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Bivariate Lupaş-Durrmeyer type operators involving Pólya distribution

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Abstract. In this paper, we consider a bivariate extension of blending type approximation by Lupaş-Durrmeyer type operators involving Pólya Distribution. We illustrate the convergence rate of these type operators using Peetre's K-functional, modulus of smoothness and for functions in a Lipschitz type space.

1. Introduction and preliminaries

Most likely used Bernstein polynomial of r^{th} degree for the continuous function $\gamma(x)$ and the interval $I = [0,1]$ is defined as

$$B_r(\gamma; x) = \sum_{k=0}^{r} P_{r,k}(x)\gamma\left(\frac{k}{r}\right),$$

where $P_{r,k}(x) = \binom{r}{k} x^k (1-x)^{r-k}$, $x \in [0,1]$.

Gupta and Rassias [15] introduced Lupaş-Durrmeyer operators for any $\gamma \in C(I)$ is given by

$$D_r^{\left[\frac{1}{r}\right]}(\gamma; x) = (r+1)\sum_{k=0}^{r} b_{r,k}^{\left[\frac{1}{r}\right]}(x)\int_0^1 b_{r,k}(t)\gamma(t)dt, \quad t \in [0,1]. \tag{1}$$

Agrawal et al. [6] constructed the bivariate modification of the linear positive operator given in (1) based on Pòlya distribution and discussed the degree of approximation and the rate of convergence for these operators. Kajla and Acar [18] proposed the blending type approximation by generalised Bernstein-Durrmeyer type operator. Kajla et al. [19] constructed a Durrmeyer type generalization of the Lupaş and

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Lupaş operators [20] involving $\tau(x)$ $(0 < \tau(x) \leq 1)$ as

$$D_{r,\tau,\rho}^{\left[\frac{1}{r}\right]}(\gamma; x) = \sum_{k=0}^{r} b_{r,k}^{\left[\frac{1}{r}\right]}(x) \int_0^1 \left(\frac{t^{k\rho+\tau(x)-1}(1-t)^{(r-k)\rho+\tau(x)-1}}{B(k\rho+\tau(x),(r-k)\rho+\tau(x))} \right) \gamma(t)dt, \tag{2}$$

where $B(l, m) = \int_0^1 t^{l-1}(1-t)^{m-1}dt$ $(l, m > 0$ and $\rho > 0)$.

Acar and Kajla [2] proposed bivariate extension of Bernstein type operators and discussed the degree of approximation for these operator. Many researchers also defined different type of generalization of these operators and discussed their approximation behaviour which we refer in (cf. [1, 3–5, 7–9, 11–14, 16, 17, 21–25, 27–30]).

Let fixed a continuous function which is strictly positive $\tau : C(I) \to C(I)$. We propose a bivariate generalization of the operator defined in (2) involving $\tau(x)(0 < \tau(x) \leq 1)$ and $\gamma : C(I^2) \to C(I^2)$, $I^2 = [0, 1] \times [0, 1]$

$$D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma; x_1, x_2) = \sum_{k_1=0}^{r_1}\sum_{k_2=0}^{r_2} b_{r_1,r_2,k_1,k_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(x_1, x_2) \int_0^1\int_0^1 \left(\frac{t^{k_1\rho_1+\tau(x_1)-1}(1-t)^{(r_1-k_1)\rho_1+\tau(x_1)-1}}{B(k_1\rho_1+\tau(x_1),(r_1-k_1)\rho_1+\tau(x_1))} \right) \times \left(\frac{s^{k_2\rho_2+\tau(x_2)-1}(1-s)^{(r_2-k_2)\rho_2+\tau(x_2)-1}}{B(k_2\rho_2+\tau(x_2),(r_2-k_2)\rho_2+\tau(x_2))} \right) \gamma(t, s)dtds.$$

$$B(l, m) = \int_0^1 t^{l-1}(1-t)^{m-1}dt \quad (l, m > 0 \ \ and \ \ \rho_1, \rho_2 > 0).$$

Lemma 1.1. *Let the bivariate test function be defined by* $e_{mn}(x_1, x_2) = x_1^m x_2^n$, $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$, *with* $m + n \leq 4$. *Then, we have*

(i) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(e_{00}; x_1, x_2) = 1;$

(ii) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(e_{10}; x_1, x_2) = \dfrac{r_1x_1\rho_1 + \tau(x_1)}{r_1\rho_1 + 2\tau(x_1)};$

(iii) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(e_{01}; x_1, x_2) = \dfrac{r_2x_2\rho_2 + \tau(x_2)}{r_2\rho_2 + 2\tau(x_2)};$

(iv) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(e_{11}; x_1, x_2) = \left(\dfrac{r_1x_1\rho_1 + \tau(x_1)}{r_1\rho_1 + 2\tau(x_1)}\right) \times \left(\dfrac{r_2x_2\rho_2 + \tau(x_2)}{r_2\rho_2 + 2\tau(x_2)}\right);$

(v) $$D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(e_{20}; x_1, x_2) = \frac{x_1^2\rho_1^2r_1^2(r_1-1)}{(r_1+1)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)} + \frac{x_1r_1\rho_1(r_1+1)(1+2\tau(x_1)+2r_1\rho_1)}{(r_1+1)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)} + \frac{\tau(x_1)(1+\tau(x_1))}{(r_1+1)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)};$$

(vi) $$D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(e_{02}; x_1, x_2) = \frac{x_2^2\rho_2^2r_2^2(r_2-1)}{(r_2+1)(r_2\rho_2+2\tau(x_2))(r_1\rho_2+2\tau(x_2)+1)} + \frac{x_1r_2\rho_2(r_2+1)(1+2\tau(x_2)+2r_2\rho_2)}{(r_2+1)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)} + \frac{\tau(x_2)(1+\tau(x_2))}{(r_2+1)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)};$$

(vii) $$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{30};x_1,x_2) = \frac{x_1^3 r_1^3 \rho_1^3 (r_1-1)(r_1-2)}{(r_1+1)(r_1+2)(r_2\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)}$$
$$+\frac{3x_1^2 r_1^2 \rho_1^2 (r_1-1)(r_1+2)(1+\tau(x_1)+2r_1\rho_1)}{(r_1+1)(r_1+2)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)}$$
$$+\frac{x_1(r_1(r_1+1)(r_1+2)(2+3\tau(x_1))(2+\tau(x_2)))\rho_1+6r_1^2(r_1+2)(1+\tau(x_1))\rho_1^2+6r_1^3\rho_1^3}{(r_1+1)(r_1+2)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)}$$
$$+\frac{\tau(x_1)(1+\tau(x_1))(2+\tau(x_1))}{(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)};$$

(viii) $$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{03};x_1,x_2) = \frac{x_2^3 r_2^3 \rho_2^3 (r_2-1)(r_2-2)}{(r_2+1)(r_2+2)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)}$$
$$+\frac{3x_2^2 r_2^2 \rho_2^2 (r_2-1)(r_2+2)(1+\tau(x_2)+2r_2\rho_2)}{(r_2+1)(r_2+2)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)}$$
$$+\frac{x_2(r_2(r_2+1)(r_2+2)(2+3\tau(x_2))(2+\tau(x_2)))\rho_2+6r_2^2(r_2+2)(1+\tau(x_2))\rho_2^2+6r_2^3\rho_2^3}{(r_2+1)(r_2+2)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)}$$
$$+\frac{\tau(x_2)(1+\tau(x_2))(2+\tau(x_2))}{(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)};$$

