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Some new refinements of numerical radius inequalities for Hilbert and semi-Hilbert space operators

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Abstract. Let T and S be bounded linear operators on a complex Hilbert space $\mathcal{H}$. In this paper, we define a new quantity $K(T)$ which is less than the numerical radius $w(T)$ of T. We employ this quantity to provide some new refinements of the numerical radii of products TS, commutators $TS - ST$, and anticommutators $TS + ST$, which give an improvement to the important results by A. Abu-Omar and F. Kittaneh (Studia Mathematica, 227 (2), (2015)). Furthermore, we extend these results to the case of semi-Hilbertian space operators in order to improve some results of A. Zamani (Linear Algebra and its Applications, 578, (2019)).

1. Introduction and preliminary

Let $\big(\mathcal{H}, \langle\cdot,\cdot\rangle\big)$ be a complex Hilbert space equiped with the norm $\|\cdot\|$. We denote by $\mathcal{B}(\mathcal{H})$ the unital C*-algebra of all bounded linear operators acting on $\mathcal{H}$, with I its unit element. Let $T \in \mathcal{B}(\mathcal{H})$, we denote by T^*, $\mathfrak{R}(T) = \dfrac{T+T^*}{2}$ and $\mathfrak{I}(T) = \dfrac{T-T^*}{2i}$ the adjoint, the real part and the imaginary part of T, respectively. The numerical range of T is given by:

$$W(T) = \Big\{\langle Tx, x\rangle \ : \ x \in \mathcal{H}, \ \|x\| = 1\Big\}.$$

It is well-known that $W(T)$ is a convex set in $\mathbb{C}$ and its closure $\overline{W(T)}$ contains the spectrum $\sigma(T)$ of T. Moreover, $W(T^*) = \Big\{\overline{\lambda} : \lambda \in W(T)\Big\}$ and $W(U^*TU) = W(T)$ for any unitary $U \in \mathcal{B}(\mathcal{H})$. The operator norm, the numerical radius and the spectral radius of T are denoted by $\|T\|$, $w(T)$ and $r(T)$ respectively and they are given by $\|T\| = \sup\Big\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\Big\}$, $w(T) = \sup\Big\{|\lambda| : \lambda \in W(T)\Big\}$ and $r(T) = \sup\Big\{|z| : z \in \sigma(T)\Big\}$. It is known that

$$\max\Big\{\frac{1}{2}\|T\|; r(T)\Big\} \leq w(T) \leq \|T\|, \tag{1}$$

thus $w(\cdot)$ defines an equivalent norm to the usual operator norm $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$. It is also easy to check that the norm $w(\cdot)$ is self-adjoint, i.e., $w(T^*) = w(T)$ for every $T \in \mathcal{B}(\mathcal{H})$. Furthermore, if T is normal then

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$w(T) = \|T\| = r(T)$. It is well-known that the norm $w(\cdot)$ satisfies the power inequality

$$w\left(T^n\right) \leq \left(w(T)\right)^n \text{ (for all } n \in \mathbb{N}).$$

Unfortunately, the norm $w(\cdot)$ is not submultiplicative. This means that, in general, the inequality

$$w\left(TS\right) \leq w(T)w(S) \text{ (for } T, S \in \mathcal{B}(\mathcal{H})),$$

does not hold. Indeed, if we take

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

it is easy to check that $w(T) = w(S) = \dfrac{1}{2}$ and $w(TS) = 1$. Moreover, the inequality

$$w\left(TS\right) \leq \|T\|w(S) \text{ (for } T, S \in \mathcal{B}(\mathcal{H})),$$

is not true in general, even if T and S commute, see [29]. In 2022, Benabdi et al.[9] proved that

$$w\left(TS\right) \leq \frac{3\sqrt{3}}{4}\|T\|w(S),$$

whenever $T, S \in \mathcal{B}(\mathcal{H})$ such that T is a positive operator, i.e., $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. They also showed that the constant $\dfrac{3\sqrt{3}}{4}$ is the smallest possible for which the inequality holds.

In addition, we have the following numerical radius inequalities.

Theorem 1.1 ([21], Chapter 2). *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

1. $$w\left(TS\right) \leq \|TS\| \leq 2\|T\|w(S) \leq 4w(T)w(S).$$

2. *If T and S commute, then*
$$w\left(TS\right) \leq 2w(T)w(S).$$

3. *If T and S doubly commute (i.e., $TS = ST$ and $TS^* = S^*T$), then*
$$w\left(TS\right) \leq \|T\|w(S).$$

4. $$w\left(TS + ST\right) \leq 4w(T)w(S).$$

5. $$w\left(TS + ST\right) \leq 2\sqrt{2}\|T\|w(S).$$

6. $$w\left(TS \pm ST^*\right) \leq 2\|T\|w(S).$$

Fuglede's Theorem asserts that if T is a normal operator commuting with S, then T and S doubly commute. In particular, whenever $TS = ST$ such that T is normal, we have

$$w\Big(TS\Big) \leq w(T)w(S).$$

In [33], Yamazaki showed the following interesting formula

$$w(T) = \sup_{\theta\in\mathbb{R}} \|\Re(e^{i\theta}T)\| = \sup_{\theta\in\mathbb{R}} \|\Im(e^{i\theta}T)\|, \tag{2}$$

for every $T \in \mathcal{B}(\mathcal{H})$. For proofs and more material about the numerical radius, we refer the reader to [21, 22] and the references therein.

We denote by $R(T)$ the numerical radius distance of T from the scalar operators:

$$R(T) = \inf_{z\in\mathbb{C}} w\Big(T - zI\Big).$$

Clearly $R(T) \leq w(T)$ and we can easily prove that $R(\cdot)$ is a semi-norm on $\mathcal{B}(\mathcal{H})$. Furthermore, $R(T) = 0$ if and only if $T = \alpha I$ for some $\alpha \in \mathbb{C}$. A straightforward calculation shows that $R(T^*) = R(T)$ and $R(U^*TU) = R(T)$ for any unitary $U \in \mathcal{B}(\mathcal{H})$. Moreover, it is well-known that $R(T)$ is the radius of the smallest disk in the complex plane containing $W(T)$, see [23, 25]. In the case when T is normal, we can easily verify that $R(T)$ is exactly the radius of the smallest disk in the complex plane containing $\sigma(T)$, because $\overline{W(T)}$ is the convex hull of $\sigma(T)$ (see, [21, Theorem 1.4-4]). The quantity $R(T)$ has appeared in several numerical radius inequalities. We present some of them.

In 1970, Stampfli [32] proved that $\|\delta_T\| = 2D(T)$, where $D(T) = \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|$ and δ_T is the inner derivation induced by T:

$$\delta_T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}),\ S \mapsto TS - ST.$$

He also showed that if T is normal, then $\|\delta_T\| = 2R(T)$. In this particular case, we have the following inequality:

$$w\Big(TS - ST\Big) \leq 2R(T)\|S\| \ \text{ (for all } S \in \mathcal{B}(\mathcal{H})). \tag{3}$$

Later, in 2011, Hirzallah et al. [23] presented several numerical radius inequalities of the commutator $TS - ST$, where $S \in \mathcal{B}(\mathcal{H})$. When T is self-adjoint, they obtained an improvement of (3), as follows:

$$w\Big(TS - ST\Big) \leq 2R(T)R(S) \ \text{ (for all } S \in \mathcal{B}(\mathcal{H})).$$

Moreover, they showed that

$$w\Big(TS - ST\Big) \leq \|T\|R(S) \ \text{ (for all } S \in \mathcal{B}(\mathcal{H})),$$

when T is positive, see [23, Corollary 4.1].

In 2015, A. Abu-Omar and F. Kittaneh [2] used the quantity $R(T)$ to establish some new estimates for the numerical radii of products TS, commutators $TS - ST$, and anticommutators $TS + ST$. These estimates improved the inequalities stated in Theorem 1.1. We present some of their results in the following theorem.

