Filomat 37:21 (2023), 6961–6975
https://doi.org/10.2298/FIL2321961H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Two point Ostrowski and Ostrowski-Gruss type inequalities on time scales with applications

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Abstract. Here, Fink type identity for two point formula on time scales has been proved, an expansion of Guessab–Schmeisser two points formula for $n-$times differentiable functions via Fink type identity is established to provide some estimates of Ostrowski and Ostrowski-Gruss type inequalities with applications.

1. introduction

The approximate value of $\int_a^b f(t)dt$ for differentiable function f, by a general one point rule

$$\int_a^b f(t)dt = (b-a)f(x) + \mathfrak{E}(f,x)$$

is due to Ostrowski in his celebrated work for differentiable function f. Later on called an Ostrowski inequality defined as [14]:

$$\left|\mathfrak{E}(f,x)\right| \leq \left[\frac{(b-a)^2}{4} + \left(x + \frac{a+b}{2}\right)^2\right] \|f'\|_\infty, \tag{1}$$

provided that $\|f'\|_\infty = \sup_{x\in[a,b]} |f'(x)| \leq \infty$ and the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller ones. Inequality (1) has lot of applications in different fields of Sciences such as Numerical analysis, Probability theory, Theory of means etc. Due to its importance many researchers generalized, extended and modified it in different ways by using innovative techniques [7, 8, 10–12]. This result was generalized and refined by Fink [8]. In [9], Guessab and Schmeisser discussed the following more general quadrature formula

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{f(x)+f(a+b-x)}{2} + \mathfrak{H}(f;x), \tag{2}$$

for each real number $x \in [a, \frac{a+b}{2}]$, where $\mathfrak{H}(f;x)$ is the remainder term. In [9], the authors have also established some estimates for the absolute value of the remainder term $\mathfrak{H}(f;x)$. It is important to mention

2020 *Mathematics Subject Classification*. Primary 30D05, 26D10, 26A42.
Keywords. Fink identity; Guessab-Schmeisser inequality; Time scales, Ostrowski inequality
Received: 13 December 2022; Accepted: 11 March 2022
Communicated by Dragan S. Djordjević
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here that Alomari [4] generalized the Guessab-Schmeisser formula (2) for n-times differentiable mapping via Fink identity and provided the bounds for the absolute value of the remainder term $\mathfrak{H}(f;x)$. In [2] the authors have extended Alomari's main result [4, Theorem 5] on time scales providing distinct bounds for the absolute value of $\mathfrak{H}(f;x)$ than those given in [4] due to the generalized polynomials and different computational techniques. It may be observed that (2) defines a family of quadrature formulae which contains the trapezoidal rule and the midpoint rule at the boundary cases $x = a$ and $x = \frac{a+b}{2}$ respectively. It also includes any other quadrature formula with two symmetric nodes; for example, it includes the two-point Maclaurin formula and the two-point Gaussian formula. In order to generalize Guessab–Schmeisser formula (2) for symmetric and non-symmetric points, the author in [3] has discussed the general quadrature rule

$$\int_a^b f(t)dt = Q(f;y,x,z) + \Im(f;y,x,z),$$

provided that:

$$Q(f;y,x,z) = (x-a)f(y) + (b-x)f(z) \ \ \forall\, x \in [y,z], \ \ a \le y \le z \le b,$$

with the error term $\Im(f;y,x,z)$. In the same paper he has calculated some estimates for the error term and deduced some generalized two-point Ostrowski type inequality and concluded some existing results. Here, our aim is to discuss an identity expressing two-point formula of Ostrowski's type via Fink approach on time scales to derive some estimates of two point Ostrowski's functional. Some particular consequences are also drawn. Among others inequalities, there is another following inequality called Čebyšev-Grüss type inequality for two $rd-$continuous and Δ integrable functions f_1, f_2 on $[a,b]$ [15]

$$\left|C(f_1,f_2)\right| \le \frac{(\phi-\varphi)(\Gamma-\gamma)}{4}, \ \ \phi \le f_1(x) \le \varphi; \ \gamma \le f_2(x) \le \Gamma \ \ \forall\, x \in [a,b], \tag{3}$$

where as the celebrated Ĉebyšev functional is defined by:

$$C(f_1,f_2) := \frac{1}{b-a}\int_a^b f_1(t)f_2(t)\Delta t - \frac{1}{b-a}\int_a^b f_1(t)\Delta t.\frac{1}{b-a}\int_a^b f_2(t)\Delta t \tag{4}$$

This paper is organized in the following way. After the Introduction, in Section 2 some preliminaries and assumptions, in Section 3 main results and in Section 4 some applications relating the topic have been discussed.

2. Some preliminaries and assumptions

A time scale is a non-empty closed subset of real numbers, it is usually denoted by a symbol $\mathbb{T}$. By $[a,b]_{\mathbb{T}}$ we always mean $[a,b] \cap \mathbb{T}$, where$[a,b] \subseteq \mathbb{R}$. For $n = 1$, the sum $\sum_{k=1}^{n-1}$ is vacuously considered to be zero. For $n \in \mathbb{N}$ and f an $n-$times differentiable function on $\mathbb{T}^{k^n}$, the delta Taylor monomial for $\mathbb{T}$ is defined as [13, Theorem 4.1]:

$$f(t) \ = \ \sum_{k=0}^{n-1} f^{\Delta^k}(s)h_k(t,s) + \int_s^t h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau, \tag{5}$$

for $t \in \mathbb{T}$ and $s \in \mathbb{T}^{k^{n-1}}$, where as we define the remainder function by $R_{-1,f}(.,s) := f(s)$, and for $n > -1$

$$R_{n,f}(s,t) \ := \ f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(s)h_k(t,s) = \int_s^t h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau \tag{6}$$

Following operations are well defined on time scales.

- The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) = \inf\{s \in \mathbb{T}; s > t\}$.
- The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s \in \mathbb{T}; s < t\}$.
- The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined as $\mu(t) = \sigma(t) - t$.

For more about time scales we refer [6]. Setting $f_1(t) \to f^{\Delta^k}(t)$ and $f_2(t) \to \mathfrak{K}_3(t; y, z)$ in (4), we have

$$
\begin{aligned}
C(f_1, f_2) :=& \frac{(\rho^{n-1}(z) - a)f(y) + (b - \rho^{n-1}(z))f(z) - (\rho^{n-1}(z) - a)[\Re(y) + \mathfrak{U}(y)]}{(b-a)} \\
&- \frac{(b - \rho^{n-1}(z))[\Re(z) + \mathfrak{U}(z)]}{(b-a)} - \frac{n}{b-a}\int_a^b f(w)\Delta w - \frac{f^{\Delta^{k-1}}(b) - f^{\Delta^{k-1}}(a)}{b-a} \\
&\times \Bigg[\frac{\sup_{t\in[a,\rho^{n-1}(y)]} h_{n-1}(y, \sigma(t)) h_2(a, \rho^{n-1}(y))}{b-a} \\
&- \frac{\sup_{t\in[\rho^{n-1}(y),\rho^{n-1}(z)]} h_{n-1}(y, \sigma(t)) h_2(\rho^{n-1}(z), \rho^{n-1}(y))}{b-a} \\
&- \frac{\sup_{t\in[\rho^{n-1}(y),\rho^{n-1}(z)]} h_{n-1}(y, \sigma(t)) h_2(\rho^{n-1}(z), \rho^{n-1}(y))}{b-a} \Bigg] =: \mathfrak{O}(f; y, z, n). \quad (7)
\end{aligned}
$$

For convenience the following significant notations and result have been used.