(ix) $$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{40};x_1,x_2)$$
$$= \frac{x_1^4 r_1^4 \rho_1^4 (r_1-1)(r_1-2)(r_1-3)}{(r_1+1)(r_1+2)(r_1+3)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)(r_1\rho_1+2\tau(x_1)+3)}$$
$$+\frac{2x_1^3 r_1^3 \rho_1^3 (r_1-1)(r_1-2)((r_1+3)(3+2\tau(x_1))+6r_1\rho_1)}{(r_1+1)(r_1+2)(r_1+3)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)(r_1\rho_1+2\tau(x_1)+3)}$$
$$+\frac{x_1^2 r_1^2 \rho_1^2((r_1-1)(r_1+2)(r_1+3)(11+6\tau(x_1)(3+\tau(x_1)))+12r_1(r_1-1)(r+3)(3+2\tau(x_1))\rho_1+2r_1\rho_1^2(1-19r_1+36r_1^2))}{(r_1+1)(r_1+2)(r_1+3)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)(r_1\rho_1+2\tau(x_1)+3)}$$
$$+\frac{2r_1x_1\rho_1\left((r_1+1)(r_1+2)(r_1+3)(3+2\tau(x_1))(1+\tau(x_1)(3+\tau(x_1)))+r_1(r_1+2)(r_1+3)(11+6\tau(x_1)(3+\tau(x_1)))\rho_1+6r_1^2(r_1+3)(3+2\tau(x_1))\rho_1^2-r_1^3\rho_1^3+13r_1\rho_1^3\right)}{(r_1+1)(r_1+2)(r_1+3)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)(r_1\rho_1+2\tau(x_1)+3)}$$
$$+\frac{\tau(x_1)(1+\tau(x_1))(2+\tau(x_1))(3+\tau(x_1))}{(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)(r_1\rho_1+2\tau(x_1)+2)(r_1\rho_1+2\tau(x_1)+3)};$$

(x) $$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{04};x_1,x_2)$$
$$= \frac{x_2^4 r_2^4 \rho_2^4 (r_2-1)(r_2-2)(r_2-3)}{(r_2+1)(r_2+2)(r_2+3)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)(r_2\rho_2+2\tau(x_2)+3)}$$
$$+\frac{2x_2^3 r_2^3 \rho_2^3 (r_2-1)(r_2-2)((r_2+3)(3+2\tau(x_2))+6r_2\rho_2)}{(r_2+1)(r_2+2)(r_2+3)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)(r_2\rho_2+2\tau(x_2)+3)}$$
$$+\frac{x_2^2 r_2^2 \rho_2^2((r_2-1)(r_2+2)(r_2+3)(11+6\tau(x_2)(3+\tau(x_2)))+12r_2(r_2-1)(r_2+3)(3+2\tau(x_2))\rho_2+2r_2\rho_2^2(1-19r_2+36r_2^2))}{(r_2+1)(r_2+2)(r_1+3)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)(r_2\rho_2+2\tau(x_2)+3)}$$
$$+\frac{2r_2x_2\rho_2\left((r_2+1)(r_2+2)(r_2+3)(3+2\tau(x_2))(1+\tau(x_2)(3+\tau(x_2)))+r_2(r_2+2)(r_2+3)(11+6\tau(x_2)(3+\tau(x_2)))\rho_2+6r_2^2(r_2+3)(3+2\tau(x_2))\rho_2^2-r_2^3\rho_2^3+13r_2\rho_2^3\right)}{(r_2+1)(r_2+2)(r_2+3)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)(r_2\rho_2+2\tau(x_2)+3)}$$
$$+\frac{\tau(x_2)(1+\tau(x_2))(2+\tau(x_2))(3+\tau(x_2))}{(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)(r_2\rho_2+2\tau(x_2)+2)(r_2\rho_2+2\tau(x_2)+3)}.$$

Lemma 1.2. *By Direct computation, we have*

(i) $$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_1-x_1);x_1,x_2) = \frac{\tau(x_1)(1-2x_1)}{r_1\rho_1+2\tau(x_1)};$$

(ii) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}((t_2-x_2);x_1,x_2)=\dfrac{\tau(x_2)(1-2x_2)}{r_2\rho_2+2\tau(x_2)};$

(iii) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}((t_1-x_1)^2;x_1,x_2)=\dfrac{x_1(1-x_1)(r_1\rho_1(1+r_1+2r_1\rho_1)-4(r_1+1)\tau^2(x_1)-2(r_1+1)\tau(x_1))}{(r_1+1)(r_1\rho_1+2\tau(x_1))(r_1\rho_1+2\tau(x_1)+1)};$

(iv) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}((t_2-x_2)^2;x_1,x_2)=\dfrac{x_2(1-x_2)(r_2\rho_2(1+r_2+2r_2\rho_2)-4(r_2+1)\tau^2(x_2)-2(r_2+1)\tau(x_2))}{(r_2+1)(r_2\rho_2+2\tau(x_2))(r_2\rho_2+2\tau(x_2)+1)};$

(v) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}((t_1-x_1)^4;x_1,x_2)=\frac{6(r_1+1)(r_1+2)(r_1+3)\tau(x_1)+11(r_1+1)(r_1+2)(r_1+3)\tau^2(x_1)+6(r_1+1)(r_1+2)(r_1+3)\tau^3(x_1)}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_2\rho_2)(1+2\tau(x_1)+r_2\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)}$

$$+\frac{(r_1+1)(r_1+2)(r_1+3)\tau^4(x_1)}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_2\rho_1)(1+2\tau(x_1)+r_1\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)}$$
$$+\frac{(x_1(-24(r_1+1)(r_1+2)(r_1+3)\tau(x_1)-52(r_1+1)(r_1+2)(r_1+3)\tau^2(x_1)-36(r_1+1)(r_1+2)(r_1+3)\tau^3(x_1))}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_1\rho_1)(1+2\tau(x_1)+r_1\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)}$$
$$\frac{x_1(-8(r_1+1)(r_1+2)(r_1+3)\tau(x_1)^4+6r_1(r_1+1)(r_1+2)(r_1+3)\rho_1+14r_1(r_1+1)(r_1+2)(r_1+3)\tau(x_1)\rho_2)}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_2\rho_1)(1+2\tau(x_1)+r_2\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)}$$
$$\frac{x_1(6r_1(r_1+1)(r_1+2)(r_1+3)\tau^2(x_1)\rho_1+22r_1^2(r_1+1)(r_1+2)(r_1+3)\rho_1^2+36r_1^2(r_1+1)(r_1+2)(r_1+3)\tau(x_1)\rho_1^2)}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_1\rho_1)(1+2\tau(x_1)+r_1\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)}$$
$$\frac{x_1(12r_1^2(r_1+1)(r_1+2)(r_1+3)\tau^2(x_1)\rho_1^2+36r_1^3(r_13)\rho_1^3+24r_1^3(3+r_1)\tau(x_1)\rho_1^3-2r_1^3\rho_1^4+26r_1^4\rho_1^4))}{(r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_1\rho_1)(1+2\tau(x_1)+r_1\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1)}$$
$$+\Big(x_1^3\Big(-24(r_1+1)(r_1+2)(r_1+3)\tau(x_1)-88(r_1+1)(r_1+2)(r_1+3)\tau^2(x_1)-96(r_1+1)(r_1+2)(r_1+3)\tau^3(x_1)$$
$$-32(r_1+1)(r_1+2)(r_1+3)\tau^4(x_1)+36r_1(1+r_1)(2+r_1)(3+r_1)\rho_1+88r_1(1+r_1)(2+r_1)(3+r_1)\tau(x_1)\rho_1$$
$$+48r_1(1+r_1)(2+r_1)(3+r_1)\tau^2(x_1)\rho_1-6(-23+r_1)r_1^2(2+r_1)(3+r_1)\rho^2+240r_1^2(2+r_1)(3+r_1)$$
$$\tau(x_1)\rho_1^2+96r_1^2(2+r_1)(3+r_1)\tau^2(x_1)\rho_1^2+120r_1^3(3+r_1)\rho_1^3-12(-10+r_1)r_1^3(3+r_1)\rho_1^3-12r_1^4(3+r_1)\rho_1^3$$
$$+192r_1^3(3+r_1)\tau\rho^3-24(-7+r_1)r_1^4\rho_1^4\Big)\Big)\,/((r_1+1)(r_1+2)(r_1+3)(2\tau(x_1)+r_2\rho_2)(1+2\tau(x_1)+r_2\rho_1)(2+2\tau(x_1)+$$
$$r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1))+\Big(x_1^4\Big(12(1+r_1)(2+r_1)(3+r_1)\tau(x_1)+44(1+r_1)(2+r_1)(3+r_1)\tau^2(x_1)$$
$$+48(1+r_1)(2+r_1)(3+r_1)\tau^3(x_1)+16(1+r_1)(2+r_1)(3+r_1)\tau^4(x_1)-18r_1(1+r_1)(2+r_1)(3+r_1)\rho_1$$
$$-44r_1(1+r_1)(2+r_1)(3+r_1)\tau(x_1)\rho_1-24r_1(1+r_1)(2+r_1)(3+r_1)\tau^2(x_1)\rho_1+3(-23+r_1)r_1^2(2+r_1)$$
$$(3+r_1)\rho_1^2-120r_1^2(2+r_1)(3+r_1)\tau(x_1)\rho_1^2-48r_1^2(2+r_1)(3+r_1)\tau^2(x_1)\rho_1^2+12(r_1-10)r_1^3(3+r_1)\rho_1^3$$
$$-96r_1^3(3+r_1)\tau(x_1)\rho_1^3+12(-7+r_1)r_1^4\rho^4\Big)\Big)\Big/\,((1+r_1)(2+r_1)(3+r_1)(2\tau(x_1)+r_1\rho_1)(1+2\tau(x_1)+r_1\rho_1)(2+2\tau(x_1)+$$
$$r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1))+\Big(x_1^2\Big(36(1+r_1)(2+r_1)(3+r_1)\tau(x_1)+96(1+r_1)(2+r_1)(3+r_1)\tau^2(x_1)$$
$$+84(1+r_1)(2+r_1)(3+r_1)\tau^3(x_1)+24(1+r_1)(2+r_1)(3+r_1)\tau^4(x_1)-24r_1(1+r_1)(2+r_1)(3+r_1)\rho_1$$
$$-58r_1(1+r_1)(2+r_1)(3+r_1)\tau(x_1)\rho_1-30r_1(1+r_1)(2+r_1)(3+r_1)\tau^2(x_1)\rho_1-22r_1^2(2+r_1)(3+r_1)\rho_1^2$$
$$+3(-23+r_1)r_1^2(2+r_1)(3+r_1)\rho_1^2-156r_1^2(2+r_1)(3+r_1)\tau(x_1)\rho_1^2-60r_1^2(2+r_1)(3+r_1)\tau^2(x_1)\rho_1^2$$
$$-156r_1^3(3+r_1)\rho_1^3+12r_1^4(3+r_1)\rho_1^3-120r_1^3(3+r_1)\tau(x_1)\rho_1^3+2r_1^3\rho^4-26r_1^4\rho^4+12(-7+r_1)r_1^4\rho_1^4\Big)\Big)\Big/\,((1+r_1)(2+$$
$$r_1)(3+r_1)(2\tau(x_1)+r_1\rho_1)(1+2\tau(x_1)+r_1\rho_1)(2+2\tau(x_1)+r_1\rho_1)(3+2\tau(x_1)+r_1\rho_1))$$