Theorem 1.2 ([2]). *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

1.

$$w\Big(TS\Big) \leq \|TS\| \leq \|T\|\Big(w(S) + R(S)\Big) \leq 2\|T\|w(S). \tag{4}$$

2.

$$w\Big(TS\Big) \leq \|TS\| \leq \Big(w(T) + R(T)\Big)\Big(w(S) + R(S)\Big) \leq 4w(T)w(S). \tag{5}$$

3. *If T and S commute, then*

$$w\Big(TS\Big) \le w(T)\Big(w(S) + R(S)\Big) \le 2w(T)w(S). \tag{6}$$

4.

$$w\Big(TS + ST\Big) \le 2w(T)\Big(w(S) + R(S)\Big) \le 4w(T)w(S). \tag{7}$$

5.

$$w\Big(TS + ST\Big) \le 2\|T\| \sqrt{w(S)^2 + R(S)^2} \le 2\sqrt{2}\|T\|w(S). \tag{8}$$

6.

$$w\Big(TS + ST\Big) \le 2\sqrt{w(T)^2 + R(T)^2}\sqrt{w(S)^2 + R(S)^2} \le 4w(T)w(S). \tag{9}$$

One of the most recent research directions in the theory of operators on Hilbert spaces is the study of numerical radius inequalities and their various improvements, see for example [1–3, 12, 24, 30] and the references therein.

The notion of numerical radius of an operator plays an important role and has received a lot of attention due to its variety of usage in different areas such as mathematics and physics, see for example [20]. This concept has been extended in several directions (e.g., see [4, 10, 13, 24, 26] and the references therein). One such direction is the A-numerical radius. This new notion was first introduced by Saddi in [31], as follows:

$$w_A(T) = \sup\Big\{|\langle ATx, x\rangle| : x \in \mathcal{H},\ \|x\|_A = 1\Big\} \ \ (\text{for } T \in \mathcal{B}(\mathcal{H})),$$

where $A \in \mathcal{B}(\mathcal{H})$ is a positive operator and $\|x\|_A = \sqrt{\langle Ax, x\rangle}$ for every $x \in \mathcal{H}$. In the last few years, many mathematicians have focused their attention on studying various types of A-numerical radius inequalities. For example, A. Zamani [34] extended the numerical radii inequalities discussed in Theorem 1.2 to the case of A-numerical radius. We refer the reader to [10, 16–19, 28, 34] and the references therein for further information and other types of A-numerical radius inequalities.

The main purpose of this paper is to provide some new refinements of the numerical radii inequalities mentioned in Theorem 1.2, and to extend it to the semi-Hilbertian space operators. We start with the Hilbert space operators case, in which we use a new quantity $K(T)$ that is less than the numerical radius $w(T)$. Then, we extend our results to the semi-Hilbertian space operators case, employing a quantity $K_A(T)$ which is similar to $K(T)$.

The paper is organized as follows. In Section 2, we first define the quantity $K(T)$ of $T \in \mathcal{B}(\mathcal{H})$. Next, we provide some properties related to this quantity, such as the continuity of the map $T \mapsto K(T)$ on $\mathcal{B}(\mathcal{H})$. We conclude this section, by studying the set of all self-adjoint operators $C \in \mathcal{B}(\mathcal{H})$ that commute with an operator $B \in \mathcal{B}(\mathcal{H})$. In Section 3, we present our improvements of Theorem 1.2. In Section 4, we start by recalling some results related to the theory of semi-Hilbertian space operators. Finally, we extend our results to this case.

2. The quantity K(T)

In order to define the quantity $K(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$, we need the following results, which are proved by Kaadoud in [25].

Lemma 2.1 ([25]). *Let K be a non-empty compact subset of $\mathbb{C}$, then there exist a unique complex number z_K and a unique positive real number R_K such that the disk of the center z_K and radius R_K is the smallest disk of the complex plane containing K. Moreover*

$$R_K = \sup_{\lambda\in K} |\lambda - z_K| = \inf_{z\in\mathbb{C}} \sup_{\lambda\in K} |\lambda - z|.$$

Lemma 2.2 ([25]). *Let K be a non-empty compact subset of $\mathbb{C}$ and $\alpha \in \mathbb{C}$. The following properties are equivalent:*

1. $z_K = \alpha$.
2. $|K - \alpha| < |K - (\alpha + \lambda)|$ *for all* $\lambda \in \mathbb{C} \setminus \{0\}$.
3. $|K - \alpha|^2 + |\lambda|^2 \le |K - (\alpha + \lambda)|^2$ *for all* $\lambda \in \mathbb{C}$.

(Here, $|K - \beta| = \sup_{z \in K} |z - \beta|$ for $\beta \in \mathbb{C}$.)

In the sequel of this section, $T \in \mathcal{B}(\mathcal{H})$. According to Lemma 2.1, there exists a unique complex number $z_{\overline{W(T)}}$ such that $R(T) = w\Big(T - z_{\overline{W(T)}} I\Big)$. In addition, the disk of the center $z_{\overline{W(T)}}$ and radius $R(T)$ is the smallest disk of the complex plane containing the numerical range $W(T)$. Furthermore, it follows from [25, Proposition 26] that $z_{\overline{W(T)}} \in \overline{W(T)}$. We write simply z_T instead of $z_{\overline{W(T)}}$.

From Lemma 2.2, we can easily obtain the following properties:

1. $z_{T^*} = \overline{z_T}$ and $z_{\lambda T} = \lambda z_T$ for all $\lambda \in \mathbb{C}$.
2. $z_{UTU^*} = z_T$ for any unitary $U \in \mathcal{B}(\mathcal{H})$.

Now, we define the quantity $K(T)$ as follows:

$$K(T) = \inf_{\lambda \in \mathbb{R}} w\Big(T - i\lambda z_T I\Big).$$

It is obvious that

$$K(T) = \begin{cases} w(T) \text{ if } z_T = 0, \\ \inf_{\lambda \in \mathbb{R}} w\Big(e^{-i(\theta + \frac{\pi}{2})} T - \lambda I\Big) \text{ otherwise,} \end{cases} \tag{10}$$

where $\theta = arg(z_T)$ is the argument of z_T, i.e., $z_T = |z_T| e^{i\theta}$. This means that $K(T)$ is simply the distance from the operator $e^{-i(\theta + \frac{\pi}{2})} T$ to the real space $\mathbb{R}I$ in the complex Banach space $\Big(\mathcal{B}(\mathcal{H}), w(\cdot)\Big)$ when $z_T \neq 0$. Moreover, the following facts follow immediately from the definition of $K(T)$:

$$K(T^*) = K(T), K(\eta T) = |\eta| K(T) \text{ and } K(U^* T U) = K(T),$$

for all $\eta \in \mathbb{C}$ and all unitary $U \in \mathcal{B}(\mathcal{H})$. Also, we have the following inequality

$$R(T) \le K(T) \le w(T). \tag{11}$$

The following example shows that the inequality (11) can be strict.

Example 2.3. *Let* $T = \begin{bmatrix} 1+i & 0 \\ 0 & i \end{bmatrix}$*, it is easy to check that* $w(T) = \sqrt{2}$*. It follows from [21, Lemma 1.1-1] that* $W(T)$ *is the segment joining the values* $1 + i$ *and* i*. This implies that* $z_T = \frac{1}{2} + i$ *and* $R(T) = \frac{1}{2}$*. Now, let* $\lambda \in \mathbb{R}$

$$\begin{aligned} w\Big(T - i\lambda z_T I\Big) &= w\left(\begin{bmatrix} (1+\lambda) + i(1 - \frac{\lambda}{2}) & 0 \\ 0 & \lambda + i(1 - \frac{\lambda}{2}) \end{bmatrix}\right) \\ &= \max\left\{\Big|(1+\lambda) + i(1 - \frac{\lambda}{2})\Big|, \Big|\lambda + i(1 - \frac{\lambda}{2})\Big|\right\} \\ &= \max\left\{\sqrt{(1+\lambda)^2 + (1 - \frac{\lambda}{2})^2}, \sqrt{\lambda^2 + (1 - \frac{\lambda}{2})^2}\right\} \\ &= \begin{cases} \sqrt{(1+\lambda)^2 + (1 - \frac{\lambda}{2})^2} \text{ if } \lambda \ge \frac{-1}{2} \\ \sqrt{\lambda^2 + (1 - \frac{\lambda}{2})^2} \text{ if } \lambda \le \frac{-1}{2}. \end{cases} \end{aligned}$$

Then, we can conclude that $K(T) = \inf_{\lambda\in\mathbb{R}} w(T - i\lambda z_T I) = w(T - i(\frac{-2}{5})z_T I) = \frac{3\sqrt{5}}{5}$*. This shows that*

$R(T) = 0.5 < K(T) \simeq 1.341 < w(T) \simeq 1.414.$

Let us now give some properties concerning the quantity $K(T)$.

