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Cohomology and deformations of twisted $\mathcal{O}$-operators on 3-Lie algebras

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Abstract. The purpose of this paper is to introduce and study twisted $\mathcal{O}$-operators on 3-Lie algebras. We construct an L_∞-algebra whose Maurer-Cartan elements are twisted $\mathcal{O}$-operators and define a cohomology of a twisted $\mathcal{O}$-operator T as the Chevalley-Eilenberg cohomology of a certain 3-Lie algebra induced by T with coefficients in a suitable representation. Then we consider infinitesimal and formal deformations of twisted $\mathcal{O}$-operators.

1. Introduction

A natural generalization of binary operations appeared first when Cayley studied cubic matrices which are ternary operations. Furthermore one may consider in general n-ary operations of associative type or Lie type. In particular, 3-Lie algebras and more generally, n-Lie algebras [21] are generalizations of Lie algebras to ternary and n-ary cases.

The first instances of ternary Lie algebras are related to Nambu Mechanics [32], which was formulated algebraically by Takhtajan in [39]. The first complete algebraic study of n-Lie algebras is due to Filippov, see [21]. We refer to [7, 9, 12] for the realizations and classifications of 3-Lie algebras and n-Lie algebras. Ternary operations turn to be useful in many mathematics and physics domains, like string theory. The quantization of the Nambu brackets in [8] was a motivation to present a general construction of $(n+1)$-Lie algebras induced by n-Lie algebras using the n-ary brackets and trace-like linear forms, see also [5–7, 26]. The structure of 3-Lie (super)algebras induced by Lie (super)algebras, classification of 3-Lie algebras and application to constructions of B.R.S. algebras have been considered in [1–4].

A deformation theory based on one-parameter formal power series was introduced first by Gerstenhaber in [22] for associative algebras and then extended to Lie algebras by Nijenhuis and Richardson in [33]. It is shown that deformations are controlled by suitable cohomologies, Hochschild cohomology in associative case and Chevalley-Eilenberg cohomology in Lie case. The same approach was used for various algebraic structures. Deformation theory of 3-Lie algebras was studied and investigated in [20].

In[11], the authors studied the solutions of 3-Lie classical Yang-Baxter equation, that lead to introduce the notion of $\mathcal{O}$-operator on 3-Lie algebras with respect to a representation. In particular, Rota-Baxter

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operators on 3-Lie algebras, introduced in [10], are $\mathcal{O}$-operators on a 3-Lie algebra with respect to the adjoint representation.

Twisted Rota-Baxter operators introduced by Uchino in the context of associative algebras [42, 43] are algebraic analogue of twisted Poisson structure introduced and studied in [27, 36]. They are also related to NS-algebras considered by Leroux in [31], see also [24]. Twisted Rota-Baxter operators on Lie algebras and Leibniz algebras were studied in [17, 19]. A cohomology of twisted Rota-Baxter operators was derived, in [17, 18], from a suitable L_∞-algebra whose Maurer-Cartan elements are given by twisted Rota-Baxter operators. Such a cohomology can be seen as the Hochschild (resp. Chevalley-Eilenberg) cohomology of a certain Lie algebra with coefficients in a suitable representation. A cohomology of a twisted relative Rota-Baxter operator as a Loday-Pirashvili cohomology of a certain Leibniz algebra was constructed in [19]. One may see [13, 30] for Hom-type version of Nijenhuis Bracket and cohomologies of relative Rota-Baxter Lie algebras.

The main purpose of this paper is to study twisted $\mathcal{O}$-operators on 3-Lie algebras. We provide some characterizations and key constructions. We construct an L_∞-algebra whose Maurer-Cartan elements are twisted $\mathcal{O}$-operators and define a cohomology of a twisted $\mathcal{O}$-operator T as the Chevalley-Eilenberg cohomology of a certain 3-Lie algebra induced by T with coefficients in a suitable representation. Furthermore, we define a cohomology of twisted $\mathcal{O}$-operators that controls their deformations.

The paper is organized as follows. In Section 2, we briefly recall basics about representations and cohomology of 3-Lie algebras. Then in Section 3, we introduce Θ-twisted $\mathcal{O}$-operators on 3-Lie algebras, provide some examples and characterization results. Section 4 is devoted to constructing an L_∞-algebra whose Maurer-Cartan elements are twisted $\mathcal{O}$-operators and defining the cohomology of a twisted $\mathcal{O}$-operator on a 3-Lie algebra with coefficients in a suitable representation. It is also compared to a cohomology of twisted $\mathcal{O}$-operators as Chevalley-Eilenberg cohomology. In Section 5, we study deformations of twisted $\mathcal{O}$-operators and show that they are controlled by the cohomology theory established in Section 4.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$ and finite-dimensional.

2. Preliminaries

In this section, we recall some basic results about 3-Lie algebras and their representations. Our main references are [14, 21, 25, 39]. A 3-Lie algebra $\mathfrak{g}$ is a vector space together with a skew-symmetric 3-linear map $[\cdot,\cdot,\cdot]_\mathfrak{g} : \wedge^3\mathfrak{g} \to \mathfrak{g}$, such that for all $x_i \in \mathfrak{g}, 1 \le i \le 5$, the following Filippov-Jacobi Identity (sometimes called fundamental identity or Nambu identity) holds

$$[x_1, x_2, [x_3, x_4, x_5]_\mathfrak{g}]_\mathfrak{g} = [[x_1, x_2, x_3]_\mathfrak{g}, x_4, x_5]_\mathfrak{g} + [x_3, [x_1, x_2, x_4]_\mathfrak{g}, x_5]_\mathfrak{g} + [x_3, x_4, [x_1, x_2, x_5]_\mathfrak{g}]_\mathfrak{g}. \tag{1}$$

For $x_1, x_2 \in \mathfrak{g}$, define $ad_{x_1,x_2} \in \mathfrak{gl}(\mathfrak{g})$ by

$$ad_{x_1,x_2}x = [x_1, x_2, x]_\mathfrak{g}, \quad \forall x \in \mathfrak{g}. \tag{2}$$

Then Filippov-Jacobi identity may be expressed as ad_{x_1,x_2} is a derivation, i.e.

$$ad_{x_1,x_2}[x_3, x_4, x_5]_\mathfrak{g} = [ad_{x_1,x_2}x_3, x_4, x_5]_\mathfrak{g} + [x_3, ad_{x_1,x_2}x_4, x_5]_\mathfrak{g} + [x_3, x_4, ad_{x_1,x_2}x_5]_\mathfrak{g}.$$

Definition 2.1. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ be a 3-Lie algebra, V be a vector space and $\rho : \wedge^2\mathfrak{g} \to gl(V)$ be a linear map. The pair (V, ρ) is called a representation of $\mathfrak{g}$ (or V is a $\mathfrak{g}$-module) if ρ satisfies, for all $x_1, x_2, x_3, x_4 \in \mathfrak{g}$,*

$$\rho(x_1, x_2)\rho(x_3, x_4) = \rho([x_1, x_2, x_3]_\mathfrak{g}, x_4) + \rho(x_3, [x_1, x_2, x_4]_\mathfrak{g}) + \rho(x_3, x_4)\rho(x_1, x_2), \tag{3}$$

$$\rho([x_1, x_2, x_3]_\mathfrak{g}, x_4) = \rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_1)\rho(x_2, x_4). \tag{4}$$

Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ be a 3-Lie algebra. The linear map $ad : \wedge^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defines a representation of $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ on itself, which is called the adjoint representation.

Let (V,ρ) be a representation of a 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$. Denote by

$$\mathfrak{C}^n_{3Lie}(\mathfrak{g};V)=Hom(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n-1}\wedge\mathfrak{g},V),\ (n\geq 1),$$

which is the space of n-cochains. Consider the differential $\partial:\mathfrak{C}^n_{3Lie}(\mathfrak{g};V)\to\mathfrak{C}^{n+1}_{3Lie}(\mathfrak{g};V)$ defined by

$$\begin{aligned}
&(\partial f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1})\\
&=\sum_{1\leq j<k\leq n}(-1)^j f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\cdots,\mathfrak{X}_{k-1},[x_j,y_j,x_k]_{\mathfrak{g}}\wedge y_k+x_k\wedge[x_j,y_j,y_k]_{\mathfrak{g}},\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_n,x_{n+1})\\
&\quad+\sum_{j=1}^{n}(-1)^j f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\cdots,\mathfrak{X}_n,[x_j,y_j,x_{n+1}]_{\mathfrak{g}})+\sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\cdots,\mathfrak{X}_n,x_{n+1})\\
&\quad+(-1)^{n+1}\Big(\rho(y_n,x_{n+1})f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(x_{n+1},x_n)f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\Big),
\end{aligned}\tag{5}$$

for all $\mathfrak{X}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g},\ i=1,2,\cdots,n$ and $x_{n+1}\in\mathfrak{g}$. It was proved in [14, 39] that $\partial\circ\partial=0$. Thus $(\oplus_{n=1}^{+\infty}\mathfrak{C}^n_{3Lie}(\mathfrak{g};V),\partial)$ is a cochain complex which is called Chevalley-Eilenberg cochain complex of 3-Lie algebras. The quotient space $H^n_{3Lie}(\mathfrak{g};V)=Z^n_{3Lie}(\mathfrak{g};V)/B^n_{3Lie}(\mathfrak{g};V)$, where $Z^n_{3Lie}(\mathfrak{g};V)=\{f\in\mathfrak{C}^n_{3Lie}(\mathfrak{g};V)|\ \partial f=0\}$ is the space of n-cocycles and $B^n_{3Lie}(\mathfrak{g};V)=\{f=\partial g|\ g\in\mathfrak{C}^{n-1}_{3Lie}(\mathfrak{g};V)\}$ is the space of n-coboundaries, is called the n^{th} cohomology group of the 3-Lie algebra $\mathfrak{g}$ with coefficients in V.