(vi) $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}((t_2-x_2)^4;x_1,x_2)=\frac{6(r_2+1)(r_2+2)(r_2+3)\tau(x_2)+11(r_2+1)(r_2+2)(r_2+3)\tau^2(x_2)+6(r_2+1)(r_2+2)(r_2+3)\tau^3(x_2)}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)}$

$$+\frac{(r_2+1)(r_2+2)(r_2+3)\tau^4(x_2)}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)}$$
$$+\frac{(x_2(-24(r_2+1)(r_2+2)(r_2+3)\tau(x_2)-52(r_2+1)(r_2+2)(r_2+3)\tau^2(x_2)-36(r_2+1)(r_2+2)(r_2+3)\tau^3(x_2))}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)}$$
$$\frac{x_2(-8(r_2+1)(r_2+2)(r_2+3)\tau(x_2)^4+6r_2(r_2+1)(r_2+2)(r_2+3)\rho_2+14r_2(r_2+1)(r_2+2)(r_2+3)\tau(x_2)\rho_2)}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)}$$

$$\frac{x_2(6r_2(r_2+1)(r_2+2)(r_2+3)\tau^2(x_2)\rho_2+22r_2^2(r_2+1)(r_2+2)(r_2+3)\rho_2^2+36r_2^2(r_2+1)(r_2+2)(r_2+3)\tau(x_2)\rho_2^2)}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)}$$
$$\frac{x_2(12r_2^2(r_2+1)(r_2+2)(r_2+3)\tau^2(x_2)\rho_2^2+36r_2^3(r_23)\rho_2^3+24r_2^3(3+r_2)\tau(x_2)\rho_2^3-2r_2^3\rho_2^4+26r_2^4\rho_2^4))}{(r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)}$$
$$+\Big(x_2^3\Big(-24(r_2+1)(r_2+2)(r_2+3)\tau(x_2)-88(r_2+1)(r_2+2)(r_2+3)\tau^2(x_2)-96(r_2+1)(r_2+2)(r_2+3)\tau^3(x_2)$$
$$-32(r_2+1)(r_2+2)(r_2+3)\tau^4(x_2)+36r(1+r_2)(2+r_2)(3+r_2)\rho_2+88r_2(1+r_2)(2+r_2)(3+r_2)\tau(x_2)\rho_2$$
$$+48r_2(1+r_2)(2+r_2)(3+r_2)\tau^2(x_2)\rho_2-6(-23+r_2)r_2^2(2+r_2)(3+r_2)\rho_2^2+240r_2^2(2+r_2)(3+r_2)$$
$$\tau(x_2)\rho_2^2+96r_2^2(2+r_2)(3+r_2)\tau^2(x_2)\rho_2^2+120r_2^3(3+r_2)\rho_2^3-12(-10+r_2)r_2^3(3+r_2)\rho_2^3-12r_2^4(3+r_2)\rho_2^3$$
$$+192r_2^3(3+r_2)\tau(x_2)\rho_2^3-24(-7+r_2)r_2^4\rho_2^4\Big)\Big)\Big/((r_2+1)(r_2+2)(r_2+3)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+$$
$$2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2))+\Big(x_2^4\Big(12(1+r_2)(2+r_2)(3+r_2)\tau(x_2)+44(1+r_2)(2+r_2)(3+r_2)\tau^2(x_2)$$
$$+48(1+r_2)(2+r_2)(3+r_2)\tau^3(x_2)+16(1+r_2)(2+r_2)(3+r_2)\tau^4(x_2)-18r_2(1+r_2)(2+r_2)(3+r_2)\rho_2-44r_2$$
$$(1+r_2)(2+r_2)(3+r_2)\tau(x_1)\rho_2-24r_2(1+r_2)(2+r_2)(3+r_2)\tau^2(x_2)\rho_2+3(-23+r_2)r_2^2(2+r_2)(3+r_2)\rho_2^2$$
$$-120r_2^2(2+r_2)(3+r_2)\tau\rho^2-48r_2^2(2+r_2)(3+r_2)\tau^2(x_2)\rho_2^2+12(-10+r_2)r_2^3(3+r_2)\rho_2^3-96r_2^3(3+r_2)\tau(x_2)\rho_2^3$$
$$+12(-7+r_2)r_2^4\rho_2^4\Big)\Big)\Big/((1+r_2)(2+r_2)(3+r_2)(2\tau(x_2)+r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+$$
$$r_2\rho_2))+\Big(x_2^2\Big(36(1+r_2)(2+r_2)(3+r_2)\tau(x_2)+96(1+r_2)(2+r_2)(3+r_2)\tau^2(x_2)+84(1+r_2)(2+r_2)$$
$$(3+r_2)\tau^3(x_2)+24(1+r_2)(2+r_2)(3+r_2)\tau^4(x_2)-24r_2(1+r_2)(2+r_2)(3+r_2)\rho_2-58r_2(1+r_2)(2+r_2)$$
$$(3+r_2)\tau(x_2)\rho_2-30r_2(1+r_2)(2+r_2)(3+r_2)\tau^2(x_2)\rho_2-22r_2^2(2+r_2)(3+r_2)\rho_2^2+3(-23+r_2)r_2^2$$
$$(2+r_2)(3+r_2)\rho_2^2-156r_2^2(2+r_2)(3+r_2)\tau(x_2)\rho_2^2-60r_2^2(2+r_2)(3+r_2)\tau^2(x_2)\rho_2^2-156r_2^3(3+r_2)\rho_2^3$$
$$+12r_2^4(3+r_2)\rho_2^3-120r_2^3(3+r_2)\tau(x_2)\rho_2^3+2r_2^3\rho_2^4-26r_2^4\rho_2^4+12(-7+r_2)r_2^4\rho_2^4\Big)\Big)\Big/((1+r_2)(2+r_2)(3+r_2)(2\tau(x_2)+$$
$$r_2\rho_2)(1+2\tau(x_2)+r_2\rho_2)(2+2\tau(x_2)+r_2\rho_2)(3+2\tau(x_2)+r_2\rho_2)).$$