Proposition 2.4. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then*

1. *There exists a real number* λ_T *such that* $K(T) = w(T - i\lambda_T z_T I)$*.*
2. *If* $z_T = 0$ *then* $K(T) = R(T) = w(T)$*.*
3. *If* $z_T \neq 0$ *then* $R(T) < K(T)$*.*
4. *If* $z_T \neq 0$ *then* $\|\Re(a_T T)\| \leq K(T)$ *and* $\|\Im(a_T T)\| \leq R(T)$*, with* $a_T = \frac{\overline{z_T}}{|z_T|}$*.*
5. $$R(T) \leq \frac{K(T) + R(T)}{2} \leq \sqrt{\frac{K(T)^2 + R(T)^2}{2}} \leq K(T) \leq w(T) \tag{12}$$
   and
   $$w(T) \leq \sqrt{K(T)^2 + R(T)^2} \leq \sqrt{2}K(T) \leq \sqrt{2}w(T). \tag{13}$$
6. *If* $R(T) = 0$ *then* $K(T) = w(T)$*.*
7. $K(T) = 0$ *if and only if* $T = 0$*.*
8. *If* T *is self-adjoint then* $K(T) = w(T)$*.*

Proof.
1. If $z_T = 0$ the result is obvious, it suffices to take $\lambda_T = 0$. Now, we assume that $z_T \neq 0$. Using the fact that $\lim w(T - i\lambda z_T I) = +\infty$ when $|\lambda|$ goes to $+\infty$, and combining it with a compactness argument, we can show that the infimum in the definition of $K(T)$ is attained at some $\lambda_T \in \mathbb{R}$. This means that $K(T) = w(T - i\lambda_T z_T I)$.
2. Clear.
3. By contradiction we assume that $K(T) = R(T)$. Then, by the uniqueness of z_T, it follows that $z_T = i\lambda_T z_T$. Since $z_T \neq 0$ we get $1 = i\lambda_T$. Which is impossible, because $\lambda_T \in \mathbb{R}$. Hence, $R(T) < K(T)$.
4. It is obvious to check that $\Im(a_T T) = \Im(a_T(T - z_T I))$ and $\Re(a_T T) = \Re(a_T(T - i\lambda_T z_T I))$. By applying the formula (2) we get that $\|\Im(a_T T)\| \leq w(T - z_T I)$ and $\|\Re(a_T T)\| \leq w(T - i\lambda_T z_T)$. Thus, $\|\Re(a_T T)\| \leq K(T)$ and $\|\Im(a_T T)\| \leq R(T)$.
5. It is sufficient to show that
   $$w(T) \leq \sqrt{K(T)^2 + R(T)^2}. \tag{14}$$
   The other inequalities in (12) and (13) are obvious. If $z_T = 0$ then the inequality is trivial. Now, we assume that $z_T \neq 0$. Let $x \in \mathcal{H}$ such that $\|x\| = 1$, then
   $$\begin{aligned} |\langle Tx, x\rangle| &= |\langle a_T Tx, x\rangle| \quad (\text{with } a_T = \frac{\overline{z_T}}{|z_T|}) \\ &= \sqrt{\Re(\langle a_T Tx, x\rangle)^2 + \Im(\langle a_T Tx, x\rangle)^2} \\ &= \sqrt{\langle \Re(a_T T)x, x\rangle^2 + \langle \Im(a_T T)x, x\rangle^2} \\ &\leq \sqrt{\|\Re(a_T T)\|^2 + \|\Im(a_T T)\|^2} \\ &\leq \sqrt{K(T)^2 + R(T)^2}. \end{aligned}$$
   This implies that $w(T) \leq \sqrt{K(T)^2 + R(T)^2}$.
6. If $R(T) = 0$ then $T = z_T I$. So, by a simple calculation, we can show that $K(T) = w(T)$.

7. If $T = 0$ then $w(T) = 0$. Hence $K(T) = 0$. Reciprocally, if $K(T) = 0$ then $R(T) = 0$. So by (6) we get that $w(T) = K(T) = 0$. Consequently, $T = 0$.
8. The result is evident when $z_T = 0$. Now, we assume that $z_T \neq 0$. Since T is self-adjoint we have $W(T)$ is a real segment. Then, $z_T \in \mathbb{R}$, this implies that $a_T = \pm 1$. By part (4) we get $w(T) = \|T\| \leq K(T)$. Thus, $K(T) = w(T)$.
□

Remark 2.5. 1. *Example 2.3 shows that the inequalities (12) and (13) can be strict.*
2. *The parts* (2) *and* (3) *show that $z_T = 0$ if and only if $K(T) = R(T)$.*
3. *The part* (8) *shows that, in general, the reciprocal of the part* (6) *is not true. In fact, let* $T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ *be a self-adjoint matrix. From part* (8) *we obtain that* $K(T) = w(T) = 2$, *but* $R(T) = \dfrac{1}{2}$.
4. *The reciprocal of the part* (8), *in general, is not true. Indeed, let* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. *It follows from [21, Example 1] that* $z_T = 0$. *Hence, from part* (2) *we get* $K(T) = w(T)$, *but* T *is not self-adjoint.*
5. *Example 2.3 shows that if the self-adjoint assumption is replaced by normal, then part* (8) *is generally false.*

Proposition 2.6. *Let* $T, S \in \mathcal{B}(\mathcal{H})$. *Then*

$$|K(T) - K(S)| \leq w(T - S) + \max\{|\lambda_T|; |\lambda_S|\} \, |z_T - z_S|, \tag{15}$$

where λ_T *and* λ_S *are real numbers which satisfy*

$$K(T) = w(T - iz_T\lambda_T I) \text{ and } K(S) = w(S - iz_S\lambda_S I).$$

Note that Proposition 2.4 asserts the existence of the real numbers λ_T and λ_S.

Proof. By the definition of $K(T)$ we get

$$\begin{aligned} K(T) \leq w(T - i\lambda_S z_T I) &= w\Big(T - S + S - i\lambda_S z_S I + i\lambda_S(z_S - z_T)I\Big) \\ &\leq w(T - S) + K(S) + |\lambda_S||z_T - z_S| \\ &\leq w(T - S) + K(S) + \max\{|\lambda_T|; |\lambda_S|\} \, |z_T - z_S|. \end{aligned}$$

Hence, by symmetry, we obtain (15). □

In [25], Kaadoud proved that the map $T \mapsto z_T$ is continuous on $\mathcal{B}(\mathcal{H})$. This implies that the map $T \mapsto R(T)$ is also continuous on $\mathcal{B}(\mathcal{H})$. The following result shows the continuity of the map $T \mapsto K(T)$.

Proposition 2.7. *Let* $\{T_n\}$ *be a sequence of elements of* $\mathcal{B}(\mathcal{H})$ *such that* $\{T_n\}$ *converges to* T *in* $\mathcal{B}(\mathcal{H})$. *Then, the sequence* $\{K(T_n)\}$ *converges to* $K(T)$.