Let $\Theta\in\mathfrak{C}^2_{3Lie}(\mathfrak{g};V)$ be a 2-cocycle in the Chevalley-Eilenberg cochain complex, that is $\Theta:\wedge^3\mathfrak{g}\to V$ is a trilinear map satisfying, for all $x_1,x_2,y_1,y_2,y_3\in\mathfrak{g}$,

$$\begin{aligned}
&\Theta(x_1,x_2,[y_1,y_2,y_3]_{\mathfrak{g}})+\rho(x_1,x_2)\Theta(y_1,y_2,y_3)-\Theta([x_1,x_2,y_1]_{\mathfrak{g}},y_2,y_3)-\Theta(y_1,[x_1,x_2,y_2]_{\mathfrak{g}},y_3)\\
&-\Theta(y_1,y_2,[x_1,x_2,y_3]_{\mathfrak{g}})-\rho(y_2,y_3)\Theta(x_1,x_2,y_1)-\rho(y_3,y_1)\Theta(x_1,x_2,y_2)-\rho(y_1,y_2)\Theta(x_1,x_2,y_3)=0.
\end{aligned}\tag{6}$$

The direct sum $\mathfrak{g}\oplus V$ carries a 3-Lie algebra structure given by

$$[(x,u),(y,v),(z,w)]_{\Theta}=\Big([x,y,z]_{\mathfrak{g}},\rho(x,y)w+\rho(z,x)v+\rho(y,z)u+\Theta(x,y,z)\Big),\tag{7}$$

which is called the Θ-twisted semi-direct product and denoted by $\mathfrak{g}\ltimes_{\Theta}V$.

3. Twisted $\mathcal{O}$-operators on 3-Lie algebras

In this section, we introduce twisted $\mathcal{O}$-operators on 3-Lie algebras. We give some constructions and provide examples. Let $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$ be a 3-Lie algebra, (V,ρ) be a representation and $\Theta\in\mathfrak{C}^2_{3Lie}(\mathfrak{g};V)$ be a 2-cocycle in the Chevalley-Eilenberg cochain complex.

Definition 3.1. *A linear map $T:V\to\mathfrak{g}$ is said to be a Θ-twisted $\mathcal{O}$-operator if T satisfies*

$$[Tu,Tv,Tw]_{\mathfrak{g}}=T\Big(\rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v+\Theta(Tu,Tv,Tw)\Big),\tag{8}$$

for all $u,v,w\in V$.

Using the Θ-twisted semi-direct product, one can characterize twisted $\mathcal{O}$-operators by their graphs.

Proposition 3.1. *A linear map $T:V\to\mathfrak{g}$ is a Θ-twisted $\mathcal{O}$-operator if and only if the graph $Gr(T)=\{(Tu,u)|\ u\in V\}$ is a subalgebra of the Θ-twisted semi-direct product $\mathfrak{g}\ltimes_{\Theta}V$.*

Proof. Let (Tu, u), (Tv, v) and $(Tw, w) \in Gr(T)$. Then we have

$$\begin{aligned}&[(Tu,u),(Tv,v),(Tw,w)]_\Theta\\ =&\Big([Tu,Tv,Tw]_\mathfrak{g}, \rho(Tu,Tv)w+\rho(Tw,Tu)v+\rho(Tv,Tw)u+\Theta(Tu,Tv,Tw)\Big).\end{aligned}$$

Assume that $Gr(T)$ is a subalgebra of the Θ-twisted semi-direct product $\mathfrak{g}\ltimes_\Theta V$, then we have

$$[Tu,Tv,Tw]_\mathfrak{g} = T\Big(\rho(Tu,Tv)w+\rho(Tw,Tu)v+\rho(Tv,Tw)u+\Theta(Tu,Tv,Tw)\Big).$$

On the other hand, if T is a Θ-twisted O-operator, then we obtain

$$\begin{aligned}&[(Tu,u),(Tv,v),(Tw,w)]_\Theta\\ =&\Big(T\Big(\rho(Tu,Tv)w+\rho(Tw,Tu)v+\rho(Tv,Tw)u+\Theta(Tu,Tv,Tw)\Big),\\ &\rho(Tu,Tv)w+\rho(Tw,Tu)v+\rho(Tv,Tw)u+\Theta(Tu,Tv,Tw)\Big)\in Gr(T).\end{aligned}$$

Hence, $Gr(T)$ is a subalgebra of the Θ-twisted semi-direct product $\mathfrak{g}\ltimes_\Theta V$. □

The set $Gr(T)$ is isomorphic to V as a vector space by the identification $(Tu, u) \cong u$. A Θ-twisted O-operator T induces a 3-Lie algebra structure on V, where the bracket is given by

$$[u,v,w]_T = \rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v+\Theta(Tu,Tv,Tw). \tag{9}$$

It is obvious that T is a 3-Lie algebra morphism, that is $T([u, v, w]_T) = [Tu, Tv, Tw]_\mathfrak{g}$.

Definition 3.2. *Let $T : V \to \mathfrak{g}$ and $T' : V' \to \mathfrak{g}'$ be Θ-twisted and Θ'-twisted O-operators. A morphism of twisted O-operators from T to T' consists of a pair (ϕ, ψ) of a 3-Lie algebra morphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ and a linear map $\psi : V \to V'$ satisfying*

$$\psi(\rho(x,y)u) = \rho'(\phi(x),\phi(y))\psi(u),\ \forall x,y\in g,\ u\in V, \tag{10}$$

$$\psi\circ\Theta = \Theta'\circ(\phi\otimes\phi\otimes\phi), \tag{11}$$

$$\phi\circ T = T'\circ\psi. \tag{12}$$

Example 3.2. *Any O-operator (in particular, Rota-Baxter operator of weight 0) on a 3-Lie algebra is a Θ-twisted O-operator with $\Theta = 0$.*

Example 3.3. *Let $\mathfrak{g}$ be a 3-Lie algebra and V be a $\mathfrak{g}$-module. Suppose $\theta : \mathfrak{g} \to V$ is an invertible 1-cochain in the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with coefficients in V. Then $T = \theta^{-1} : V \to \mathfrak{g}$ is a Θ-twisted O-operator with $\Theta = -\partial\theta$. The proof follows from the fact that*

$$\begin{aligned}\Theta(Tu,Tv,Tw) &= -(\partial\theta)(Tu,Tv,Tw)\\ &= -\rho(Tu,Tv)\theta(Tw)-\rho(Tv,Tw)\theta(Tu)-\rho(Tw,Tu)\theta(Tv)+\theta([Tu,Tv,Tw]_\mathfrak{g}).\end{aligned} \tag{13}$$

By applying T to both sides of (13)*, we get the identity* (8)*.*

Example 3.4. *Let $N : \mathfrak{g} \to \mathfrak{g}$ be a Nijenhuis operator on a 3-Lie algebra $\mathfrak{g}$, i.e. N satisfies the identity*

$$\begin{aligned}[Nx,Ny,Nz]_\mathfrak{g} = N\Big(&[Nx,Ny,z]_\mathfrak{g}+[Nx,y,Nz]_\mathfrak{g}+[x,Ny,Nz]_\mathfrak{g}\\ &-N([Nx,y,z]_\mathfrak{g}+[x,Ny,z]_\mathfrak{g}+[x,y,Nz]_\mathfrak{g})+N^2[x,y,z]_\mathfrak{g}\Big),\quad \forall x,y,z\in\mathfrak{g}.\end{aligned} \tag{14}$$

In this case, $\mathfrak{g}$ carries a new 3-Lie algebra structure given by the following bracket

$$[x,y,z]_N = [Nx,Ny,z]_\mathfrak{g}+[Nx,y,Nz]_\mathfrak{g}+[x,Ny,Nz]_\mathfrak{g}-N([Nx,y,z]_\mathfrak{g}$$

$$+ [x, Ny, z]_{\mathfrak{g}} + [x, y, Nz]_{\mathfrak{g}} - N[x, y, z]_{\mathfrak{g}}).$$

We denote this 3-Lie algebra by $\mathfrak{g}_N$. Moreover, the 3-Lie algebra $\mathfrak{g}_N$ has a representation on $\mathfrak{g}$ given by $\rho(x, y)z = [Nx, Ny, z]_{\mathfrak{g}}$, for all $x, y, z \in \mathfrak{g}$. With this representation, the map $\Theta : \wedge^3 \mathfrak{g}_N \to \mathfrak{g}$ defined by

$$\Theta(x, y, z) = -N([Nx, y, z]_{\mathfrak{g}} + [x, Ny, z]_{\mathfrak{g}} + [x, y, Nz]_{\mathfrak{g}} - N[x, y, z]_{\mathfrak{g}})$$

is a 2-cocycle in the Chevalley-Eilenberg cohomology of $\mathfrak{g}_N$ with coefficients in $\mathfrak{g}$. Then it is easy to observe that the identity map $id : \mathfrak{g} \to \mathfrak{g}_N$ is a Θ-twisted $\mathcal{O}$-operator.

Given a Θ-twisted $\mathcal{O}$-operator T and a 1-cochain θ, we construct a $(\Theta+\partial\theta)$-twisted $\mathcal{O}$-operator under certain condition. First we have the following observation.