$$
\|S(x)\|_{\infty,[a,b]} := ess\sup_{a\le t\le b} |S(x,t)|; \quad \|S(x)\|_{\alpha,[a,b]} := \sqrt[\alpha]{\int_a^b |S(x,t)|^\alpha \Delta t}, \ \alpha \ge 1.
$$

$$
\Delta^n f(t) = \sum_{m=0}^{n} (-1)^m E^{n-m} f(t) \binom{n}{m}; \ E \text{ is a displacement operator.}
$$

$$
\mathfrak{Z}_q(h; a, b) := \sqrt[q]{\frac{(h-a)^{nq-q+1} + \left(\frac{a+b}{2} - h\right)^{nq-q+1}}{[(n-1)!]^q}}
$$

$$
\alpha := \frac{(\sigma(\rho^{n-1}(y)) - \sigma(a))(\rho^{n-1}(y) - a)^{n-1}}{(n-1)!}
$$

$$
\beta := \frac{(\sigma(\rho^{n-1}(z)) - \sigma(\rho^{n-1}(y)))(\rho^{n-1}(z) - \rho^{n-1}(y))^{n-1}}{(n-1)!}
$$

$$
\gamma := \frac{(\sigma(b) - \sigma(\rho^{n-1}(z)))(b - \rho^{n-1}(z))^{n-1}}{(n-1)!}
$$

$$
\mathfrak{W}(y, z; a, b) := \frac{\alpha + \beta + |\alpha - \beta| + 2\gamma + |\alpha + \beta + |\alpha - \beta| - 2\gamma|}{4}
$$

$$
\begin{aligned}
\mathfrak{F}(y, z; a, b; s, u, q) :=& \frac{[\sigma(\rho^{n-1}(z)) - \sigma(\rho^{n-1}(y))]^s [\rho^{n-1}(z) - \rho^{n-1}(y)]^{\frac{u}{p}} (\rho^{n-1}(z) - \rho^{n-1}(y))^{\frac{np-1}{p}}}{(n-1)!} \\
&+ \frac{[\rho^{n-1}(y) - a]^{\frac{u}{p}} [\sigma(\rho^{n-1}(y)) - \sigma(a)]^s (y-a)^{\frac{np-1}{p}} + [\sigma(b) - \sigma(\rho^{n-1}(z))]^s [b - \rho^{n-1}(z)]^{\frac{u}{p}} (b - \rho^{n-1}(z))^{\frac{np-1}{p}}}{(n-1)!}
\end{aligned}
$$

$$
\mathfrak{Y}(h; a, b) := \frac{\sigma(b) - \sigma(\frac{a+b}{2}) + |2\sigma(\rho^{n-1}(a+b-h)) - \sigma\left(\frac{a+b}{2}\right) - \sigma(b)|}{2}
$$

$$\mathfrak{X}(h;a,b) := \frac{\sigma\left(\frac{a+b}{2}\right) - \sigma(a) + |2\sigma(\rho^{n-1}(h)) - \sigma\left(\frac{a+b}{2}\right) - \sigma(a)|}{2}$$

$$\mathfrak{D}(y,z;a,b) := 3\sigma(\rho^{n-1}(z)) - \sigma(a) - 2\sigma(b) + |2\sigma(\rho^{n-1}(y)) - \sigma(a) - \sigma(\rho^{n-1}(z))|$$

$$\mathfrak{V}(h;a,b) := [\sigma(\rho^{n-1}(h)) - \sigma(a)]\sqrt[p]{\rho^{n-1}(h) - a} + \left[\sigma\left(\frac{a+b}{2}\right) - \sigma(\rho^{n-1}(h))\right]\sqrt[p]{\frac{a+b}{2} - \rho^{n-1}(h)}$$

$$\begin{aligned}\mathfrak{P}(h;a,b) \quad := \quad & \left[\sigma(\rho^{n-1}(a+b-h)) - \sigma\left(\frac{a+b}{2}\right)\right]\sqrt[p]{\rho^{n-1}(a+b-h) - \frac{a+b}{2}} \\ & + [\sigma(b) - \sigma(\rho^{n-1}(a+b-h))]\sqrt[p]{b - \rho^{n-1}(a+b-h)}\end{aligned}$$

$$\mathfrak{U}(\mathfrak{x}) := \sum_{k=1}^{n-1} \frac{n-k}{b-a} \int_a^b \mu(p) h_{k-1}(\mathfrak{x}, p) f^{\Delta^k}(p) \Delta p$$

$$\mathfrak{R}(\mathfrak{y}) := \sum_{k=1}^{n-1} \frac{n-k}{b-a} \left\{ h_k(\mathfrak{y}, b) f^{\Delta^{k-1}}(b) - h_k(\mathfrak{y}, a) f^{\Delta^{k-1}}(a) \right\}$$

$$\mathfrak{F}^c(y,z;a,b;0,0,q) := \frac{\sqrt[q]{(y-a)^{nq-q+1}} + \sqrt[q]{(z-y)^{nq-q+1}} + \sqrt[q]{(b-z)^{nq-q+1}}}{(n-1)!}$$

$$\mathfrak{F}^c(y,z;a,b;1,1,q) := \frac{(y-a)^{n+1} + (z-y)^{n+1} + (b-z)^{n+1}}{(n-1)!}$$

$$\begin{aligned}\mathfrak{W}^c(y,z;a,b) \quad := \quad & \frac{(y-a)^n + (z-y)^n + |(y-a)^n - (z-y)^n| + 2(b-z)^n}{4(n-1)!} \\ & + \frac{|(y-a)^n + (z-y)^n + |(y-a)^n - (z-y)^n| - 2(b-z)^n|}{4(n-1)!}\end{aligned}$$

$$\mathfrak{R}^c(\mathfrak{y}) := \sum_{k=1}^{n-1} \frac{(n-k)[(\mathfrak{y}-b)^k f^{(k-1)}(b) - (\mathfrak{y}-a)^k f^{(k-1)}(a)]}{(b-a)k!}$$

$$\mathfrak{A}^c(f,z,y,z;n) := \frac{(z-a)f(y) + (b-z)f(z) - (z-a)\mathfrak{R}^c(y) - (b-z)\mathfrak{R}^c(z)}{n(b-a)}$$

$$\mathfrak{A}\left(f, \frac{a+b}{2}, h, a+b-h; n\right) = \frac{f(h) + f(a+b-h)}{2n} - \frac{\mathfrak{S} + \mathfrak{T}}{2n}$$

$$\mathfrak{S} = \sum_{k=1}^{n-1} \frac{n-k}{b-a} \left[f^{\Delta^{k-1}}(b)\{h_k(a+b-h,b) + h_k(h,b)\} - f^{\Delta^{k-1}}(a)\{h_k(a+b-h,a) + h_k(h,a)\} \right]$$

$$\mathfrak{T} = \sum_{k=1}^{n-1} \frac{n-k}{b-a} \int_a^b \mu(w) f^{\Delta^k}(w)[h_{k-1}(h,w) + h_{k-1}(a+b-h,w)]\Delta w$$

$$\mathfrak{A}^d(f, z-n+1, y, z; n) =: \frac{(z-n+1-a)\{f(y) - \mathfrak{R}^d(y) + \mathfrak{U}^d(y)\} + (b-z+n-1)\{f(z) - \mathfrak{R}^d(z) + \mathfrak{U}^d(z)\}}{n(b-a)}$$

$$\mathfrak{U}^d(\mathfrak{x}) := \sum_{p=a}^{b-1}\sum_{k=1}^{n-1}\sum_{m=0}^{k}(-1)^m E^{k-m} f(p)\binom{k}{m}\binom{\mathfrak{x}-p}{k-1}\frac{n-k}{b-a}.$$

$$\mathfrak{R}^d(\mathfrak{y}) := \sum_{k=1}^{n-1}\sum_{m=0}^{k-1}\frac{n-k}{b-a}(-1)^m\binom{k-1}{m}\left\{\binom{\mathfrak{y}-b}{k}E^{k-1-m}f(b)-\binom{\mathfrak{y}-a}{k}E^{k-1-m}f(a)\right\}$$

$$\mathfrak{F}^d(y,z;a,b;s,u,q) := \frac{(z-y)^{\frac{sp+u+np-1}{p}}+(b-z+n-1)^{\frac{sp+u+np-1}{p}}}{(n-1)!}+\frac{(y-n-a+1)^{\frac{sp+u}{p}}(y-a)^{\frac{np-1}{p}}}{(n-1)!}$$

$$\mathfrak{D}^d(y,z;a,b) := 3z-3n+3-a-2b+|2y-z-a-n+1|$$

$$\mathfrak{W}^d(y,z;a,b) := \frac{(y-a+1-n)^n+(z-y)^n+2(b-z+n-1)^n+|(y-a+1-n)^n-(z-y)^n|}{4(n-1)!}$$
$$+\frac{|(y-a+1-n)^n+(z-y)^n-2(b-z+n-1)^n+|(y-a+1-n)^n-(z-y)^n||}{4(n-1)!}$$

$$\mathfrak{S}^c := \sum_{k=1}^{n-1}\frac{(n-k)\{(a-h)^k+(h-b)^k\}\{f^{(k-1)}(b)+(-1)^{k+1}f^{(k-1)}(a)\}}{(b-a)k!}$$

$$\mathfrak{Y}^c(h;a,b) := \frac{b-a+|3a-b-4h-4n+4|}{4}$$

$$\mathfrak{X}^c(h;a,b) := \frac{b-a+|4h-4n+4-3a-b|}{4}$$

$$\mathfrak{V}^c(h;a,b) := (h-n-a+1)^{\frac{1+p}{p}}+\left(\frac{a+b-2h+2n-2}{2}\right)^{\frac{1+p}{p}}$$