Proof. If $z_T = 0$, then by Proposition 2.4, we have $w(T) = R(T) = K(T)$. On the other hand, the inequality (11) implies that

$$R(T_n) \leq K(T_n) \leq w(T_n), \quad (\text{for all } n \in \mathbb{N}). \tag{16}$$

By using the facts $\lim\limits_{n\to+\infty} R(T_n) = R(T)$ and $\lim\limits_{n\to+\infty} w(T_n) = w(T)$, together with the inequality (16), we get $\lim\limits_{n\to+\infty} K(T_n) = K(T)$.

Now, we suppose that $z_T \neq 0$. We first show that there exists $M > 0$ such that $|\lambda_{T_n}| \leq M$ for all $n \in \mathbb{N}$. By contradiction, we suppose that $\{\lambda_{T_n}\}$ is an unbounded sequence. Taking a subsequence if necessary, we may assume that $\lim\limits_{n\to+\infty} |\lambda_{T_n}| = +\infty$. Since the sequence $\{z_{T_n}\}$ converges to z_T we have $z_{T_n} \neq 0$ for all sufficiently large $n \in \mathbb{N}$. Then, $\lim\limits_{n\to+\infty} |\lambda_{T_n} z_{T_n}| = +\infty$. This implies that $\lim\limits_{n\to+\infty} K(T_n) = +\infty$, which contradicts the fact that

$\{w(T_n)\}$ is a bounded sequence. Consequently, there exists $M > 0$ such that $|\lambda_{T_n}| \le M$ for all $n \in \mathbb{N}$. It follows from (15) that

$$\begin{aligned} |K(T) - K(T_n)| &\le w(T - T_n) + \max\{|\lambda_T|; |\lambda_{T_n}|\}\, |z_T - z_{T_n}| \\ &\le w(T - T_n) + \max\{|\lambda_T|; M\}\, |z_T - z_{T_n}|, \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that, $\lim\limits_{n \mapsto +\infty} K(T_n) = K(T)$. Thus, the map $T \mapsto K(T)$ is continuous on $\mathcal{B}(\mathcal{H})$. □

The following result shows that $K(\cdot)$ satisfies the triangular inequality for special cases.

Proposition 2.8. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If one of the following conditions*

(C_1) $z_{T+S} = 0$,
(C_2) z_{T+S}, z_T *and* z_S *are real numbers,*

is satisfied, then

$$K(T + S) \le K(T) + K(S).$$

Proof. Let λ_T and λ_S be the real numbres such that

$$K(T) = w(T - i\lambda_T z_T I) \text{ and } K(S) = w(S - i\lambda_S z_S I).$$

If the condition (C_1) holds, then

$$K(T + S) = w(T + S) \le w(T + S - \mu I) \text{ for all } \mu \in \mathbb{C}.$$

In particular, for $\mu = i\lambda_T z_T + i\lambda_S z_S$ we obtain

$$\begin{aligned} K(T + S) &\le w\Big(T + S - i\lambda_T z_T I - i\lambda_S z_S I\Big) \\ &\le w\Big(T - i\lambda_T z_T I\Big) + w\Big(S - i\lambda_S z_S I\Big) \\ &= K(T) + K(S). \end{aligned}$$

Now, we assume that the condition (C_2) holds. By the formula (10) we get

$$\begin{aligned} K(T + S) = dist_w\Big(\pm i(T + S); \mathbb{R}I\Big) &\le dist_w\Big(\pm iT; \mathbb{R}I\Big) + dist_w\Big(\pm iS; \mathbb{R}I\Big) \\ &= K(T) + K(S), \end{aligned}$$

where $dist_w$ means the distance in the sense of $w(\cdot)$. This completes the proof. □

Remark 2.9. *In general, z_{T+S} is not equal to $z_T + z_S$. Indeed, if we take the matrices*

$$T = \begin{bmatrix} 3+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 2+i \end{bmatrix} \text{ and } S = \begin{bmatrix} -1+4i & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -2-i \end{bmatrix}.$$

Then, we have

$$T + S = \begin{bmatrix} 2+5i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is obvious that T, S and $T + S$ are normal operators. Hence, from [21, Theorem 1.4-4], $W(T)$ is the segment joining the values $1 + i$ and $3 + i$, $W(S)$ is the right triangle with vertices $\{-1 + 4i; -1 - i; -2 - i\}$ and $W(T + S)$ is the segment *joining the values* 0 *and* $2 + 5i$. *This implies that* $z_T = 2 + i$, $z_S = \frac{-3}{2} + \frac{3}{2}i$ *and* $z_{T+S} = 1 + \frac{5}{2}i$. *Thus,* $z_{T+S} \neq z_T + z_S$. *However, in this case we have*

$$K(T + S) \leq K(T) + K(S),$$

because, $K(T) = w(T - i(\frac{-1}{5})z_T I) = \frac{7\sqrt{5}}{5}$, $K(S) = w(S - i(\frac{-2}{3})z_S I) = \sqrt{13}$ *and* $K(T + S) = w(T + S) = \sqrt{29}$. *This fact combined with proposition 2.8, make us wonder: Does* $K(\cdot)$ *satisfy the triangle inequality?*

For an operator $B \in \mathcal{B}(\mathcal{H})$ we define the set $\mathscr{S}(B)$ by

$$\mathscr{S}(B) = \{C \in \mathcal{B}(\mathcal{H}) \mid C \text{ is self-adjoint and } BC = CB\}.$$

It is clear that $\mathscr{S}(B)$ is a closed real linear space. Moreover, $\mathscr{S}(B^*) = \mathscr{S}(B)$. Now, we give some properties concerning the real space $\mathscr{S}(B)$.

Proposition 2.10. *Let* $B, C \in \mathcal{B}(\mathcal{H})$. *If* $\{C_n\}$ *is a seuquence of elements of* $\mathscr{S}(B)$ *such that* $\lim_{n \mapsto +\infty} \langle C_n x, x\rangle = \langle Cx, x\rangle$ *for all* $x \in \mathcal{H}$, *then* $C \in \mathscr{S}(B)$.

Proof. We have $\langle C_n x, x\rangle \in \mathbb{R}$ for all $n \in \mathbb{N}$ and all $x \in \mathcal{H}$, thus $\langle Cx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. Hence, C is self-adjoint. It is, therefore, sufficient to show that $BC = CB$. From the Polarization Identity, we get

$$\langle Cx, y\rangle = \lim_{n \mapsto +\infty} \langle C_n x, y\rangle \text{ for all } x, y \in \mathcal{H}.$$

This implies that

$$\begin{aligned} \langle BCx, x\rangle &= \langle Cx, B^* x\rangle \\ &= \lim_{n \mapsto +\infty} \langle C_n x, B^* x\rangle \\ &= \lim_{n \mapsto +\infty} \langle BC_n x, x\rangle \\ &= \lim_{n \mapsto +\infty} \langle C_n Bx, x\rangle \quad (\text{because, } C_n \in \mathscr{S}(B)) \\ &= \langle CBx, x\rangle, \end{aligned}$$

for every $x \in \mathcal{H}$. Consequently, $BC = CB$. □

Remark 2.11. 1. $\mathscr{S}(I) = \mathscr{S}(0) = \{C \in \mathcal{B}(\mathcal{H}) \mid C \text{ is self-adjoint}\}$.

2. *By the Cartesian Decomposition, we get that* $\mathcal{B}(\mathcal{H}) = \mathscr{S}(I) + i\mathscr{S}(I)$.
3. *Proposition 2.10 asserts that the real space* $\mathscr{S}(B)$ *is closed in the topological space* $(\mathcal{B}(\mathcal{H}), \tau)$, *with* τ *is the coarsest topology on* $\mathcal{B}(\mathcal{H})$ *for which every map* Φ_x $(x \in \mathcal{H})$ *is continuous on* $\mathcal{B}(\mathcal{H})$, *where*

$$\Phi_x(T) = \langle Tx, x\rangle \quad (\text{for all } T \in \mathcal{B}(\mathcal{H})).$$

Proposition 2.12. *Let* $B \in \mathcal{B}(\mathcal{H})$. *If* $C \in \mathscr{S}(B)$ *and* $f : J \longrightarrow \mathbb{R}$ *is a continuous function on an interval* J *such that* $\sigma(C) \subset J$, *then* $f(C) \in \mathscr{S}(B)$.