Proposition 3.5. *Let $\mathfrak{g}$ be a 3-Lie algebra and V be a $\mathfrak{g}$-module. For any 2-cocycle $\Theta \in \mathfrak{C}^2_{3Lie}(\mathfrak{g}; V)$ and 1-cochain $\theta \in \mathfrak{C}^1_{3Lie}(\mathfrak{g}; V)$, we have a 3-Lie algebra isomorphism*

$$\mathfrak{g} \ltimes_{\Theta} V \cong \mathfrak{g} \ltimes_{\Theta+\partial\theta} V.$$

Proof. Define $\psi_\theta : \mathfrak{g} \ltimes_{\Theta} V \to \mathfrak{g} \ltimes_{\Theta+\partial\theta} V$ by $\psi_\theta(x, u) = (x, u - \theta(x))$, for all $(x, u) \in \mathfrak{g} \oplus V$. Then we have,

$$\begin{aligned}
&\psi_\theta([(x, u), (y, v), (z, w)]_{\Theta}) \\
&= \Big([x, y, z]_{\mathfrak{g}}, \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z) - \theta([x, y, z]_{\mathfrak{g}})\Big) \\
&= \Big([x, y, z]_{\mathfrak{g}}, \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z) \\
&- \rho(x, y)\theta(z) - \rho(z, x)\theta(y) - \rho(y, z)\theta(x) + (\partial\theta)(x, y, z)\Big) \\
&= [(x, u - \theta(x)), (y, v - \theta(y)), (z, w - \theta(z))]_{\Theta+\partial\theta},
\end{aligned}$$

for all $(x, u), (y, v), (z, w) \in \mathfrak{g} \oplus V$. This proves the result. □

Proposition 3.6. *Let $T : V \to \mathfrak{g}$ be a Θ-twisted $\mathcal{O}$-operator. For any 1-cochain $\theta \in \mathfrak{C}^1_{3Lie}(\mathfrak{g}; V)$, if the linear map $(Id_V - \theta \circ T) : V \to V$ is invertible, then the linear map $T \circ (Id_V - \theta \circ T)^{-1} : V \to \mathfrak{g}$ is a $(\Theta + \partial\theta)$-twisted $\mathcal{O}$-operator.*

Proof. Consider the subalgebra $Gr(T) \subset \mathfrak{g} \ltimes_{\Theta} V$ of the Θ-twisted semi-direct product. Thus by Proposition 3.5, we get that

$$\psi_\theta(Gr(T)) = \{(Tu, u - (\theta \circ T)(u)|\ u \in V\} \subset \mathfrak{g} \ltimes_{\Theta+\partial\theta} V$$

is a subalgebra. Since the map $(Id_V - \theta \circ T) : V \to V$ is invertible, we have $\psi_\theta(Gr(T))$ is the graph of the linear map $T \circ (Id_V - \theta \circ T)^{-1}$. In this case, it follows from Proposition 3.1 that $T \circ (Id_V - \theta \circ T)^{-1}$ is a $(\Theta + \partial\theta)$-twisted $\mathcal{O}$-operator. □

Next, we give a construction of a new Θ-twisted $\mathcal{O}$-operator out of an old one and a suitable 1-cocycle. Let $T : V \to \mathfrak{g}$ be a Θ-twisted $\mathcal{O}$-operator. Suppose $\theta \in \mathfrak{C}^1_{3Lie}(\mathfrak{g}; V)$ is a 1-cocycle in the Chevalley-Eilenberg cochain complex of $\mathfrak{g}$ with coefficients in V. Then θ is said to be T-admissible if the linear map $(Id_V + \theta \circ T) : V \to V$ is invertible.

Proposition 3.7. *Let $\theta \in \mathfrak{C}^1_{3Lie}(\mathfrak{g}; V)$ be a T-admissible 1-cocycle. Then $T \circ (Id_V + \theta \circ T)^{-1} : V \to \mathfrak{g}$ is a Θ-twisted $\mathcal{O}$-operator.*

Proof. Consider the deformed subspace

$$\tau_\theta(Gr(T)) = \{(Tu, u + (\theta \circ T)(u))|\ u \in V\} \subset \mathfrak{g} \ltimes_{\Theta} V.$$

Since θ is a 1-cocycle, $\tau_\theta(Gr(T)) \subset \mathfrak{g} \ltimes_{\Theta} V$ turns out to be a subalgebra. Furthermore, since the map $(Id_V + \theta \circ T)$ is invertible, then $\tau_\theta(Gr(T))$ is the graph of the map $T \circ (Id_V + \theta \circ T)^{-1}$. Hence the result follows from Proposition 3.1. □

The Θ-twisted $\mathcal{O}$-operator in the above proposition is called the gauge transformation of T associated with θ. We denote this Θ-twisted $\mathcal{O}$-operator simply by T_θ.

Proposition 3.8. *Let T be a Θ-twisted $\mathcal{O}$-operator and θ be a T-admissible 1-cocycle. Then the 3-Lie algebra structures on V induced from the Θ-twisted $\mathcal{O}$-operators T and T_θ are isomorphic.*

Proof. Consider the linear isomorphism $(Id_V + \theta\circ T) : V \to V$. For any $u, v, w \in V$, we have

$$\begin{aligned}
&[(Id_V + \theta\circ T)(u), (Id_V + \theta\circ T)(v), (Id_V + \theta\circ T)(w)]_{T_B}\\
&= \rho(Tv, Tw)(Id_V + \theta\circ T)(u) + \rho(Tw, Tu)(Id_V + \theta\circ T)(v)\\
&+ \rho(Tu, Tv)(Id_V + \theta\circ T) + \Theta(Tu, Tv, Tw)\\
&= \rho(Tv, Tw)u + \rho(Tw, Tu)v + \rho(Tu, Tv)w + \Theta(Tu, Tv, Tw)\\
&+ \rho(Tv, Tw)(\theta\circ T)(u) + \rho(Tw, Tu)(\theta\circ T)(v) + \rho(Tu, Tv)(\theta\circ T)(w)\\
&= [u, v, w]_T + \theta([Tu, Tv, Tw]_{\mathfrak{g}})\\
&= [u, v, w]_T + \theta\circ T([u, v, w]_T) = (Id_V + \theta\circ T)([u, v, w]_T).
\end{aligned}$$

This shows that $(Id_V + \theta\circ T) : (V, [\cdot,\cdot,\cdot]_T) \to (V, [\cdot,\cdot,\cdot]_{T_\theta})$ is a 3-Lie algebra isomorphism. □

4. Cohomology of twisted $\mathcal{O}$-operators

In this section, we construct an L_∞-algebra whose Maurer-Cartan elements are Θ-twisted $\mathcal{O}$-operators on 3-Lie algebras. Such characterization of Θ-twisted $\mathcal{O}$-operator T allows us to introduce a cohomology of T. Next, we show that the cohomology of T is equivalently described by the Chevalley-Eilenberg cohomology of V with coefficients in a suitable representation on $\mathfrak{g}$.

4.1. Maurer-Cartan characterization and cohomology

Let $\mathfrak{g}$ be a vector space. Consider the graded vector space

$$C^*(\mathfrak{g},\mathfrak{g}) = \oplus_{n\geq 0} C^n(\mathfrak{g},\mathfrak{g}) = \oplus_{n\geq 0} Hom(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n}\wedge\mathfrak{g},\mathfrak{g}).$$

The degree of elements in $C^n(\mathfrak{g},\mathfrak{g})$ is defined to be n. Then the graded vector space $C^*(\mathfrak{g},\mathfrak{g})$ equipped with the graded commutator bracket

$$[P, Q]_{3Lie} = P\circ Q - (-1)^{pq} Q\circ P, \quad \forall\, P \in C^p(\mathfrak{g},\mathfrak{g}), Q \in C^q(\mathfrak{g},\mathfrak{g}), \tag{15}$$

is a graded Lie algebra, with $P\circ Q \in C^{p+q}(\mathfrak{g},\mathfrak{g})$ defined by

$$\begin{aligned}
&(P\circ Q)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p+q},x)\\
&=\sum_{k=1}^{p}(-1)^{(k-1)q}\sum_{\sigma\in\mathcal{S}(k-1,q)}(-1)^\sigma P(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(k-1)},Q(\mathfrak{X}_{\sigma(k)},\cdots,\mathfrak{X}_{\sigma(k+q-1)},x_{k+q})\wedge y_{k+q},\mathfrak{X}_{k+q+1},\cdots,\mathfrak{X}_{p+q},x)\\
&+\sum_{k=1}^{p}(-1)^{(k-1)q}\sum_{\sigma\in\mathcal{S}(k-1,q)}(-1)^\sigma P(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(k-1)},x_{k+q}\wedge Q(\mathfrak{X}_{\sigma(k)},\cdots,\mathfrak{X}_{\sigma(k+q-1)},y_{k+q}),\mathfrak{X}_{k+q+1},\cdots,\mathfrak{X}_{p+q},x)\\
&+\sum_{\sigma\in\mathcal{S}(p,q)}(-1)^{pq}(-1)^\sigma P(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(p)},Q(\mathfrak{X}_{\sigma(p+1)},\cdots,\mathfrak{X}_{\sigma(p+q-1)},\mathfrak{X}_{\sigma(p+q)},x)),
\end{aligned}$$

where $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2\mathfrak{g}$, $i = 1, 2, \cdots, p+q$ and $x \in \mathfrak{g}$. See [35] for more details.

We recall from [35] the following result.

Proposition 4.1. *Let $\mathfrak{g}$ be a vector space. Then $\pi \in C^1(\mathfrak{g},\mathfrak{g}) = Hom(\wedge^3\mathfrak{g},\mathfrak{g})$ defines a 3-Lie algebra structure on $\mathfrak{g}$ if and only if π is a Maurer-Cartan element of the graded Lie algebra $(C^*(\mathfrak{g},\mathfrak{g}), [\cdot,\cdot]_{3Lie})$, i.e. it satisfies the Maurer-Cartan equation $[\pi,\pi]_{3Lie} = 0$. Moreover, $(C^*(\mathfrak{g},\mathfrak{g}), [\cdot,\cdot]_{3Lie}, \mathrm{d}_\pi)$ is a differential graded Lie algebra, where d_π is defined by*

$$\mathrm{d}_\pi := [\pi,\cdot]_{3Lie}. \tag{16}$$

The notion of an L_∞-algebra was introduced by Schlessinger and Stasheff in [37, 38]. See [28, 29] for more details.

