{m}\sqrt[m]{z^{m-j}\widehat{z}^{j-1}}>0$ and $\sum\limits_{\varsigma=1}^{r}\sqrt[r]{z^{r-\varsigma}\widehat{z}^{\varsigma-1}}>0$, which imply that

$$(\sqrt[m]{z}-\sqrt[m]{\widehat{z}})(\sqrt[r]{z}-\sqrt[r]{\widehat{z}})=\frac{(z-\widehat{z})^2}{\Big(\sum\limits_{j=1}^{m}\sqrt[m]{z^{m-j}\widehat{z}^{j-1}}\Big)\Big(\sum\limits_{\varsigma=1}^{r}\sqrt[r]{z^{r-\varsigma}\widehat{z}^{\varsigma-1}}\Big)}>0.$$

Owing to the fact that n,p and l are odd natural numbers, it can be easily seen that $\sum\limits_{j=1}^{p}w^{n-j}\widehat{w}^{j-1}\geq 0$, $\sum\limits_{\delta=1}^{p}w^{p-\delta}\widehat{w}^{\delta-1}\geq 0$, $\sum\limits_{j'=1}^{l}x^{l-j'}\widehat{x}^{j'-1}\geq 0$ and $\sum\limits_{j''=1}^{l}y^{l-j''}\widehat{y}^{j''-1}\geq 0$. In virtue of the fact that $\varrho,\beta,\gamma,\theta>0$, thanks to the above-mentioned arguments and making use of (1), it follows that

$$\langle M(A)-M(B),J_2(\eta(A,B))\rangle\geq 0,\quad\forall A,B\in H_2(\mathbb{C}),$$

which means that M is an η-accretive mapping.

Let us define the functions $f,g,h:\mathbb{R}\to\mathbb{R}$ by

$$f(u):=u^2+\frac{1}{u^2+\alpha},g(u):=2|u|+u^k+\sigma\text{ and }h(u):=u^{2l}+u^l,\quad\forall u\in\mathbb{R},$$

respectively. Then, for any $A=\begin{pmatrix} z & x-iy\\ x+iy & w\end{pmatrix}\in H_2(\mathbb{C})$, we yeild

$$
\begin{aligned}
(P_1+M)(A) &= (P_1+M)\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) \\
&= \begin{pmatrix} z^2+\frac{1}{z^2+\alpha} & x^{2l}+x^l-i(y^{2l}+y^l) \\ x^{2l}+x^l+i(y^{2l}+y^l) & 2|w|+w^k+\sigma \end{pmatrix} \\
&= \begin{pmatrix} f(z) & h(x)-ih(y) \\ h(x)+ih(y) & g(w) \end{pmatrix}.
\end{aligned}
$$

It can be easily observed that Range(f) = $(0,+\infty)$ and Range(g) = $[\sigma,+\infty)$ because of the constant k is an even natural number. In the meanwhile, in the light of the fact that

$$
u^{2l}+u^l=(u^l+\frac{1}{2})^2-\frac{1}{4}\geq -\frac{1}{4}, \quad \forall u\in\mathbb{R},
$$

we deduce that Range(h) = $[-\frac{1}{4},+\infty)$. In view of these facts, it follows that $(P_1+M)(H_2(\mathbb{C}))\neq H_2(\mathbb{C})$, which guarantees that the mapping P_1+M is not surjective and so M is not a P_1-η-accretive mapping. Now, let $\lambda>0$ be chosen arbitrarily but fixed and assume that the functions $\widetilde{f},\widetilde{g},\widetilde{h}:\mathbb{R}\to\mathbb{R}$ are defined, respectively, by

$$
\widetilde{f}(u):=|u-a|+|u-b|+u^\delta+\lambda\varrho\sqrt[m]{u},\ \widetilde{g}(u):=u+\frac{e^u}{e^u+1}+\lambda\theta u^n \text{ and } \widetilde{h}(u):=u+\lambda u^l,
$$

for all $u\in\mathbb{R}$. Then, for any $A=\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\in H_2(\mathbb{C})$, yields

$$
\begin{aligned}
(P_2+\lambda M)(A) &= (P_2+\lambda M)\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) \\
&= \begin{pmatrix} |z-a|+|z-b|+z^\delta+\lambda\varrho\sqrt[m]{z} & x+\lambda x^l-i(y+\lambda y^l) \\ x+\lambda x^l+i(y+\lambda y^l) & w+\frac{e^w}{e^w+1}+\lambda\theta w^n \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{f}(z) & \widetilde{h}(x)-i\widetilde{h}(y) \\ \widetilde{h}(x)+i\widetilde{h}(y) & \widetilde{g}(w) \end{pmatrix}.
\end{aligned}
$$

Considering the fact that m,n,l and δ are odd natural numbers, it is easy to see that Range($\widehat{f}$) = Range($\widehat{g}$) = Range($\widehat{h}$) = $\mathbb{R}$, which implies that $(P_2+\lambda M)(H_2(\mathbb{C}))=H_2(\mathbb{C})$, that is, the mapping $P_2+\lambda M$ is surjective. Taking into account the arbitrariness in the choice of $\lambda>0$, we conclude that M is a P_2-η-accretive mapping.

It is important to emphasize that if $P=I$, the identity mapping on E, then the definition of P-η-accretive mapping is that of generalized m-accretive mapping. In fact, the class of P-η-accretive mappings has close relation with that of generalized m-accretive mappings in the framework of Banach spaces. On the other hand, as we have seen from Example 2.8, for given mappings $P:E\to E$ and $\eta:E\times E\to E$, a P-η-accretive mapping need not be generalized m-accretive. The following conclusion provides us the sufficient conditions for a P-η-accretive mapping M to be generalized m-accretive.

Lemma 2.11 ([57]). *Let E be a real q-uniformly smooth Banach space, $\eta:E\times E\to E$ be a vector-valued mapping, $P:E\to E$ be a strictly η-accretive mapping, $M:E\to 2^E$ be a P-η-accretive mapping, and let $x,u\in E$ be two given points. If $\langle u-v,J_q(\eta(x,y))\rangle\geq 0$ holds for all $(y,v)\in$ Graph(M), then $(x,u)\in$ Graph(M).*

In the light of Example 2.9, for given mappings $P:E\to E$ and $\eta:E\times E\to E$, a generalized m-accretive mapping may not be P-η-accretive. A natural question to ask is under which conditions a generalized m-accretive mapping is P-η-accretive. Before giving an answer to this question, we need to recall the following notions.

Definition 2.12. *Let E be a real q-uniformly smooth Banach space. A mapping $P : E \to E$ is said to be coercive if*

$$\lim_{\|x\|\to+\infty} \frac{\langle P(x), J_q(x)\rangle}{\|x\|} = +\infty.$$

Definition 2.13. *Let E be a real q-uniformly smooth Banach space and $P : E \to E$ be a single-valued mapping. P is said to be*

(i) *bounded, if $P(A)$ is a bounded subset of E, for every bounded subset A of E.*
(ii) *hemi-continuous if for any fixed points $x, y, z \in E$, the function $t \longmapsto \langle P(x + ty), J_q(z)\rangle$ is continuous at 0^+.*

Theorem 2.14. *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, and $P : E \to E$ be a bounded, coercive, hemi-continuous and η-accretive mapping. If $M : E \to 2^E$ is a generalized m-accretive mapping, then M is P-η-accretive.*

Proof. Taking into account that the mapping P is bounded, coercive, hemi-continuous and η-accretive, from Theorem 3.1 of Guo [31, P.401], it follows that $P + \lambda M$ is surjective for every $\lambda > 0$, i.e., $\text{Range}(P + \lambda M)(E) = E$ holds for every $\lambda > 0$. Accordingly, M is a P-η-accretive mapping. The proof is finished. □

Theorem 2.15. *Suppose that E is a real q-uniformly smooth Banach space, $\eta : E \times E \to E$ is a vector-valued mapping, $P : E \to E$ is a strictly η-accretive mapping, and $M : E \to 2^E$ is an η-accretive mapping. Then, the mapping $(P + \lambda M)^{-1} : \text{Range}(P + \lambda M) \to E$ is single-valued for every constant $\lambda > 0$.*

Proof. Choose constant $\lambda > 0$ and point $u \in \text{Range}(P + \lambda M)$ arbitrarily but fixed. Then for any $x, y \in (P + \lambda M)^{-1}(u)$, we have $u = (P + \lambda M)(x) = (P + \lambda M)(y)$, from which we deduce that

$$\lambda^{-1}(u - P(x)) \in M(x) \text{ and } \lambda^{-1}(u - P(y)) \in M(y).$$

Since M is η-accretive, it follows that

$$0 \le \langle \lambda^{-1}(u - P(x)) - \lambda^{-1}(u - P(y)), J_q(\eta(x, y))\rangle = \lambda^{-1}\langle P(x) - P(y), J_q(\eta(x, y))\rangle.$$

Thanks to the fact that the mapping P is strictly η-accretive, the preceding inequality ensures that $x = y$ and so the mapping $(P + \lambda M)^{-1}$ from $\text{Range}(P + \lambda M)$ into E is single-valued. This gives the desired result. □

The following assertion due to Kazmi and Khan [44] is an immediate consequence of the above theorem.

Lemma 2.16 ([57]). *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \to E$ be a vector-valued mapping, $P : E \to E$ be a strictly η-accretive mapping, and $M : E \to 2^E$ be a P-η-accretive mapping. Then, the mapping $(P + \lambda M)^{-1} : E \to E$ is single-valued for every real constant $\lambda > 0$.*

The resolvent operator $R^{P,\eta}_{M,\lambda}$ associated with P, η, M and given constant $\lambda > 0$ is defined based on Lemma 2.16 as follows.

Definition 2.17 ([44, 57]). *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \to E$ be a single-valued mapping, $P : E \to E$ be a strictly η-accretive mapping, $M : E \to 2^E$ be a P-η-accretive mapping, and $\lambda > 0$ be an arbitrary real constant. The resolvent operator $R^{P,\eta}_{M,\lambda} : E \to E$ associated with P, η, M and λ is defined by*

$$R^{P,\eta}_{M,\lambda}(u) = (P + \lambda M)^{-1}(u), \quad \forall u \in E.$$

We conclude this section with an important result due to Peng and Zhu [57] related to the Lipschitz continuity of the resolvent operator $R^{P,\eta}_{M,\lambda}$. Before proceeding to it, let us recall the following concept.

Definition 2.18. *A vector-valued mapping $\eta : E \times E \to E$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \le \tau\|x - y\|$, for all $x, y \in E$.*

Lemma 2.19 ([57]). *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \to E$ be a τ-Lipschitz continuous mapping, $P : E \to E$ be a γ-strongly η-accretive mapping, $M : E \to 2^E$ be a P-η-accretive mapping, and $\lambda > 0$ be an arbitrary real constant. Then, the resolvent operator $R^{P,\eta}_{M,\lambda} : E \to E$ is Lipschitz continuous with a constant $\frac{\tau^{q-1}}{\gamma}$, i.e.,*

$$\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\| \le \frac{\tau^{q-1}}{\gamma}\|u - v\|, \quad \forall u, v \in E.$$

3. System of Generalized Variational-like Inclusions: Existence and Uniqueness of Solution

This section is concerned with the introduction of a new system of variational-like inclusions involving P-η-accretive mappings in the setting of real q-uniformly smooth Banach spaces. At the same time, with the help of the resolvent operator technique, under some suitable conditions, the existence and uniqueness of solution for our considered system is proved.

Let for each $i \in \{1,2\}$, E_i be a real q_i-uniformly smooth Banach space with a norm $\|.\|_i$ and $q_i > 1$, $P_i, f_i, g_i : E_i \to E_i$, $\eta_i : E_i \times E_i \to E_i$, $F : E_1 \times E_2 \to E_1$, $G : E_1 \times E_2 \to E_2$, $S : E_2 \times E_1 \to E_1$ and $T : E_1 \times E_2 \to E_2$ be nonlinear mappings. Suppose further that $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ are two multi-valued nonlinear mappings such that for each $z \in E_1$, $M(.,z) : E_1 \to 2^{E_1}$ is a P_1-η_1-accretive mapping with $g_1(E_1) \cap \mathrm{dom}M(.,z) \neq \emptyset$, and $N(.,t) : E_2 \to 2^{E_2}$ is a P_2-η_2-accretive mapping for all $t \in E_2$ with $g_2(E_2) \cap \mathrm{dom}N(.,t) \neq \emptyset$. We consider the problem of finding $(x,y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in F(x,y) + S(y, x - f_1(x)) + M(g_1(x), x), \\ 0 \in G(x,y) + T(x, y - f_2(y)) + N(g_2(y), y), \end{cases} \tag{2}$$

which is called a *system of generalized nonlinear variational-like inclusions* (SGNVLI) with P-η-accretive mappings.