Proof. Since f is a real-valued function, it follows from continuous function calculus that $f(C)$ is self-adjoint. Hence, it is sufficient to show that $Bf(C) = f(C)B$. By Stone-Weierstrass theorem, there exists a sequence of polynomials $\{p_n\}$, with real coefficients, which converges uniformly to f in $\sigma(C)$. Then, the sequence $\{p_n(C)\}$ converges to $f(C)$. Hence, from the closedness of $\mathscr{S}(B)$, it is enough to prove that $Bp(C) = p(C)B$ for every polynomial p with real coefficients. However, this can be deduced directly from the fact that $\mathscr{S}(B)$ is a real linear space and $C^k \in \mathscr{S}(B)$ for all $k \in \mathbb{N}$. □

3. Numerical radius inequalities for operators

In this section, we present our improvements. We start with the following result which gives a refinement of part (6) in Theorem 1.1.

Theorem 3.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w\Big(TS + ST^*\Big) &\leq 2\, dist\Big(T, i\mathscr{S}(S)\Big)\, w(S) \\ &\leq 2 \inf_{\mu\in\mathbb{R}} \|T - i\mu C\|\, w(S) \ \Big(\text{for all } C \in \mathscr{S}(S)\Big) \\ &\leq 2\|T\|w(S), \end{aligned} \tag{17}$$

with $dist\Big(T, i\mathscr{S}(S)\Big) = \inf\Big\{\|T - iC\| \mid C \in \mathscr{S}(S)\Big\}$.

Proof. We only need to establish the first inequality in (17), since the other inequalities in (17) are results that will automatically follow. Let $C \in \mathscr{S}(S)$, then

$$w\Big(TS + ST^*\Big) = w\Big((T - iC)S + S(T - iC)^*\Big).$$

By applying Theorem 1.1 (6), we get

$$w\Big(TS + ST^*\Big) \leq 2\|T - iC\|w(S).$$

Hence, by taking the infimum over all $C \in \mathscr{S}(S)$, we have

$$w\Big(TS + ST^*\Big) \leq 2\, dist\Big(T, i\mathscr{S}(S)\Big)\, w(S).$$

□

Remark 3.2. *If we replace T by iT in (17), we get*

$$\begin{aligned} w\Big(TS - ST^*\Big) &\leq 2\, dist\Big(T, \mathscr{S}(S)\Big)\, w(S) \\ &\leq 2\|T\|w(S). \end{aligned}$$

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w\Big(TS + ST^*\Big) \leq 2k(T)w(S) \ \text{ and } \ w\Big(TS - ST^*\Big) \leq 2k'(T)w(S),$$

where $k(T) = \inf_{\mu\in\mathbb{R}} \|T - i\mu I\|$ *and* $k'(T) = \inf_{\mu\in\mathbb{R}} \|T - \mu I\|$.

The example below highlights that the inequalities in Theorem 3.1 are considerable improvements.

Example 3.4. *Let* $T = \begin{bmatrix} 1+2i & 0 \\ 0 & -i \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *We can easily find that* $\|T\| = \sqrt{5}$, $w(S) = 1$ *and* $w\Big(TS + ST^*\Big) = 2$. *On the other hand, it is not difficult to show that any self-adjoint* 2×2 *matrix C has the form* $C = \begin{bmatrix} a & z \\ \overline{z} & b \end{bmatrix}$, *where* $a, b \in \mathbb{R}$ *and* $z \in \mathbb{C}$. *This implies that* $\mathscr{S}(S) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. *A simple calculation shows that*

$$dist(T, i\mathscr{S}(S)) = \inf_{a,b\in\mathbb{R}} \Big(\max\Big\{ \sqrt{1 + (2-a)^2}; |1 + b| \Big\}\Big).$$

Since $1 \leq \sqrt{1 + (2-a)^2}$ *for all* $a \in \mathbb{R}$, *we get that* $1 \leq dist(T, i\mathscr{S}(S))$. *For* $a = 2$ *and* $b = 0$, *we obtain* $dist(T, i\mathscr{S}(S)) \leq 1$. *Thus,* $dist(T, i\mathscr{S}(S)) = 1$. *This gives that*

$$w(TS + ST^*) = 2\, dist(T, i\mathscr{S}(S))\, w(S) = 2 < 2\, \|T\|\, w(S) \simeq 4,4721.$$

In the remainder of this section, T and S are elements of $\mathcal{B}(\mathcal{H})$ such that $z_T \neq 0$ and $z_S \neq 0$. We denote by $a_T = \dfrac{\overline{z_T}}{|z_T|}$ and $b_S = \dfrac{\overline{z_S}}{|z_S|}$.

The following result gives a refinement of the inequalities (6) and (7). We use some ideas of [2, Theorem 2.5].

Theorem 3.5. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w(TS+ST) &\leq 2\Big(dist\Big(\mathfrak{R}(b_S S), i\mathscr{S}(T)\Big) + dist\Big(\mathfrak{I}(b_S S), i\mathscr{S}(T)\Big)\Big)\, w(T) \\ &\leq 2\Big(k\Big(\mathfrak{R}(b_S S)\Big) + k\Big(\mathfrak{I}(b_S S)\Big)\Big)\, w(T) \\ &\leq 2\Big(\|\mathfrak{R}(b_S S)\| + \|\mathfrak{I}(b_S S)\|\Big)\, w(T) \\ &\leq 2\Big(K(S) + R(S)\Big)\, w(T) \\ &\leq 2\Big(w(S) + R(S)\Big)\, w(T). \end{aligned}$$

In particular, if T and S commute then

$$\begin{aligned} w(TS) &\leq \Big(dist\Big(\mathfrak{R}(b_S S), i\mathscr{S}(T)\Big) + dist\Big(\mathfrak{I}(b_S S), i\mathscr{S}(T)\Big)\Big)\, w(T) \\ &\leq \Big(k\Big(\mathfrak{R}(b_S S)\Big) + k\Big(\mathfrak{I}(b_S S)\Big)\Big)\, w(T) \\ &\leq \Big(\|\mathfrak{R}(b_S S)\| + \|\mathfrak{I}(b_S S)\|\Big)\, w(T) \\ &\leq \Big(K(S) + R(S)\Big)\, w(T) \\ &\leq \Big(w(S) + R(S)\Big)\, w(T). \end{aligned}$$

Proof. It is clear that

$$\begin{aligned} w(TS+ST) &= w\Big(T\mathfrak{R}(b_S S) + \mathfrak{R}(b_S S)T + i\Big(T\mathfrak{I}(b_S S) + \mathfrak{I}(b_S S)T\Big)\Big) \\ &\leq w\Big(T\mathfrak{R}(b_S S) + \mathfrak{R}(b_S S)T\Big) + w\Big(T\mathfrak{I}(b_S S) + \mathfrak{I}(b_S S)T\Big). \end{aligned}$$

So, by applying Theorem 3.1 we get

$$\begin{aligned} w(TS+ST) &\leq 2\Big(dist\Big(\mathfrak{R}(b_S S), i\mathscr{S}(T)\Big) + dist\Big(\mathfrak{I}(b_S S), i\mathscr{S}(T)\Big)\Big) w(T) \\ &\leq 2\Big(k\Big(\mathfrak{R}(b_S S)\Big) + k\Big(\mathfrak{I}(b_S S)\Big)\Big)\, w(T) \\ &\leq 2\Big(\|\mathfrak{R}(b_S S)\| + \|\mathfrak{I}(b_S S)\|\Big)\, w(T) \\ &\leq 2\Big(K(S) + R(S)\Big)\, w(T) \quad \text{[by Proposition 2.4 (4)]} \\ &\leq 2\Big(w(S) + R(S)\Big)\, w(T). \end{aligned}$$

This gives the desired results. □

The following example shows that, using our inequalities in Theorem 3.5, we get considerably smaller quantities to the ones obtained by (6) and (7).