Definition 4.1. *An L_∞-algebra is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k$ equipped with a collection of linear maps $l_k : \otimes^k\mathfrak{g} \to \mathfrak{g}$ of degree 1 $(k \geq 1)$ with the property that, for any homogeneous elements $x_1, \cdots, x_n \in \mathfrak{g}$, we have*

(i) (graded symmetry) *for every $\sigma \in \mathbb{S}_n$,*

$$l_n(x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \cdots, x_{n-1}, x_n),$$

(ii) (generalized Jacobi identity) *for all $n \geq 1$,*

$$\sum_{i=1}^{n}\sum_{\sigma\in\mathbb{S}_{(i,n-i)}}\varepsilon(\sigma)l_{n-i+1}(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$

Definition 4.2. *A Maurer-Cartan element of an L_∞-algebra $(\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k, \{l_i\}_{i=1}^{+\infty})$ is an element $\alpha \in \mathfrak{g}^0$ satisfying the Maurer-Cartan equation*

$$\sum_{n=1}^{+\infty}\frac{1}{n!}l_n(\alpha, \cdots, \alpha) = 0. \tag{17}$$

Let α be a Maurer-Cartan element of an L_∞-algebra $(\mathfrak{g}, \{l_i\}_{i=1}^{+\infty})$. For all $k \geq 1$ and $x_1, \cdots, x_n \in \mathfrak{g}$, define a series of linear maps $l_k^\alpha : \otimes^k\mathfrak{g} \to \mathfrak{g}$ of degree 1 by

$$l_k^\alpha(x_1, \cdots, x_k) = \sum_{n=0}^{+\infty}\frac{1}{n!}l_{n+k}\{\underbrace{\alpha, \cdots, \alpha}_{n}, x_1, \cdots, x_k\}. \tag{18}$$

Theorem 4.2. *[23] With the above notations, $(\mathfrak{g}, \{l_i^\alpha\}_{i=1}^{+\infty})$ is an L_∞-algebra, obtained from the L_∞-algebra $(\mathfrak{g}, \{l_i\}_{i=1}^{+\infty})$ by twisting with the Maurer-Cartan element α. Moreover, $\alpha + \alpha'$ is a Maurer-Cartan element of $(\mathfrak{g}, \{l_i\}_{i=1}^{+\infty})$ if and only if α' is a Maurer-Cartan element of the twisted L_∞-algebra $(\mathfrak{g}, \{l_i^\alpha\}_{i=1}^{+\infty})$.*

Let (V, ρ) be a representation of a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ and let Θ be a 2-cocycle in the cohomology of $\mathfrak{g}$ with coefficients in V. For convenience, we use $\pi : \bigwedge^3\mathfrak{g} \to \mathfrak{g}$ to indicate the 3-Lie bracket on $\mathfrak{g}$. Then $\pi + \rho + \Theta$ corresponds to the semi-direct product 3-Lie algebra structure on $\mathfrak{g} \oplus V$ given by

$$[x + u, y + v, z + w]_\Theta = [x, y, z]_\mathfrak{g} + \rho(x, y)w + \rho(z, x)v + \rho(y, z)u + \Theta(x, y, z). \tag{19}$$

Therefore, we have

$$[\pi + \rho + \Theta, \pi + \rho + \Theta]_{3Lie} = 0.$$

Consider the graded vector space

$$C^*(V, \mathfrak{g}) = \oplus_{n\geq 0}C^n(V, \mathfrak{g}) = \oplus_{n\geq 0}Hom(\underbrace{\wedge^2 V \otimes \cdots \otimes \wedge^2 V}_{n\geq 0} \wedge V, \mathfrak{g}).$$

Define

$$l_3 : C^m(V, \mathfrak{g}) \times C^n(V, \mathfrak{g}) \times C^p(V, \mathfrak{g}) \to C^{m+n+p+1}(V, \mathfrak{g}),$$
$$l_4 : C^m(V, \mathfrak{g}) \times C^n(V, \mathfrak{g}) \times C^p(V, \mathfrak{g}) \times C^q(V, \mathfrak{g}) \to C^{m+n+p+q+1}(V, \mathfrak{g})$$

by

$$l_3(P, Q, R) = [[[\pi + \rho, P]_{3Lie}, Q]_{3Lie}, R]_{3Lie},$$

$$l_4(P,Q,R,S) = [[[[\Theta,P]_{3Lie},Q]_{3Lie},R]_{3Lie},S]_{3Lie}.$$

One method for constructing explicit L_∞-algebras is given by Voronov's higher derived brackets [44]. Moreover, using the above method, the ternary bracket l_3 and the 4-ary bracket l_4 are compatible in the sense of L_∞-algebra. This follows since Θ is a 2-cocycle. In summary, we obtain the following result.

Theorem 4.3. *Let (V,ρ) be a representation of a 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_\mathfrak{g})$ and Θ be a 2-cocycle in the cohomology of $\mathfrak{g}$ with coefficients in V. Then the graded vector space $C^*(V,\mathfrak{g})$ is an L_∞-algebra with*

$$l_1 = l_2 = 0, \quad l_3(\cdot,\cdot,\cdot), \quad l_4(\cdot,\cdot,\cdot,\cdot), \tag{20}$$

and higher brackets are trivial. A linear map $T: V \to \mathfrak{g}$ is a Θ-twisted O-operator if and only if T is a solution of the Maurer-Cartan equation of the L_∞-algebra $(C^*(V,\mathfrak{g}), l_3, l_4)$, i.e.

$$\frac{1}{3!}l_3(T,T,T) + \frac{1}{4!}l_4(T,T,T,T) = 0.$$

Proof. Using the above discussion, the first part follows. For the second part, we have that for any $T \in Hom(V,\mathfrak{g})$,

$$l_4(T,T,T,T)(u,v,w) = -24T(\Theta(Tu,Tv,Tw)). \tag{21}$$

Next, as in [40, Theorem 3.4] proof we have

$$l_3(T,T,T)(u,v,w) = 6\Big([Tu,Tv,Tw]_\mathfrak{g} - T\big(\rho(Tu,Tv)w + \rho(Tv,Tw)u + \rho(Tw,Tu)v\big)\Big). \tag{22}$$

Hence from Eqs. (21) and (22), we get

$$\begin{aligned}
&\Big(\frac{1}{3!}l_3(T,T,T) + \frac{1}{4!}l_4(T,T,T,T)\Big)(u,v,w)\\
=&[Tu,Tv,Tw]_\mathfrak{g} - T\big(\rho(Tu,Tv)w + \rho(Tv,Tw)u + \rho(Tw,Tu)v\big) - T(\Theta(Tu,Tv,Tw)).
\end{aligned}$$

Thus, a linear map $T \in Hom(V,\mathfrak{g})$ is a Θ-twisted O-operator of a 3-Lie algebra $\mathfrak{g}$ with respect to a representation ρ if and only if T is a Maurer-Cartan element of the L_∞-algebra $(C^*(V,\mathfrak{g}), l_3, l_4)$. □

Proposition 4.4. *Let T be a Θ-twisted O-operator of a 3-Lie algebra $\mathfrak{g}$ with respect to a representation ρ. Then $C^*(V,\mathfrak{g})$ carries a twisted L_∞-algebra structure given by*

$$l_1^T(P) = \frac{1}{2}l_3(T,T,P) + \frac{1}{6}l_4(T,T,T,P), \tag{23}$$

$$l_2^T(P,Q) = l_3(T,P,Q) + \frac{1}{2}l_4(T,T,P,Q), \tag{24}$$

$$l_3^T(P,Q,R) = l_3(P,Q,R) + l_4(T,P,Q,R), \tag{25}$$

$$l_4^T(P,Q,R,S) = l_4(P,Q,R,S), \tag{26}$$

$$l_k^T = 0, \quad k \geq 5, \tag{27}$$

where $P \in C^p(V,\mathfrak{g}), Q \in C^q(V,\mathfrak{g}), R \in C^r(V,\mathfrak{g})$ and $\in C^s(V,\mathfrak{g})$. Moreover, for any linear map $T' : V \to \mathfrak{g}$, the sum $T + T'$ is a Θ-twisted O-operator if and only if T' is a Maurer-Cartan element in the twisted L_∞-algebra $(C^*(V,\mathfrak{g}), l_1^T, l_2^T, l_3^T, l_4^T)$, that is T' satisfies

$$l_1^T(T') + \frac{1}{2!}l_2^T(T',T') + \frac{1}{3!}l_3^T(T',T',T') + \frac{1}{4!}l_4^T(T',T',T',T') = 0.$$

Proof. For the first part, since T is a Maurer-Cartan element of the L_∞-algebra $(C^*(V,\mathfrak{g}), l_3, l_4)$, by Theorem 4.2, we have that $C^*(V,\mathfrak{g})$ carries a twisted L_∞-algebra structure. For the second part, by Theorem 4.3, $T+T'$ is a Θ-twisted O-operator if and only if

$$\frac{1}{3!}l_3(T+T',T+T',T+T')+\frac{1}{4!}l_4(T+T',T+T',T+T',T+T')=0. \tag{28}$$

Applying $\frac{1}{3!}l_3(T,T,T)+\frac{1}{4!}l_4(T,T,T,T)=0$, the above condition is equivalent to

$$\begin{aligned}&\frac{1}{3!}\Big(3l_3(T,T,T')+3l_3(T,T',T')+l_3(T',T',T')\Big)\\&+\frac{1}{4!}\Big(4l_4(T,T,T,T')+6l_4(T,T,T',T')+4l_4(T,T',T',T')+l_4(T',T',T',T')\Big)=0.\end{aligned}$$

That is, $l_1^T(T')+\frac{1}{2!}l_2^T(T',T')+\frac{1}{3!}l_3^T(T',T',T')+\frac{1}{4!}l_4^T(T',T',T',T')=0$, which implies that T' is a Maurer-Cartan element of the twisted L_∞-algebra $(C^*(V,\mathfrak{g}), l_1^T, l_2^T, l_3^T, l_4^T)$. □

The above characterization of a Θ-twisted O-operator T allows us to define a cohomology associated to T. More precisely, we define $C_T^n(V,\mathfrak{g})=Hom(\underbrace{\wedge^2V\otimes\cdots\otimes\wedge^2V}_{n\geq 0}\wedge V,\mathfrak{g})$, for $n\geq 0$ and the differential operator $d_T: C_T^n(V,\mathfrak{g})\to C_T^{n+1}(V,\mathfrak{g})$ by

$$d_T(f)=\frac{1}{2}l_3(T,T,f)+\frac{1}{6}l_4(T,T,T,f),\quad f\in C_T^n(V,\mathfrak{g}). \tag{29}$$

The corresponding cohomology groups are

$$H_T^n(V,\mathfrak{g})=\frac{Z_T^n(V,\mathfrak{g})}{B_T^n(V,\mathfrak{g})}=\frac{\{f\in C_T^n(V,\mathfrak{g})|d_T(f)=0\}}{\{d_T(g)|g\in C_T^{n-1}(V,\mathfrak{g})\}}.$$

4.2. *Cohomology of twisted O-operators as Chevalley-Eilenberg cohomology*

In this subsection, we define a cohomology of a Θ-twisted O-operator as the Chevalley-Eilenberg cohomology of the 3-Lie algebra $(V,[\cdot,\cdot,\cdot]_T)$ given by Eq. (9) with coefficients in a suitable representation on $\mathfrak{g}$. This cohomology will be used in Section 5 to study formal deformations of T.