If for $i = 1,2$, $g_i \equiv I_i$, the identity mapping on E_i, $f_i = S = T \equiv 0$, $M : E_1 \to 2^{E_1}$ and $N : E_2 \to 2^{E_2}$ are two univariate multi-valued nonlinear mappings, then the SGNVLI (2) reduces to the problem of finding $(x,y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in F(x,y) + M(x), \\ 0 \in G(x,y) + N(y), \end{cases} \tag{3}$$

which was introduced and studied by Peng and Zhu [57].

It should be remarked that for suitable choices of the mappings $P_i, \eta_i, f_i, g_i, F, G, S, T, M, N$ and the underlying spaces E_i ($i = 1,2$), the SGNVLI (2) collapses to various classes of variational inclusions and variational inequalities, see, for example, [27, 37, 53, 65, 67, 70] and the references therein.

The following conclusion which tells the SGNVLI (2) is equivalent to a fixed point problem and follows directly from Definition 2.17 and some simple arguments, provides us a characterization of the solution of the SGNVLI (2).

Lemma 3.1. *Let $E_i, P_i, \eta_i, f_i, g_i, F, G, S, T, M, N$ ($i = 1,2$) be the same as in the* SGNVLI (2) *such that for each $i \in \{1,2\}$, P_i is a strictly η_i-accretive mapping with $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. Then $(x,y) \in E_1 \times E_2$ is a solution of the* SGNVLI (2) *if and only if*

$$\begin{cases} g_1(x) = R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))], \\ g_2(y) = R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho(G(x,y) + T(x, y - f_2(y)))], \end{cases} \tag{4}$$

where $R^{P_1,\eta_1}_{M(.,x),\lambda} = (P_1 + \lambda M(.,x))^{-1}$, $R^{P_2,\eta_2}_{N(.,y),\rho} = (P_2 + \rho N(.,y))^{-1}$, and $\lambda, \rho > 0$ are two constants.

In order to present the main result of this section, we need to define the following concepts.

Definition 3.2. *Let E be a real q-uniformly smooth Banach space. A mapping $T : E \to E$ is said to be (γ,μ)-relaxed cocoercive if there exist two constants $\gamma, \mu > 0$ such that*

$$\langle T(x) - T(y), J_q(x-y)\rangle \geq -\gamma\|T(x) - T(y)\|^q + \mu\|x-y\|^q, \quad \forall x, y \in E.$$

Definition 3.3. *Suppose that E is a real q-uniformly smooth Banach space and let $F : E \times E \to E$ and $T : E \to E$ be two nonlinear mappings. For a given point $(a,b) \in E \times E$, the mapping*

(i) *$F(a,.)$ is said to be k-strongly accretive with respect to T (or T-strongly accretive with constant k) if there exists a constant $k > 0$ such that*

$$\langle F(a,x) - F(a,y), J_q(T(x) - T(y))\rangle \geq k\|x-y\|^q, \quad \forall x, y \in E;$$

(ii) *$F(.,b)$ is said to be r-strongly accretive with respect to T (or T-strongly accretive with constant r) if there exists a constant $r > 0$ such that*

$$\langle F(x,b) - F(y,b), J_q(T(x) - T(y))\rangle \geq r\|x-y\|^q, \quad \forall x, y \in E;$$

(iii) *$F(a,.)$ is said to be ϱ-Lipschitz continuous if there exists a constant $\varrho > 0$ such that*

$$\|F(a,x) - F(a,y)\| \leq \varrho\|x-y\|, \quad \forall x, y \in E;$$

(iv) *$F(.,b)$ is said to be δ-Lipschitz continuous if there exists a constant $\delta > 0$ such that*

$$\|F(x,b) - F(y,b)\| \leq \delta\|x-y\|, \quad \forall x, y \in E;$$

(v) *$F(.,.)$ is said to be (ξ,ς)-mixed Lipschitz continuous in the first and second arguments if there exist two constants $\xi, \varsigma > 0$ such that*

$$\|F(x,y) - F(x',y')\| \leq \xi\|x-x'\| + \varsigma\|y-y'\|, \quad \forall x, x', y, y' \in E.$$

Theorem 3.4. *Let for each $i \in \{1,2\}$, E_i be a real q_i-uniformly smooth Banach space with $q_i > 1$ and a norm $\|.\|_i$. Suppose that for each $i \in \{1,2\}$, $\eta_i : E_i \times E_i \to E_i$ is a τ_i-Lipschitz continuous mapping, $P_i : E_i \to E_i$ is a γ_i-strongly η_i-accretive and δ_i-Lipschitz continuous mapping, $f_i : E_i \to E_i$ is a (ϖ_i, ϱ_i)-relaxed cocoercive and ζ_i-Lipschitz continuous mapping, and $g_i : E_i \to E_i$ is a (σ_i, π_i)-relaxed cocoercive and θ_i-Lipschitz continuous mapping such that $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. Let $F : E_1 \times E_2 \to E_1$ and $G : E_1 \times E_2 \to E_2$ be two nonlinear mappings such that for any given point $(a,b) \in E_1 \times E_2$, $F(.,b)$ is r_1-strongly accretive with respect to $P_1 \circ g_1$ and s_1-Lipschitz continuous, $F(a,.)$ is ξ_1-Lipschitz continuous, $G(a,.)$ is r_2-strongly accretive with respect to $P_2 \circ g_2$ and s_2-Lipschitz continuous and $G(.,b)$ is ξ_2-Lipschitz continuous. Assume that $S : E_2 \times E_1 \to E_1$ and $T : E_1 \times E_2 \to E_2$ are (μ_1,ν_1)-mixed Lipschitz and (μ_2,ν_2)-mixed Lipschitz continuous in the first and second arguments. Suppose that $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ are two multi-valued nonlinear mappings such that for each $z \in E_1$, $M(.,z) : E_1 \to 2^{E_1}$ is a P_1-η_1-accretive mapping with $g_1(E_1) \cap \mathrm{dom}M(.,z) \neq \emptyset$, and $N(.,t) : E_2 \to 2^{E_2}$ is a P_2-η_2-accretive mapping for all $t \in E_2$ with $g_2(E_2) \cap \mathrm{dom}N(.,t) \neq \emptyset$. Assume further that there exist constants $\varsigma_i > 0$ $(i = 1,2)$ such that*

$$\|R^{P_1,\eta_1}_{M(.,u),\lambda}(w) - R^{P_1,\eta_1}_{M(.,v),\lambda}(w)\| \leq \varsigma_1\|u-v\|_1, \quad \forall u, v, w \in E_1, \tag{5}$$

$$\|R^{P_2,\eta_2}_{N(.,u),\rho}(w) - R^{P_2,\eta_2}_{N(.,v),\rho}(w)\| \leq \varsigma_2\|u-v\|_2, \quad \forall u, v, w \in E_2. \tag{6}$$

If there exist two constants $\lambda, \rho > 0$ such that

$$\begin{aligned} &\varsigma_1 + \sqrt[q_1]{1 - q_1\pi_1 + (c_{q_1} + q_1\sigma_1)\theta_1^{q_1}} + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big(\sqrt[q_1]{\delta_1^{q_1}\theta_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1}c_{q_1}s_1^{q_1}} \\ &+ \lambda\nu_1\sqrt[q_1]{1 - q_1\varrho_1 + (c_{q_1} + q_1\varpi_1)\zeta_1^{q_1}}\Big) + \frac{\rho\tau_2^{q_2-1}}{\gamma_2}(\xi_2 + \mu_2) < 1, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\varsigma_2 + \sqrt[q_2]{1 - q_2\pi_2 + (c_{q_2} + q_2\sigma_2)\theta_2^{q_2}} + \frac{\tau_2^{q_2-1}}{\gamma_2}\Big(\sqrt[q_2]{\delta_2^{q_2}\theta_2^{q_2} - q_2\rho r_2 + \rho^{q_2}c_{q_2}s_2^{q_2}} \\ &+ \rho\nu_2\sqrt[q_2]{1 - q_2\varrho_2 + (c_{q_2} + q_2\varpi_2)\zeta_2^{q_2}}\Big) + \frac{\lambda\tau_1^{q_1-1}}{\gamma_1}(\xi_1 + \mu_1) < 1, \end{aligned} \tag{8}$$

where c_{q_1} and c_{q_2} are constants guaranteed by Lemma 2.3, and for the case when q_1 and q_2 are even natural numbers, in addition to (7) and (8), the following conditions hold:

$$\begin{cases} q_i\pi_i < 1 + (c_{q_i} + q_i\sigma_i)\theta_i^{q_i}, & (i = 1,2), \\ q_i\varrho_i < 1 + (c_{q_i} + q_i\varpi_i)\zeta_i^{q_i}, & (i = 1,2), \\ q_1\lambda r_1 < \delta_1^{q_1}\theta_1^{q_1} + \lambda^{q_1}c_{q_1}s_1^{q_1}, \\ q_2\rho r_2 < \delta_2^{q_2}\theta_2^{q_2} + \rho^{q_2}c_{q_2}s_2^{q_2}, \end{cases} \tag{9}$$

then the SGNVLI (2) *admits a unique solution.*

Proof. For any given $\lambda, \rho > 0$, define the mappings $\Psi_\lambda : E_1 \times E_2 \to E_1$ and $\Phi_\rho : E_1 \times E_2 \to E_2$, respectively, by

$$\Psi_\lambda(x,y) = x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))] \tag{10}$$

and

$$\Phi_\rho(x,y) = y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho(G(x,y) + T(x, y - f_2(y)))], \tag{11}$$

for all $(x,y) \in E_1 \times E_2$. Moreover, for any $\lambda, \rho > 0$, define a mapping $Q_{\lambda,\rho} : E_1 \times E_2 \to E_1 \times E_2$ by

$$Q_{\lambda,\rho}(x,y) = (\Psi_\lambda(x,y), \Phi_\rho(x,y)), \quad \forall (x,y) \in E_1 \times E_2. \tag{12}$$

It follows from Lemma 2.19 and (5) that for all $(x,y), (x',y') \in E_1 \times E_2$,

$$\begin{aligned}
\|\Psi_\lambda(x,y) - \Psi_\lambda(x',y')\|_1 &= \|x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))] - (x' - g_1(x') \\
&\quad + R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x')) - \lambda(F(x',y') + S(y', x' - f_1(x')))]\big)\|_1 \\
&\le \|x - x' - (g_1(x) - g_1(x'))\|_1 + \|R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))] \\
&\quad - R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))]\|_1 \\
&\quad + \|R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))] \\
&\quad - R^{P_1,\eta_1}_{M(.,x'),\lambda}[P_1(g_1(x')) - \lambda(F(x',y') + S(y', x' - f_1(x')))]\|_1 \\
&\le \|x - x' - (g_1(x) - g_1(x'))\|_1 + \varsigma_1\|x - x'\|_1 + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big(\|P_1(g_1(x)) - P_1(g_1(x')) \\
&\quad - \lambda(F(x,y) - F(x',y'))\|_1 + \lambda\|S(y, x - f_1(x)) - S(y', x' - f_1(x'))\|_1\Big) \\
&\le \|x - x' - (g_1(x) - g_1(x'))\|_1 + \varsigma_1\|x - x'\|_1 \\
&\quad + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big(\|P_1(g_1(x)) - P_1(g_1(x')) - \lambda(F(x,y) - F(x',y))\|_1 \\
&\quad + \lambda\|F(x',y) - F(x',y')\|_1 + \lambda\|S(y, x - f_1(x)) - S(y', x' - f_1(x'))\|_1\Big).
\end{aligned} \tag{13}$$

Invoking Lemma 2.3, there exists a constant $c_{q_1} > 0$ such that

$$\|x - x' - (g_1(x) - g_1(x'))\|_1^{q_1} \le \|x - x'\|_1^{q_1} - q_1\langle g_1(x) - g_1(x'), J_{q_1}(x - x')\rangle + c_{q_1}\|g_1(x) - g_1(x')\|_1^{q_1}.$$

From (σ_1, π_1)-relaxed cocoercivity and θ_1-Lipschitz continuity of g_1, it follows that

$$\begin{aligned}
\|x - x' - (g_1(x) - g_1(x'))\|_1^{q_1} &\le \|x - x'\|_1^{q_1} - q_1\pi_1\|x - x'\|_1^{q_1} + (c_{q_1} + q_1\sigma_1)\theta_1^{q_1}\|x - x'\|_1^{q_1} \\
&= (1 - q_1\pi_1 + (c_{q_1} + q_1\sigma_1)\theta_1^{q_1})\|x - x'\|_1^{q_1},
\end{aligned}$$

whence we deduce that

$$\|x - x' - (g_1(x) - g_1(x'))\|_1 \le \sqrt[q_1]{1 - q_1\pi_1 + (c_{q_1} + q_1\sigma_1)\theta_1^{q_1}}\,\|x - x'\|_1. \tag{14}$$