Example 3.6. *Let* $T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1+i & 0 \\ 0 & i \end{bmatrix}$. *It is obvious that* $TS = ST$, $w(T) = 2$, $w(S) = \sqrt{2}$ *and* $w(TS) = 2\sqrt{2}$. *We have shown in Example 2.3 that* $z_S = \frac{1}{2} + i$, $R(S) = \frac{1}{2}$ *and* $K(S) = \frac{3}{\sqrt{5}}$. *A straightforward* *calculation gives that*

$$\mathfrak{R}(b_S S) = \begin{bmatrix} \frac{3}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} \end{bmatrix}, \ \mathfrak{I}(b_S S) = \begin{bmatrix} \frac{-1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \text{ and } \mathscr{S}(T) = \Big\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \Big\},$$

where $b_S = \frac{\overline{z_S}}{|z_S|} = \frac{1}{\sqrt{5}} - i\frac{2}{\sqrt{5}}$*. From this we obtain*

$$\begin{aligned} \|\mathfrak{R}(b_S S)\| = dist(\mathfrak{R}(b_S S), i\mathscr{S}(T)) &= \inf_{a,b\in\mathbb{R}} \Big(\max\Big\{ \sqrt{\frac{9}{5} + a^2}; \sqrt{\frac{4}{5} + b^2} \Big\} \Big) = \frac{3}{\sqrt{5}}. \\ \|\mathfrak{I}(b_S S)\| = dist(\mathfrak{I}(b_S S), i\mathscr{S}(T)) &= \inf_{a,b\in\mathbb{R}} \Big(\max\Big\{ \sqrt{\frac{1}{5} + a^2}; \sqrt{\frac{1}{5} + b^2} \Big\} \Big) = \frac{1}{\sqrt{5}}. \end{aligned}$$

This implies that

$$\begin{aligned} w(TS) \simeq 2.828 &< \Big(\|\mathfrak{R}(b_S S)\| + \|\mathfrak{I}(b_S S)\| \Big) w(T) \simeq 3.577 \\ &< \Big(K(S) + R(S) \Big) w(T) \simeq 3.683 \\ &< \Big(w(S) + R(S) \Big) w(T) \simeq 3.828. \end{aligned}$$

The following result gives a refinement of the inequalities (4) and (5). Our approach is based on [2, Theorem 2.3].

Theorem 3.7. *Let* $T, S \in \mathcal{B}(\mathcal{H})$*. Then*

$$\begin{aligned} w(TS) \le \|TS\| &\le \|T\| \Big[K(S) + R(S) \Big] \\ &\le \|T\| \Big[w(S) + R(S) \Big], \end{aligned} \tag{18}$$

and,

$$\begin{aligned} w(TS) \le \|TS\| &\le \Big[K(T) + R(T) \Big] \Big[K(S) + R(S) \Big] \\ &\le \Big[w(T) + R(T) \Big] \Big[K(S) + R(S) \Big] \\ &\le \Big[w(T) + R(T) \Big] \Big[w(S) + R(S) \Big]. \end{aligned} \tag{19}$$

Proof. We show the inequalities (18).

$$\|TS\| = \|T(b_S S)\| = \|T\mathfrak{R}(b_S S) + iT\mathfrak{I}(b_S S)\| \le \|T\| \Big[\|\mathfrak{R}(b_S S)\| + \|\mathfrak{I}(b_S S)\| \Big]$$

It follows from Proposition 2.4 (4) that

$$\begin{aligned} w(TS) \le \|TS\| &\le \|T\| \Big[K(S) + R(S) \Big] \\ &\le \|T\| \ \Big[w(S) + R(S) \Big]. \end{aligned}$$

The inequalities (19) result immediately from (18) and the fact that

$$\|T\| = \|a_T T\| \le \|\mathfrak{R}(a_T T)\| + \|\mathfrak{I}(a_T T)\|.$$

□

As an immediate consequence of Theorem 3.7, we have the following result.

Corollary 3.8. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w(TS) \leq 2\|T\|K(S) \leq 2\|T\|w(S),$$

and,

$$w(TS) \leq 4K(T)K(S) \leq 4w(T)w(S).$$

The inequalities (18) and (19) are non-trivial improvements, as shown in the following example.

Example 3.9. *Let $S = T = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It is obvious that $\|S\| = w(S) = \sqrt{2}$, $\|TS\| = w(TS) = 2$ and $W(S)$ is the right triangle with vertices $\{1+i; 1; 0\}$. This gives that $z_S = \frac{1}{2} + \frac{i}{2}$ and $R(S) = \frac{\sqrt{2}}{2}$. Now, let $\lambda \in \mathbb{R}$ then*

$$\begin{aligned} w(S - i\lambda z_S I) &= \max\left\{ \sqrt{1 + (1-\lambda)^2};\ \sqrt{1+\lambda^2}; |\lambda| \right\} \\ &= \max\left\{ \sqrt{1 + (1-\lambda)^2};\ \sqrt{1+\lambda^2} \right\} \ \left(\text{since}, |\lambda| < \sqrt{1+\lambda^2}\right) \\ &= \begin{cases} \sqrt{1+\lambda^2} \ \text{ if } \ \lambda \geq \dfrac{1}{2} \\ \sqrt{1+(1-\lambda)^2} \ \text{ if } \ \lambda \leq \dfrac{1}{2}. \end{cases} \end{aligned}$$

This implies that $K(S) = w\left(S - i\frac{z_S}{2} I\right) = \frac{\sqrt{5}}{2}$. It follows from inequalities (18) and (19) that

$$\begin{aligned} w(TS) = \|TS\| = 2 &< \|T\|\left[K(S) + R(S)\right] \simeq 2.581 \\ &< \|T\|\left[w(S) + R(S)\right] = 3, \end{aligned}$$

and,

$$\begin{aligned} w(TS) = \|TS\| = 2 &< \left[K(T) + R(T)\right]\left[K(S) + R(S)\right] \simeq 3.331 \\ &< \left[w(T) + R(T)\right]\left[K(S) + R(S)\right] \simeq 3.871 \\ &< \left[w(T) + R(T)\right]\left[w(S) + R(S)\right] = 4.5. \end{aligned}$$

The example below demonstrates that the inequalities in Corollary 3.8 are significant improvements.

Example 3.10. *From Example 3.9, we have*

$$w(TS) = 2 < 2\|T\|K(S) \simeq 3.162 < 2\|T\|w(S) = 4,$$

and

$$w(TS) = 2 < 4K(T)K(S) = 5 < 4w(T)w(S) = 8.$$

Let us now give a refinement of the inequalities (6), (8) and (9). In order to achieve this goal, we need the following lemmas which have been proved by Abu-Omar and Kittaneh in [1, 3].

Lemma 3.11 ([3]). *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$w\left(TS \pm ST\right) \leq \sqrt{\|T^*T + TT^*\|}\sqrt{\|S^*S + SS^*\|}.$$

Lemma 3.12 ([1]). *Let $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} r\Big(A_1B_1 + A_2B_2\Big) &\leq r\left(\begin{bmatrix} w(B_1A_1) & \sqrt{\|B_1A_2\|\|B_2A_1\|} \\ \sqrt{\|B_1A_2\|\|B_2A_1\|} & w(B_2A_2) \end{bmatrix}\right) \\ &= \frac{1}{2}\Big(w(B_1A_1) + w(B_2A_2)\Big) \\ &\quad + \frac{1}{2}\sqrt{\Big(w(B_1A_1) - w(B_2A_2)\Big)^2 + 4\|B_1A_2\|\|B_2A_1\|}. \end{aligned}$$

Now, we can prove the following result, which improves the inequalities (8) and (9). We use some ideas of [2, Remark 2.8].