$$\mathfrak{P}^c(h;a,b) := (h+n-1-a)^{\frac{1+p}{p}}+\left(\frac{a+b-2h-2n+2}{2}\right)^{\frac{1+p}{p}}.$$

$$\mathfrak{S}^d := \sum_{k=1}^{n-1}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\frac{n-k}{b-a}\left[E^{k-1-m}f(b)\left\{\binom{a-h}{k}+\binom{h-b}{k}\right\}-E^{k-1-m}f(a)\left\{\binom{b-h}{k}+\binom{h-a}{k}\right\}\right]$$

$$\mathfrak{T}^d := \sum_{k=1}^{n-1}\sum_{p=a}^{b-1}\sum_{m=0}^{k}\frac{n-k}{b-a}(-1)^m E^{k-m}f(p)\binom{k}{m}\left\{\binom{h-p}{k-1}+\binom{a+b-h-p}{k-1}\right\}$$

$$\mathfrak{Y}^d(h;a,b) := \frac{b-a+|3a+b-4h-4n+4|}{4}$$

$$\mathfrak{X}^d(h;a,b) := \frac{b-a+|3a+b-4h+4n-4|}{4}$$

$$\mathfrak{K}_1(t;x,y) := \begin{cases}\sigma(t)-a, & a\le\sigma(t)\le\rho^{n-1}(y)\\ \sigma(t)-x, & \rho^{n-1}(y)\le\sigma(t)\le x\le b.\end{cases}$$

$$\mathfrak{K}_2(t;x,z) := \begin{cases}\sigma(t)-x, & a\le x\le\sigma(t)\le\rho^{n-1}(z)\\ \sigma(t)-b, & \rho^{n-1}(z)\le\sigma(t)\le b.\end{cases}$$

$$\mathfrak{K}_3(t;y,z) := \begin{cases}(\sigma(t)-a)h_{n-1}(y,\sigma(t)), & a\le\sigma(t)<\rho^{n-1}(y)\\ (\sigma(t)-\rho^{n-1}(z))h_{n-1}(y,\sigma(t)), & \rho^{n-1}(y)\le\sigma(t)\le\rho^{n-1}(z)\\ (\sigma(t)-b)h_{n-1}(z,\sigma(t)), & \rho^{n-1}(z)\le\sigma(t)\le b.\end{cases} \tag{8}$$

$$\mathfrak{K}_4(t;y,z) := \begin{cases}\sigma(t)-a, & a\le\sigma(t)<\rho^{n-1}(y)\\ \sigma(t)-\rho^{n-1}(z), & \rho^{n-1}(y)\le\sigma(t)\le\rho^{n-1}(z)\\ \sigma(t)-b, & \rho^{n-1}(z)\le\sigma(t)\le b\end{cases} \tag{9}$$

3. main results

Lemma 3.1. *Let $\mathbb{T}$ be a time scale; let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be such that $f \in C_{rd}^{(n)}([a,b])$ for some $n \in N_0$, $h_n(x,t)$ a generalized polynomial, $\mu(p)$ a graininess function such that $\mathfrak{K}_1(t;x,y)h_{n-1}(y,\sigma(t))f^{\Delta^k}(t) \in C_{rd}([a,x])$ and $\mathfrak{K}_2(t;x,z)h_{n-1}(z,\sigma(t))f^{\Delta^k}(t) \in C_{rd}([x,b])$, then*

$$(x-a)f(y) + (b-x)f(z) - (x-a)[\mathfrak{R}(y) + \mathfrak{U}(y)] - (b-x)[\mathfrak{R}(z) + \mathfrak{U}(z)] - n\int_a^b f(w)\Delta w$$
$$= \int_a^x \mathfrak{K}_1(t;x,y)h_{n-1}(y,\sigma(t))f^{\Delta^k}(t)\Delta t + \int_x^b \mathfrak{K}_2(t;x,z)h_{n-1}(z,\sigma(t))f^{\Delta^k}(t)\Delta t. \quad (10)$$

Proof. By Fink Identity [2, Lemma 1]

$$\frac{f(x) - \mathfrak{R}(x) - \mathfrak{U}(x)}{n} - \frac{1}{b-a}\int_a^b f(w)\Delta w = \frac{1}{n(b-a)}\int_a^b h_{n-1}(x,\sigma(t))f^{\Delta^k}(t)\mathfrak{K}_2(t;a,x)\Delta t. \quad (11)$$

Equivalently, equation (11) is rewritten as:

$$\frac{f(x) - \mathfrak{R}(x) - \mathfrak{U}(x)}{n} - \frac{1}{b-a}\int_a^b f(w)\Delta w = \frac{1}{n(b-a)}\int_a^b \Delta s \int_s^{\rho^{n-1}(x)} h_{n-1}(x,\sigma(t))f^{\Delta^k}(t)\Delta t. \quad (12)$$

By fixing $y, z \in [a,b]$ and replacing x by y; b by x in (12), the following holds:

$$\frac{f(y) - \mathfrak{R}(y) - \mathfrak{U}(y)}{n} - \frac{1}{x-a}\int_a^b f(w)\Delta w = \frac{1}{n(x-a)}\int_a^x \Delta s \int_s^{\rho^{n-1}(y)} h_{n-1}(y,\sigma(t))f^{\Delta^k}(t)\Delta t. \quad (13)$$

Similarly, replacing x by z; a by x in (12), the following holds:

$$\frac{f(z) - \mathfrak{R}(z) - \mathfrak{U}(z)}{n} - \frac{1}{b-x}\int_x^b f(w)\Delta w = \frac{1}{n(b-x)}\int_x^b \Delta s \int_s^{\rho^{n-1}(z)} h_{n-1}(z,\sigma(t))f^{\Delta^k}(t)\Delta t. \quad (14)$$

Multiplications to equations (13) and (14), respectively, by $x-a$ and $b-x$, then addition of the resulting equations yields the following:

$$\frac{(x-a)f(y) + (b-x)f(z)}{n} - (x-a)\frac{\mathfrak{R}(y) + \mathfrak{U}(y)}{n} - (b-x)\frac{\mathfrak{R}(z) + \mathfrak{U}(z)}{n} - \int_a^b f(w)\Delta w$$
$$= \frac{1}{n}\left[\int_a^x \Delta s \int_s^{\rho^{n-1}(y)} h_{n-1}(y,\sigma(t))f^{\Delta^k}(t)\Delta t + \int_x^b \Delta s \int_s^{\rho^{n-1}(z)} h_{n-1}(z,\sigma(t))f^{\Delta^k}(t)\Delta t\right] \quad (15)$$

But,

$$\begin{aligned}
\int_a^x \Delta s \int_s^{\rho^{n-1}(y)} \Delta t &= \int_a^{\rho^{n-1}(y)} \Delta s \int_s^{\rho^{n-1}(y)} \Delta t + \int_{\rho^{n-1}(y)}^x \Delta s \int_s^{\rho^{n-1}(y)} \Delta t \\
&= \int_a^{\rho^{n-1}(y)} \Delta t \int_a^{\sigma(t)} \Delta s - \int_{\rho^{n-1}(y)}^x \Delta s \int_{\rho^{n-1}(y)}^s \Delta t \\
&= \int_a^{\rho^{n-1}(y)} \Delta t \int_a^{\sigma(t)} \Delta s - \int_{\rho^{n-1}(y)}^x \Delta t \int_{\sigma(t)}^x \Delta s. \quad (16)
\end{aligned}$$

$$\begin{aligned}\int_x^b \Delta s \int_s^{\rho^{n-1}(z)} \Delta t &= \int_x^{\rho^{n-1}(z)} \Delta s \int_s^{\rho^{n-1}(z)} \Delta t + \int_{\rho^{n-1}(z)}^b \Delta s \int_s^{\rho^{n-1}(z)} \Delta t \\ &= \int_x^{\rho^{n-1}(z)} \Delta t \int_x^{\sigma(t)} \Delta s - \int_x^b \Delta s \int_{\rho^{n-1}(z)}^s \Delta t \\ &= \int_x^{\rho^{n-1}(z)} \Delta t \int_x^{\sigma(t)} \Delta s - \int_{\rho^{n-1}(z)}^b \Delta t \int_{\sigma(t)}^b \Delta s. \end{aligned} \tag{17}$$