By using Lemma 2.3 and in virtue of the facts that the mappings P_1 and g_1 are δ_1-Lipschitz continuous and θ_1-Lipschitz continuous, respectively, and the mapping F is r_1-strongly accretive with respect to $P_1 \circ g_1$ and

s_1-Lipschitz continuous in the first argument, we obtain

$$\begin{aligned}
&\|P_1(g_1(x)) - P_1(g_1(x')) - \lambda(F(x,y) - F(x',y))\|_1^{q_1} \\
&\quad \leq \|P_1(g_1(x)) - P_1(g_1(x'))\|_1^{q_1} - q_1\lambda\langle F(x,y) - F(x',y), J_{q_1}(P_1(g_1(x)) - P_1(g_1(x')))\rangle \\
&\qquad + \lambda^{q_1} c_{q_1}\|F(x,y) - F(x',y)\|_1^{q_1} \\
&\quad \leq \delta_1^{q_1}\|g_1(x) - g_1(x')\|_1^{q_1} - q_1\lambda r_1\|x - x'\|_1^{q_1} + \lambda^{q_1} c_{q_1} s_1^{q_1}\|x - x'\|_1^{q_1} \\
&\quad \leq (\delta_1^{q_1}\theta_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1})\|x - x'\|_1^{q_1},
\end{aligned}$$

which implies that

$$\|P_1(g_1(x)) - P_1(g_1(x')) - \lambda(F(x,y) - F(x',y))\|_1 \leq \sqrt[q_1]{\delta_1^{q_1}\theta_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1}}\|x - x'\|_1. \tag{15}$$

Since F is ξ_1-Lipschitz continuous in the second argument and S is (μ_1, ν_1)-mixed Lipschitz continuous in the first and second arguments, it follows that

$$\|F(x',y) - F(x',y')\|_1 \leq \xi_1\|y - y'\|_2 \tag{16}$$

and

$$\|S(y, x - f_1(x)) - S(y', x' - f_1(x'))\|_1 \leq \mu_1\|y - y'\|_2 + \nu_1\|x - x' - (f_1(x) - f_1(x'))\|_1. \tag{17}$$

Making use of Lemma 2.3 and taking into account of the assumptions, in a similar way to the proof of (14), we get

$$\|x - x' - (f_1(x) - f_1(x'))\|_1 \leq \sqrt[q_1]{1 - q_1\varrho_1 + (c_{q_1} + q_1\varpi_1)\zeta_1^{q_1}}\|x - x'\|_1. \tag{18}$$

Substituting (14)–(18) into (13), we yield

$$\begin{aligned}
\|\Psi_\lambda(x,y) - \Psi_\lambda(x',y')\|_1 \leq\ & \sqrt[q_1]{1 - q_1\pi_1 + (c_{q_1} + q_1\sigma_1)\theta_1^{q_1}}\|x - x'\|_1 + \varsigma_1\|x - x'\|_1 \\
& + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big(\sqrt[q_1]{\delta_1^{q_1}\theta_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1}}\|x - x'\|_1 \\
& + \lambda\xi_1\|y - y'\|_2 + \lambda\mu_1\|y - y'\|_2 \\
& + \lambda\nu_1\sqrt[q_1]{1 - q_1\varrho_1 + (c_{q_1} + q_1\varpi_1)\zeta_1^{q_1}}\|x - x'\|_1\Big) \\
=\ & \varphi_1\|x - x'\|_1 + \vartheta_1\|y - y'\|_2,
\end{aligned} \tag{19}$$

where

$$\varphi_1 = \varsigma_1 + \sqrt[q_1]{1 - q_1\pi_1 + (c_{q_1} + q_1\sigma_1)\theta_1^{q_1}} + \frac{\tau_1^{q_1-1}}{\gamma_1}\Big(\sqrt[q_1]{\delta_1^{q_1}\theta_1^{q_1} - q_1\lambda r_1 + \lambda^{q_1} c_{q_1} s_1^{q_1}} + \lambda\nu_1\sqrt[q_1]{1 - q_1\varrho_1 + (c_{q_1} + q_1\varpi_1)\zeta_1^{q_1}}\Big)$$

and $\vartheta_1 = \frac{\lambda\tau_1^{q_1-1}}{\gamma_1}(\xi_1 + \mu_1)$. By an argument analogous to the previous inequality (19), one can prove that

$$\|\Phi_\rho(x,y) - \Phi_\rho(x',y')\|_2 \leq \varphi_2\|x - x'\|_1 + \vartheta_2\|y - y'\|_2, \tag{20}$$

where

$$\vartheta_2 = \varsigma_2 + \sqrt[q_2]{1 - q_2\pi_2 + (c_{q_2} + q_2\sigma_2)\theta_2^{q_2}} + \frac{\tau_2^{q_2-1}}{\gamma_2}\Big(\sqrt[q_2]{\delta_2^{q_2}\theta_2^{q_2} - q_2\rho r_2 + \rho^{q_2} c_{q_2} s_2^{q_2}} + \rho\nu_2\sqrt[q_2]{1 - q_2\varrho_2 + (c_{q_2} + q_2\varpi_2)\zeta_2^{q_2}}\Big)$$

and $\varphi_2 = \frac{\rho\tau_2^{q_2-1}}{\gamma_2}(\xi_2 + \mu_2)$. Let us define a norm $\|.\|_*$ on $E_1 \times E_2$ by

$$\|(x,y)\|_* = \|x\|_1 + \|y\|_2, \quad \forall (x,y) \in E_1 \times E_2. \tag{21}$$

It can be easily seen that $(E_1 \times E_2, \|.\|_*)$ is a Banach space. Employing (19) and (20), we infer that

$$\begin{aligned} \|\Psi_\lambda(x,y) - \Psi_\lambda(x',y')\|_1 + \|\Phi_\rho(x,y) - \Phi_\rho(x',y')\|_2 &\leq (\varphi_1 + \varphi_2)\|x - x'\|_1 + (\vartheta_1 + \vartheta_2)\|y - y'\|_2 \\ &\leq k\|(x,y) - (x',y')\|_*, \end{aligned} \tag{22}$$

where $k = \max\{\varphi_1 + \varphi_2, \vartheta_1 + \vartheta_2\}$. Clearly, (7) and (8) imply that $k \in (0,1)$ and it follows from (20) that $Q_{\lambda,\rho}$ is a contraction mapping. According to Banach fixed point theorem, there exists a unique point $(x^*, y^*) \in E_1 \times E_2$ such that $Q_{\lambda,\rho}(x^*, y^*) = (x^*, y^*)$. Applying (10)–(**??**), we conclude that

$$\begin{cases} g_1(x^*) = R^{P_1,\eta_1}_{M(.,x^*),\lambda}[P_1(g_1(x^*)) - \lambda(F(x^*,y^*) + S(y^*, x^* - f_1(x^*)))], \\ g_2(y^*) = R^{P_2,\eta_2}_{N(.,y^*),\rho}[P_2(g_2(y^*)) - \rho(G(x^*,y^*) + T(x^*, y^* - f_2(y^*)))]. \end{cases}$$

Now, Lemma 3.1 guarantees that (x^*, y^*) is the unique solution of the SGNVLI (2). This completes the proof. □

Corollary 3.5. [57, Theorem 4.1] *Let E_1 and E_2 be real q-uniformly smooth Banach spaces. For each $i \in \{1,2\}$, let $\eta_i : E_i \times E_i \to E_i$ be Lipschitz continuous with constant τ_i, and $P_i : E_i \to E_i$ be strongly η_i-accretive and Lipschitz continuous with constants γ_i and δ_i, respectively. Let $F : E_1 \times E_2 \to E_1$ be a nonlinear operator such that for any given $(a,b) \in E_1 \times E_2$, $F(.,b)$ is P_1-strongly accretive and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a,.)$ is Lipschitz continuous with constant ξ_1. Let $G : E_1 \times E_2 \to E_2$ be a nonlinear operator such that for any given $(x,y) \in E_1 \times E_2$, $G(x,.)$ is P_2-strongly accretive and Lipschitz continuous with constants r_2 and s_2, respectively, and $G(.,y)$ is Lipschitz continuous with constant ξ_2. Assume that $M : E_1 \to 2^{E_1}$ is a P_1-η_1-accretive operator and $N : E_2 \to 2^{E_2}$ is a P_2-η_2-accretive operator. If there exist constants $\lambda, \rho > 0$ such that*

$$\frac{\tau_1^{q-1}}{\gamma_1}\sqrt[q]{\delta_1^q - q\lambda r_1 + c_q\lambda^q s_1^q} + \frac{\xi_2\rho\tau_2^{q-1}}{\gamma_2} < 1, \tag{23}$$

and

$$\frac{\tau_2^{q-1}}{\gamma_2}\sqrt[q]{\delta_2^q - q\rho r_2 + c_q\rho^q s_2^q} + \frac{\xi_1\lambda\tau_1^{q-1}}{\gamma_1} < 1, \tag{24}$$

where c_q is a constant guaranteed by Lemma 2.3, and for the case when q is an even natural number, in addition to (23) *and* (24)*, the following conditions hold:*

$$q\lambda r_1 < \delta_1^q + c_q\lambda^q s_1^q \text{ and } q\rho r_2 \leq \delta_2^q + c_q\rho^q s_2^q, \tag{25}$$

then the problem (3) *admits a unique solution.*

Proof. Since for $i = 1,2$, $g_i \equiv I_i$, the identity mapping on E_i, it follows that for $i = 1,2$, $\theta_i = 1$ and

$$\|x - x' - (g_1(x) - g_1(x'))\|_1 = \|y - y' - (g_2(y) - g_2(y'))\|_2 = 0.$$

In the light of the assumptions, taking $q_i = q$, $f_i = S = T \equiv 0$ and $\varpi_i = \varrho_i = \zeta_i = \mu_i = \nu_i = \varsigma_i = 0$ for $i = 1,2$, (7) and (8) reduce to (23) and (24), respectively. Now, the statement follows immediately using Theorem 3.4. □

Remark 3.6. It is significant to mention that by a careful reading the proof of Theorem 4.1 in [57], we noted that the conditions mentioned in the context of [57, Theorem 4.1] do not guarantee the existence of a unique solution for the problem (3). Indeed, for the case when q is an even natural number, then in addition to (23) and (24), conditions (25) must be also added to the context of [57, Theorem 4.1], as we have done in the context of Corollary 3.5.

For a given real normed space E with a norm $\|.\|$, let us recall that a nonlinear mapping $T : E \to E$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in E$. Due to the existence of a deep and close relation between the class of nonexpansive mappings and the classes of monotone and accretive operators, since the appearance of the theory of nonexpansive mapping in sixties, it has been intensively studied by many mathematicians. Because of its many diverse applications in the theory of fixed points, in recent decades, several interesting generalizations of the notion of nonexpansive mapping in the setting of different spaces have been appeared in the literature. Some classes of generalized nonexpansive mappings appeared in the literature are recalled in the following.

Definition 3.7. *A nonlinear mapping $T : E \to E$ is said to be*

(i) *L-Lipschitzian if there exists a constant $L > 0$ such that*

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in E;$$

(ii) *uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\|, \quad \forall x, y \in E;$$

(iii) *asymptotically nonexpansive [30] if there exists a sequence $\{a_n\} \subset (0, +\infty)$ with $\lim\limits_{n\to\infty} a_n = 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq (1 + a_n)\|x - y\|, \quad \forall x, y \in E.$$

Equivalently, T is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \quad \forall x, y \in E;$$

(iv) *total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive) [3] if there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in E$,*

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n\phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

It is significant to emphasize that every uniformly L-Lipschitzian mapping is L-Lipschitzian, but the converse is not true necessarily. In other words, the class of L-Lipschitzian mappings is broader than the class of uniformly L-Lipschitzian mappings. For illustration this fact, the following example is provided.