Theorem 3.13. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w\Big(TS \pm ST\Big) &\leq \sqrt{2}\|T\|\sqrt{\mu_1(S)} \\ &\leq \sqrt{2}\|T\|\sqrt{\mu_2(S)} \\ &\leq \sqrt{2}\|T\|\sqrt{\mu_3(S)} \\ &\leq 2\|T\|\sqrt{K(S)^2 + R(S)^2} \\ &\leq 2\|T\|\sqrt{w(S)^2 + R(S)^2}, \end{aligned} \tag{20}$$

and,

$$\begin{aligned} w\Big(TS \pm ST\Big) &\leq \sqrt{\mu_1(T)}\sqrt{\mu_1(S)} \\ &\leq \sqrt{\mu_2(T)}\sqrt{\mu_2(S)} \\ &\leq \sqrt{\mu_3(T)}\sqrt{\mu_3(S)} \\ &\leq 2\sqrt{K(T)^2 + R(T)^2}\sqrt{K(S)^2 + R(S)^2} \\ &\leq 2\sqrt{w(T)^2 + R(T)^2}\sqrt{w(S)^2 + R(S)^2}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \mu_1(T) &= \Big(\|\mathfrak{R}(a_TT)\|^2 + \|\mathfrak{I}(a_TT)\|^2\Big) \\ &\quad + \sqrt{\Big(\|\mathfrak{R}(a_TT)\|^2 - \|\mathfrak{I}(a_TT)\|^2\Big)^2 + 4\|\mathfrak{R}(a_TT)\mathfrak{I}(a_TT)\|^2}. \\ \mu_2(T) &= \Big(K(T)^2 + R(T)^2\Big) + \sqrt{\Big(K(T)^2 - R(T)^2\Big)^2 + 4\|\mathfrak{R}(a_TT)\mathfrak{I}(a_TT)\|^2}. \\ \mu_3(T) &= \Big(K(T)^2 + R(T)^2\Big) + \sqrt{\Big(K(T)^2 - R(T)^2\Big)^2 + 4\|\mathfrak{R}(a_TT)\|^2\,\|\mathfrak{I}(a_TT)\|^2}. \end{aligned}$$

Proof. We first show the inequalities (21). By using the fact that

$$T^*T + TT^* = (a_TT)^*(a_TT) + (a_TT)(a_TT)^* = 2\Big((\mathfrak{R}(a_TT))^2 + (\mathfrak{I}(a_TT))^2\Big),$$

and by letting $A_1 = B_1 = \mathfrak{R}(a_TT), A_2 = B_2 = \mathfrak{I}(a_TT)$ in Lemma 3.12, we get

$$\|T^*T + TT^*\| \leq \mu_1(T),$$

because $\|\mathfrak{R}(a_TT)\mathfrak{I}(a_TT)\| = \|(\mathfrak{R}(a_TT)\mathfrak{I}(a_TT))^*\| = \|\mathfrak{I}(a_TT)\mathfrak{R}(a_TT)\|$.

It is easy to check that $M_1 \leq M_2$, with

$$M_1 = \begin{bmatrix} \|\mathfrak{R}(a_TT)\|^2 & \|\mathfrak{R}(a_TT)\mathfrak{I}(a_TT)\| \\ \|\mathfrak{R}(a_TT)\mathfrak{I}(a_TT)\| & \|\mathfrak{I}(a_TT)\|^2 \end{bmatrix},$$

and

$$M_2 = \begin{bmatrix} K(T)^2 & \|\Re(a_T T)\Im(a_T T)\| \\ \|\Re(a_T T)\Im(a_T T)\| & R(T)^2 \end{bmatrix}.$$

Hence $\|M_1\| \le \|M_2\|$, with $\|M_1\| = \dfrac{\mu_1(T)}{2}$ and $\|M_2\| = \dfrac{\mu_2(T)}{2}$. This implies that

$$\begin{aligned}\sqrt{\|T^*T+TT^*\|} &\le \sqrt{\mu_1(T)}\\ &\le \sqrt{\mu_2(T)}\\ &\le \sqrt{\mu_3(T)}\\ &\le \sqrt{2}\,\sqrt{K(T)^2+R(T)^2}\\ &\le \sqrt{2}\,\sqrt{w(T)^2+R(T)^2}.\end{aligned} \tag{22}$$

By symmetry, we obtain the same inequalities (22) for S. Using Lemma 3.11 and the inequalities (22), we derive the inequalities (21).

For the inequalities (20), it suffices to observe that

$$\sqrt{\|T^*T+TT^*\|} \le \sqrt{\|T^*T\|+\|TT^*\|} = \sqrt{2}\|T\|.$$

Hence, the required inequalities are obtained immediately by employing Lemma 3.11 and (22). □

The inequalities (20) and (21) are important refinements, as the following example shows.

Example 3.14. *From Example 3.6, we have*

$$\begin{aligned} w(TS+ST) &= \sqrt{2}\|T\|\sqrt{\mu_1(S)} \simeq 5,656\\ &< \sqrt{2}\|T\|\sqrt{\mu_2(S)}\\ &= \sqrt{2}\|T\|\sqrt{\mu_3(S)} \simeq 5,664\\ &< 2\|T\|\sqrt{K(S)^2+R(S)^2} \simeq 5,727\\ &< 2\|T\|\sqrt{w(S)^2+R(S)^2} = 6.\end{aligned}$$

Since $T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ *is self-adjoint, we obtain* $\mu_1(T) = \mu_2(T) = \mu_3(T) = 2\|T\|^2$ *and* $K(T) = w(T) = \|T\|$. *Moreover, in this case we have* $R(T) = \dfrac{1}{2}$. *This implies that*

$$\begin{aligned} w(TS+ST) \simeq 5,656 &< 2\sqrt{K(T)^2+R(T)^2}\,\sqrt{K(S)^2+R(S)^2} \simeq 5,903\\ &< 2\sqrt{w(T)^2+R(T)^2}\,\sqrt{w(S)^2+R(S)^2} \simeq 6,184.\end{aligned}$$

The inequality (6) can be improved as follows.

Theorem 3.15. *Let* $T, S \in \mathcal{B}(\mathcal{H})$. *If* T *and* S *commute then*

$$\begin{aligned} w(TS) &\le \eta_1(T,S)\\ &\le \eta_2(T,S)\\ &\le w(T)\Big[K(S)+R(S)\Big]\\ &\le 2w(T)K(S)\\ &\le 2w(T)w(S),\end{aligned}$$

where

$$\eta_1(T,S) = \frac{1}{2}\Big(w(T)[K(S)+R(S)] + \sqrt{w(T)^2\Big(K(S)-R(S)\Big)^2 + 4\sup_{\theta\in\mathbb{R}}\|\Re(b_S S)\Im(e^{i\theta}T)\|\|\Im(b_S S)\Re(e^{i\theta}T)\|}\Big),$$

$$\eta_2(T,S) = \frac{w(T)}{2}\Big([K(S)+R(S)] + \sqrt{\Big(K(S)-R(S)\Big)^2 + 4\|\Re(b_S S)\|\|\Im(b_S S)\|}\Big).$$

Proof. It suffices to show the first inequality, as the others follow. Let $\theta \in \mathbb{R}$, by the commutativity of T and S we get that

$$\Re(e^{i\theta}T(b_S S)) = \Re(e^{i\theta}T)\Re(b_S S) - \Im(e^{i\theta}T)\Im(b_S S).$$

Hence, by taking $A_1 = \Re(e^{i\theta}T)$, $B_1 = \Re(b_S S)$, $A_2 = -\Im(e^{i\theta}T)$, $B_2 = \Im(b_S S)$ in Lemma 3.12, and by the norm monotonicity of matrices with nonnegative entries, we have

$$\begin{aligned} r\Big(\Re(e^{i\theta}T(b_S S))\Big) &\leq \left\|\begin{bmatrix} \|\Re(b_S S)\|\,\|\Re(e^{i\theta}T)\| & \sqrt{m(\theta,T,S)} \\ \sqrt{m(\theta,T,S)} & \|\Im(b_S S)\|\|\Im(e^{i\theta}T)\| \end{bmatrix}\right\| \\ &\leq \left\|\begin{bmatrix} K(S)w(T) & \sqrt{m(\theta,T,S)} \\ \sqrt{m(\theta,T,S)} & R(S)w(S) \end{bmatrix}\right\| \\ &= \frac{1}{2}\Big(w(T)[K(S)+R(S)]\Big) \\ &\quad + \frac{1}{2}\sqrt{w(T)^2(K(S)-R(S))^2 + 4m(\theta,T,S)}, \end{aligned}$$

where $m(\theta,T,S) = \|\Re(b_S S)\Im(e^{i\theta}T)\|\|\Im(b_S S)\Re(e^{i\theta}T)\|$. By taking the supremum over all $\theta \in \mathbb{R}$, we get

$$w(TS) \leq \eta_1(T,S).$$

□

We finish this section with the following example, which shows that the inequalities in Theorem 3.15 are significant improvements.