Lemma 1.3. *From Lemma 1.2, we have*

$$\lim_{r_1\to\infty} r_1 D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_1-x_1)^2;x_1,x_2) = \frac{1}{\rho_1}(1+2\rho_1)x_1(1-x_1);$$
$$\lim_{r_2\to\infty} r_2 D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_2-x_2)^2;x_1,x_2) = \frac{1}{\rho_2}(1+2\rho_2)x_2(1-x_2);$$
$$\lim_{r_1\to\infty} r_1^2 D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_1-x_1)^4;x_1,x_2) = \left(\frac{3x_1^2(1-x_1)^2(1+2\rho_1)^2}{\rho_1^2}\right);$$
$$\lim_{r_2\to\infty} r_2^2 D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_2-x_2)^4;x_1,x_2) = \left(\frac{3x_2^2(1-x_2)^2(1+2\rho_2)^2}{\rho_2^2}\right).$$

Lemma 1.4. *We have*

$$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_1-x_1)^2;x_1,x_2) = \frac{A_1}{r_1}\left\{x_1(1-x_1)+\frac{1}{r_1}\right\} = \Phi^2_{r_1,\rho_1}(x_1);$$
$$D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_2-x_2)^2;x_1,x_2) = \frac{A_2}{r_2}\left\{x_2(1-x_2)+\frac{1}{r_2}\right\} = \Psi^2_{r_2,\rho_2}(x_2),$$

where $A_i = A_i(\rho_i), i = 1,2,$ is a positive constant.

Theorem 1.5. *Suppose $\gamma \in C(I^2)$. Then,*

$$\lim_{r_1,r_2\to\infty} \|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma) - \gamma\| = 0.$$

Proof. From the Lemma 1.1, we have

$$\lim_{r_1,r_2\to\infty} D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{mn}; x_1, x_2) = e_{mn}, (m, n) \in \{(0, 0), (1, 0), (0, 1)\}$$

and

$$\lim_{r_1,r_2\to\infty} D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{02} + e_{20}; x_1, x_2) = e_{02} + e_{20}$$

uniformly on I^2. Hence, the result follows in view of the theorem given by Volkov [31] □

Let $C^n(I^2), n \in \mathbb{N}$ denote the space of $n - times$ partially differentiable continuous function in I^2.

2. Voronovskaja Type Theorem

In the next theorem, we establish a Voronovskaja type asympototic result.

Theorem 2.1. *If $\gamma \in C^2(I^2)$, then*

$$\begin{aligned}\lim_{r\to\infty} r\Big(D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma; x_1, x_2) - \gamma(x_1, x_2)\Big) &= \frac{\tau(x_1)(1-2x_1)}{\rho_1}\gamma'_{x_1}(x_1, x_2) + \frac{\tau(x_2)(1-2x_2)}{\rho_2}\gamma'_{x_2}(x_1, x_2)\\ &+ \frac{1}{2\rho_1}(1+2\rho_1)x_1(1-x_1)\gamma''_{x_1x_1}(x_1, x_2) + \frac{1}{2\rho_2}(1+2\rho_2)x_2(1-x_2)\gamma''_{x_2x_2}(x_1, x_2),\end{aligned}$$

uniformly in $(x_1, x_2) \in I^2$.

Proof. Using Taylor's series expansion, we get

$$\begin{aligned}\gamma(u, v) &= \gamma(x_1, x_2) + \gamma'_{x_1}(x_1, x_2)(u - x_1) + \gamma'_{x_2}(v - x_2) + \frac{1}{2}\{\gamma''_{x_1x_1}(x_1, x_2)(u - x_1)^2 + 2\gamma''_{x_1x_2}(x_1, x_2)(u - x_1)(v - x_2)\\ &\quad + \gamma''_{x_1x_2}(x_1, x_2)(v - x_2)^2\} + \beta(u, v; x_1, x_2)\sqrt{(u - x_1)^4 + (v - x_2)^4},\end{aligned}$$

where $\beta(u, v; x_1, x_2) \in C(I^2)$ and $\beta(u, v; x_1, x_2) \to 0$ as $(u, v) \to (x_1, x_2)$.

Applying positivity and linearity of $D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma; x_1, x_2)$ on the above Taylor's expansion, we have

$$\begin{aligned}D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma; x_1, x_2) &= \gamma(x_1, x_2) + \gamma'_{x_1}(x_1, x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1\rho_2}((u - x_1); x_1, x_2) + \gamma'_{x_2}(x_1, x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v - x_2); x_1, x_2)\\ &\quad + \frac{1}{2}(\gamma''_{x_1x_1}(x_1, x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u - x_1)^2; x_1, x_2) + \gamma''_{x_2x_2}(x_1, x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v - x_2)^2; x_1, x_2)\\ &\quad + 2\gamma''_{x_1x_2}(x_1, x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u - x_1); x_1, x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v - x_2); x_1, x_2)\end{aligned}$$

$$+D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}\left(\beta(u,v;x_1,x_2)\sqrt{(u-x_1)^4+(v-x_2)^4};x_1,x_2\right)$$

From Lemma 1.2, we have

$$\begin{aligned}\lim_{r\to\infty} r\left(D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}(\gamma;x_1,x_2)-\gamma(x_1,x_2)\right) &= \frac{\tau(x_1)(1-2x_1)}{\rho_1}\gamma'_{x_1}(x_1,x_2)+\frac{\tau(x_2)(1-2x_2)}{\rho_2}\gamma'_{x_2}(x_1,x_2) \qquad (3)\\ &+\frac{1}{2\rho_1}(1+2\rho_1)x_1(1-x_1)\gamma''_{x_1x_1}(x_1,x_2)+\frac{1}{2\rho_2}(1+2\rho_2)x_2(1-x_2)\gamma''_{x_2x_2}(x_1,x_2)\\ &+\lim_{r\to\infty} rD_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}\left(\beta(u,v;x_1,x_2)\sqrt{(u-x_1)^4+(v-x_2)^4};x_1,x_2\right), \qquad (4)\end{aligned}$$

uniformly in $(x_1,x_2)\in I^2$. By using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}rD_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}\left(\beta(u,v;x_1,x_2)\sqrt{(u-x_1)^4+(v-x_2)^4};x_1,x_2\right) &\leq \left(D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}(\beta^2(u,v;x_1,x_2);x_1,x_2)\right)^{\frac{1}{2}}\\ &(r^2D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}((u-x_1)^4;x_1,x_2)+r^2D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}((v-x_2)^4;x_1,x_2))^{\frac{1}{2}}\end{aligned}$$

and $\beta^2(u,v;x_1,x_2)\to 0$, and $(u,v)\to(x_1,x_2)$, using Theorem 1.5,

$$\lim_{r\to\infty} D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}(\beta^2(u,v;x_1,x_2);x_1,x_2)=0,$$

uniformly in $(x_1,x_2)\in I^2$. By using Lemma 1.3, we may write

$$(r^2D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}((u-x_1)^4;x_1,x_2)+r^2D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}((v-x_2)^4;x_1,x_2))=O(1),$$

uniformly in $(x_1,x_2)\in I^2$, as $r\to\infty$. Hence,

$$\lim_{r\to\infty} rD_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}\left(\beta(u,v;x_1,x_2)\sqrt{(u-x_1)^4+(v-x_2)^4};x_1,x_2\right)=0,$$

uniformly in $(x_1,x_2)\in I^2$. Therefore, by (6) we obtain the desired result. □

3. Grüss Voronovskaja type theorem

The following theorem shows the non-multiplicative behaviour of operator $D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}$.