Example 3.8. Let $k < 0$ be an arbitrary constant and consider $E = [k, +\infty)$ with the Euclidean norm $\|.\| = |.|$ defined on $\mathbb{R}$. Suppose further that the self-mapping T of E is defined by

$$T(x) = \begin{cases} x, & \text{if } x \in [k, 0), \\ lx, & \text{if } x \in [0, +\infty), \end{cases}$$

where $l > 1$ is an arbitrary constant. In the light of the facts that

(i) for all $x, y \in [k, 0]$,

$$|T(x) - T(y)| = |x - y| < l|x - y|;$$

(ii) for all $x, y \in [0, +\infty)$,

$$|Tx - T(y)| = |lx - ly| \leq l|x - y|;$$

(iii) for all $x \in [k, 0]$ and $y \in [0, +\infty)$

$$|T(x) - T(y)| = |x - ly| < l|x - y|,$$

it follows that T is an l-Lipschitzian mapping. But, in virtue of the fact that $l > 1$ we infer that for all $n \in \mathbb{N}\backslash\{1\}$,

$$|T^n(x) - T^n(y)| = l^n|x-y| > l|x-y|, \quad \forall x, y \in [0, +\infty).$$

This fact ensures that T is not a uniformly l-Lipschitzian mapping.

It is also remarkable that every asymptotically nonexpansive mapping is total asymptotically nonexpansive with $b_n = 0$ (or equivalently $b_n = 0$ and $a_n = k_n - 1$) for all $n \in \mathbb{N}$ and $\phi(t) = t$ for all $t \geq 0$, but the converse need not be true. The following example shows that the class of total asymptotically nonexpansive mappings is essentially wider than the class of asymptotically nonexpansive mappings.

Example 3.9. For $1 \leq p < \infty$, consider the classical space

$$l^p = \Big\{x = \{x_n\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\Big\},$$

consisting of all p-power summable sequences, with the p-norm $\|.\|_p$ defined on it by

$$\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}, \quad \forall x = \{x_n\}_{n\in\mathbb{N}} \in l^p.$$

Moreover, let B denote the closed unit ball in the real Banach space l^p and consider the subset $E := [\alpha, \beta] \times B$ of $\mathbb{R} \times l^p$ with the norm $\|.\|_E = |.|_{\mathbb{R}} + \|.\|_p$ defined on $\mathbb{R} \times l^p$, where $\alpha < 0$ and $\beta \geq 1$ are arbitrary real constants. Let further that the self-mapping T of E be defined by

$$T(u, x) = \begin{cases} (u, \widehat{x}), & \text{if } u \in [\alpha, 0), \\ (\varrho, \widehat{x}), & \text{if } u = 0, \\ (\varrho u, \widehat{x}), & \text{if } u \in (0, \beta], \end{cases}$$

where

$$\begin{aligned}\widehat{x} = (&\underbrace{0, 0, \ldots, 0}_{\sigma \text{ times}}, \gamma \sin|x_1|^{t_1}, 0, \frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_2|^{q_1} - |x_2|^{s_1}), 0, \frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_3|^{\lambda_1} - \sin^{k_1}|x_3|),\\ &0, \gamma\sin|x_4|^{t_2}, 0, \frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_5|^{q_2} - |x_5|^{s_2}), 0, \frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_6|^{\lambda_2} - \sin^{k_2}|x_6|), \ldots, 0,\\ &\gamma\sin|x_m|^{t_{\frac{m+2}{3}}}, 0, \frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_{m+1}|^{q_{\frac{m+2}{3}}} - |x_{m+1}|^{s_{\frac{m+2}{3}}}), 0, \frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_{m+2}|^{\lambda_{\frac{m+2}{3}}}\\ &- \sin^{k_{\frac{m+2}{3}}}|x_{m+2}|), 0, \gamma x_{m+3}, 0, \gamma x_{m+4}, \ldots),\end{aligned}$$

$\gamma, \varrho \in (0, 1)$ are arbitrary real constants, $m \in \{3\nu - 2 | \nu \in \mathbb{N}\}$, $\sigma \geq m+2$ and $\lambda_i, k_i, s_i, t_i, q_i \in \mathbb{N}\backslash\{1\}$ $(i = 1, 2, \ldots, \frac{m+2}{3})$ are arbitrary but fixed natural numbers. Indeed, the element $\widehat{x}$ of l^p can be written as $\widehat{x} = \{\widehat{x_n}\}_{n=1}^{\infty}$, where $\widehat{x_i} = \widehat{x}_{\sigma+2j} = 0$ for all $1 \leq i \leq \sigma$ and $j \in \mathbb{N}$,

$$\widehat{x}_{\sigma+2i-1} = \begin{cases} \gamma\sin|x_i|^{t_{\frac{i+2}{3}}}, & \text{if } i \in \{3r-2 | r = 1, 2, \ldots, \frac{m+2}{3}\}, \\ \frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_i|^{q_{\frac{i+1}{3}}} - |x_i|^{s_{\frac{i+1}{3}}}), & \text{if } i \in \{3r-1 | r = 1, 2, \ldots, \frac{m+2}{3}\}, \\ \frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_i|^{\lambda_{\frac{i}{3}}} - \sin^{k_{\frac{i}{3}}}|x_i|), & \text{if } i \in \{3r | r = 1, 2, \ldots, \frac{m+2}{3}\}, \end{cases}$$

and $\widehat{x}_{\sigma+2m+j} = \gamma x_{m+\frac{j+1}{2}}$ for all $j \in \{2\delta + 3 | \delta \in \mathbb{N}\}$. In virtue of the fact that the mapping T is discontinuous at the points $(0, x)$ for all $x \in B$, it follows that T is not Lipschitzian and so it is not an asymptotically nonexpansive mapping.

It can be easily proved that for all $(u,x),(v,y)\in[\alpha,0)\times B$,

$$\begin{aligned}
&\|T(u,x)-T(v,y)\|_E=\|(u-v,(\underbrace{0,0,\dots,0}_{\sigma\ times},\gamma(\sin|x_1|^{t_1}-\sin|y_1|^{t_1}),\\
&0,\frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_2|^{q_1}-\sin|y_2|^{q_1}-(|x_2|^{s_1}-|y_2|^{s_1})),0,\frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_3|^{\lambda_1}-|y_3|^{\lambda_1}\\
&-(\sin^{k_1}|x_3|-\sin^{k_1}|y_3|)),0,\gamma(\sin|x_4|^{t_2}-\sin|y_4|^{t_2}),0,\frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_5|^{q_2}-\sin|y_5|^{q_2}\\
&-(|x_5|^{s_2}-|y_5|^{s_2})),0,\frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_6|^{\lambda_2}-|y_6|^{\lambda_2}-(\sin^{k_2}|x_6|-\sin^{k_2}|y_6|)),\dots,0,\\
&\gamma(\sin|x_m|^{t_{\frac{m+2}{3}}}-\sin|y_m|^{t_{\frac{m+2}{3}}}),0,\frac{\gamma}{\sqrt[p]{2^{p+1}}}(\sin|x_{m+1}|^{q_{\frac{m+2}{3}}}-\sin|y_{m+1}|^{q_{\frac{m+2}{3}}}\\
&-(|x_{m+1}|^{s_{\frac{m+2}{3}}}-|y_{m+1}|^{s_{\frac{m+2}{3}}})),0,\frac{\gamma}{\sqrt[p]{2^{p+1}}}(|x_{m+2}|^{\lambda_{\frac{m+2}{3}}}-|y_{m+2}|^{\lambda_{\frac{m+2}{3}}}\\
&-(\sin^{k_{\frac{m+2}{3}}}|x_{m+2}|-\sin^{k_{\frac{m+2}{3}}}|y_{m+2}|)),0,\gamma(x_{m+3}-y_{m+3}),0,\gamma(x_{m+4}-y_{m+4}),\dots))\|_X\\
&=|u-v|+\Big(\gamma^p\sum_{i=1}^{\frac{m+2}{3}}\Big|\sin|x_{3i-2}|^{t_i}-\sin|y_{3i-2}|^{t_i}\Big|^p+(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^p\sum_{i=1}^{\frac{m+2}{3}}\Big|\sin|x_{3i-1}|^{q_i}-\sin|y_{3i-1}|^{q_i}\\
&-(|x_{3i-1}|^{s_i}-|y_{3i-1}|^{s_i})\Big|^p+(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^p\sum_{i=1}^{\frac{m+2}{3}}\Big||x_{3i}|^{\lambda_i}-|y_{3i}|^{\lambda_i}-(sin^{k_i}|x_{3i}|-sin^{k_i}|y_{3i}|)\Big|^p+\gamma^p\sum_{i=m+3}^{\infty}|x_i-y_i|^p\Big)^{\frac{1}{p}}\\
&\le|u-v|+\gamma\max\Big\{\sum_{r''=1}^{t_i}|x_{3i-2}|^{t_i-r''}|y_{3i-2}|^{r''-1},\sum_{s'=1}^{q_i}|x_{3i-1}|^{q_i-s'}|y_{3i-1}|^{s'-1},\sum_{j=1}^{s_i}|x_{3i-1}|^{s_i-j}|y_{3i-1}|^{j-1},\\
&\sum_{r=1}^{\lambda_i}|x_{3i}|^{\lambda_i-r}|y_{3i}|^{r-1},\sum_{r'=1}^{k_i}|x_{3i}|^{k_i-r'}|y_{3i}|^{r'-1},1:i=1,2,\dots,\frac{m+2}{3}\Big\}\|x-y\|_p.
\end{aligned}\tag{26}$$

Taking into account that $x,y\in B$, it follows that $0\le|x_{3i-2}|^{t_i-r''},|y_{3i-2}|^{r''-1}\le 1$ for each $r''\in\{1,2,\dots,t_i\}$, $0\le|x_{3i-1}|^{q_i-s'},|y_{3i-1}|^{s'-1}\le 1$ for each $s'\in\{1,2,\dots,q_i\}$, $0\le|x_{3i-1}|^{s_i-j},|y_{3i-1}|^{j-1}\le 1$ for each $j\in\{1,2,\dots,s_i\}$, $0\le|x_{3i}|^{\lambda_i-r},|y_{3i}|^{r-1}\le 1$ for each $r\in\{1,2,\dots,\lambda_i\}$, and $0\le|x_{3i}|^{k_i-r'},|y_{3i}|^{r'-1}\le 1$ for each $r'\in\{1,2,\dots,k_i\}$ and $i\in\{1,2,\dots,\frac{m+2}{3}\}$. Relying on these facts, we deduce that

$$0\le\sum_{r''=1}^{t_i}|x_{3i-2}|^{t_i-r''}|y_{3i-2}|^{r''-1}\le t_i,\quad 0\le\sum_{s'=1}^{q_i}|x_{3i-1}|^{q_i-s'}|y_{3i-1}|^{s'-1}\le q_i,$$

$$0\le\sum_{j=1}^{s_i}|x_{3i-1}|^{s_i-j}|y_{3i-1}|^{j-1}\le s_i,\quad 0\le\sum_{r=1}^{\lambda_i}|x_{3i}|^{\lambda_i-r}|y_{3i}|^{r-1}\le\lambda_i$$

and

$$0\le\sum_{r'=1}^{k_i}|x_{3i}|^{k_i-r'}|y_{3i}|^{r'-1}\le k_i,$$

for each $i\in\{1,2,\dots,\frac{m+2}{3}\}$. These facts and (26) imply that for all $(u,x),(v,y)\in[\alpha,0]\times B$,

$$\|T(u,x)-T(v,y)\|_E\le|u-v|+\gamma\xi\|x-y\|_p,\tag{27}$$

where $\xi=\max\{\lambda_i,k_i,q_i,s_i,t_i:i=1,2,\dots,\frac{m+2}{3}\}$. Using the same arguments as for (26) and (27), on can prove that

(i) for all $(u,x),(v,y)\in\{0\}\times B$,

$$\begin{aligned}\|T(u,x)-T(v,y)\|_E &\le \gamma\xi\|x-y\|_p\\ &\le |u-v|+\gamma\xi\|x-y\|_p;\end{aligned}\tag{28}$$

(ii) for all $(u,x),(v,y)\in(0,\beta]\times B$,

$$\begin{aligned}\|T(u,x)-T(v,y)\|_E &= \|(\varrho(u-v),\widehat{x}-\widehat{y})\|_E\\ &\le \varrho|u-v|+\gamma\xi\|x-y\|_p\\ &\le |u-v|+\gamma\xi\|x-y\|_p+\varrho\beta;\end{aligned}\tag{29}$$

(iii) for the case when $(u,x)\in[\alpha,0)\times B$ and $(v,y)\in(0,\beta]\times B$,

$$\begin{aligned}\|T(u,x)-T(v,y)\|_E &= \|(u-\varrho v,\widehat{x}-\widehat{y})\|_E\\ &\le |u-\varrho v|+\gamma\xi\|x-y\|_p\\ &\le |u|+\varrho|v|+\gamma\xi\|x-y\|_p\\ &\le |u-v|+\gamma\xi\|x-y\|_p+\varrho\beta;\end{aligned}\tag{30}$$