Proposition 4.5. *Let T be a Θ-twisted O-operator on a 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_\mathfrak{g})$ with respect to a representation (V,ρ). Define $\rho_\Theta:\wedge^2V\to gl(\mathfrak{g})$ by*

$$\rho_\Theta(u,v)x=[Tu,Tv,x]_\mathfrak{g}-T\Big(\rho(Tv,x)u+\rho(x,Tu)v+\Theta(x,Tu,Tv)\Big),\ \forall u,v\in V,x\in\mathfrak{g}. \tag{30}$$

Then $(\mathfrak{g},\rho_\Theta)$ is a representation of the 3-Lie algebra $(V,[\cdot,\cdot,\cdot]_T)$ on the vector space $\mathfrak{g}$.

Proof. By a direct calculation using the definition of ρ_Θ, we get

$$\begin{aligned}&\rho_\Theta(u_1,u_2)\rho_\Theta(u_3,u_4)x-\rho_\Theta([u_1,u_2,u_3]_T,u_4)x\\&-\rho_\Theta(u_3,[u_1,u_2,u_4]_T)x-\rho_\Theta(u_3,u_4)\rho_T(u_1,u_2)x\\&=[Tu_1,Tu_2,[Tu_3,Tu_4,x]_\mathfrak{g}]_\mathfrak{g}+[Tu_1,Tu_2,T\rho(Tu_3,x)u_4]_\mathfrak{g}-[Tu_1,Tu_2,T\rho(Tu_4,x)u_3]_\mathfrak{g}\\&-[Tu_1,Tu_2,T\Theta(x,Tu_3,Tu_4)]_\mathfrak{g}+T\rho(Tu_1,[Tu_3,Tu_4,x]_\mathfrak{g})u_2+T\rho(Tu_1,T\rho(Tu_3,x)u_4)u_2\\&-T\rho(Tu_1,T\rho(Tu_4,x)u_3)u_2-T\rho(Tu_1,T\Theta(x,Tu_3,Tu_4))u_2-T\rho(Tu_2,[Tu_3,Tu_4,x]_\mathfrak{g})u_1\\&-T\rho(Tu_2,T\rho(Tu_3,x)u_4)u_1+T\rho(Tu_2,T\rho(Tu_4,x)u_3)u_1+T\rho(Tu_2,T\Theta(x,Tu_3,Tu_4))u_1\\&-T\Theta(T\rho(Tu_3,x)u_4,Tu_1,Tu_2)+T\Theta(T\rho(Tu_4,x)u_3,Tu_1,Tu_2)\\&+T\Theta(T\Theta(x,Tu_3,T_4),Tu_1,Tu_2)-T\Theta([Tu_3,Tu_4,x]_\mathfrak{g},Tu_1,Tu_2)\end{aligned}$$

$$
\begin{aligned}
&- [[Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, Tu_4, x]_{\mathfrak{g}} - T(\rho([Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, x)u_4 + T\rho(Tu_4, x)\rho(Tu_1, Tu_2)u_3\\
&+ T\rho(Tu_4, x)\rho(Tu_2, Tu_3)u_1 + T\rho(Tu_4, x)\rho(Tu_3, Tu_1)u_2 + T\rho(Tu_4, x)\Theta(Tu_1, Tu_2, Tu_3)\\
&+ T\Theta(x, [Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, Tu_4) - [Tu_3, [Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}, x]_{\mathfrak{g}} - T\rho(Tu_3, x)\rho(Tu_1, Tu_2)u_4\\
&- T\rho(Tu_3, x)\rho(Tu_2, Tu_4)u_1 - T\rho(Tu_3, x)\rho(Tu_4, Tu_1)u_2 - T\rho(Tu_3, x)\Theta(Tu_1, Tu_2, Tu_4)\\
&+ T(\rho([Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}, x)u_3 + T\Theta(x, Tu_3, [Tu_1, Tu_2, Tu_4]_{\mathfrak{g}})\\
&- [Tu_3, Tu_4, [Tu_1, Tu_2, x]_{\mathfrak{g}}]_{\mathfrak{g}} - [Tu_3, Tu_4, T\rho(Tu_1, x)u_2]_{\mathfrak{g}} + [Tu_3, Tu_4, T\rho(Tu_2, x)u_1]_{\mathfrak{g}}\\
&+ [Tu_3, Tu_4, T\Theta(x, Tu_1, Tu_2)]_{\mathfrak{g}} - T\rho(Tu_3, [Tu_1, Tu_2, x]_{\mathfrak{g}})u_4 - T\rho(Tu_3, T\rho(Tu_1, x)u_2)u_4\\
&+ T\rho(Tu_3, T\rho(Tu_2, x)u_1)u_4 + T\rho(Tu_3, T\Theta(x, Tu_1, Tu_2))u_4 + T\rho(Tu_4, [Tu_1, Tu_2, x]_{\mathfrak{g}})u_3\\
&+ T\rho(Tu_4, T\rho(Tu_1, x)u_2)u_3 - T\rho(Tu_4, T\rho(Tu_2, x)u_1)u_3 - T\rho(Tu_4, T\Theta(x, Tu_1, Tu_2))u_3\\
&+ T\Theta(T\rho(Tu_1, x)u_2, Tu_3, Tu_4) - T\Theta(T\rho(Tu_2, x)u_1, Tu_3, Tu_4)\\
&- T\Theta(T\Theta(x, Tu_1, Tu_2), Tu_3, Tu_4) + T\Theta([Tu_1, Tu_2, x]_{\mathfrak{g}}, Tu_3, Tu_4)\\
\overset{(1)+(3)+(8)}{=}& -T\Big(\Theta(Tu_1, Tu_2, [Tu_3, Tu_4, x]_{\mathfrak{g}}) + \rho(Tu_1, Tu_2)\Theta(Tu_3, Tu_4, x)\\
&- \rho(Tu_4, x)\Theta(Tu_1, Tu_2, Tu_3) - \Theta([Tu_1, Tu_2, Tu_3]_{\mathfrak{g}}, Tu_4, x)\\
&- \rho(x, Tu_3)\Theta(Tu_1, Tu_2, Tu_4) - \Theta(Tu_3, [Tu_1, Tu_2, Tu_4]_{\mathfrak{g}}, x)\\
&- \rho(Tu_3, Tu_4)\Theta(Tu_1, Tu_2, x) - \Theta(Tu_3, Tu_4, [Tu_1, Tu_2, x]_{\mathfrak{g}})\Big)\\
\overset{(6)}{=}& -T\Big((\partial\Theta)(Tu_1, Tu_2, Tu_3, Tu_4, x)\Big) = 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\rho_\Theta([u_1, u_2, u_3]_T, u_4)x - \rho_\Theta(u_1, u_2)\rho_\Theta(u_3, u_4)x\\
&- \rho_\Theta(u_2, u_3)\rho_\Theta(u_1, u_4)x - \rho_\Theta(u_3, u_1)\rho_\Theta(u_2, u_4)x = 0.
\end{aligned}
$$

Hence the result follows. □

Let $\partial_\Theta : \mathfrak{C}^n_{3Lie}(V;\mathfrak{g}) \to \mathfrak{C}^{n+1}_{3Lie}(V;\mathfrak{g})$, $(n \geq 1)$ be the corresponding coboundary operator of the 3-Lie algebra $(V, [\cdot,\cdot,\cdot]_T)$ with coefficients in the representation $(\mathfrak{g}, \rho_\Theta)$. More precisely, $\partial_\Theta : \mathfrak{C}^n_{3Lie}(V;\mathfrak{g}) \to \mathfrak{C}^{n+1}_{3Lie}(V;\mathfrak{g})$ is given by

$$
\begin{aligned}
&(\partial_\Theta f)(\mathfrak{U}_1, \cdots, \mathfrak{U}_n, u_{n+1})\\
&= \sum_{1\leq j<k\leq n} (-1)^j f(\mathfrak{U}_1, \cdots, \widehat{\mathfrak{U}_j}, \cdots, \mathfrak{U}_{k-1}, [u_j, v_j, u_k]_T \wedge v_k + u_k \wedge [u_j, v_j, v_k]_T, \mathfrak{U}_{k+1}, \cdots, \mathfrak{U}_n, u_{n+1})\\
&+ \sum_{j=1}^n (-1)^j f(\mathfrak{U}_1, \cdots, \widehat{\mathfrak{U}_j}, \cdots, \mathfrak{U}_n, [u_j, v_j, u_{n+1}]_T) + \sum_{j=1}^n (-1)^{j+1} \rho_\Theta(u_j, v_j) f(\mathfrak{U}_1, \cdots, \widehat{\mathfrak{U}_j}, \cdots, \mathfrak{U}_n, u_{n+1})\\
&+ (-1)^{n+1}\Big(\rho_\Theta(v_n, u_{n+1}) f(\mathfrak{U}_1, \cdots, \mathfrak{U}_{n-1}, u_n) + \rho_\Theta(u_{n+1}, u_n) f(\mathfrak{U}_1, \cdots, \mathfrak{U}_{n-1}, v_n)\Big),
\end{aligned}
\tag{31}
$$

for all $\mathfrak{U}_i = u_i \wedge v_i \in \wedge^2 V$, $i = 1, 2, \cdots, n$ and $u_{n+1} \in V$.
It is obvious that $f \in \mathfrak{C}^1_{3Lie}(V;\mathfrak{g})$ is closed if and only if

$$
\begin{aligned}
&[Tu, Tv, f(w)]_{\mathfrak{g}} + [f(u), Tv, Tw]_{\mathfrak{g}} + [Tu, f(v), Tw]_{\mathfrak{g}}\\
&- f\Big(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw)\Big)\\
&- T\Big(\rho(Tv, f(w))u + \rho(f(w), Tu)v + \Theta(f(w), Tu, Tv)\Big)\\
&- T\Big(\rho(Tw, f(u))v + \rho(f(u), Tv)w + \Theta(f(u), Tv, Tw)\Big)\\
&- T\Big(\rho(Tu, f(v))w + \rho(f(v), Tw)u + \Theta(f(v), Tw, Tu)\Big) = 0.
\end{aligned}
$$