Theorem 3.1. *For $\gamma,\zeta\in C^2(I^2)$, there holds the following equality*

$$\begin{aligned}\lim_{r\to\infty} r\{D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}(\gamma\zeta;x_1,x_2)-D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}(\gamma;x_1,x_2)D_{r,r,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r},\frac{1}{r}\right]}(\zeta;x_1,x_2)\} &= \frac{1}{\rho_1}(1+2\rho_1)x_1(1-x_1)\gamma'_{x_1}(x_1,x_2)\zeta'_{x_1}(x_1,x_2)\\ &+\frac{1}{\rho_2}(1+\rho_2)x_2(1-x_2)\gamma'_{x_2}(x_1,x_2)\zeta'_{x_2}(x_1,x_2),\end{aligned}$$

uniformly in $(x_1,x_2)\in I^2$.

Proof.

$$\lim_{r\to\infty} r\{D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma\zeta;x_1,x_2) - D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\zeta;x_1,x_2)\}$$

$$= r\Bigg(D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma\zeta;x_1,x_2) - \gamma(x_1,x_2)\zeta(x_1,x_2) - (\gamma(x_1,x_2)\zeta'_{x_1}(x_1,x_2) + \zeta(x_1,x_2)\gamma'_{x_1}(x_1,x_2)) + \zeta(x_1,x_2)\gamma'_{x_1}(x_1,x_2))$$

$$D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1);x_1,x_2) - (\gamma(x_1,x_2)\zeta'_{x_2}(x_1,x_2) + \zeta(x_1,x_2)\gamma'_{x_2}(x_1,x_2))D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2);x_1,x_2)$$

$$-\frac{1}{2}(\gamma(x_1,x_2)\zeta''_{x_1x_1}(x_1,x_2) + 2\gamma'_{x_1}(x_1,x_2)\zeta'_{x_1}(x_1,x_2) + \zeta(x_1,x_2)\gamma''_{x_1x_1}(x_1,x_2))D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)^2;x_1,x_2)$$

$$-(\gamma(x_1,x_2)\zeta''_{x_1x_2}(x_1,x_2) + \gamma'_{x_1}(x_1,x_2)\zeta'_{x_2}(x_1,x_2) + \gamma'_{x_2}(x_1,x_2)\zeta'_{x_1}(x_1,x_2)$$

$$+\zeta(x_1,x_2)\gamma''_{x_1x_2}(x_1,x_2))D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)(v-x_2);x_1,x_2)$$

$$-\frac{1}{2}(\gamma(x_1,x_2)\zeta''_{x_2x_2}(x_1,x_2) + 2\gamma'_{x_2}(x_1,x_2)\zeta'_{x_2}(x_1,x_2) + \zeta(x_1,x_2)\gamma''_{x_2x_2}(x_1,x_2))D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2)^2;x_1,x_2)$$

$$-\zeta(x_1,x_2)(D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2) - \gamma(x_1,x_2) - \gamma'_{x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1);x_1,x_2)$$

$$-\gamma'_{x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_1);x_1,x_2) - \frac{1}{2}\gamma''_{x_1x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)^2;x_1,x_2)$$

$$-\gamma''_{x_1x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)(v-x_2);x_1,x_2)$$

$$-\frac{1}{2}\gamma''_{x_2x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2)^2;x_1,x_2)) - \gamma(x_1,x_2)(D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\zeta;x_1,x_2) - \zeta(x_1,x_2)$$

$$-\zeta'_{x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1);x_1,x_2) - \zeta'_{x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2);x_1,x_2)$$

$$-\frac{1}{2}\zeta''_{x_1x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)^2;x_1,x_2)$$

$$-\zeta''_{x_1x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)(v-x_2);x_1,x_2) - \frac{1}{2}\zeta''_{x_2x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2)^2;x_1,x_2))$$

$$+\zeta'_{x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1);x_1,x_2)(\gamma(x_1,x_2) - D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)) + \zeta'_{x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2);x_1,x_2)$$

$$(\gamma(x_1,x_2) - D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)) + \zeta''_{x_1x_1}\frac{D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1}((u-x_1)^2;x_1,x_2)}{2}(\gamma(x_1,x_2) - D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2))$$

$$+\zeta''_{x_2x_2}(x_1,x_2)\frac{D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_2}((v-x_2)^2;x_1,x_2)}{2}(\gamma(x_1,x_2)-D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2))$$

$$+\zeta''_{x_1x_2}(x_1,x_2)D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1}((u-x_1);x_1,x_2)D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_2}((v-x_2);x_1,x_2)$$

$$(\gamma(x_1,x_2)-D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2))+\gamma'_{x_1}(x_1,x_2)\zeta'_{x_1}(x_1,x_2)D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1}((u-x_1)^2;x_1,x_2)+\gamma'_{x_1}(x_1,x_2)\zeta'_{x_2}(x_1,x_2)$$

$$\left.D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1}((u-x_1);x_1,x_2)D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_2}((v-x_2);x_1,x_2)+\gamma'_{x_2}(x_1,x_2)\zeta'_{x_2}(x_1,x_2)D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1}((v-x_2)^2;x_1,x_2)\right),$$

from Theorem (2.1), for each $\gamma\in C^2(I^2)$ it follows that

$$\lim_{r\to\infty} r\left\{D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)-\gamma(x_1,x_2)-\gamma'_{x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1);x_1,x_2)\right.$$

$$+\gamma'_{x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2);x_1,x_2)-\frac{1}{2}(\gamma''_{x_1x_1}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1)^2;x_1,x_2)$$

$$\left.+\gamma''_{x_2x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2)^2;x_1,x_2)-2\gamma''_{x_1x_2}(x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((u-x_1);x_1,x_2)D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}((v-x_2);x_1,x_2))\right\}$$

$$=\lim_{r\to\infty} r\left\{D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}\left(\beta(u,v;x_1,x_2)\sqrt{(u-x_1)^4+(v-x_2)^4};x_1,x_2\right)\right\}=0,$$

and from Theorem (1.5), $D^{\left[\frac{1}{r},\frac{1}{r}\right]}_{r,r,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)\to\gamma(x_1,x_2)$, as $r\to\infty$, uniformly in $(x_1,x_2)\in I^2$, hence applying Lemma 1.3, we get the required result. □

4. Degree of approximation

Next, we shall prove a qualitative result by using the Lipschitz class. For $0<\alpha_1,\alpha_2\le 1$, $Lip_k^{(\alpha_1,\alpha_2)}$ is the the Lipschitz class for the bivariate case as follows:

$$|\gamma(t_1,t_2)-\gamma(x_1,x_2)|\le K|t_1-x_1|^{\alpha_1}|t_2-x_2|^{\alpha_2}$$

where K is some positive constant.