(iv) if $(u,x)\in[\alpha,0)\times B$ and $(v,y)\in\{0\}\times B$, then

$$\begin{aligned}\|T(u,x)-T(v,y)\|_E &= \|(u-\varrho,\widehat{x}-\widehat{y})\|_E\\ &\le |u-\varrho|+\gamma\xi\|x-y\|_p\\ &\le |u|+\gamma\xi\|x-y\|_p+\varrho\\ &= |u-v|+\gamma\xi\|x-y\|_p+\varrho;\end{aligned}\tag{31}$$

(v) for all $(u,x)\in\{0\}\times B$ and $(v,y)\in(0,\beta]\times B$,

$$\begin{aligned}\|T(u,x)-T(v,y)\|_E &= \|(\varrho-\varrho v,\widehat{x}-\widehat{y})\|_E\\ &\le |\varrho-\varrho v|+\gamma\xi\|x-y\|_p\\ &\le \varrho+\varrho|v|+\gamma\xi\|x-y\|_p\\ &= \varrho|u-v|+\gamma\xi\|x-y\|_p+\varrho\\ &< |u-v|+\gamma\xi\|x-y\|_p+\varrho.\end{aligned}\tag{32}$$

Making use of (27)–(32) and taking into account that $\varrho>0$ and $\beta\ge 1$, it follows that for all $(u,x),(v,y)\in E$,

$$\begin{aligned}\|T(u,x)-T(v,y)\|_E &\le |u-v|+\gamma\xi\|x-y\|_p+\varrho\beta\\ &\le |u-v|+\|x-y\|_p+\gamma\xi(|u-v|+\|x-y\|_p)+\varrho\beta.\end{aligned}\tag{33}$$

For all $(u,x)\in[\alpha,0)\times B$ and $n\ge 2$, we have $T^n(u,x)=(u,\widehat{x})$, where

$$\begin{aligned}\widehat{x}=\Big(&\underbrace{0,0,\dots,0}_{(2^n-1)\sigma\ times},\gamma^n\sin|x_1|^{t_1},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^n(\sin|x_2|^{q_1}-|x_2|^{s_1}),\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\\ &(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^n(|x_3|^{\lambda_1}-\sin^{k_1}|x_3|),\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\gamma^n\sin|x_4|^{t_2},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^n(\sin|x_5|^{q_2}\\ &-|x_5|^{s_2}),\underbrace{0,0,\dots,0}_{(2^n-1)\ times},(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^n(|x_6|^{\lambda_2}-\sin^{k_2}|x_6|),\dots,\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\gamma^n\sin|x_m|^{t_{\frac{m+2}{3}}},\\ &\underbrace{0,0,\dots,0}_{(2^n-1)\ times},(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^n(\sin|x_{m+1}|^{q_{\frac{m+2}{3}}}-|x_{m+1}|^{s_{\frac{m+2}{3}}}),\underbrace{0,0,\dots,0}_{(2^n-1)\ times},(\frac{\gamma}{\sqrt[p]{2^{p+1}}})^n(|x_{m+2}|^{\lambda_{\frac{m+2}{3}}}\\ &-\sin^{k_{\frac{m+2}{3}}}|x_{m+2}|),\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\gamma^n x_{m+3},\underbrace{0,0,\dots,0}_{(2^n-1)\ times},\gamma^n x_{m+4},\dots\Big).\end{aligned}$$

At the same time, for each $n \in \mathbb{N}$, $T^n(u,x) = (\varrho^n, \widehat{x})$ and $T^n(u,x) = (\varrho^n u, \widehat{x})$ for all $(u,x) \in \{0\} \times B$ and $(u,x) \in (0,\beta] \times B$, respectively. Then, by an argument analogous to those of (26)–(32), one can show that for all $(u,x),(v,y) \in E$ and $n \geq 2$,

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_E &\leq |u-v| + \gamma^n \xi \|x-y\|_p + \varrho^n \beta \\ &\leq |u-v| + \|x-y\|_p + \gamma^n \xi(|u-v| + \|x-y\|_p) + \varrho^n \beta.\end{aligned} \tag{34}$$

Utilizing (33) and (34), for all $(u,x),(v,y) \in E$ and $n \in \mathbb{N}$, we yield

$$\begin{aligned}\|T^n(u,x) - T^n(v,y)\|_E &\leq |u-v| + \|x-y\|_p + \gamma^n \xi(|u-v| + \|x-y\|_p) + \varrho^n \beta \\ &= \|(u,x)-(v,y)\|_E + \gamma^n \xi \|(u,x)-(v,y)\|_E + \varrho^n \beta.\end{aligned} \tag{35}$$

Let us now take $\mu_n = \gamma^n$ and $b_n = \varrho^n \beta$ for each $n \in \mathbb{N}$. Then the fact that $\gamma, \varrho \in (0,1)$ implies that $\mu_n, b_n \to 0$, as $n \to \infty$. Defining the mapping $\phi : [0,+\infty) \to [0,+\infty)$ as $\phi(w) = \xi w$ for all $w \in [0,+\infty)$, from (35) it follows that for all $(u,x),(v,y) \in E$ and $n \in \mathbb{N}$,

$$\|T^n(u,x) - T^n(v,y)\|_E \leq \|(u,x)-(v,y)\|_E + \mu_n \phi(\|(u,x)-(v,y)\|_E) + b_n,$$

i.e., T is a $(\{\gamma^n\}, \{\varrho^n \beta\}, \phi)$-total asymptotically nonexpansive mapping.

Lemma 3.10. *Let, for each $i \in \{1,2\}$, E_i be a real Banach space with the norm $\|.\|_i$, and let $S_i : E_i \to E_i$ be an $\left(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i\right)$-total asymptotically nonexpansive mapping. Suppose further that Q and ϕ are self-mappings of $E_1 \times E_2$ and $\mathbb{R}^+$, respectively, defined by*

$$Q(x_1,x_2) = (S_1 x_1, S_2 x_2), \quad \forall (x_1,x_2) \in E_1 \times E_2 \tag{36}$$

and

$$\phi(t) = \max\{\phi_i(t) : i = 1,2\}, \quad \forall t \in \mathbb{R}^+. \tag{37}$$

Then Q is an $\left(\{a_{n,1}+a_{n,2}\}_{n=1}^{\infty}, \{b_{n,1}+b_{n,2}\}_{n=1}^{\infty}, \phi\right)$-total asymptotically nonexpansive mapping.

Proof. In virtue of the fact that for each $i \in \{1,2\}$, S_i is an $\left(\{a_{n,i}\}_{n=1}^{\infty}, \{b_{n,i}\}_{n=1}^{\infty}, \phi_i\right)$-total asymptotically nonexpansive mapping and $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function, for all $(x,x'),(y,y') \in E_1 \times E_2$ and $n \in \mathbb{N}$, we yield

$$\begin{aligned}\|Q^n(x,x') - Q^n(y,y')\|_* &= \|(S_1^n x, S_2^n x') - (S_1^n y, S_2^n y')\|_* \\ &= \|(S_1^n x - S_1^n y, S_2^n x' - S_2^n y')\|_* \\ &= \|S_1^n x - S_1^n y\|_1 + \|S_2^n x' - S_2^n y'\|_2 \\ &\leq \|x-y\|_1 + a_{n,1}\phi_1(\|x-y\|_1) + b_{n,1} + \|x'-y'\|_2 + a_{n,2}\phi_2(\|x'-y'\|_2) + b_{n,2} \\ &\leq \|x-y\|_1 + \|x'-y'\|_2 + a_{n,1}\phi_1(\|x-y\|_1 + \|x'-y'\|_2) \\ &\quad + a_{n,2}\phi_2(\|x-y\|_1 + \|x'-y'\|_2) + b_{n,1} + b_{n,2} \\ &\leq \|(x,x')-(y,y')\|_* + (a_{n,1}+a_{n,2})\phi(\|(x,x')-(y,y')\|_*) + b_{n,1} + b_{n,2},\end{aligned} \tag{38}$$

where $\|.\|_*$ is a norm on $E_1 \times E_2$ defined as (21). From (38) it follows that Q is an $\left(\{a_{n,1}+a_{n,2}\}_{n=1}^{\infty}, \{b_{n,1}+b_{n,2}\}_{n=1}^{\infty}, \phi\right)$-total asymptotically nonexpansive mapping. The proof is finished. □

4. Iterative Algorithm and Convergence Analysis

In this section, applying two total asymptotically nonexpansive mappings S_1 and S_2 and by using the resolvent operator technique associated with P-η-accretive mappings, a new perturbed p-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the total asymptotically nonexpansive mapping $Q = (S_1, S_2)$ which is the unique solution of the SGNVLI (2) is constructed. The convergence and stability of the iterative sequence generated by our proposed iterative algorithm under some suitable conditions are also proved.

Assume that for each $i \in \{1,2\}$, E_i is a real q_i-uniformly smooth Banach space with $q_i > 1$ and the norm $\|.\|_i$, and $S_i : E_i \to E_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total asymptotically nonexpansive mapping. Moreover, let Q be a self-mapping of $E_1 \times E_2$ defined as (36). Denote by $\mathrm{Fix}(S_i)$ $(i = 1,2)$ and $\mathrm{Fix}(Q)$ the sets of all the fixed points of S_i $(i = 1,2)$ and Q, respectively. Denote further by Ω the set of all the solutions of the SGNVLI (2) where for each $i \in \{1,2\}$, the nonlinear mapping P_i is a strictly η_i-accretive mapping with $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. Utilizing (36), we infer that for any $(x_1, x_2) \in E_1 \times E_2$, $(x_1, x_2) \in \mathrm{Fix}(Q)$ if and only if $x_i \in \mathrm{Fix}(S_i)$ for each $i \in \{1,2\}$, that is, $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1, S_2) = \mathrm{Fix}(S_1) \times \mathrm{Fix}(S_2)$. If $(x, y) \in \mathrm{Fix}(Q) \cap \Omega$, then making use of Lemma 3.1, it can be easily observed that for each $n \in \mathbb{N}$,

$$\begin{cases} x = S_1^n x = x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))] \\ \quad = S_1^n\Big(x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x,y) + S(y, x - f_1(x)))]\Big), \\ y = S_2^n y = y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho(G(x,y) + T(x, y - f_2(y)))] \\ \quad = S_2^n\Big(y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho(G(x,y) + T(x, y - f_2(y)))]\Big). \end{cases} \tag{39}$$

With the help of the fixed point formulation (39), we are able to construct an iterative algorithm for finding a common element of the two sets of $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1, S_2)$ and Ω as follows.

Algorithm 4.1. *Let $E_i, P_i, \eta_i, f_i, g_i, F, G, S, T, M, N$ $(i = 1,2)$ be the same as in the* SGNVLI (2) *such that for each $i \in \{1,2\}$, P_i is a strictly η_i-accretive mapping with* $\mathrm{dom}(P_i) \cap g_i(E_i) \neq \emptyset$. *Suppose further that for each $i \in \{1,2\}$, $S_i : E_i \to E_i$ is an $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$-total asymptotically nonexpansive mapping. For any given $(x_0, y_0) \in E_1 \times E_2$, define the iterative sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ in $E_1 \times E_2$ in the following way:*

$$\begin{cases} x_{n+1} = \alpha_{n,1} x_n + (1 - \alpha_{n,1}) S_1^n\{z_{n,1} - g_1(z_{n,1}) + R^{P_1,\eta_1}_{M(.,z_{n,1}),\lambda}(\Delta(z_{n,1}, t_{n,1}))\} + \alpha_{n,1} e_{n,1} + l_{n,1}, \\ y_{n+1} = \alpha_{n,1} y_n + (1 - \alpha_{n,1}) S_2^n\{t_{n,1} - g_2(t_{n,1}) + R^{P_2,\eta_2}_{N(.,t_{n,1}),\rho}(\Psi(z_{n,1}, t_{n,1}))\} + \alpha_{n,1} \hat{e}_{n,1} + \hat{l}_{n,1}, \\ z_{n,i} = \alpha_{n,i+1} x_n + (1 - \alpha_{n,i+1}) S_1^n\{z_{n,i+1} - g_1(z_{n,i+1}) + R^{P_1,\eta_1}_{M(.,z_{n,i+1}),\lambda}(\Delta(z_{n,i+1}, t_{n,i+1}))\} + \alpha_{n,i+1} e_{n,i+1} + l_{n,i+1}, \\ t_{n,i} = \alpha_{n,i+1} y_n + (1 - \alpha_{n,i+1}) S_2^n\{t_{n,i+1} - g_2(t_{n,i+1}) + R^{P_2,\eta_2}_{N(.,t_{n,i+1}),\rho}(\Psi(z_{n,i+1}, t_{n,i+1}))\} + \alpha_{n,i+1} \hat{e}_{n,i+1} + \hat{l}_{n,i+1}, \\ \dots \\ z_{n,p-1} = \alpha_{n,p} x_n + (1 - \alpha_{n,p}) S_1^n\{x_n - g_1(x_n) + R^{P_1,\eta_1}_{M(.,x_n),\lambda}(\Delta(x_n, y_n))\} + \alpha_{n,p} e_{n,p} + l_{n,p}, \\ t_{n,p-1} = \alpha_{n,p} y_n + (1 - \alpha_{n,p}) S_2^n\{y_n - g_2(y_n) + R^{P_2,\eta_2}_{N(.,y_n),\rho}(\Psi(x_n, y_n))\} + \alpha_{n,p} \hat{e}_{n,p} + \hat{l}_{n,p}, \end{cases} \tag{40}$$