For all $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$, we define $\delta(\mathfrak{X}) : V \to \mathfrak{g}$ by

$$\delta(\mathfrak{X})(v) = T\Big(\rho(\mathfrak{X})v + \Theta(\mathfrak{X}, Tv)\Big) - [\mathfrak{X}, Tv]_{\mathfrak{g}}, \ \forall v \in V. \tag{32}$$

Proposition 4.6. *Let T be a Θ-twisted O-operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ). Then $\delta(\mathfrak{X})$ is a 1-cocycle on the 3-Lie algebra $(V, [\cdot,\cdot,\cdot]_T)$ with coefficients in $(\mathfrak{g}, \rho_\Theta)$.*

Proof. For all $u, v, w \in V$, we have

$$\begin{aligned}
&(\partial_\Theta \delta(\mathfrak{X}))(u,v,w)\\
&= [Tu, Tv, \delta(\mathfrak{X})(w)]_{\mathfrak{g}} + [\delta(\mathfrak{X})(u), Tv, Tw]_{\mathfrak{g}} + [Tu, \delta(\mathfrak{X})(v), Tw]_{\mathfrak{g}}\\
&- \delta(\mathfrak{X})\Big(\rho(Tu,Tv)w + \rho(Tv,Tw)u + \rho(Tw,Tu)v + \Theta(Tu,Tv,Tw)\Big)\\
&- T\Big(\rho(Tv, \delta(\mathfrak{X})(w))u + \rho(\delta(\mathfrak{X})(w), Tu)v + \Theta(\delta(\mathfrak{X})(w), Tu, Tv)\Big)\\
&- T\Big(\rho(Tw, \delta(\mathfrak{X})(u))v + \rho(\delta(\mathfrak{X})(u), Tv)w + \Theta(\delta(\mathfrak{X})(u), Tv, Tw)\Big)\\
&- T\Big(\rho(Tu, \delta(\mathfrak{X})(v))w + \rho(\delta(\mathfrak{X})(v), Tw)u + \Theta(\delta(\mathfrak{X})(v), Tw, Tu)\Big)\\
&= [Tu, Tv, T\rho(\mathfrak{X})w]_{\mathfrak{g}} + [Tu, Tv, T\Theta(\mathfrak{X}, w)]_{\mathfrak{g}} - [Tu, Tv, [\mathfrak{X}, Tw]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&+ [T\rho(\mathfrak{X})u, Tv, Tw]_{\mathfrak{g}} + [T\Theta(\mathfrak{X}, Tu), Tv, Tw]_{\mathfrak{g}} - [[\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv, Tw]_{\mathfrak{g}}\\
&+ [Tu, T\rho(\mathfrak{X})v, Tw]_{\mathfrak{g}} + [Tu, T\Theta(\mathfrak{X}, Tv), Tw]_{\mathfrak{g}} - [Tu, [\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw]_{\mathfrak{g}}\\
&- T\rho(\mathfrak{X})\Big(\rho(Tu,Tv)w + \rho(Tv,Tw)u + \rho(Tw,Tu)v + \Theta(Tu,Tv,Tw)\Big)\\
&- T\Theta(\mathfrak{X}, [Tu,Tv,Tw]_{\mathfrak{g}}) + [\mathfrak{X}, [Tu,Tv,Tw]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&- T\Big(\rho(Tv, T\rho(\mathfrak{X})w)u + \rho(Tv, T\Theta(\mathfrak{X}, Tw))u - \rho(Tv, [\mathfrak{X}, Tw]_{\mathfrak{g}})u\\
&+ \rho(T\rho(\mathfrak{X})w, Tu)v + \rho(T\Theta(\mathfrak{X}, Tw), Tu)v - \rho([\mathfrak{X}, Tw]_{\mathfrak{g}}, Tu)v\\
&+ \Theta(T\rho(\mathfrak{X})w, Tu, Tv) + \Theta(T\Theta(\mathfrak{X}, Tw), Tu, Tv) - \Theta([\mathfrak{X}, Tw]_{\mathfrak{g}}, Tu, Tv)\Big)\\
&- T\Big(\rho(Tw, T\rho(\mathfrak{X})u)v + \rho(Tw, T\Theta(\mathfrak{X}, Tu))v - \rho(Tw, [\mathfrak{X}, Tu]_{\mathfrak{g}})v\\
&+ \rho(T\rho(\mathfrak{X})u, Tv)w + \rho(T\Theta(\mathfrak{X}, Tu), Tv)w - \rho([\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv)w\\
&+ \Theta(T\rho(\mathfrak{X})u, Tv, Tw) + \Theta(T\Theta(\mathfrak{X}, Tu), Tv, Tw) - \Theta([\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv, Tw)\Big)\\
&- T\Big(\rho(Tu, T\rho(\mathfrak{X})v)w + \rho(Tu, T\Theta(\mathfrak{X}, Tv))w - \rho(Tu, [\mathfrak{X}, Tv]_{\mathfrak{g}})w\\
&+ \rho(T\rho(\mathfrak{X})v, Tw)u + \rho(T\Theta(\mathfrak{X}, Tv), Tw)u - \rho([\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw)u\\
&+ \Theta(T\rho(\mathfrak{X})v, Tw, Tu) + \Theta(T\Theta(\mathfrak{X}, Tv), Tw, Tu) - \Theta([\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw, Tu)\Big)\\
&\overset{(8)+(1)+(3)}{=} -T\Big(\Theta(\mathfrak{X}, [Tu,Tv,Tw]_{\mathfrak{g}}) + T\rho(\mathfrak{X})\Theta(Tu,Tv,Tw)\\
&- T\rho(Tu,Tv)\Theta(\mathfrak{X}, Tw) - T\rho(Tv,Tw)\Theta(\mathfrak{X}, Tu) - T\rho(Tw,Tu)\Theta(\mathfrak{X}, Tv)\\
&- T\Theta([\mathfrak{X}, Tw]_{\mathfrak{g}}, Tu, Tv) - T\Theta([\mathfrak{X}, Tu]_{\mathfrak{g}}, Tv, Tw) - T\Theta([\mathfrak{X}, Tv]_{\mathfrak{g}}, Tw, Tu)\Big)\\
&= -T\Big((\partial\Theta)(\mathfrak{X}, Tu, Tv, Tw)\Big) = 0.
\end{aligned}$$

Thus, we deduce that $\partial_\Theta \delta(\mathfrak{X}) = 0$. □

Now, we give a cohomology of Θ-twisted O-operators on 3-Lie algebras.

Definition 4.3. *Let T be a Θ-twisted O-operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ). Define the set of n-cochains by*

$$\mathfrak{C}^n_\Theta(V; \mathfrak{g}) = \begin{cases} \mathfrak{C}^n_{3Lie}(V; \mathfrak{g}), & n \geq 1,\\ \mathfrak{g} \wedge \mathfrak{g}, & n = 0. \end{cases} \tag{33}$$

Define $D_\Theta : \mathfrak{C}^n_\Theta(V;\mathfrak{g}) \to \mathfrak{C}^{n+1}_\Theta(V;\mathfrak{g})$ *by*

$$D_\Theta = \begin{cases} \partial_\Theta, & n \geq 1, \\ \delta, & n = 0. \end{cases} \tag{34}$$

Denote the set of n-cocycles by $\mathcal{Z}^n_\Theta(V;\mathfrak{g})$ and the set of n-coboundaries by $\mathcal{B}^n_\Theta(V;\mathfrak{g})$. Denote by

$$\mathcal{H}^n_\Theta(V;\mathfrak{g}) = \mathcal{Z}^n_\Theta(V;\mathfrak{g})/\mathcal{B}^n_\Theta(V;\mathfrak{g}),\ n \geq 0$$

the n^{th} cohomology group which will be taken to be the n^{th} cohomology group for the Θ-twisted $\mathcal{O}$-operator T.