Theorem 4.1. *Let $\gamma\in Lip_K^{(\alpha_1,\alpha_2)}$. Then, for each $(x_1,x_2)\in I^2$ we have*

$$|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)-\gamma(x_1,x_2)|\le K\Phi^{\alpha_1}_{r_1,\rho_1}(x_1)\Psi^{\alpha_2}_{r_2,\rho_2}(x_2)$$

where $\Phi_{r_1,\rho_1}(x_1)$ and $\Psi_{r_2,\rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. Using the operator $D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(.;x_1,x_2)$, we may write

$$\begin{aligned}|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)-\gamma(x_1,x_2)| &\leq D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(|(\gamma(t_1,t_2)-\gamma(x_1,x_2)|;x_1,x_2)|)\\ &\leq KD^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}|(t_1-x_1|^{\alpha_1}|t_2-x_2|^{\alpha_2};x_1,x_2)\\ &= KD^{\left[\frac{1}{r_1}\right]}_{r_1,\tau,\rho_1,}(|t_1-x_1|^{\alpha_1};x_1,x_2)D^{\left[\frac{1}{r_2}\right]}_{r_2,\tau,\rho_2}(|t_2-x_2|^{\alpha_2};x_1,x_2)\end{aligned}$$

By the Hölder's inequality with $v_1=\dfrac{2}{\alpha_1}, u_1=\dfrac{2}{2-\alpha_1}$ and $v_2=\dfrac{2}{\alpha_2}, u_2=\dfrac{2}{2-\alpha_2}$ and using Lemma 1.4, we get

$$\begin{aligned}|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)-\gamma(x_1,x_2)| &\leq K.\Big(D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1,}(|t_1-x_1|^2;x_1,x_2)^{\dfrac{\alpha_1}{2}}\Big)\Big((D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1,}(e_{00};x_1,x_2)\Big)^{\dfrac{2-\alpha_1}{2}}\\ &\quad\times\Big(D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1,}(|t_2-x_2|^2;x_1,x_2)\Big)^{\dfrac{\alpha_2}{2}}\Big(D^{\left[\frac{1}{r}\right]}_{r,\tau,\rho_1,}(e_{00};x_1,x_2)\Big)^{\dfrac{2-\alpha_1}{2}}\\ &\leq \Phi^{\alpha_1}_{r,\rho_1}(x_1)\Psi^{\alpha_2}_{r,\rho_2}(x_2).\end{aligned}$$

□

In these theorem, we determine the convergence rate of $D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma)$ to γ for $\gamma\in C^1(I^2)$.

Theorem 4.2. *Let $\gamma\in C^1(I^2)$ and $(x_1,x_2)\in I^2$. Then, we get*

$$|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)-\gamma(x_1,x_2)|\leq\|\gamma'_{x_1}\|_{C(I^2)}\Phi_{r_1,\rho_1}(x_1)+\|\gamma'_{x_2}\|_{C(I^2)}\Psi_{r_2,\rho_2}(x_2)$$

where $\Phi_{r_1,\rho_1}(x_1)$ and $\Psi_{r_2,\rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. For any $(x_1,x_2)\in I^2$, we may write

$$\gamma(t_1,t_2)-\gamma(x_1,x_2)=\int_{x_1}^{t_1}\gamma'_w(w,t_2)dw+\int_{x_2}^{t_2}\gamma'_h(t_1,h)dh \tag{5}$$

operating $D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(.;x_1,x_2)$ on both sides in equation (5), we get

$$|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma,x_1,x_2)-\gamma(x_1,x_2)|\leq D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\int_{x_1}^{t_1}\gamma'_w(w,t_2)dw;x_1,x_2\Big)+D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\int_{x_2}^{t_2}\gamma'_h(t_1,h)dh;x_1,x_2\Big)$$

Hence using the inequalities
$\left|\int_{x_1}^{t_1}\gamma'_w(w,t_2)dw\right|\leq\|\gamma'_{x_1}\|_{C(I^2)}|t_1-x_1|$ and $\left|\int_{x_2}^{t_2}\gamma'_h(t_1,h)dh\right|\leq\|\gamma'_{x_2}\|_{C(I^2)}|t_2-x_2|$, we have

$$|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma,x_1,x_2)-\gamma(x_1,x_2)|\leq\|\gamma'_{x_1}\|_{C(I^2)}D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(|t_1-x_1|;x_1,\Big)+\|\gamma'_{x_2}\|_{C(I^2)}D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(|t_2-x_2|;x_2\Big).$$

By Lemma 1.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma,x_1,x_2)-\gamma(x_1,x_2)| &\leq \|\gamma'_{x_1}\|_{C(I^2)}\Big(D^{\left[\frac{1}{r_1}\right]}_{r_1,\tau,\rho_1}\big(|t_1-x_1|^2;x_1,\Big)^{\frac{1}{2}}\Big(D^{\left[\frac{1}{r_1}\right]}_{r_1,\tau,\rho_1}(e_{00};x_1,x_2)\Big)^{\frac{1}{2}} \\ &+\|\gamma'_{x_2}\|_{C(I^2)}\Big(D^{\left[\frac{1}{r_2}\right]}_{r_2,\tau,\rho_2}(|t_2-x_2|^2;x_1,x_2)\Big)^{\frac{1}{2}}\Big(D^{\left[\frac{1}{r_2}\right]}_{r_2,\tau,\rho_2}(e_{00};x_1,x_2)\Big)^{\frac{1}{2}} \\ &\leq \|\gamma'_{x_1}\|C(I^2)\Phi_{r_1,\rho_1}(x_1)+\|\gamma'_{x_2}\|C(I^2)\Psi_{r_2,\rho_2}(x_2). \end{aligned}$$

Hence the proof is completed. □

For $\gamma\in C(I^2)$ and any $\varsigma_1,\varsigma_2>0$, for the bivariate case the complete modulus of continuity of γ is defined as

$$\bar{\omega}(\gamma;\varsigma_1,\varsigma_2)=\sup\{|\gamma(t_1,t_2)-\gamma(x_1,x_2)|:(t_1,t_2),(x_1,x_2)\in I^2 \textit{ and } |t_1-x_1|\leq\varsigma_1,|t_2-x_2|\leq\varsigma_2\}.$$

For $\gamma\in C(I^2)$ and any $\zeta>0$, the partial moduli of continuity of γ with respect to x_1 and x_2 is given by

$$\bar{\omega}^1(\gamma,\zeta)=\sup\{|\gamma(x_{11},x_2)-\gamma(x_{12},x_2)|:y\in I \textit{ and } |x_{11}-x_{12}|\leq\zeta\}$$

$$\bar{\omega}^2(\gamma,\zeta)=\sup\{|\gamma(x_1,x_{21})-\gamma(x_1,x_{22})|:x\in I \textit{ and } |x_{21}-x_{22}|\leq\zeta\}.$$

Evidently, these moduli of continuity satisfy the properties of the usual modulus of continuity.

Theorem 4.3. *For $\gamma\in C(I^2)$ and $(x_1,x_2)\in I^2$, we have the following inequality :*

$$\Big|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma,x_1,x_2)-\gamma(x_1,x_2)\Big|\leq 4\bar{\omega}(\gamma;\Phi_{r_1,\rho_1}(x_1),\Psi_{r_2,\rho_2}(x_2))$$

where $\Phi_{r_1,\rho_1}(x_1)$ and $\Psi_{r_2,\rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. By using the theorem (4.1) of [26] and Lemma 1.4, for any $\delta_1,\delta_2>0$, we get

$$\begin{aligned} \Big|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma,x_1,x_2)-\gamma(x_1,x_2)\Big| &\leq \bar{\omega}(\gamma;\delta_1,\delta_2)(1+\delta_1^{-1}\sqrt{D^{\frac{1}{r}}_{r,\tau,\rho_1}((e_{10}-x_1)^2;x_1,x_2)} \\ &+\delta_2^{-1}\sqrt{D^{\frac{1}{r}}_{r,\tau,\rho_2}((e_{01}-x_2)^2;x_1,x_2)} \\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{D^{\frac{1}{r}}_{r,\tau,\rho_1}((e_{10}-x_1)^2;x_1,x_2)}\sqrt{D^{\frac{1}{r}}_{r,\tau,\rho_2}((e_{01}-x_2)^2;x_1,x_2)}). \end{aligned}$$

Taking the value of $\delta_1=\Phi_{r_1,\rho_1}(x_1)$ and $\delta_2=\Psi_{r_2,\rho_2}(x_2)$, we get the desired result.