In the light of relations (16) and (17), the followings hold true:

$$\begin{aligned}\int_a^x \Delta s \int_s^{\rho^{n-1}(y)} h_{n-1}(y,\sigma(t)) f^{\Delta^k}(t)\Delta t = \int_a^{\rho^{n-1}(y)} \Delta t \int_a^{\sigma(t)} h_{n-1}(y,\sigma(t)) f^{\Delta^k}(t)\Delta s \\ - \int_{\rho^{n-1}(y)}^x \Delta t \int_{\sigma(t)}^x h_{n-1}(y,\sigma(t)) f^{\Delta^k}(t)\Delta s. \end{aligned} \tag{18}$$

$$\begin{aligned}\int_x^b \Delta s \int_s^{\rho^{n-1}(z)} h_{n-1}(z,\sigma(t)) f^{\Delta^k}(t)\Delta t = \int_x^{\rho^{n-1}(z)} \Delta t \int_x^{\sigma(t)} h_{n-1}(z,\sigma(t)) f^{\Delta^k}(t)\Delta s \\ - \int_{\rho^{n-1}(z)}^b \Delta t \int_{\sigma(t)}^b h_{n-1}(z,\sigma(t)) f^{\Delta^k}(t)\Delta s. \end{aligned} \tag{19}$$

A combination of the relations (15), (18) and (19), yields the following:

$$\begin{aligned}&\frac{(x-a)f(y)+(b-x)f(z)}{n} - (x-a)\frac{\Re(y)+\mathfrak{U}(y)}{n} - (b-x)\frac{\Re(z)+\mathfrak{U}(z)}{n} - \int_a^b f(w)\Delta w \\ &= \frac{1}{n}\Bigg[\int_a^{\rho^{n-1}(y)} (\sigma(t)-a)h_{n-1}(y,\sigma(t))f^{\Delta^k}(t)\Delta t - \int_{\rho^{n-1}(y)}^x (x-\sigma(t))h_{n-1}(y,\sigma(t))f^{\Delta^k}(t)\Delta t \\ &\quad + \int_x^{\rho^{n-1}(z)} (\sigma(t)-x)h_{n-1}(z,\sigma(t))f^{\Delta^k}(t)\Delta t - \int_{\rho^{n-1}(z)}^b (b-\sigma(t))h_{n-1}(z,\sigma(t))f^{\Delta^k}(t)\Delta t\Bigg], \end{aligned} \tag{20}$$

which yields the desired result (10). □

Remark 3.2. *Identity* (10) *in Lemma 3.1 is no more than Fink representation of general two point Ostrowski's formula. Either for $x \to a$ or $x \to b$ Lemma 3.1 reduces to [2, Lemma 1].*

For generalized polynomial to be symmetric, in the first component with respect to $\frac{a+b}{2}$, the identity (10) *in Lemma 3.1 reduces to the identity* (11) *in [2, Theorem 1]. Furthermore, by use of the triangle inequality and employing some known norm inequalities, it reduces to [2, Theorem 2]*

In particular, for $x \to \rho^{n-1}(z)$, identity (10) reduces to the following Ostrowski type identity:

Lemma 3.3. *Let $\mathbb{T}$ be a time scale.Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that $f \in C_{rd}^{(n)}([a,b])$ for some $n \in N_0$. Let $h_n(x,t)$ be a generalized polynomial and $\mu(p)$ a graininess function such that $\mathfrak{K}_3(t;y,z)f^{\Delta^k}(t) \in C_{rd}([a,b])$, then*

$$\begin{aligned}&\frac{(\rho^{n-1}(z)-a)f(y)+(b-\rho^{n-1}(z))f(z)-(\rho^{n-1}(z)-a)[\Re(y)+\mathfrak{U}(y)]}{n(b-a)} \\ &\quad - \frac{(b-\rho^{n-1}(z))[\Re(z)+\mathfrak{U}(z)]}{n(b-a)} - \frac{1}{b-a}\int_a^b f(w)\Delta w = \frac{1}{n(b-a)}\int_a^b \mathfrak{K}_3(t;y,z)f^{\Delta^k}(t)\Delta t. \end{aligned} \tag{21}$$

Theorem 3.4. *Let $\mathbb{T}$ be a time scale. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ such that $f \in C_{rd}^{(n)}([a,b])$ for some $n \in N_0$. Let $h_k(x,t)$ be a generalized polynomial and $\mu(p)$ is a graininess function such that $\mathfrak{K}_1(t;x,y)h_{n-1}(y,\sigma(t))f^{\Delta^k}(t) \in C_{rd}([a,x])$ and $\mathfrak{K}_2(t;x,z)h_{n-1}(z,\sigma(t))f^{\Delta^k}(t) \in C_{rd}([x,b])$, $h \in [a, \frac{a+b}{2}]$; let $p > 1$ be a real number such that $\frac{p}{q} = p-1$, then*

$$\left|\frac{f(h)+f(a+b-h)-\mathfrak{S}-\mathfrak{T}}{2n} - \frac{1}{b-a}\int_a^b f(w)\Delta w\right| \leq \frac{\mathfrak{Z}_q(h;a,b)\urcorner(h;a,b)}{n(b-a)} \tag{22}$$

$$\urcorner(h;a,b) := \begin{cases} \mathfrak{X}(h;a,b)\|f^{\Delta^k}\|_{p,[a,\frac{a+b}{2}]} + \mathfrak{Y}(h;a,b)\|f^{\Delta^k}\|_{p,[\frac{a+b}{2},b]}, & q \in (1,\infty) \\ \mathfrak{V}(h;a,b)\|f^{\Delta^k}\|_{\infty,[a,\frac{a+b}{2}]} + \mathfrak{P}(h;a,b)\|f^{\Delta^k}\|_{\infty,[\frac{a+b}{2},b]}, & q \in (1,\infty); \\ \mathfrak{X}(h;a,b)\|f^{\Delta^k}\|_{\infty,[a,\frac{a+b}{2}]} + \mathfrak{Y}(h;a,b)\|f^{\Delta^k}\|_{\infty,[\frac{a+b}{2},b]}, & q = 1. \end{cases}$$

Proof. Utilizing the triangle integral inequality on the identity (10) for $x \to \frac{a+b}{2}$, $z \to a+b-y$, $y \to h$ and employing some known norm inequalities, we get

$$\left|(b-a)\frac{f(h)+f(a+b-h)-\mathfrak{S}-\mathfrak{T}}{2n} - \int_a^b f(w)\Delta w\right|$$
$$\leq \begin{cases} \left\|\mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{\infty,[a,\frac{a+b}{2}]} \times \left\|f^{\Delta^k}\right\|_{p,[a,\frac{a+b}{2}]} \times \sqrt[q]{\int_a^{\frac{a+b}{2}} |h_{n-1}(h,\sigma(t))|^q\Delta t} \\ + \left\|\mathfrak{K}_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{\infty,[\frac{a+b}{2},b]} \times \left\|f^{\Delta^k}\right\|_{p,[\frac{a+b}{2},b]} \times \sqrt[q]{\int_{\frac{a+b}{2}}^b |h_{n-1}(h,\sigma(t))|^q\Delta t}, \\ \left\|f^{\Delta^k}\right\|_{\infty,[a,\frac{a+b}{2}]} \times \left\|\mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{p,[a,\frac{a+b}{2}]} \times \sqrt[q]{\int_a^{\frac{a+b}{2}} |h_{n-1}(h,\sigma(t))|^q\Delta t} \\ + \left\|f^{\Delta^k}\right\|_{\infty,[\frac{a+b}{2},b]} \times \left\|\mathfrak{K}_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{p,[\frac{a+b}{2},b]} \times \sqrt[q]{\int_{\frac{a+b}{2}}^b |h_{n-1}(h,\sigma(t))|^q\Delta t}; \\ \left\|\mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{\infty,[a,\frac{a+b}{2}]} \times \left\|f^{\Delta^k}\right\|_{\infty,[a,\frac{a+b}{2}]} \times \int_a^{\frac{a+b}{2}} |h_{n-1}(h,\sigma(t))|\Delta t + \\ \left\|\mathfrak{K}_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{\infty,[\frac{a+b}{2},b]} \times \left\|f^{\Delta^k}\right\|_{\infty,[\frac{a+b}{2},b]} \times \int_{\frac{a+b}{2}}^b |h_{n-1}(a+b-h,\sigma(t))|\Delta t. \end{cases} \tag{23}$$

For $1 < q < \infty$, consider

$$\begin{aligned} \mathfrak{L}_{q,[a,\frac{a+b}{2}]}(h) &:= \int_a^{\frac{a+b}{2}} |h_{n-1}(h,\sigma(t))|^q\Delta t \\ &= \int_a^h |h_{n-1}(h,\sigma(t))|^q\Delta t + \int_h^{\frac{a+b}{2}} |h_{n-1}(h,\sigma(t))|^q\Delta t \end{aligned} \tag{24}$$

$$\begin{aligned} \mathfrak{L}_{q,[\frac{a+b}{2},b]}(h) &:= \int_{\frac{a+b}{2}}^b |h_{n-1}(a+b-h,\sigma(t))|^q\Delta t \\ &= \int_{\frac{a+b}{2}}^{a+b-h} |h_{n-1}(a+b-h,\sigma(t))|^q\Delta t + \int_{a+b-h}^b |h_{n-1}(a+b-h,\sigma(t))|^q\Delta t \end{aligned} \tag{25}$$