Theorem 4.7. *Let T be a Θ-twisted $\mathcal{O}$-operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g})$ with respect to a representation (V,ρ). Then we have*

$$d_T(f) = (-1)^{n-1} D_\Theta(f), \quad \forall f \in Hom(\underbrace{\wedge^2 V \otimes \cdots \otimes \wedge^2 V}_{n-1} \wedge V, \mathfrak{g}),\ n = 1, 2, \cdots. \tag{35}$$

Proof. According to [40, Theorem 4.5], we have

$$\begin{aligned}
&\frac{1}{2} l_3(T,T,f)(\mathfrak{U}_1,\cdots,\mathfrak{U}_n,u_{n+1}) \\
=&(-1)^{n-1}\Big\{(-1)^{n+1}\Big([Tv_n, Tu_{n+1}, f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},u_n)]_\mathfrak{g} - T\rho(Tu_{n+1}, f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},u_n))v_n \\
&- T\rho(f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},u_n),Tv_n)u_{n+1}\Big) \\
&+ (-1)^{n+1}\Big([Tu_{n+1}, f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},v_n)]_\mathfrak{g} - T\rho(Tu_n, f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},v_n))u_{n+1} \\
&- T\rho(f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},v_n),Tu_{n+1})u_n\Big) + \sum_{j=1}^{n}(-1)^{j+1}\Big([Tu_j,Tv_j,f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,u_{n+1})]_\mathfrak{g} \\
&- T\rho(Tv_j, f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,u_{n+1}))u_j - T\rho(f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,u_{n+1}),Tu_j)v_j\Big) \\
&+ \sum_{1\leq j<k\leq n}(-1)^j f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_{k-1},\Big(\rho(Tu_j,Tv_j)u_k + \rho(Tv_j,Tu_k)u_j + \rho(Tu_k,Tu_j)v_j\Big)\wedge v_k \\
&+ u_k \wedge \Big(\rho(Tu_j,Tv_j)v_k + \rho(Tv_j,Tv_k)u_j + \rho(Tv_k,Tu_j)v_j\Big),\mathfrak{U}_{k+1},\cdots,\mathfrak{U}_n,u_{n+1}) \\
&+ \sum_{j=1}^{n}(-1)^j f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,\rho(Tu_j,Tv_j)u_{n+1} + \rho(Tv_j,Tu_{n+1})u_j + \rho(Tu_{n+1},Tu_j)v_j)\Big\}.
\end{aligned}$$

Here we observe that

$$\begin{aligned}
&l_4(T,T,T,f)(\mathfrak{U}_1,\cdots,\mathfrak{U}_n,u_{n+1}) \\
=&[\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie},f]_{3-Lie}(\mathfrak{U}_1,\cdots,\mathfrak{U}_n,u_{n+1}) \\
=&[\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie}(f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},u_n)\wedge v_n,u_{n+1}) \\
&+ [\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie}(u_n\wedge f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},v_n),u_{n+1}) \\
&+ \sum_{j=1}^{n}(-1)^{n-1}(-1)^{j-1}[\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie}\Big(\mathfrak{U}_j,f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,u_{n+1})\Big) \\
&- (-1)^{n-1}\sum_{k=1}^{n-1}\sum_{j=1}^{k}(-1)^{j+1}
\end{aligned}$$

$$f\Big(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_k,[\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie}(\mathfrak{U}_j,u_{k+1})\wedge v_{k+1},\mathfrak{U}_{k+2},\cdots,\mathfrak{U}_n,u_{n+1}\Big)$$
$$-(-1)^{n-1}\sum_{k=1}^{n-1}\sum_{j=1}^{k}(-1)^{j+1}$$
$$f\Big(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_k,u_{k+1}\wedge[\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie}(\mathfrak{U}_j,v_{k+1}),\mathfrak{U}_{k+2},\cdots,\mathfrak{U}_n,u_{n+1}\Big)$$
$$-(-1)^{n-1}\sum_{j=1}^{n}(-1)^{j+1}f\Big(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,[\Theta,T]_{3-Lie},T]_{3-Lie},T]_{3-Lie}(\mathfrak{U}_j,u_{n+1})\Big)$$
$$=(-1)^{n-1}6\Big\{(-1)^{n+1}\Big(-T\Theta(f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},u_n),Tv_n,Tu_{n+1})-T\Theta(f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},v_n),Tu_{n+1},Tu_n)\Big)$$
$$-\sum_{j=1}^{n}(-1)^{j+1}T\Theta(f(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,u_{n+1}),Tu_j,Tv_j)$$
$$+\sum_{j=1}^{n}(-1)^{j}f\Big(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_n,\Theta(Tu_j,Tv_j,Tu_{n+1})\Big)$$
$$+\sum_{1\le j<k\le n}(-1)^{j}f\Big(\mathfrak{U}_1,\cdots,\widehat{\mathfrak{U}_j},\cdots,\mathfrak{U}_{k-1},\Theta(Tu_j,Tv_j,Tu_k)\wedge v_k+u_k\wedge\Theta(Tu_j,Tv_j,Tv_k),\mathfrak{U}_{k+1},\cdots,\mathfrak{U}_n,u_{n+1}\Big)\Big\}.$$

Hence $d_T(f)=\frac{1}{2}l_3(T,T,f)+\frac{1}{6}l_4(T,T,T,f)=(-1)^{n-1}D_\Theta(f)$. The proof is finished. □

5. Deformations of twisted $\mathcal{O}$-operators

In this section, we study infinitesimal and formal deformations of a Θ-twisted $\mathcal{O}$-operator. For deformations of Rota-Baxter and $\mathcal{O}$-operators, see [15, 16, 34, 41].

5.1. *Infinitesimal deformations*

Let $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$ be a 3-Lie algebra, (V,ρ) be a representation of $\mathfrak{g}$, and $\Theta\in\mathfrak{C}^2_{3Lie}(\mathfrak{g};V)$ be a 2-cocycle in the Chevalley-Eilenberg cochain complex. Let $T:V\to\mathfrak{g}$ be a Θ-twisted $\mathcal{O}$-operator.

Definition 5.1. *An infinitesimal deformation of T consists of a parametrized sum $T_t=T+tT_1$, for some $T_1\in Hom(V,\mathfrak{g})$ such that T_t is a Θ-twisted $\mathcal{O}$-operator for all values of t. In this case, we say that T_1 generates an infinitesimal deformation of T.*

Suppose that T_1 generates an infinitesimal deformation of T. Then we have

$$[T_tu,T_tv,T_tw]_{\mathfrak{g}}$$
$$=T_t\Big(\rho(T_tu,T_tv)w+\rho(T_tv,T_tw)u+\rho(T_tw,T_tu)v+\Theta(T_tu,T_tv,T_tw)\Big),$$

for $u,v,w\in V$. This is equivalent to the following conditions

$$[Tu,Tv,T_1w]_{\mathfrak{g}}+[Tu,T_1v,Tw]_{\mathfrak{g}}+[T_1u,Tv,Tw]_{\mathfrak{g}}$$
$$=T\Big(\rho(Tu,T_1v)w+\rho(T_1u,Tv)w+\rho(Tv,T_1w)u+\rho(T_1v,Tw)u$$
$$+\rho(Tw,T_1u)v+\rho(T_1w,Tu)v+\Theta(Tu,Tv,T_1w)+\Theta(Tu,T_1v,Tw)$$
$$+\Theta(T_1u,Tv,Tw)\Big)$$
$$+T_1\Big(\rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v+\Theta(Tu,Tv,Tw)\Big),\tag{36}$$

$$[Tu, T_1v, T_1w]_{\mathfrak{g}} + [T_1u, Tv, T_1w]_{\mathfrak{g}} + [T_1u, T_1v, Tw]_{\mathfrak{g}}$$
$$= T\Big(\rho(T_1u, T_1v)w + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v$$
$$+ \Theta(Tu, T_1v, T_1w) + \Theta(T_1u, Tv, T_1w) + \Theta(T_1u, T_1v, Tw)\Big)$$
$$+ T_1\Big(\rho(Tu, T_1v)w + \rho(T_1u, Tv)w + \rho(Tv, T_1w)u + \rho(T_1v, Tw)u$$
$$+ \rho(Tw, T_1u)v + \rho(T_1w, Tu)v + \Theta(Tu, Tv, T_1w) + \Theta(Tu, T_1v, Tw)$$
$$+ \Theta(T_1u, Tv, Tw)\Big), \tag{37}$$

$$[T_1u, T_1v, T_1w]_{\mathfrak{g}} = T\Big(\Theta(T_1u, T_1v, T_1w)\Big)$$
$$+ T_1\Big(\rho(T_1u, T_1v)w + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v$$
$$+ \Theta(Tu, T_1v, T_1w) + \Theta(T_1u, Tv, T_1w) + \Theta(T_1u, T_1v, Tw)\Big) \tag{38}$$

and

$$T_1\Big(\Theta(T_1u, T_1v, T_1w)\Big) = 0. \tag{39}$$

Note that the identity (36) implies that T_1 is a 1-cocycle with respect to the cohomology of T. Hence, T_1 defines a cohomology class in $\mathcal{H}^1_{\Theta}(V; \mathfrak{g})$.

Definition 5.2. *Two infinitesimal deformations $T_t = T + tT_1$ and $T'_t = T + tT'_1$ of a Θ-twisted O-operator T are said to be equivalent if there exists an element $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$ such that the pair*

$$\Big(\phi_t = Id_{\mathfrak{g}} + t[\mathfrak{X}, -]_{\mathfrak{g}},\ \psi_t = Id_V + t\Big(\rho(\mathfrak{X})(-) + \Theta(\mathfrak{X}, T-)\Big)\Big) \tag{40}$$

defines a morphism of Θ-twisted O-operators from T_t to T'_t.

An infinitesimal deformation $T_t = T + tT_1$ of a Θ-twisted O-operator is said to be trivial if T_t is equivalent to $T'_t = T$.

The condition that $\phi_t = Id_{\mathfrak{g}} + t[\mathfrak{X}, -]_{\mathfrak{g}}$ is a 3-Lie algebra morphism of $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ is equivalent to

$$\begin{cases} [z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}]_{\mathfrak{g}} + [[\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2, [\mathfrak{X}, z_3]_{\mathfrak{g}}]_{\mathfrak{g}} \\ +[[\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, z_3]_{\mathfrak{g}} = 0, \\ [[\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \quad \text{for all } z_1, z_2, z_3 \in \mathfrak{g}. \end{cases} \tag{41}$$

The condition $\psi_t(\rho(z_1, z_2)u) = \rho(\phi_t(z_1), \phi_(z_2))\psi_t(u)$ implies that

$$\begin{cases} \Theta(\mathfrak{X}, T\rho(z_1, z_2)u) = \rho(z_1, z_2)\Theta(\mathfrak{X}, Tu), \\ \Big(\rho(z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}) + \rho([\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2)\Big)\Big(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)\Big) \\ +\rho([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}})u = 0, \\ \rho([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}})\Big(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)\Big) = 0, \end{cases} \tag{42}$$

Finally, conditions $\psi_t \circ \Theta = \Theta \circ (\phi_t \otimes \phi_t \otimes \phi_t)$ and $\phi_t \circ T_t = T'_t \circ \psi_t$ are respectively equivalent to

$$\begin{cases} \rho(\mathfrak{X})\Theta(z_1, z_2, z_3) + \Theta(\mathfrak{X}, T\Theta(z_1, z_2, z_3)) = \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2, z_3) \\ +\Theta(z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}, z_3) + \Theta(z_1, z_2, [\mathfrak{X}, z_3]_{\mathfrak{g}}), \\ \Theta(z_1, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}) + \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, z_2, [\mathfrak{X}, z_3]_{\mathfrak{g}}) \\ +\Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, z_3) = 0, \\ \Theta([\mathfrak{X}, z_1]_{\mathfrak{g}}, [\mathfrak{X}, z_2]_{\mathfrak{g}}, [\mathfrak{X}, z_3]_{\mathfrak{g}}) = 0, \end{cases} \tag{43}$$

$$\begin{cases} T_1u + [\mathfrak{X}, Tu]_{\mathfrak{g}} = T\Big(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)\Big) + T_1^{'}u, \\ [\mathfrak{X}, T_1u]_{\mathfrak{g}} = T_1^{'}\Big(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)\Big). \end{cases} \tag{44}$$

Note that the above identities hold for all $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$, $z_1, z_2, z_3 \in \mathfrak{g}$ and $u \in V$.