In the next theorem, we estimate the degree of approximation of γ by $D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma)$ in terms of partial moduli of continuity. □

Theorem 4.4. *For $\gamma\in C(I^2)$ and $(x_1,x_2)\in I^2$, we have the following inequality*

$$\Big|D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2)-\gamma(x_1,x_2)\Big|\leq 2\Big(\omega(\gamma;\Phi_{r_1,\rho_1}(x_1))+\omega^2(\gamma;\Psi_{r_2,\rho_2}(x_2))\Big)$$

where $\Phi_{r_1,\rho_1}(x_1)$ and $\Psi_{r_2,\rho_2}(x_2)$ are defined as in Lemma 1.4.

Proof. Using Cauchy-Schwarz inequality, Lemma 1.4 and definition of the partial moduli of continuity, we have

$$\begin{aligned}
\left|D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma,x_1,x_2)-\gamma(x_1,x_2)\right| &\leq D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(|\gamma(t_1,t_2)|-\gamma(x_1,x_2)|;x_1,x_2)\\
&\leq D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(|\gamma(t_1,x_2)-\gamma(x_1,x_2)|;x_1,x_2)+D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(|\gamma(x_1,t_2)-\gamma(x_1,x_2)|;x_1,x_2)\\
&\leq D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}\bar{\omega}^1(\gamma;|t_1-x_1|);x_1,x_2)+D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\bar{\omega}^2(\gamma;|t_2-x_2|);x_1,x_2)\\
&\leq \bar{\omega}^1(\gamma;\Phi_{r_1,\rho_1}(x_1)\Big[1+\frac{1}{\Phi_{r_1,\rho_1}(x_1)}D_{r_1,\tau,\rho_1}^{\frac{1}{r_1}}(|t_1-x_1|;x_1,x_2)\Big]\\
&\quad+\bar{\omega}^2(\gamma;\Psi_{r_2,\rho_2}(x_2)\Big[1+\frac{1}{\Phi_{r_2,\rho_2}(x_2)}D_{r_2,\tau,\rho_2}^{\frac{1}{r_2}}(|t_2-x_2|;x_1,x_2)\Big]\\
&\leq \bar{\omega}^1(\gamma;\Phi_{r_1,\rho_1}(x_1)\Big[1+\frac{1}{\Phi_{r_1,\rho_1}(x_1)}\Big(D_{r_1,\tau,\rho_1}^{\frac{1}{r_1}}((e_{10}-x_1)^2;x_1,x_2)\Big)^{\frac{1}{2}}\Big]\\
&\quad+\bar{\omega}^2(\gamma;\Psi_{r_2,\rho_2}(x_2)\Big[1+\frac{1}{\Phi_{r_2,\rho_2}(x_2)}\Big(D_{r_2,\tau,\rho_2}^{\frac{1}{r_2}}((e_{01}-x_2)^2;x_1,x_2)\Big)^{\frac{1}{2}}\Big]
\end{aligned}$$

from which the desired result is directly proof. □

Let $\gamma^{i,j}(x_1,x_2)=\Big(\frac{\partial^{i+j}}{\partial x_1^i\partial x_2^j}\Big)\gamma(x_1,x_2)$ $i,j\in\mathbb{N}_0$ such that $0\leq i+j\leq 2$ and $C^2(I^2)$ be endowed with the norm

$$\|\gamma\|_{C^2(I^2)}=\|\gamma\|_{C(I^2)}+\|\gamma\|^1_{C(I^2)}+\|\gamma\|^2_{C(I^2)},$$

where

$$\|\gamma\|^1_{C(I^2)}=\sup_{(x_1,x_2)\in I^2}\Big\{|\gamma(x_1,x_2)|,|\gamma^{(1,0)}(x_1,x_2)|,|\gamma^{(0,1)}(x_1,x_2)|\Big\}$$

and

$$\|\gamma\|^2_{C(I^2)}=\sup\{|\gamma(x,x_2)|,|\gamma^{(1,0)}(x_1,x_2)|,|\gamma^{(0,1)}(x_1,x_2)|,|\gamma^{(2,0)}(x_1,x_2)|,|\gamma^{(1,1)}(x_1,x_2)|,|\gamma^{(0,2)}(x_1,x_2)|\}.$$

Now, we proceed to determine the approximation order of the sequence $D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma;.)$ to the function $\gamma\in C(I^2)$ in terms of the Peetre's K-function given by

$$\kappa(\gamma;\delta)=\inf_{\mu\in C^2(I^2)}\Big\{\|\gamma-\mu\|_{C(I^2)}+\delta\|\mu\|_{C^2(I^2)},\delta>0\Big\}$$

It is known [27] that the inequality

$$\kappa(\gamma;\delta)\leq M_1\{\bar{\omega}_2(\gamma;\sqrt{\delta})+\min\{1,\delta\}\|\gamma\|_{C(I^2)}\Big\}$$

holds for each $\delta>0$, where the constant M_1 is independent of γ,δ and $\bar{\omega}_2(\gamma;\sqrt{\delta})$ is defined as second order modulus of continuity for the bivariate extension.

Theorem 4.5. *For any $(x_1,y)\in I^2$, we have*

$$\left|D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma,x_1,x_2)-\gamma(x_1,x_2)\right| \leq M\Big\{\bar{\omega}_2\Big(\gamma;\sqrt{\xi_{r_1,r_2,\tau,\rho_1,\rho_2}(x_1,x_2)}\Big)$$

$$+\min\left\{1, \frac{\xi_{r_1,r_2,\tau,\rho_1,\rho_2}(x_1,x_2)}{4}\right\}_{\|\gamma\|_{C(I^2)}}\Bigg\}$$
$$+\bar{\omega}\left(\gamma; \left|\frac{\tau(x_1)(1-2x_1)}{(r_1\rho_1+2\tau(x_1))}\right|, \left|\frac{\tau(x_2)(1-2x_2)}{(r_2\rho_2+2\tau(x_2))}\right|\right),$$

where

$$\xi_{r_1,r_2,\tau,\rho_1,\rho_2}(x_1,x_2) = \left\{\Phi_{r_1,\rho_1}(x_1)\Psi_{r_2,\rho_2}(x_2) + \left|\frac{\tau(x_1)(1-2x_1)}{(r_1\rho_1+2\tau(x_1))}\right| \times \left|\frac{\tau(x_2)(1-2x_2)}{(r_2\rho_2+2\tau(x_2))}\right|\right\},$$

$\Phi_{r_1,\rho_1}(x_1)$ *and* $\Psi_{r_2,\rho_2}(x_2)$ *are defined in Lemma 1.4.*

Proof. The following auxiliary operator are defined as follows:

$$\overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2) = \overset{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(\gamma;x_1,x_2) - \gamma\left(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}, \frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right) + \gamma(x_1,x_2). \tag{6}$$

Then, by Lemma 1.1,

$$\overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{00};x_1,x_2) = 1$$

$$\overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{10};x_1,x_2) = x_1 \;\; \textit{and} \;\; \overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{01};x_1,x_2) = x_2.$$

$$\text{Hence } \overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_1-x_1);x_1,x_2) = 0, \;\; \overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}((t_2-x_2);x_1,x_2) = 0. \tag{7}$$

Let $\lambda \in C^2(I^2)$ and $(t_1,t_2) \in I^2$ be arbitrary. Then, by Taylor's formula

$$\begin{aligned}\lambda(t_1,t_2) - \lambda(x_1,x_2) &= \lambda(t_1,x_2) + \lambda(x_1,x_2) + \lambda(t_1,t_2) - \tau(t_1,x_2)\\ &= \frac{\partial\lambda(x_1,x_2)}{\partial x_1}(t_1-x_1) + \int_{x_1}^{t_1}(t_1-u)\frac{\partial^2\lambda(u,x_2)}{\partial u^2} + \frac{\partial\lambda(x_1,x_2)}{\partial x_2}(t_2-x_2) + \int_{x_2}^{t_2}(t_2-v)\frac{\partial^2\lambda(x_1,v)}{\partial v^2}dv\\ &\quad + \int_{x_2}^{t_2}\int_{x_1}^{t_1}\frac{\partial^2\lambda(t_1,t_2)}{\partial u\partial v}dudv.\end{aligned} \tag{8}$$