By using [5, Theorem 4.2] and the fact that σ is an increasing function, the following inclusions hold

$$0 \leq h_{n-1}(h,\sigma(t)) \leq \frac{(h-\sigma(t))^{n-1}}{(n-1)!} \leq \frac{(h-t)^{n-1}}{(n-1)!} \leq \frac{(h-a)^{n-1}}{(n-1)!} \tag{26}$$

$$0 \leq h_{n-1}(h,\sigma(t)) \leq \frac{(\frac{a+b}{2}-\sigma(t))^{n-1}}{(n-1)!} \leq \frac{(\frac{a+b}{2}-t)^{n-1}}{(n-1)!} \leq \frac{(\frac{a+b}{2}-h)^{n-1}}{(n-1)!} \tag{27}$$

$$0 \le h_{n-1}(a+b-h,\sigma(t)) \le \frac{(a+b-h-\sigma(t))^{n-1}}{(n-1)!} \le \frac{(a+b-h-t)^{n-1}}{(n-1)!} \le \frac{(\frac{a+b}{2}-h)^{n-1}}{(n-1)!} \tag{28}$$

$$0 \le h_{n-1}(a+b-h,\sigma(t)) \le \frac{(b-\sigma(t))^{n-1}}{(n-1)!} \le \frac{(b-t)^{n-1}}{(n-1)!} \le \frac{(h-a)^{n-1}}{(n-1)!} \tag{29}$$

In the light of the relations (24)-(29), we have

$$\mathfrak{L}_{q,[a,\frac{a+b}{2}]}(h), \mathfrak{L}_{q,[\frac{a+b}{2},b]}(h) \le \frac{(h-a)^{nq-q+1} + (\frac{a+b}{2}-h)^{nq-q+1}}{[(n-1)!]^q} \tag{30}$$

But, $\sigma - a$, $\sigma - \frac{a+b}{2}$ are increasing on respective intervals $[a, \rho^{n-1}(h)]$, $[\rho^{n-1}(h), \frac{a+b}{2}]$ and $\sigma - \frac{a+b}{2}$, $\sigma - b$ are increasing on respective intervals $[\frac{a+b}{2}, \rho^{n-1}(a+b-h)]$, $[\rho^{n-1}(a+b-h), b]$ and hence by [1]

$$\begin{aligned}
&\left\| \mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{p,\left[a,\frac{a+b}{2}\right]}\\
&\le \left\| \mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{p,\left[a,\rho^{n-1}(h)\right]} + \left\| \mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{p,\left[\rho^{n-1}(h),\frac{a+b}{2}\right]}\\
&= \|\sigma(t)-a\|_{p,\left[a,\rho^{n-1}(h)\right]} + \left\|\sigma(t)-\frac{a+b}{2}\right\|_{p,\left[\rho^{n-1}(h),\frac{a+b}{2}\right]}\\
&\le \left[\sigma(\rho^{n-1}(h))-\sigma(a)\right]\sqrt[p]{\rho^{n-1}(h)-a} + \left[\sigma\left(\frac{a+b}{2}\right)-\sigma(\rho^{n-1}(h))\right]\\
&\quad \times \sqrt[p]{\frac{a+b}{2}-\rho^{n-1}(h)}
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\left\| \mathfrak{K}_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{p,\left[\frac{a+b}{2},b\right]}\\
&\le \left\| \mathfrak{K}_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{p,\left[\frac{a+b}{2},\rho^{n-1}(a+b-h)\right]} + \left\| \mathfrak{K}_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{p,\left[\rho^{n-1}(a+b-h),b\right]}\\
&= \left\|\sigma(t)-\frac{a+b}{2}\right\|_{p,\left[\frac{a+b}{2},\rho^{n-1}(a+b-h)\right]} + \|\sigma(t)-b\|_{p,\left[\rho^{n-1}(a+b-h),b\right]}\\
&\le \left[\sigma(\rho^{n-1}(a+b-h))-\sigma\left(\frac{a+b}{2}\right)\right]\sqrt[p]{\rho^{n-1}(a+b-h)-\frac{a+b}{2}} +\\
&\quad \left[\sigma(b)-\sigma(\rho^{n-1}(a+b-h))\right]\sqrt[p]{b-\rho^{n-1}(a+b-h)}
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\left\| \mathfrak{K}_1\left(.;\frac{a+b}{2},h\right)\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}\\
\le\ & \|\sigma(t)-a\|_{\infty,\left[a,\rho^{n-1}(h)\right]} + \left\|\sigma(t)-\frac{a+b}{2}\right\|_{\infty,\left[\rho^{n-1}(h),\frac{a+b}{2}\right]}\\
\le\ & \max\left\{\sigma(\rho^{n-1}(h))-\sigma(a), \sigma\left(\frac{a+b}{2}\right)-\sigma(\rho^{n-1}(h))\right\}\\
=\ & \frac{\sigma(\frac{a+b}{2})-\sigma(a)+|2\sigma(\rho^{n-1}(h))-\sigma(\frac{a+b}{2})-\sigma(a)|}{2}
\end{aligned} \tag{33}$$

$$
\begin{aligned}
&\left\|\Re_2\left(.;\frac{a+b}{2},a+b-h\right)\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\\
\leq\ & \left\|\sigma(t)-\frac{a+b}{2}\right\|_{\infty,\left[\frac{a+b}{2},\rho^{n-1}(a+b-h)\right]}+\|\sigma(t)-b\|_{\infty,[\rho^{n-1}(a+b-h),b]}\\
\leq\ & \max\left\{\sigma(\rho^{n-1}(a+b-h))-\sigma\left(\frac{a+b}{2}\right),\sigma(b)-\sigma(\rho^{n-1}(a+b-h))\right\}\\
=\ & \frac{\sigma(b)-\sigma(\frac{a+b}{2})+|2\sigma(\rho^{n-1}(a+b-h))-\sigma\left(\frac{a+b}{2}\right)-\sigma(b)|}{2} \qquad (34)
\end{aligned}
$$

Here, we have used the identity, $\max\{a,b\}=\frac{a+b+|a-b|}{2}$.

A combination of (23) and (30)-(34), yields the desired result (22) □

Theorem 3.5. *Let $\mathbb{T}$ be a time scale.Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ such that $f\in C_{rd}^{(n)}([a,b])$ for some $n\in N_0$. Let $h_n(x,t)$ be a generalized polynomial and $\mu(p)$ a graininess function such that $\Re_3(t;y,z)f^{\Delta^k}(t)\in C_{rd}([a,b])$; let $p>1$ be a real number such that $\frac{p}{q}=p-1$, then*

$$
\left|\mathfrak{U}(f,\rho^{n-1}(z),y,z;n)-\frac{1}{b-a}\int_a^b f(w)\Delta w\right| \qquad (35)
$$

$$
\leq\begin{cases}
\mathfrak{W}(y,z;a,b)\|f^{\Delta^k}\|_{1,[a,b]}.\\
\frac{\{\mathfrak{D}(y,z;a,b)-4\sigma(\rho^{n-1}(z))+4\sigma(b)+|\mathfrak{D}(y,z;a,b)|\}\mathfrak{L}_1(y,z)\|f^{\Delta^k}\|_{\infty,[a,b]}}{4}.\\
\frac{\{\mathfrak{D}(y,z;a,b)-4\sigma(\rho^{n-1}(z))+4\sigma(b)+|\mathfrak{D}(y,z;a,b)|\}\|f^{\Delta^k}\|_{p,[a,b]}}{4}\\
\times\mathfrak{F}(y,z;a,b;0,0,q), & 1<p,q<\infty\\
\mathfrak{F}(y,z;a,b;1,1,q)\|f^{\Delta^k}\|_{\infty,[a,b]}, & 1<p,q<\infty
\end{cases}
$$