From the first condition of (44), we have

$$T_1u - T_1^{'}u = T\Big(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)\Big) - [\mathfrak{X}, Tu]_{\mathfrak{g}} = D_{\Theta}(\mathfrak{X})(u).$$

Therefore, we get the following theorem.

Theorem 5.1. *Let $T_t = T + tT_1$ and $T_t^{'} = T + tT_1^{'}$ be two equivalent infinitesimal deformations of a Θ-twisted $\mathcal{O}$-operator T. Then T_1 and $T_1^{'}$ define the same cohomology class in $\mathcal{H}^1_{\Theta}(V;\mathfrak{g})$.*

5.2. *Formal deformations*

Now we consider a more general situation by using formal power series. Let $\mathfrak{g}$ be a 3-Lie algebra, V be a $\mathfrak{g}$-module and Θ be a 2-cocycle in the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in V. Let $T : V \to \mathfrak{g}$ be a Θ-twisted $\mathcal{O}$-operator.

Let $\mathbb{K}[[t]]$ be the power series ring in one variable t. For any $\mathbb{K}$-linear space V, denote by $V[[t]]$ the vector space of formal power series in t with coefficients in V. If $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ is a 3-Lie algebra over $\mathbb{K}$, then there is a 3-Lie algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$ given by

$$\Big[\sum_{i=0}^{+\infty} x_i t^i, \sum_{j=0}^{+\infty} y_j t^j, \sum_{k=0}^{+\infty} z_k t^k\Big]_{\mathfrak{g}} = \sum_{s=0}^{+\infty} \sum_{i+j+k=s} [x_i, y_j, z_k]_{\mathfrak{g}} t^s, \ \forall x_i, y_j, z_k \in \mathfrak{g}. \tag{45}$$

For any representation (V, ρ) of $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$, there is a natural representation of the 3-Lie algebra $\mathfrak{g}[[t]]$ on the $\mathbb{K}[[t]]$-module $V[[t]]$, which is given by

$$\rho\Big(\sum_{i=0}^{+\infty} x_i t^i, \sum_{j=0}^{+\infty} y_j t^j\Big)\Big(\sum_{k=0}^{+\infty} v_k t^k\Big) = \sum_{s=0}^{+\infty} \sum_{i+j+k=s} \rho(x_i, y_j) v_k t^s, \ \forall x_i, y_j \in \mathfrak{g}, \ v_k \in V. \tag{46}$$

Similarly, the 2-cocycle Θ can be extended to a 2-cocycle (denoted also by Θ) on the 3-Lie algebra $\mathfrak{g}[[t]]$ with coefficients in $V[[t]]$. Consider a power series

$$T_t = \sum_{i=0}^{+\infty} T_i t^i, \ T_i \in Hom_{\mathbb{K}}(V;\mathfrak{g}), \tag{47}$$

that is, $T_t \in Hom_{\mathbb{K}}(V;\mathfrak{g})[[t]] = Hom_{\mathbb{K}}(V;\mathfrak{g}[[t]])$. Extend it to be a $\mathbb{K}[[t]]$-module map from $V[[t]]$ to $\mathfrak{g}[[t]]$ which is still denoted by T_t.

Definition 5.3. *If $T_t = \sum_{i=0}^{+\infty} T_i t^i$ with $T_0 = T$ satisfies*

$$\begin{aligned} &[T_t u, T_t v, T_t w]_{\mathfrak{g}} \\ &= T_t\Big(\rho(T_t u, T_t v)w + \rho(T_t v, T_t w)u + \rho(T_t w, T_t u)v + \Theta(T_t u, T_t v, T_t w)\Big), \end{aligned} \tag{48}$$

we say that T_t is a formal deformation of the Θ-twisted $\mathcal{O}$-operator T.

Recall that a formal deformation of a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ is a formal power series $\omega_t = \sum_{k=0}^{+\infty} \omega_k t^k$, where $\omega_k \in Hom(\wedge^3\mathfrak{g};\mathfrak{g})$ such that $\omega_0(x, y, z) = [x, y, z]_{\mathfrak{g}}$ for any $x, y, z \in \mathfrak{g}$ and ω_t defines a 3-Lie algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$.

Based on the relationship between Θ-twisted $\mathcal{O}$-operators and 3-Lie algebras, we have the following construction.

Proposition 5.2. *Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ be a formal deformation of a Θ-twisted O-operator T on the 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ). Then $[\cdot,\cdot,\cdot]_{T_t}$ defined by*

$$[u,v,w]_{T_t} = \sum_{s=0}^{+\infty}\Big(\sum_{i+j=s}\Big(\rho(T_iu, T_jv)w + \rho(T_iv, T_jw)u + \rho(T_iw, T_ju)v\Big) + \sum_{i+j+k=s}\Theta(T_iu, T_jv, T_kw)\Big)t^s$$

for all $u, v, w \in V$, is a formal deformation of the 3-Lie algebra $(V, [\cdot,\cdot,\cdot]_T)$ defined in (9).

By applying Eqs.(45)-(47) to expand Eq.(48) and collecting coefficients of t^s, we see that Eq.(48) is equivalent to the system of equations, for $s = 0, 1, 2, \cdots$,

$$\begin{aligned}&\sum_{i+j+k=s}[T_iu, T_jv, T_kw]_{\mathfrak{g}}\\ &= \sum_{i+j+k=s} T_i\Big(\rho(T_ju, T_kv)w + \rho(T_jv, T_kw)u + \rho(T_jw, T_ku)v\Big)\\ &+ \sum_{i+j+k+m=s} T_i\Big(\Theta(T_ju, T_kv, T_mw)\Big).\end{aligned} \tag{49}$$

Note that (49) holds for $s = 0$ since $T_0 = T$ is a Θ-twisted O-operator. For $s = 1$, we get

$$\begin{aligned}&[Tu, Tv, T_1w]_{\mathfrak{g}} + [Tu, T_1v, Tw]_{\mathfrak{g}} + [T_1u, Tv, Tw]_{\mathfrak{g}}\\ &= T\Big(\rho(Tu, T_1v)w + \rho(T_1u, Tv)w + \rho(Tv, T_1w)u + \rho(T_1v, Tw)u\\ &+ \rho(Tw, T_1u)v + \rho(T_1w, Tu)v + \Theta(Tu, Tv, T_1w) + \Theta(Tu, T_1v, Tw)\\ &+ \Theta(T_1u, Tv, Tw)\Big)\\ &+ T_1\Big(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \Theta(Tu, Tv, Tw)\Big),\end{aligned}$$

which is exactly Eq. (36). This implies that $(D_\Theta(T_1))(u, v, w) = 0$. Hence the linear term T_1 is a 1-cocycle with respect to the cohomology of T. It is called the infinitesimal of the deformation T_t.
In the sequel, we discuss equivalent formal deformations.

Definition 5.4. *Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ and $T'_t = \sum_{i=0}^{+\infty} T'_i t^i$ be two formal deformations of a Θ-twisted O-operator $T = T_0 = T'_0$ on a 3-Lie algebra $\mathfrak{g}$ with respect to a representation (V, ρ). They are said to be equivalent if there exist an element $\mathfrak{X} \in \mathfrak{g} \wedge \mathfrak{g}$, $\phi_i \in \mathfrak{gl}(\mathfrak{g})$ and $\psi_i \in \mathfrak{gl}(V)$, $i \geq 2$, such that the pair*

$$\Big(\phi_t = Id_{\mathfrak{g}} + t[\mathfrak{X}, -]_{\mathfrak{g}} + \sum_{i=2}^{+\infty}\phi_i t^i,\ \psi_t = Id_V + t\Big(\rho(\mathfrak{X})(-) + \Theta(\mathfrak{X}, T-)\Big) + \sum_{i=2}^{+\infty}\psi_i t^i\Big), \tag{50}$$

is a morphism of Θ-twisted O-operators from T_t to T'_t.

Theorem 5.3. *If two formal deformations of a Θ-twisted O-operator T on a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation (V, ρ) are equivalent, then their infinitesimals are in the same cohomology class.*

Proof. Let (ϕ_t, ψ_t) be the two maps defined by Eq.(50) which gives an equivalence between two deformations $T_t = \sum_{i=0}^{+\infty} T_i t^i$ and $T'_t = \sum_{i=0}^{+\infty} T'_i t^i$ of a Θ-twisted O-operator T. By $\phi_t \circ T_t = T'_t \circ \psi_t$, we have

$$T_1u = T'_1u + T(\rho(\mathfrak{X})u + \Theta(\mathfrak{X}, Tu)) - [\mathfrak{X}, Tu]_{\mathfrak{g}}$$

$$= T_1^{'}u + (D_\Theta(\mathfrak{X}))(u), \ \forall u \in V,$$

which implies that T_1 and $T_1^{'}$ are in the same cohomology class. □

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