where $i = 1, 2, \dots, p-2$; for all $n \in \mathbb{N} \cup \{0\}$ and $i = 1, 2, \dots, p-1$,

$$\begin{aligned} &\Delta(z_{n,i}, t_{n,i}) = P_1(g_1(z_{n,i})) - \lambda(F(z_{n,i}, t_{n,i}) + S(t_{n,i}, z_{n,i} - f_1(z_{n,i}))), \\ &\Psi(z_{n,i}, t_{n,i}) = P_2(g_2(t_{n,i})) - \rho(G(z_{n,i}, t_{n,i}) + T(z_{n,i}, t_{n,i} - f_2(t_{n,i}))), \\ &\Delta(x_n, y_n) = P_1(g_1(x_n)) - \lambda(F(x_n, y_n) + S(y_n, x_n - f_1(x_n))), \\ &\Psi(x_n, y_n) = P_2(g_2(y_n)) - \rho(G(x_n, y_n) + T(x_n, y_n - f_2(y_n))), \end{aligned}$$

$\lambda, \rho > 0$ are two constants, $\{\alpha_{n,i}\}_{n=0}^{\infty}$ $(i = 1, 2, \dots, p)$ are sequences in $[0,1)$ such that $\sum\limits_{n=0}^{\infty} \prod\limits_{i=1}^{p} (1 - \alpha_{n,i}) = \infty$, and $\{e_{n,i}\}_{n=0}^{\infty}, \{\hat{e}_{n,i}\}_{n=0}^{\infty}, \{l_{n,i}\}_{n=0}^{\infty}$ and $\{\hat{l}_{n,i}\}_{n=0}^{\infty}$ $(i = 1, 2, \dots, p)$ are $4p$ sequences to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions: For $i = 1, 2, \dots, p$, $\{(e_{n,i}, \hat{e}_{n,i})\}_{n=0}^{\infty}$ and $\{(l_{n,i}, \hat{l}_{n,i})\}_{n=0}^{\infty}$ are $2p$ sequences in $E_1 \times E_2$ such that for all $n \in \mathbb{N} \cup \{0\}$ and $i = 1, 2, \dots, p$,

$$\begin{cases} e_{n,i} = e'_{n,i} + e''_{n,i}, \hat{e}_{n,i} = \hat{e}'_{n,i} + \hat{e}''_{n,i}, \\ \lim\limits_{n \to \infty} \|(e'_{n,i}, \hat{e}'_{n,i})\|_* = 0, \\ \sum\limits_{n=0}^{\infty} \|(e''_{n,i}, \hat{e}''_{n,i})\|_* < \infty, \sum\limits_{n=0}^{\infty} \|(l_{n,i}, \hat{l}_{n,i})\|_* < \infty. \end{cases} \tag{41}$$

Let $\{(u_n, v_n)\}_{n=0}^{\infty}$ *be any sequence in* $E_1 \times E_2$ *and define* $\{\epsilon_n\}_{n=0}^{\infty}$ *by*

$$\begin{cases} \epsilon_n = \|(u_{n+1}, v_{n+1}) - (L_n, D_n)\|_*, \\ L_n = \alpha_{n,1}u_n + (1-\alpha_{n,1})S_1^n\{\nu_{n,1} - g_1(\nu_{n,1}) + R^{P_1,\eta_1}_{M(.,\nu_{n,1}),\lambda}(\Delta(\nu_{n,1}, \omega_{n,1}))\} + \alpha_{n,1}e_{n,1} + l_{n,1}, \\ D_n = \alpha_{n,1}v_n + (1-\alpha_{n,1})S_2^n\{\omega_{n,1} - g_2(\omega_{n,1}) + R^{P_2,\eta_2}_{N(.,\omega_{n,1}),\rho}(\Psi(\nu_{n,1}, \omega_{n,1}))\} + \alpha_{n,1}\hat{e}_{n,1} + \hat{l}_{n,1}, \\ \nu_{n,1} = \alpha_{n,2}u_n + (1-\alpha_{n,2})S_1^n\{\nu_{n,2} - g_1(\nu_{n,2}) + R^{P_1,\eta_1}_{M(.,\nu_{n,2}),\lambda}(\Delta(\nu_{n,2}, \omega_{n,2}))\} + \alpha_{n,2}e_{n,2} + l_{n,2}, \\ \omega_{n,1} = \alpha_{n,2}v_n + (1-\alpha_{n,2})S_2^n\{\omega_{n,2} - g_2(\omega_{n,2}) + R^{P_2,\eta_2}_{N(.,\omega_{n,2}),\rho}(\Psi(\nu_{n,2}, \omega_{n,2}))\} + \alpha_{n,2}\hat{e}_{n,2} + \hat{l}_{n,2}, \\ \dots \\ \nu_{n,p-2} = \alpha_{n,p-1}u_n + (1-\alpha_{n,p-1})S_1^n\{\nu_{n,p-1} - g_1(\nu_{n,p-1}) \\ \qquad\qquad + R^{P_1,\eta_1}_{M(.,\nu_{n,p-1}),\lambda}(\Delta(\nu_{n,p-1}, \omega_{n,p-1}))\} + \alpha_{n,p-1}e_{n,p-1} + l_{n,p-1}, \\ \omega_{n,p-2} = \alpha_{n,p-1}v_n + (1-\alpha_{n,p-1})S_2^n\{\omega_{n,p-1} - g_2(\omega_{n,p-1}) \\ \qquad\qquad + R^{P_2,\eta_2}_{N(.,\omega_{n,p-1}),\rho}(\Psi(\nu_{n,p-1}, \omega_{n,p-1}))\} + \alpha_{n,p-1}\hat{e}_{n,p-1} + \hat{l}_{n,p-1}, \\ \nu_{n,p-1} = \alpha_{n,p}u_n + (1-\alpha_{n,p})S_1^n\{u_n - g_1(u_n) \\ \qquad\qquad + R^{P_1,\eta_1}_{M(.,u_n),\lambda}(\Delta(u_n, v_n))\} + \alpha_{n,p}e_{n,p} + l_{n,p}, \\ \omega_{n,p-1} = \alpha_{n,p}v_n + (1-\alpha_{n,p})S_2^n\{v_n - g_2(v_n) \\ \qquad\qquad + R^{P_2,\eta_2}_{N(.,v_n),\rho}(\Psi(u_n, v_n))\} + \alpha_{n,p}\hat{e}_{n,p} + \hat{l}_{n,p}, \end{cases} \tag{42}$$

where for all $n \in \mathbb{N} \cup \{0\}$ *and* $i = 1, 2, \dots, p-1$,

$$\begin{aligned} &\Delta(\nu_{n,i}, \omega_{n,i}) = P_1(g_1(\nu_{n,i})) - \lambda(F(\nu_{n,i}, \omega_{n,i}) + S(\omega_{n,i}, \nu_{n,i} - f_1(\nu_{n,i}))), \\ &\Psi(\nu_{n,i}, \omega_{n,i}) = P_2(g_2(\omega_{n,i})) - \rho(G(\nu_{n,i}, \omega_{n,i}) + T(\nu_{n,i}, \omega_{n,i} - f_2(\omega_{n,i}))), \\ &\Delta(u_n, v_n) = P_1(g_1(u_n)) - \lambda(F(u_n, v_n) + S(v_n, u_n - f_1(u_n))), \\ &\Psi(u_n, v_n) = P_2(g_2(v_n)) - \rho(G(u_n, v_n) + T(u_n, v_n - f_2(v_n))). \end{aligned}$$

Definition 4.2. *Let* E_i $(i = 1, 2)$ *be real Banach spaces and* T *a self-mapping of* $E_1 \times E_2$. *Suppose that* $(x_0, y_0) \in E_1 \times E_2$ *and* $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ *defines an iterative procedure which yields a sequence of points* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *in* $E_1 \times E_2$. *Assume that* $\mathrm{Fix}(T) = \{(x, y) \in E_1 \times E_2 : (x, y) = T(x, y)\} \neq \emptyset$ *and* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *converges to some* $(x^*, y^*) \in \mathrm{Fix}(T)$. *Furthermore, let* $\{(z_n, w_n)\}_{n=0}^{\infty}$ *be an arbitrary sequence in* $E_1 \times E_2$ *and* $\epsilon_n = \|(z_{n+1}, w_{n+1}) - f(T, z_n, w_n)\|$ *for each* $n \in \mathbb{N} \cup \{0\}$. *If* $\lim_{n\to\infty} \epsilon_n = 0$ *implies that* $\lim_{n\to\infty}(z_n, w_n) = (x^*, y^*)$, *then the iterative procedure defined by* $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ *is said to be T-stable or stable with respect to T.*

Remark 4.3. It should be pointed out that in the last twenty years, there has been an increasing interest in establishing some stability results of the iteration procedures for variational inequalities and variational inclusions, see, for example, [2, 4, 9, 11, 33, 36, 43, 45, 46, 50, 52] and the references therein.

We are ready to prove the convergence and stability of the iterative sequence generated by our suggested perturbed p-step iterative algorithm under some appropriate conditions. Before dealing with it, let us recall the following lemma which follows directly from Lemma 2 in [51].

Lemma 4.4. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying the following conditions: there exists a natural number* n_0 *such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n=0}^{\infty} c_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Theorem 4.5. *For* $i = 1, 2$, *let* $E_i, \eta_i, P_i, f_i, g_i, F, G, S, T, M, N$ *be the same as in Theorem 3.4 and let all the conditions of Theorem 3.4 hold. Suppose that there exist constants* $\varsigma_i, \lambda, \rho > 0$ $(i = 1, 2)$ *such that* (5)–(8) *hold and for the cases when* q_1 *and* q_2 *are even natural numbers, in addition to* (5)–(8), (9) *holds. Assume further that for each* $i \in \{1, 2\}$, $S_i : E_i \to E_i$ *is an* $(\{a_{n,i}\}_{n=0}^{\infty}, \{b_{n,i}\}_{n=0}^{\infty}, \phi_i)$*-total asymptotically nonexpansive mapping and* Q *is a self-mapping of* $E_1 \times E_2$ *defined by* (36) *such that* $\mathrm{Fix}(Q) \cap \Omega \neq \emptyset$. *Then*

(i) *the iterative sequence* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *generated by Algorithm 4.1, converges strongly to the only element* (x, y) *of* $\mathrm{Fix}(Q) \cap \Omega$.

(ii) *If, in addition, there exists a constant* $\alpha > 0$ *such that* $\prod_{i=1}^{p}(1 - \alpha_{n,i}) \geq \alpha$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* $\lim_{n\to\infty}(u_n, v_n) = (x, y)$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$, *where* $\{(u_n, v_n)\}_{n=0}^{\infty}$ *is any sequence in* $E_1 \times E_2$ *defined by* (42).