Proof. Utilizing the triangle integral inequality on the identity (21) and employing some known norm inequalities, we get

$$
\begin{aligned}
&\left|\frac{(\rho^{n-1}(z)-a)f(y)+(b-\rho^{n-1}(z))f(z)-(\rho^{n-1}(z)-a)\Re(y)-(b-\rho^{n-1}(z))\Re(z)}{n(b-a)}\right.\\
&\left.-\frac{(b-\rho^{n-1}(z))\mathfrak{U}(z)+(\rho^{n-1}(z)-a)\mathfrak{U}(y)}{n(b-a)}-\frac{1}{b-a}\int_a^b f(w)\Delta w\right|\\
&\leq\frac{1}{n(b-a)}\int_a^b|f^{\Delta^k}(t)|\times|\Re_3(t;y,z)|\Delta t
\end{aligned}
$$

$$
\leq\begin{cases}
\|\Re_3(y,z)\|_{\infty,[a,b]}\times\|f^{\Delta^k}\|_{1,[a,b]}.\\
\|\Re_4(y,z)\|_{\infty,[a,b]}\times\|f^{\Delta^k}\|_{\infty,[a,b]}\times\mathfrak{L}_1(y,z).\\
\|\Re_4(y,z)\|_{\infty,[a,b]}\times\|f^{\Delta^k}\|_{p,[a,b]}\times\left\{\sqrt[q]{\frac{(y-a)^{nq-q+1}}{[(n-1)!]^q}}\right.\\
\left.+\sqrt[q]{\frac{(\rho^{n-1}(z)-\rho^{n-1}(y))^{nq-q+1}}{[(n-1)!]^q}}+\sqrt[q]{\frac{(b-\rho^{n-1}(z))^{nq-q+1}}{[(n-1)!]^q}}\right\}, & 1<p,q<\infty\\
\|f^{\Delta^k}\|_{\infty,[a,b]}\times\left\{[\sigma(\rho^{n-1}(y))-\sigma(a)]\sqrt[q]{\frac{(y-a)^{nq-q+1}}{[(n-1)!]^q}}\sqrt[p]{\rho^{n-1}(y)-a}\right.\\
+[\sigma(\rho^{n-1}(z))-\sigma(\rho^{n-1}(y))]\sqrt[p]{\rho^{n-1}(z)-\rho^{n-1}(y)}\\
\times\sqrt[q]{\frac{(\rho^{n-1}(z)-\rho^{n-1}(y))^{nq-q+1}}{[(n-1)!]^q}}+[\sigma(b)-\sigma(\rho^{n-1}(z))]\\
\left.\times\sqrt[p]{b-\rho^{n-1}(z)}\sqrt[q]{\frac{(b-\rho^{n-1}(z))^{nq-q+1}}{[(n-1)!]^q}}\right\}, & 1<p,q<\infty
\end{cases} \qquad (36)
$$

For $1 < q < \infty$, consider

$$\mathfrak{L}_q(y,z) \leq \int_a^{\rho^{n-1}(z)} |h_{n-1}(y,\sigma(t))|^q \Delta t + \int_{\rho^{n-1}(z)}^b |h_{n-1}(z,\sigma(t))|^q \Delta t \tag{37}$$

By using [5, Theorem 4.2] and the fact that σ is an increasing function, the following inclusions hold

$$0 \leq h_{n-1}(y,\sigma(t)) \leq \frac{(\rho^{n-1}(y)-\sigma(t))^{n-1}}{(n-1)!} \leq \frac{(\rho^{n-1}(y)-t)^{n-1}}{(n-1)!} \leq \frac{(\rho^{n-1}(y)-a)^{n-1}}{(n-1)!} \tag{38}$$

$$0 \leq h_{n-1}(y,\sigma(t)) \leq \frac{(\rho^{n-1}(z)-\sigma(t))^{n-1}}{(n-1)!} \leq \frac{(\rho^{n-1}(z)-t)^{n-1}}{(n-1)!} \leq \frac{(\rho^{n-1}(z)-\rho^{n-1}(y))^{n-1}}{(n-1)!} \tag{39}$$

$$0 \leq h_{n-1}(z,\sigma(t)) \leq \frac{(b-\sigma(t))^{n-1}}{(n-1)!} \leq \frac{(b-t)^{n-1}}{(n-1)!} \leq \frac{(b-\rho^{n-1}(z))^{n-1}}{(n-1)!} \tag{40}$$

But, $\sigma - a$, $\sigma - \rho^{n-1}(z)$ and $\sigma - b$ are increasing on respective intervals $[a, \rho^{n-1}(y)]$, $[\rho^{n-1}(y), \rho^{n-1}(z)]$ and $[\rho^{n-1}(z), b]$ and hence by [1]

$$\begin{aligned}
&\|\mathfrak{K}_4(y,z)\|_{p,[a,b]} \leq \|\mathfrak{K}_4(y,z)\|_{p,[a,\rho^{n-1}(y)]} + \|\mathfrak{K}_4(y,z)\|_{p,[\rho^{n-1}(y),\rho^{n-1}(z)]} + \|\mathfrak{K}_4(y,z)\|_{p,[\rho^{n-1}(z),b]} \\
&= \|\sigma(t)-a\|_{p,[a,\rho^{n-1}(y)]} + \left\|\sigma(t)-\rho^{n-1}(z)\right\|_{p,[\rho^{n-1}(y),\rho^{n-1}(z)]} + \|\sigma(t)-b\|_{p,[\rho^{n-1}(z),b]} \\
&\leq [\sigma(\rho^{n-1}(y))-\sigma(a)]\sqrt[p]{\rho^{n-1}(y)-a} + [\sigma(\rho^{n-1}(z))-\sigma(\rho^{n-1}(y))] \\
&\times \sqrt[p]{\rho^{n-1}(z)-\rho^{n-1}(y)} + [\sigma(b)-\sigma(\rho^{n-1}(z))]\sqrt[p]{b-\rho^{n-1}(z)}
\end{aligned} \tag{41}$$

In the light of the relations (38)-(40), relations (37) and (8) reduce to

$$\mathfrak{L}_q(y,z) \leq \frac{(y-a)^{nq-q+1} + (\rho^{n-1}(z)-\rho^{n-1}(y))^{nq-q+1} + (b-\rho^{n-1}(z))^{nq-q+1}}{[(n-1)!]^q}. \tag{42}$$

$$\begin{aligned}
\|\mathfrak{K}_3(y,z)\|_{\infty,[a,b]} &\leq \|(\sigma(t)-a)h_{n-1}(y,\sigma(t))\|_{\infty,[a,\rho^{n-1}(y)]} + \|(\sigma(t)-b)h_{n-1}(z,\sigma(t))\|_{\infty,[\rho^{n-1}(z),b]} \\
&\quad + \left\|(\sigma(t)-\rho^{n-1}(z))h_{n-1}(y,\sigma(t))\right\|_{\infty,[\rho^{n-1}(y),\rho^{n-1}(z)]} \\
&\leq \|(\sigma(t)-a)\|_{\infty,[a,\rho^{n-1}(y)]}\frac{(\rho^{n-1}(y)-a)^{n-1}}{(n-1)!} + \left\|(\sigma(t)-\rho^{n-1}(z))\right\|_{\infty,[\rho^{n-1}(y),\rho^{n-1}(z)]} \\
&\quad \times \frac{(\rho^{n-1}(z)-\rho^{n-1}(y))^{n-1}}{(n-1)!} + \|(\sigma(t)-b)\|_{\infty,[\rho^{n-1}(z),b]}\frac{(b-\rho^{n-1}(z))^{n-1}}{(n-1)!} \\
&\leq \max\left\{\frac{(\sigma(\rho^{n-1}(y))-\sigma(a))(\rho^{n-1}(y)-a)^{n-1}}{(n-1)!},\right. \\
&\quad \frac{(\sigma(\rho^{n-1}(z))-\sigma(\rho^{n-1}(y)))(\rho^{n-1}(z)-\rho^{n-1}(y))^{n-1}}{(n-1)!}, \\
&\quad \left.\frac{(\sigma(b)-\sigma(\rho^{n-1}(a+b-x)))(b-\rho^{n-1}(z))^{n-1}}{(n-1)!}\right\} \\
&= \frac{\alpha+\beta+|\alpha-\beta|+2\gamma+|\alpha+\beta+|\alpha-\beta|-2\gamma|}{4}
\end{aligned} \tag{43}$$

$$
\begin{aligned}
\|\mathfrak{K}_4(y,z)\|_{\infty,[a,b]} &\le \|\sigma(t)-a\|_{\infty,[a,\rho^{n-1}(y)]} + \left\|\sigma(t)-\rho^{n-1}(z)\right\|_{\infty,[\rho^{n-1}(y),\rho^{n-1}(z)]} + \|\sigma(t)-b\|_{\infty,[\rho^{n-1}(z),b]} \\
&\le \max\{(\sigma(\rho^{n-1}(y))-\sigma(a)),(\sigma(\rho^{n-1}(z))-\sigma(\rho^{n-1}(y))),(\sigma(b)-\sigma(\rho^{n-1}(z)))\} \\
&= \frac{-\sigma(\rho^{n-1}(z))-\sigma(a)+2\sigma(b)+|2\sigma(\rho^{n-1}(y))-\sigma(\rho^{n-1}(z))-\sigma(a)|}{4} \\
&+ \frac{|3\sigma(\rho^{n-1}(z))-\sigma(a)-2\sigma(b)+|2\sigma(\rho^{n-1}(y))-\sigma(\rho^{n-1}(z))-\sigma(a)||}{4} \\
&= \frac{\mathfrak{D}(y,z;a,b)-4\sigma(\rho^{n-1}(z))+4\sigma(b)+|\mathfrak{D}(y,z;a,b)|}{4}
\end{aligned}
\tag{44}
$$