Applying the operator $\overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(e_{00};x_1,x_2)$ on both sides to the equation (8), from (6) we obtain

$$\begin{aligned}&\overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}(\lambda;x_1,x_2) - \lambda(x_1,x_2)\\ &= \overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}\left(\int_{x_1}^{t_1}(t_1-u)\frac{\partial^2\lambda(u,x_2)}{\partial u^2}du;x_1,x_2\right)\\ &\quad + \overset{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}{D}_{r_1,r_2,\tau,\rho_1,\rho_2}\left(\int_{x_1}^{t_1}(t_2-v)\frac{\partial^2\lambda(x_1,v)}{\partial v^2}dv;x_1,x_2\right)\end{aligned}$$

$$+D^{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\int_{x_2}^{t_2}\int_{x_1}^{t_1}(t_1-u)\frac{\partial^2\lambda(u,x_2)}{\partial u\partial v}dudv;x_1,x_2\Big)$$

$$= D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\int_{x_1}^{t_1}(t_1-u)\frac{\partial^2\lambda(u,x_2)}{\partial u^2}du;x_1,x_2\Big)$$
$$-\int_{x_1}^{\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}}\Big(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}-u\Big)\frac{\partial^2\lambda(u,x_2)}{\partial u^2}du$$
$$+D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\int_{x_2}^{t_2}(t_2-v)\frac{\partial^2\lambda(x_1,v)}{\partial v^2}dv;x_1,x_2\Big)$$
$$-\int_{x_2}^{\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}}\Big(\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}-v\Big)\frac{\partial^2\lambda(x_1,v)}{\partial v^2}du$$
$$+D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\int_{x_2}^{t_2}\int_{x_1}^{t_1}\frac{\partial^2\lambda(u,x_2)}{\partial u\partial v}dudv;x_1,x_2\Big)$$
$$-\int_{x_1}^{\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}}\int_{x_2}^{\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}}\frac{\partial^2\lambda(t_1,t_2)}{\partial u\partial v}dudv.$$

Hence Cauchy-Schwarz theorem and by Lemma 1.2 and, we have

$$|D^{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}(\lambda;x_1,x_2)-\lambda(x_1,x_2)|$$

$$\leq D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\Big|\int_{x_1}^{t_1}|(t_1-u)|\Big|\frac{\partial^2\lambda(u,x_2)}{\partial u^2}\Big|du|;x_1,x_2\Big)$$
$$+\Big|\int_{x_1}^{\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}}\Big|$$
$$\Big|\Big(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}-u\Big)\Big|\frac{\partial^2\lambda(u,x_2)}{\partial u^2}\Big|du\Big|$$
$$+D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\Big|\int_{x_2}^{t_2}\Big|(t_2-v)\Big|\Big|\frac{\partial^2\lambda(x_1,v)}{\partial v^2}\Big|dv\Big|;x_1,x_2\Big)$$
$$+\Big|\int_{x_2}^{\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}}\Big|\Big(\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}$$
$$-v\Big)\Big|\Big|\frac{\partial^2\lambda(x_1,v)}{\partial v^2}\Big|dv\Big|+D^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}_{r_1,r_2,\tau,\rho_1,\rho_2}\Big(\Big|\int_{x_2}^{t_2}\int_{x_1}^{t_1}\Big|\frac{\partial^2\lambda(u,x_2)}{\partial u\partial v}\Big|dudv|;x_1,x_2\Big)$$
$$+\Big|\int_{x_1}^{\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}}\int_{x_2}^{\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}}\Big|\frac{\partial^2\lambda(t_1,t_2)}{\partial u\partial v}|dudv|$$

$$\leq \left\{ {}_{x_1}D_{r_1,\tau,\rho_1}^{\left[\frac{1}{r_1}\right]}((t_1-x_1)^2;x_1,x_2) + \left(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)} - x_1\right)^2 \right\}$$

$$\|\lambda\|_{C^2(I^2)} + \left\{ {}_{x_2}D_{r_2,\tau,\rho_2}^{\left[\frac{1}{r_2}\right]}((t_2-x_2)^2;x_1,x_2) + \left(\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)} - x_2\right)^2 \right\}\|\lambda\|_{C^2(I^2)}$$

$$+\left\{ {}_{x_1}D_{r_1,\tau,\rho_1}^{\left[\frac{1}{r_1}\right]}(|t_1-x_1|;x_1,x_2)\,{}_{x_2}D_{r_2,\tau,\rho_2}^{\left[\frac{1}{r_2}\right]}(|t_2-x_2|;x_1,x_2)\right.$$

$$\left.+\left|\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}\right|\left|\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right|\right.$$

$$\|\tau\|_{C^2(I^2)}$$

$$= \xi_{r_1,r_2,\tau,\rho_1,\rho_2}(x_1,x_2)\|\lambda\|_{C^2(I^2)}.$$

Also, from (7) and Lemma 1.1,

$$\begin{aligned}
|D_{r_1,r_2,\tau,\rho_1,\rho_2}^{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma;x_1,x_2)| &\leq |D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma;x_1,x_2)| \\
&+\left|\gamma\left(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)},\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right)\right| \\
&+|\gamma(x_1,x_2)| \\
&\leq 3\|\gamma\|_{C(I^2)}.
\end{aligned}$$

Therefore, for $\gamma \in C(I^2)$ and $\lambda \in C^2(I^2)$, using eqn (7), we get

$$\begin{aligned}
&|D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\lambda;x_1,x_2)-\lambda(x_1,x_2)| \\
\leq\ & |D_{r_1,r_2,\tau,\rho_1,\rho_2}^{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\lambda;x_1,x_2)-\lambda(x_1,x_2)| + \left|\gamma\left(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)},\right.\right. \\
&\left.\left.\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right)-\gamma(x_1,x_2)\right| \\
\leq\ & |D_{r_1,r_2,\tau,\rho_1,\rho_2}^{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\gamma-\lambda;x_1,x_2)| + |D_{r_1,r_2,\tau,\rho_1,\rho_2}^{*\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\lambda;x_1,x_2)-\lambda(x_1,x_2)| \\
&+|\lambda(x_1,x_2)-\gamma(x_1,x_2)| + \left|\gamma\left(\frac{r_1x_1\rho_1+2\tau(x_1)x_1+(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)},\right.\right. \\
&\left.\left.\frac{r_2x_2\rho_2+2\tau(x_2)x_2+(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right)-\gamma(x_1,x_2)\right| \\
\leq\ & 4\|\gamma-\lambda\|_{C(I^2)} + \xi_{r_1,r_2,\tau,\rho_1,\rho_2}\|\lambda\|_{C^2(I^2)} + \bar{\omega}\left(\gamma;\left|\frac{(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}\right|,\left|\frac{(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right|\right).
\end{aligned}$$

On the right-hand side we take infimum for each $\lambda \in C^2(I^2)$ and by equation (7), we have

$$\begin{aligned}
|D_{r_1,r_2,\tau,\rho_1,\rho_2}^{\left[\frac{1}{r_1},\frac{1}{r_2}\right]}(\lambda;x_1,x_2)-\lambda(x_1,x_2)| &\leq 4\kappa\left(\gamma;\frac{\xi_{r_1,r_2,\tau,\rho_1,\rho_2}}{4}\right)+\bar{\omega}\left(\gamma;\frac{(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)},\frac{(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\right) \\
&\leq M\left\{\bar{\omega}_2\left(\gamma;\frac{\sqrt{\xi_{r_1,r_2,\tau,\rho_1,\rho_2}}}{2}\right)+\min\{1,\frac{\xi_{r_1,r_2,\tau,\rho_1,\rho_2}}{4}\}\|\gamma\|_{C(I^2)}\right\}
\end{aligned}$$

$$+\bar{\omega}\Big(\gamma; \frac{(1-2x_1)\tau(x_1)}{r_1\rho_1+2\tau(x_1)}, \frac{(1-2x_2)\tau(x_2)}{r_2\rho_2+2\tau(x_2)}\Big).$$

This completes the proof. □

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