Proof. Taking into account that all the conditions of Theorem 3.4 hold, the existence of a unique solution $(x, y) \in E_1 \times E_2$ for the SGNVLI (2) follows from Theorem 3.4. In the light of Lemma 3.1, we obtain

$$\begin{cases} x = x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x, y) + S(y, x - f_1(x)))], \\ y = y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho(G(x, y) + T(x, y - f_2(y)))]. \end{cases} \tag{43}$$

Since Ω is a singleton set and $\mathrm{Fix}(Q) \cup \Omega \neq \emptyset$, we infer that $x \in \mathrm{Fix}(S_1)$ and $y \in \mathrm{Fix}(S_2)$. Thus, making use of (43), for each $n \geq 0$, one can write

$$\begin{cases} x = S_1^n x = S_1^n\Big(x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}[P_1(g_1(x)) - \lambda(F(x, y) + S(y, x - f_1(x)))]\Big), \\ y = S_2^n y = S_2^n\Big(y - g_2(y) + R^{P_2,\eta_2}_{N(.,y),\rho}[P_2(g_2(y)) - \rho(G(x, y) + T(x, y - f_2(y)))]\Big). \end{cases} \tag{44}$$

Using (40), (44), Lemma 2.19 and the assumptions, it follows that

$$\begin{aligned} \|x_{n+1} - x\|_1 &\leq \alpha_{n,1}\|x_n - x\|_1 + (1 - \alpha_{n,1})\|S_1^n(z_{n,1} - g_1(z_{n,1}) + R^{P_1,\eta_1}_{M(.,z_{n,1}),\lambda}(\Delta(z_{n,1}, t_{n,1}))) \\ &\quad - S_1^n(x - g_1(x) + R^{P_1,\eta_1}_{M(.,x),\lambda}(\Delta(x, y)))\|_1 + \alpha_{n,1}\|e_{n,1}\|_1 + \|l_{n,1}\|_1 \\ &\leq \alpha_{n,1}\|x_n - x\|_1 + (1 - \alpha_{n,1})\Big(\varphi_1\|z_{n,1} - x\|_1 + \vartheta_1\|t_{n,1} - y\|_2 + a_{n,1}\phi_1\Big(\varphi_1\|z_{n,1} - x\|_1 \\ &\quad + \vartheta_1\|t_{n,1} - y\|_2\Big) + b_{n,1}\Big) + \alpha_{n,1}\|e'_{n,1}\|_1 + \|e''_{n,1}\|_1 + \|l_{n,1}\|_1, \end{aligned} \tag{45}$$

where φ_1 and ϑ_1 are the same as in (19) and for all $(x, y) \in E_1 \times E_2$,

$$\Delta(x, y) = P_1(g_1(x)) - \lambda(F(x, y) + S(y, x - f_1(x)).$$

In a similar way, employing (40), (44), Lemma 2.19 and the assumptions, one can prove that for all $n \geq 0$,

$$\begin{aligned} \|y_{n+1} - y\|_2 &\leq \alpha_{n,1}\|y_n - y\|_2 + (1 - \alpha_{n,1})\Big(\varphi_2\|z_{n,1} - x\|_1 + \vartheta_2\|t_{n,1} - y\|_2 + a_{n,2}\phi_2\Big(\varphi_2\|z_{n,1} - x\|_1 \\ &\quad + \vartheta_2\|t_{n,1} - y\|_2\Big) + b_{n,2}\Big) + \alpha_{n,1}\|\hat{e}'_{n,1}\|_2 + \|\hat{e}''_{n,1}\|_2 + \|\hat{l}_{n,1}\|_2, \end{aligned} \tag{46}$$

where φ_2 and ϑ_2 are the same as in (20). Making use of (45) and (46), we obtain

$$\begin{aligned} \|(x_{n+1}, y_{n+1}) - (x, y)\|_* &\leq \alpha_{n,1}\|(x_n, y_n) - (x, y)\|_* + (1 - \alpha_{n,1})\Big((\varphi_1 + \varphi_2)\|z_{n,1} - x\|_1 \\ &\quad + (\vartheta_1 + \vartheta_2)\|t_{n,1} - y\|_2 + a_{n,1}\phi_1\Big(\varphi_1\|z_{n,1} - x\|_1 + \vartheta_1\|t_{n,1} - y\|_2\Big) \\ &\quad + a_{n,2}\phi_2\Big(\varphi_2\|z_{n,1} - x\|_1 + \vartheta_2\|t_{n,1} - y\|_2\Big) + b_{n,1} + b_{n,2}\Big) \\ &\quad + \alpha_{n,1}\|(e'_{n,1}, \hat{e}'_{n,1})\|_* + \|(e''_{n,1}, \hat{e}''_{n,1})\|_* + \|(l_{n,1}, \hat{l}_{n,1})\|_* \\ &\leq \alpha_{n,1}\|(x_n, y_n) - (x, y)\|_* + (1 - \alpha_n)\Big(k\|(z_{n,1}, t_{n,1}) - (x, y)\|_* \\ &\quad + a_{n,1}\phi(k\|(z_{n,1}, t_{n,1}) - (x, y)\|_*) + a_{n,2}\phi(k\|(z_{n,1}, t_{n,1}) - (x, y)\|_*) \\ &\quad + b_{n,1} + b_{n,2}\Big) + \alpha_{n,1}\|(e'_{n,1}, \hat{e}'_{n,1})\|_* + \|(e''_{n,1}, \hat{e}''_{n,1})\|_* + \|(l_{n,1}, \hat{l}_{n,1})\|_*, \end{aligned} \tag{47}$$

where ϕ is a self-mapping of $\mathbb{R}^+$ defined by (37), and k is the same as in (22). By similar arguments to the proof of (45)–(47), for $i = 1, 2, \dots, p-2$, we can establish that

$$\begin{aligned}\|(z_{n,i}, t_{n,i}) - (x, y)\|_* \leq\ & \alpha_{n,i+1}\|(x_n, y_n) - (x, y)\|_* + (1 - \alpha_{n,i+1})\Big(k\|(z_{n,i+1}, t_{n,i+1}) - (x, y)\|_* \\ & + a_{n,1}\phi(k\|(z_{n,i+1}, t_{n,i+1}) - (x, y)\|_*) + a_{n,2}\phi(k\|(z_{n,i+1}, t_{n,i+1}) - (x, y)\|_*) \\ & + b_{n,1} + b_{n,2}\Big) + \alpha_{n,i+1}\|(e'_{n,i+1}, \hat{e}'_{n,i+1})\|_* + \|(e''_{n,i+1}, \hat{e}''_{n,i+1})\|_* + \|(l_{n,i+1}, \hat{l}_{n,i+1})\|_*\end{aligned} \tag{48}$$

and

$$\begin{aligned}\|(z_{n,p-1}, t_{n,p-1}) - (x, y)\|_* \leq\ & \alpha_{n,p}\|(x_n, y_n) - (x, y)\|_* + (1 - \alpha_{n,p})\Big(k\|(x_n, y_n) - (x, y)\|_* \\ & + a_{n,1}\phi(k\|(x_n, y_n) - (x, y)\|_*) + a_{n,2}\phi(k\|(x_n, y_n) - (x, y)\|_*) \\ & + b_{n,1} + b_{n,2}\Big) + \alpha_{n,p}\|(e'_{n,p}, \hat{e}'_{n,p})\|_* + \|(e''_{n,p}, \hat{e}''_{n,p})\|_* + \|(l_{n,p}, \hat{l}_{n,p})\|_*.\end{aligned} \tag{49}$$

By using (48) and (49) one can prove that for all $n \geq 0$,

$$\begin{aligned}\|(z_{n,1}, t_{n,1}) - (x, y)\|_* \leq\ & \alpha_{n,2}\|(x_n, y_n) - (x, y)\|_* + (1 - \alpha_{n,2})\Big(k\|(z_{n,2}, t_{n,2}) - (x, y)\|_* + a_{n,1}\phi(k\|(z_{n,2}, t_{n,2}) \\ & - (x, y)\|_*) + a_{n,2}\phi(k\|(z_{n,2}, t_{n,2}) - (x, y)\|_*) + b_{n,1} + b_{n,2}\Big) + \alpha_{n,2}\|(e'_{n,2}, \hat{e}'_{n,2})\|_* \\ & + \|(e''_{n,2}, \hat{e}''_{n,2})\|_* + \|(l_{n,2}, \hat{l}_{n,2})\|_* \\ \leq\ & \Big(\alpha_{n,2} + (1 - \alpha_{n,2})\alpha_{n,3}k + (1 - \alpha_{n,2})(1 - \alpha_{n,3})\alpha_{n,4}k^2 + \cdots + \prod_{i=2}^{p-1}(1 - \alpha_{n,i})\alpha_{n,p}k^{p-2} \\ & + \prod_{i=2}^{p}(1 - \alpha_{n,i})k^{p-1}\Big)\|(x_n, y_n) - (x, y)\|_* + \prod_{i=2}^{p}(1 - \alpha_{n,i})a_{n,1}k^{p-2}\phi(k\|(x_n, y_n) - (x, y)\|_*) \\ & + \prod_{i=2}^{p-1}(1 - \alpha_{n,i})a_{n,1}k^{p-3}\phi(k\|(z_{n,p-1}, t_{n,p-1}) - (x, y)\|_*) + (1 - \alpha_{n,2})a_{n,1}\phi(k\|(z_{n,2}, t_{n,2}) - (x, y)\|_*) \\ & + \prod_{i=2}^{p-2}(1 - \alpha_{n,i})a_{n,1}k^{p-4}\phi(k\|(z_{n,p-2}, t_{n,p-2}) - (x, y)\|_*) + \dots \\ & + (1 - \alpha_{n,2})(1 - \alpha_{n,3})(1 - \alpha_{n,4})a_{n,1}k^2\phi(k\|(z_{n,4}, t_{n,4}) - (x, y)\|_*) \\ & + (1 - \alpha_{n,2})(1 - \alpha_{n,3})a_{n,1}k\phi(k\|(z_{n,3}, t_{n,3}) - (x, y)\|_*) \\ & + \prod_{i=2}^{p}(1 - \alpha_{n,i})a_{n,2}k^{p-2}\phi(k\|(x_n, y_n) - (x, y)\|_*) \\ & + \prod_{i=2}^{p-1}(1 - \alpha_{n,i})a_{n,2}k^{p-3}\phi(k\|(z_{n,p-1}, t_{n,p-1}) - (x, y)\|_*) \\ & + \prod_{i=2}^{p-2}(1 - \alpha_{n,i})a_{n,2}k^{p-4}\phi(k\|(z_{n,p-2}, t_{n,p-2}) - (x, y)\|_*) + \dots \\ & + (1 - \alpha_{n,2})(1 - \alpha_{n,3})(1 - \alpha_{n,4})a_{n,2}k^2\phi(k\|(z_{n,4}, t_{n,4}) - (x, y)\|_*) \\ & + (1 - \alpha_{n,2})(1 - \alpha_{n,3})a_{n,2}k\phi(k\|(z_{n,3}, t_{n,3}) - (x, y)\|_*) \\ & + \Big(\prod_{i=2}^{p}(1 - \alpha_{n,i})k^{p-2} + \prod_{i=2}^{p-1}(1 - \alpha_{n,i})k^{p-3} + \prod_{i=2}^{p-2}(1 - \alpha_{n,i})k^{p-4} + \dots \\ & + (1 - \alpha_{n,2})(1 - \alpha_{n,3})(1 - \alpha_{n,4})k^2 + (1 - \alpha_{n,2})(1 - \alpha_{n,3})k + 1 - \alpha_{n,2}\Big)(b_{n,1} + b_{n,2})\end{aligned} \tag{50}$$

$$
\begin{aligned}
&+\prod_{i=2}^{p-1}(1-\alpha_{n,i})\alpha_{n,p}k^{p-2}\|(e'_{n,p},\hat{e}'_{n,p})\|_* + \prod_{i=2}^{p-2}(1-\alpha_{n,i})\alpha_{n,p-1}k^{p-3}\|(e'_{n,p-1},\hat{e}'_{n,p-1})\|_* \\
&+\prod_{i=2}^{p-3}(1-\alpha_{n,i})\alpha_{n,p-2}k^{p-2}\|(e'_{n,p-2},\hat{e}'_{n,p-2})\|_* + \cdots + (1-\alpha_{n,2})(1-\alpha_{n,3})\alpha_{n,4}k^2\|(e'_{n,4},\hat{e}'_{n,4})\|_* \\
&+(1-\alpha_{n,2})\alpha_{n,3}k\|(e'_{n,3},\hat{e}'_{n,3})\|_* + \alpha_{n,2}\|(e'_{n,2},\hat{e}'_{n,2})\|_* + \prod_{i=2}^{p-1}(1-\alpha_{n,i})k^{p-2}\|(e''_{n,p},\hat{e}''_{n,p})\|_* \\
&+\prod_{i=2}^{p-2}(1-\alpha_{n,i})k^{p-3}\|(e''_{n,p-1},\hat{e}''_{n,p-1})\|_* + \prod_{i=2}^{p-3}(1-\alpha_{n,i})k^{p-4}\|(e''_{n,p-2},\hat{e}''_{n,p-2})\|_* + \dots \\
&+(1-\alpha_{n,2})(1-\alpha_{n,3})k^2\|(e''_{n,4},\hat{e}''_{n,4})\|_* + (1-\alpha_{n,2})k\|(e''_{n,3},\hat{e}''_{n,3})\|_* + \|(e''_{n,2},\hat{e}''_{n,2})\|_* \\
&+\prod_{i=2}^{p-1}(1-\alpha_{n,i})k^{p-2}\|(l_{n,p},\hat{l}_{n,p})\|_* + \prod_{i=2}^{p-2}(1-\alpha_{n,i})k^{p-3}\|(l_{n,p-1},\hat{l}_{n,p-1})\|_* \\
&+\prod_{i=2}^{p-3}(1-\alpha_{n,i})k^{p-4}\|(l_{n,p-2},\hat{l}_{n,p-2})\|_* + \cdots + (1-\alpha_{n,2})(1-\alpha_{n,3})k^2\|(l_{n,4},\hat{l}_{n,4})\|_* \\
&+(1-\alpha_{n,2})k\|(l_{n,3},\hat{l}_{n,3})\|_* + \|(l_{n,2},\hat{l}_{n,2})\|_*.
\end{aligned}
$$