Here, we have used the identity

$$
\max\{a,b,c\} = \frac{a+b+|a-b|+2c+|a+b+|a-b|-2c|}{4}.
$$

□

Theorem 3.6. *Let $\mathbb{T}$ be a time scale, $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ a function such that $f \in C_{rd}^{(n)}([a,b])$ for some $n \in N_0$; let $h_n(x,t)$ be a generalized polynomial, $\mu(y)$ a graininess function such that $\mathfrak{K}_3(t;y,z)f^{\Delta^k}(t) \in C_{rd}([a,b])$. Moreover, if $\phi \le \mathfrak{K}_3(t;y,z) \le \varphi$ and $\gamma \le f^{\Delta^k}(t) \le \Gamma$, then*

$$
|\mathfrak{O}(f;y,z,n)| \le \frac{(\phi-\varphi)(\Gamma-\gamma)}{4} \tag{45}
$$

Proof. By using the relations (3), (4), (7) and (46), we get the desired result (45)

$$
\begin{aligned}
\int_a^b \mathfrak{K}_4(t;y,z)\Delta t &= \int_a^{\rho^{n-1}(y)} (\sigma(t)-a)\Delta t + \int_{\rho^{n-1}(y)}^{\rho^{n-1}(z)} (\sigma(t)-\rho^{n-1}(z))\Delta t + \int_{\rho^{n-1}(z)}^b (\sigma(t)-b)\Delta t \\
&= \int_a^{\rho^{n-1}(y)} [t^2]^{\Delta}\Delta t + \int_{\rho^{n-1}(z)}^b [t^2]^{\Delta}\Delta t + \int_{\rho^{n-1}(y)}^{\rho^{n-1}(z)} [t^2]^{\Delta}\Delta t \\
&\quad - \int_a^{\rho^{n-1}(y)} t\Delta t - \int_{\rho^{n-1}(z)}^b t\Delta t - \int_{\rho^{n-1}(y)}^{\rho^{n-1}(z)} t\Delta t + a[a-\rho^{n-1}(y)] \\
&\quad + \rho^{n-1}(z)[\rho^{n-1}(y)-\rho^{n-1}(z)] + b[\rho^{n-1}(z)-b] \\
&= h_2(a,\rho^{n-1}(y)) - h_2(\rho^{n-1}(z),\rho^{n-1}(y)) - h_2(b,\rho^{n-1}(z))
\end{aligned}
\tag{46}
$$

□

Remark 3.7. *By setting $h \to a, b$ and $h \to \frac{a+b}{2}$ in Theorem 3.4, respectively, we can find the bounds of the trapezoidal type and midpoint type inequalities. Inequalities* (35) *and* (45)*, respectively, provide different bounds for two point Ostrowski type and Čebyšev-Gruss type functionals.*

4. Applications

For the approximation of $\int_a^b f(w)\Delta w$, the general quadrature rule is defined by:

$$
\frac{1}{b-a}\int_a^b f(w)\Delta w = \mathfrak{A}(f,x,y,z;n) + \mathfrak{E}(f,x,y,z;n), \tag{47}
$$

for $a \le \rho^{n-1}(y) \le x \le \rho^{n-1}(z) \le b$, provided that:

$$
\mathfrak{A}(f,x,y,z;n) := \frac{1}{n(b-a)}\{(x-a)f(y) + (b-x)f(z) - (x-a)[\mathfrak{R}(y)+\mathfrak{U}(y)] - (b-x)[\mathfrak{R}(z)+\mathfrak{U}(z)]\}
$$

$$\mathfrak{E}(f,x,y,z;n) := -\frac{1}{n(b-a)}\left[\int_a^x \mathfrak{K}_1(t;x,y)h_{n-1}(y,\sigma(t))f^{\Delta^k}(t)\Delta t + \int_x^b \mathfrak{K}_2(t;x,z)h_{n-1}(z,\sigma(t))f^{\Delta^k}(t)\Delta t\right] \quad (48)$$

For particular choices for $y \to h$ and $z \to a+b-h$, so that $h \in [\rho^{n-1}(a), \rho^{n-1}(x)]$, in equation (47), yields the following Ostrowski's type formula via Fink approach by generalized polynomial

$$\frac{1}{b-a}\int_a^b f(w)\Delta w = \mathfrak{A}(f,x,h,a+b-h;n) + \mathfrak{E}(f,x,h,a+b-h;n), \quad (49)$$

In particular, for $x \to \frac{a+b}{2}$ and $h_n(h,.) = h_n(a+b-h,.)$ it reduces to the Guessab-Schmeisser quadrature formula [2, Theorem 1] and hence utilizing the triangle integral inequality and employing some known norm inequalities we can retrieve [2, Theorem 2]. Particular consequences for different values of h produce different quadrature rule. For instance,

$$\frac{1}{b-a}\int_a^b f(w)\Delta w = \mathfrak{A}\left(f,\frac{a+b}{2},a,b;n\right) + \mathfrak{E}\left(f,\frac{a+b}{2},a,b;n\right) \quad \text{(Trapezoidal rule)}$$

$$\frac{1}{b-a}\int_a^b f(w)\Delta w = \mathfrak{A}\left(f,\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};n\right) + \mathfrak{E}\left(f,\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};n\right) \quad \text{(Midpoint rule)}$$

Example 4.1. *(Continuous case) Under the conditions of Theorem 3.5 for $\mathbb{T} = \mathbb{R}$, inequality* (35) *reduces to*

$$\left|\mathfrak{A}^c(f,z,y,z;n) - \frac{1}{b-a}\int_a^b f(w)dw\right| \quad (50)$$
$$\leq \begin{cases} \mathfrak{W}^c(y,z;a,b)\left\|f^{(k)}\right\|_{1,[a,b]}^c, \\ \frac{\{2b-a-z+|2y-a-z|+|3z-a-2b+|2y-a-z||\}}{4} \\ \times\mathfrak{F}^c(y,z;a,b;0,0,1)\left\|f^{(k)}\right\|_{\infty,[a,b]}^c, \\ \frac{\{2b-a-z+|2y-a-z|+|3z-a-2b+|2y-a-z||\}\left\|f^{(k)}\right\|_{p,[a,b]}^c}{4} \\ \times\mathfrak{F}^c(y,z;a,b;0,0,q), & 1<p,q<\infty \\ \mathfrak{F}^c(y,z;a,b;1,1,q)\left\|f^{(k)}\right\|_{\infty,[a,b]}^c, & 1<p,q<\infty \end{cases}$$

Example 4.2. *(Discrete case) Under the conditions of Theorem 3.5 for $\mathbb{T} = \mathbb{Z}$, inequality* (35) *reduces to*

$$\left|\mathfrak{A}^d(f,z-n+1,y,z;n) - \frac{1}{b-a}\sum_{p=0}^{b-1} f(p)\right| \quad (51)$$
$$\leq \begin{cases} \mathfrak{W}^d(y,z;a,b)\left\|f^{\Delta^k}\right\|_{1,[a,b]}^d, \\ \frac{\{n-z+2b-a-1+|2y-n-z-a+1|+|\mathfrak{D}^d(y,z;a,b)|\}}{4(n-1)!} \\ \times[(y-a)^n+(z-y)^n+(b-z+n-1)^n]\left\|f^{\Delta^k}\right\|_{\infty,[a,b]}^d, \\ \frac{\{n-z+2b-a-1+|2y-n-z-a+1|+|\mathfrak{D}^d(y,z;a,b)|\}\left\|f^{\Delta^k}\right\|_{p,[a,b]}^d}{4} \\ \times\mathfrak{F}^d(y,z;a,b;0,0,q), & 1<p,q<\infty \\ \mathfrak{F}^d(y,z;a,b;1,1,q)\left\|f^{\Delta^k}\right\|_{\infty,[a,b]}^d, & 1<p,q<\infty \end{cases}$$

Example 4.3. *(Continuous case) Under the conditions of Theorem 3.4 for $\mathbb{T} = \mathbb{R}$, inequality* (22) *reduces to*

$$\left|\frac{f(h)+f(a+b-h)-\mathfrak{S}^c}{2n} - \frac{1}{b-a}\int_a^b f(w)dw\right| \leq \frac{\mathfrak{Z}_q(h;a,b)\urcorner^c(h;a,b)}{n(b-a)} \quad (52)$$

$$\daleth^c(h;a,b) := \begin{cases} \mathfrak{X}^c(h;a,b)\left\|f^{(k)}\right\|^c_{p,[a,\frac{a+b}{2}]} + \mathfrak{Y}^c(h;a,b)\left\|f^{(k)}\right\|^c_{p,[\frac{a+b}{2},b]}, \ q \in (1,\infty) \\ \mathfrak{B}^c(h;a,b)\left\|f^{(k)}\right\|^c_{\infty,[a,\frac{a+b}{2}]} + \mathfrak{P}^c(h;a,b)\left\|f^{(k)}\right\|^c_{\infty,[\frac{a+b}{2},b]}, \ q \in (1,\infty); \\ \mathfrak{X}^c(h;a,b)\left\|f^{(k)}\right\|^c_{\infty,[a,\frac{a+b}{2}]} + \mathfrak{Y}^c(h;a,b)\left\|f^{(k)}\right\|^c_{\infty,[\frac{a+b}{2},b]}, \ q = 1. \end{cases}$$

Example 4.4. *(Discrete case) Under the conditions of Theorem 3.4 for $\mathbb{T} = \mathbb{Z}$, inequality* (22) *reduces to*

$$\left| \frac{f(h) + f(a+b-h)}{2n} - \frac{\mathfrak{S}^d + \mathfrak{T}^d}{2n} - \frac{1}{b-a}\sum_{p=0}^{b-1} f(p) \right| \le \frac{\mathfrak{Z}_q(h;a,b)\daleth^d(h;a,b)}{n(b-a)} \tag{53}$$

$$\daleth^d(h;a,b) := \begin{cases} \mathfrak{X}(h;a,b)\left\|f^{\Delta^k}\right\|_{p,[a,\frac{a+b}{2}]} + \mathfrak{Y}(h;a,b)\left\|f^{\Delta^k}\right\|_{p,[\frac{a+b}{2},b]}, \ q \in (1,\infty) \\ \mathfrak{B}(h;a,b)\left\|f^{\Delta^k}\right\|_{\infty,[a,\frac{a+b}{2}]} + \mathfrak{P}(h;a,b)\left\|f^{\Delta^k}\right\|_{\infty,[\frac{a+b}{2},b]}, \ q \in (1,\infty); \\ \mathfrak{X}(h;a,b)\left\|f^{\Delta^k}\right\|_{\infty,[a,\frac{a+b}{2}]} + \mathfrak{Y}(h;a,b)\left\|f^{\Delta^k}\right\|_{\infty,[\frac{a+b}{2},b]}, \ q = 1. \end{cases}$$

5. Graphical Analysis

In this section , We discuss the graphical representation of our main results, which are helpful to understand theoretical results.
If we choose , $f(y) = y^2$, $y = z$, $n = 2$, and $[a,b] = [0,1]$ in Theorem 3.5 for $\mathbb{T} = \mathbb{R}$, then we have

$$\left| \frac{y^2 - y + 0.44}{2} \right| \le \begin{cases} \frac{2y^2-2y+1+|2y-1|}{2}. \\ (1+|2y-1|)\left(2y^2-2y+1\right). \\ 0.577\,(1+|2y-1|)\left(y^{\frac{3}{2}} + (1-y)^{\frac{3}{2}}\right), p = q = 2 \\ 2\left(y^3 + (1-y)^3\right), \qquad p = q = 2 \end{cases} \tag{54}$$

Comparison of left and right sides of Theorem 3.5

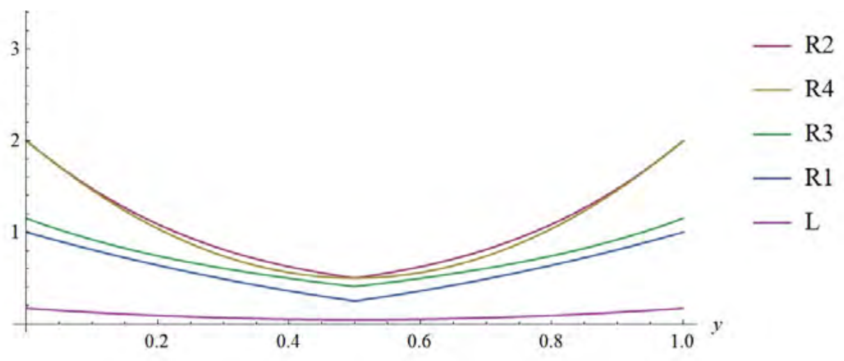

Figure 1 This is an image showing the comparison between left and right sides of Theorem 3.5.

If we choose, $f(h) = h^2$, $n = 2$ for $[a,b] = [-1,5]$ in Theorem 3.4 for $\mathbb{T} = \mathbb{R}$, then we have

$$\left|\frac{h^2-4h+7}{2}\right| \le \begin{cases} \left[\frac{2|h+3|+|2h-3|+38}{24}\right]\sqrt{(h+1)^3+(2-h)^3}, & q=2 \\ \left[\frac{h^{\frac{3}{2}}+(3-h)^{\frac{3}{2}}+2.5\left((h+2)^{\frac{3}{2}}+(1-h)^{\frac{3}{2}}\right)}{3}\right]\sqrt{(h+1)^3+(2-h)^3}, & q=2 \\ \left[\frac{5|h+3|+|2h-3|+10.5}{6}\right]\left((h+1)^2+(2-h)^2\right), & q=1. \end{cases} \tag{55}$$

Comparison of left and right sides of Theorem 3.4

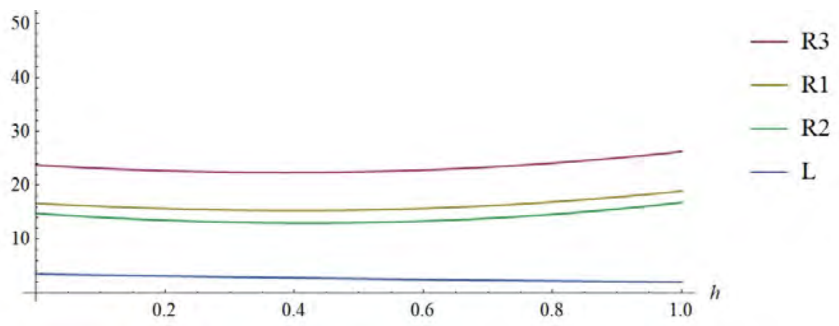

Figure 2 This is an image showing the comparison between left and right sides of Theorem 3.4.

References

[1] A. A. Aljinović,*Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral*, J. Math., Vol. 2014, Article ID 503195, 1-6.

[2] S. Hussain, I. Urooj, S. S. Dragomir, M. A. Latif,*Guessab-Schmeisser inequality via Fink Identity on time scales*, RACSAM, 114(48)(2020), 1-14.

[3] M. W. Alomari,*Two-point Ostrowski's inequality*, Results in Mathematics 72(3)(2017), 1499–1523.

[4] M. W. Alomari,*Generalizations of Guessab-Schmeisser formula via Fink type identity with applications to Quadrature rules*, arxi :1703 .01535v1 [math.CA] 4(3)(2017).

[5] M. Bohner, G. Sh. Guseinov,*The Convolution on Time Scales*, Abst. Appl. Anal., Volume 2007, Article ID 58373, 1-24.

[6] M. Bohner, A. Peterson,*Dynamic equations on time scales*, Boston, MA: Birkhäuser Boston Inc., 2001.

[7] L. Dedić, M. Matić, J. Pečarić,*On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl., 3(3)(2000), 337-353.

[8] A. M. Fink,*Bounds of the deviation of a function from its avereges*, Czechoslovak Math. J., 42(**117**)(1992), 289-310.

[9] A. Guessab, G. Schmeisser,*Sharp integral inequalities of the Hermite-Hadamard type*, J. Approx. Th., 115(2002), 260-288.

[10] S. Hussain,*Generalization of Ostrowski and Čebyšev type inequalities involving many functions*, Aequat. Math. 85(2013), 409-419.

[11] Z. Liu, *Some companions of an Ostrowski type inequality and applications*, J. Ineq. Pure and Appl. Math., 10(**2**)(2009), Article No. 52.

[12] G. V. Milovanović and J. E. Pečarić, *On generalization of the inequality of A. Ostrowski and some related applications*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 544-576(1976), 155-158.

[13] D. Mozyrska, E. Pawłuszewicz,*Delta and Nabla Monomials and generalized polynomial series on time scales*, DOI: 10.1090/conm/514/10108.

[14] A. M. Ostrowski,*On an integral inequality*, Aequat. Math., **4**(1970), 358-373.

[15] M. Z. Sarikaya, N. Aktan, H. Yildirim, *On weight Čebyšev-Grüss type inequalities on time scales*, J. Math. Inequal., 2(**2**)(2008), 185-195.