Using (47), (50) and the fact that $0 < \alpha \leq \prod_{i=1}^{p}(1-\alpha_{n,i})$, for all $n \in \mathbb{N} \cup \{0\}$, we can get

$$
\begin{aligned}
&\|(x_{n+1},y_{n+1})-(x,y)\|_* \leq \alpha_{n,1}\|(x_n,y_n)-(x,y)\|_* + (1-\alpha_{n,1})k\|(z_{n,1},t_{n,1})-(x,y)\|_* \\
&\quad + (1-\alpha_{n,1})a_{n,1}\phi(k\|(z_{n,1},t_{n,1})-(x,y)\|_*) + (1-\alpha_{n,1})a_{n,2}\phi(k\|(z_{n,1},t_{n,1})-(x,y)\|_*) \\
&\quad + (1-\alpha_{n,1})(b_{n,1}+b_{n,2}) + \alpha_{n,1}\|(e'_{n,1},\hat{e}'_{n,1})\|_* + \|(e''_{n,1},\hat{e}''_{n,1})\|_* + \|(l_{n,1},\hat{l}_{n,1})\|_* \\
&\leq \Big(1-k^{p-1}(1-k)\prod_{i=1}^{p}(1-\alpha_{n,i})\Big)\|(x_n,y_n)-(x,y)\|_* + k^{p-1}(1-k)\prod_{i=1}^{p}(1-\alpha_{n,i})\frac{L}{k^{p-1}(1-k)\alpha} \\
&\quad + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^i\|(e''_{n,i+1},\hat{e}''_{n,i+1})\|_* + \|(e''_{n,1},\hat{e}''_{n,1})\|_* + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^i\|(l_{n,i+1},\hat{l}_{n,i+1})\|_* + \|(l_{n,1},\hat{l}_{n,1})\|_*,
\end{aligned} \tag{51}
$$

where

$$
\begin{aligned}
L &= \Big(\prod_{i=1}^{p}(1-\alpha_{n,i})k^{p-1}\phi(k\|(x_n,y_n)-(x,y)\|_*) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}\phi(k\|(z_{n,i},t_{n,i})-(x,y)\|_*)\Big)a_{n,1} \\
&\quad + \Big(\prod_{i=1}^{p}(1-\alpha_{n,i})k^{p-1}\phi(k\|(x_n,y_n)-(x,y)\|_*) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}\phi(k\|(z_{n,i},t_{n,i})-(x,y)\|_*)\Big)a_{n,2} \\
&\quad + \sum_{i=1}^{p}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}(b_{n,1}+b_{n,2}) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})\alpha_{n,i+1}k^i\|(e'_{n,i+1},\hat{e}'_{n,i+1})\|_* + \alpha_{n,1}\|(e'_{n,1},\hat{e}'_{n,1})\|_*.
\end{aligned}
$$

Thanks to Definition 3.7(viii), we know that $a_{n,i}, b_{n,i} \to 0$ as $n \to \infty$ for $i = 1, 2$. In the light of (44), it is clear that the conditions of Lemma 4.4 are satisfied. Now, Lemma 4.4 and (51) guarantee that $(x_{n+1}, y_{n+1}) \to (x, y)$ as $n \to \infty$. Therefore, the iterative sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.1 converges strongly to the unique solution (x, y) of the SGNVLI (2), that is, the only element of $\mathrm{Fix}(Q) \cap \Omega$.

Now, we prove the conclusion (ii). Making use of (45), we get

$$\begin{aligned}\|(u_{n+1},v_{n+1})-(x,y)\|_* &\leq \|(u_{n+1},v_{n+1})-(L_n,D_n)\|_* + \|(L_n,D_n)-(x,y)\|_* \\ &= \epsilon_n + \|L_n - x\|_1 + \|D_n - y\|_2.\end{aligned} \tag{52}$$

By following the same arguments as in the proof of (48) and (49) with suitable changes, we obtain

$$\begin{aligned}\|L_n - x\|_1 \leq\, & \alpha_{n,1}\|u_n - x\|_1 + (1-\alpha_{n,1})\Big(\varphi_1\|\nu_{n,1}-x\|_1 + \vartheta_1\|\omega_{n,1}-y\|_2 + a_{n,1}\phi_1(\varphi_1\|\nu_{n,1}-x\|_1 \\ &+ \vartheta_1\|\omega_{n,1}-y\|_2\Big) + b_{n,1}\Big) + \alpha_{n,1}\|e'_{n,1}\|_1 + \|e''_{n,1}\|_1 + \|l_{n,1}\|_1\end{aligned} \tag{53}$$

and

$$\begin{aligned}\|D_n - y\|_2 \leq\, & \alpha_{n,1}\|v_n - y\|_2 + (1-\alpha_{n,1})\Big(\varphi_2\|\nu_{n,1}-x\|_1 + \vartheta_2\|\omega_{n,1}-y\|_2 + a_{n,2}\phi_2(\varphi_2\|\nu_{n,1}-x\|_1 \\ &+ \vartheta_2\|\omega_{n,1}-y\|_2\Big) + b_{n,2}\Big) + \alpha_{n,1}\|\hat{e}'_{n,1}\|_2 + \|\hat{e}''_{n,1}\|_2 + \|\hat{l}_{n,1}\|_2,\end{aligned} \tag{54}$$

where φ_1, ϑ_1 are the same as in (19) and φ_2, ϑ_2 are the same as in (20).

Since $0 < \alpha < \prod_{i=1}^{p}(1-\alpha_{n,i})$, for all $n \in \mathbb{N}\cup\{0\}$, employing (52)–(54), as the proof of (51), we get

$$\begin{aligned}\|(u_{n+1},v_{n+1})-(x,y)\|_* \leq\, & \Big(1 - k^{p-1}(1-k)\prod_{i=1}^{p}(1-\alpha_{n,i})\Big)\|(u_n,v_n)-(x,y)\|_* \\ &+ k^{p-1}(1-k)\prod_{i=1}^{p}(1-\alpha_{n,i})\frac{L'}{k^{p-1}(1-k)\alpha} \\ &+ \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^i\|(e''_{n,i+1},\hat{e}''_{n,i+1})\|_* + \|(e''_{n,1},\hat{e}''_{n,1})\|_* \\ &+ \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^i\|(l_{n,i+1},\hat{l}_{n,i+1})\|_* + \|(l_{n,1},\hat{l}_{n,1})\|_*,\end{aligned} \tag{55}$$

where

$$\begin{aligned}L' =\, & \Big(\prod_{i=1}^{p}(1-\alpha_{n,i})k^{p-1}\phi(k\|(u_n,v_n)-(x,y)\|_*) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}\phi(k\|(\nu_{n,i},\omega_{n,i})-(x,y)\|_*)\Big)a_{n,1} \\ &+ \Big(\prod_{i=1}^{p}(1-\alpha_{n,i})k^{p-1}\phi(k\|(u_n,v_n)-(x,y)\|_*) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}\phi(k\|(\nu_{n,i},\omega_{n,i})-(x,y)\|_*)\Big)a_{n,2} \\ &+ \sum_{i=1}^{p}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}(b_{n,1}+b_{n,2}) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})\alpha_{n,i+1}k^i\|(e_{n,i+1},\hat{e}'_{n,i+1})\|_* + \alpha_{n,1}\|(e'_{n,1},\hat{e}'_{n,1})\|_* + \epsilon_n.\end{aligned}$$

Let $\lim_{n\to\infty}\epsilon_n = 0$. Then from (41), (55) and Lemma 4.4 it follows that $\lim_{n\to\infty}(u_n,v_n) = (x,y)$.

Conversely, suppose that $\lim_{n\to\infty}(u_n, v_n) = (x, y)$. Utilizing (53) and (54), we yield

$$\begin{aligned}
\epsilon_n &= \|(u_{n+1}, v_{n+1}) - (L_n, D_n)\|_* \\
&\le \|(u_{n+1}, v_{n+1}) - (x, y)\|_* + \|(L_n, D_n) - (x, y)\|_* \\
&\le \|(u_{n+1}, v_{n+1}) - (x, y)\|_* + (1 - k^{p-1}(1-k)\prod_{i=1}^{p}(1-\alpha_{n,i})\|(u_n, v_n) - (x, y)\|_* \\
&\quad + k^{p-1}(1-k)\prod_{i=1}^{p}(1-\alpha_{n,i})\frac{L''}{k^{p-1}(1-k)\alpha} + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^i\|(e''_{n,i+1}, \hat{e}''_{n,i+1})\|_* + \|(e''_{n,1}, \hat{e}''_{n,1})\|_* \\
&\quad + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^i\|(l_{n,i+1}, \hat{l}_{n,i+1})\|_* + \|(l_{n,1}, \hat{l}_{n,1})\|_*,
\end{aligned} \tag{56}$$

where

$$\begin{aligned}
L'' &= \Big(\prod_{i=1}^{p}(1-\alpha_{n,i})k^{p-1}\phi(k\|(u_n, v_n) - (x, y)\|_*) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}\phi(k\|(\nu_{n,i}, \omega_{n,i}) - (x, y)\|_*)\Big)a_{n,1} \\
&\quad + \Big(\prod_{i=1}^{p}(1-\alpha_{n,i})k^{p-1}\phi(k\|(u_n, v_n) - (x, y)\|_*) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}\phi(k\|(\nu_{n,i}, \omega_{n,i}) - (x, y)\|_*)\Big)a_{n,2} \\
&\quad + \sum_{i=1}^{p}\prod_{j=1}^{i}(1-\alpha_{n,j})k^{i-1}(b_{n,1} + b_{n,2}) + \sum_{i=1}^{p-1}\prod_{j=1}^{i}(1-\alpha_{n,j})\alpha_{n,i+1}k^i\|(e_{n,i+1}, \hat{e}'_{n,i+1})\|_* + \alpha_{n,1}\|(e'_{n,1}, \hat{e}'_{n,1})\|_*.
\end{aligned}$$

Clearly (41) implies that for each $i \in \{1, 2, \ldots, p\}$, $\lim_{n\to\infty}\|(e''_{n,i}, \hat{e}''_{n,i})\|_* = \lim_{n\to\infty}\|(l_{n,i}, \hat{l}_{n,i})\|_* = 0$. Now, the facts that $\lim_{n\to\infty} a_{n,i} = \lim_{n\to\infty} b_{n,i} = 0$ for $i = 1, 2$, and $\lim_{n\to\infty}\|(e'_{n,i}, \hat{e}'_{n,i})\|_* = 0$ for $i = 1, 2, \ldots, p$, ensure that the right-hand side of (56) tends to zero as $n \to \infty$. This completes the proof. □

5. Conclusion

The interest in the study of nonlinear equations of evolution in the setting of Banach spaces gave rise to the appearance and study of accretive and m-accretive mappings. These operators were introduced and studied intensively in the late 1960s and early 1970s. Inspired by their wide applications, the introduction and study of a variety of generalizations of the notion of m-accretive mapping in the setting of different spaces have been the focus of much research in the last two decades. In one of these attempts, Kazmi and Khan [44] and Peng and Zhu [57] introduced and studied the concept of P-η-accretive mapping in the context of real q-uniformly smooth Banach spaces and defined the resolvent operator associated with such a mapping. In this paper, we have studied the system of generalized nonlinear variational-like inclusions (for short, SGNVLI) (2) involving P-η-accretive mappings. We have used the resolvent operator method and established a new equivalence relationship between SGNVLI (2) and a class of fixed-point problems. We have employed the obtained equivalence relationship and constructed a new iterative algorithm for finding a common element of the solutions set of SGNVLI (2) and the fixed points set of a given total asymptotically nonexpansive mapping. Finally, under some new approximate conditions imposed on the parameter and mappings, we have proved the strong convergence and stability of the sequence generated by our suggested iterative algorithm to a common point of the two sets mentioned above.

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