Example 3.16. *From Example 3.6, we have*

$$\begin{aligned} w(TS) \simeq 2.828 &< \eta_1(T,S) \simeq 3.604 \\ &= \eta_2(T,S) \\ &< w(T)[K(S)+R(S)] \simeq 3.683 \\ &< 2w(T)K(S) \simeq 5.366 \\ &< 2w(T)w(S) \simeq 5.656. \end{aligned}$$

4. A-numerical raduis inequalities for operators

In all what follows, $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, i.e., $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We denote by $A^{\frac{1}{2}}$ the square root of A and by $\langle\cdot,\cdot\rangle_A$ the positive semi-definite sesquilinear form induced by A:

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C},\ (x,y) \mapsto \langle x,y\rangle_A = \langle Ax, y\rangle.$$

The seminorm induced by $\langle\cdot,\cdot\rangle_A$ on $\mathcal{H}$ is given by $\|x\|_A = \sqrt{\langle x,x\rangle_A}$ for every $x \in \mathcal{H}$. We can easily show that $\|\cdot\|_A$ is a norm if and only if A is injective, and that $\Big(\mathcal{H}, \|\cdot\|_A\Big)$ is a complete space if and only if the range

$\mathcal{R}(A)$ of A is closed in $\mathcal{H}$. We denote by P the orthogonal projection onto $\overline{\mathcal{R}(A)}$, where $\overline{\mathcal{R}(A)}$ is the closure of $\mathcal{R}(A)$.

Let $T \in \mathcal{B}(\mathcal{H})$, an operator $R \in \mathcal{B}(\mathcal{H})$ is called A-adjoint of T if for every $x, y \in \mathcal{H}$, we have $\langle Tx, y\rangle_A = \langle x, Ry\rangle_A$, i.e., $AR = T^*A$. This kind of equation is studied by Douglas in [14]. In general, the existence and uniqueness of A-adjoint are not guaranteed, see [16]. We denote by $\mathcal{B}_A(\mathcal{H})$ the set of all operators in $\mathcal{B}(\mathcal{H})$ which admit A-adjoints. From Douglas Theorem [14], we have

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\},$$

where $\mathcal{R}(S)$ is the range of an operator $S \in \mathcal{B}(\mathcal{H})$. Notice that $\mathcal{B}_A(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$, see [6, p.1463]. If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^\sharp$ and satisfies $\mathcal{R}(T^\sharp) \subseteq \overline{\mathcal{R}(A)}$. Note that, $T^\sharp = A^\dagger T^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A, see [5]. It is important to note that if $T \in \mathcal{B}_A(\mathcal{H})$ then $T^\sharp \in \mathcal{B}_A(\mathcal{H})$, $\left(\left(T^\sharp\right)^\sharp\right)^\sharp = T^\sharp$ and $\left(T^\sharp\right)^\sharp = PTP$. Moreover, $\left(TS\right)^\sharp = S^\sharp T^\sharp$ and $\left(T + \alpha S\right)^\sharp = T^\sharp + \overline{\alpha} S^\sharp$ for all $S \in \mathcal{B}_A(\mathcal{H})$ and all $\alpha \in \mathbb{C}$.

An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is, $AT = T^*A$. In general, the fact that T is A-selfadjoint does not imply that $T^\sharp = T$, see [34, p.161]. However, if $T \in \mathcal{B}_A(\mathcal{H})$ then $T = T^\sharp$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$ (see, [34]). On the other hand, for every $T \in \mathcal{B}_A(\mathcal{H})$, the following operators are A-selfadjoint: $TT^\sharp$, $T^\sharp T$, $\mathfrak{R}_A(T) = \dfrac{T + T^\sharp}{2}$ and $\mathfrak{I}_A(T) = \dfrac{T - T^\sharp}{2i}$. Before we move on, it should be mentioned that if T is A-selfadjoint then $T^\sharp$ is also A-selfadjoint and $(T^\sharp)^\sharp = T^\sharp$, see [16, Lemma 1]. In this case, we have $\mathfrak{R}_A(T^\sharp) = T^\sharp$ and $\mathfrak{I}_A(T^\sharp) = 0$.

For more information about the class of operators defined on semi-Hilbertian spaces, we refer the reader to [5, 6, 11, 16–19, 28, 34] and the references therein.

Let $T \in \mathcal{B}(\mathcal{H})$, the A-operator seminorm $\|T\|_A$, the A-numerical radius $w_A(T)$ and the A-spectral radius $r_A(T)$ of T are respectively given by:

$$\begin{aligned} \|T\|_A &= \sup\left\{\frac{\|Tw\|_A}{\|w\|_A} : w \in \overline{\mathcal{R}(A)},\ w \neq 0\right\}, \\ w_A(T) &= \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}, \\ r_A(T) &= \limsup_{n\to+\infty} \|T^n\|_A^{\frac{1}{n}}. \end{aligned}$$

It is possible that either $\|T\|_A = +\infty$ or $w_A(T) = +\infty$ or $r_A(T) = +\infty$ for some operator $T \in \mathcal{B}(\mathcal{H})$, see [17]. However, Douglas Theorem [14] asserts that for every $T \in \mathcal{B}_A(\mathcal{H})$ we have $\|T\|_A < +\infty$. In particular, $\|\cdot\|_A$ is a seminorm on $\mathcal{B}_A(\mathcal{H})$. On the other hand, it was shown in [15] that for every $T \in \mathcal{B}_A(\mathcal{H})$:

$$\begin{aligned} \|T\|_A &= \sup\{\|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1\} \\ &= \sup\{|\langle Tx, y\rangle_A| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}. \end{aligned}$$

This implies that if $T \in \mathcal{B}_A(\mathcal{H})$ then $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$, and $\|T\|_A = 0$ if and only if $AT = 0$. In addition, if $T, S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A \leq \|T\|_A\|S\|_A$. It should be mentioned that $w_A(\cdot)$ is a seminorm on $\mathcal{B}_A(\mathcal{H})$ which is equivalent to $\|\cdot\|_A$. More precisely, we have the following inequality

$$\max\left\{r_A(T); \frac{1}{2}\|T\|_A\right\} \leq w_A(T) \leq \|T\|_A,$$

for all $T \in \mathcal{B}_A(\mathcal{H})$, see [8, 17]. It is not difficult to check that $\|T\| = \|T^\sharp\|_A$, $w_A(T) = w_A(T^\sharp)$ and $r_A(T) = r_A(T^\sharp)$ for all $T \in \mathcal{B}_A(\mathcal{H})$. We also have the following properties.

Theorem 4.1 ([15, 17, 31, 34]). *Let $T, S \in \mathcal{B}_A(H)$, the following assertions hold:*

1. $\|TT^\sharp\|_A = \|T^\sharp T\|_A = \|T\|^2 = \|T^\sharp\|_A^2$.
2. *If T is A-selfadjoint then* $r_A(T) = w_A(T) = \|T\|_A$.
3. $w_A(T) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}_A(e^{i\theta}T)\|_A = \sup_{\theta\in\mathbb{R}} \|\mathfrak{I}_A(e^{i\theta}T)\|_A$.
4. $w_A(T) = \sup_{\alpha^2+\beta^2=1} \|\alpha\mathfrak{R}_A(T) + \beta\mathfrak{I}_A(T)\|_A$.
5. $r_A(T) = \inf_{n\in\mathbb{N}^*} \|T^n\|_A^{\frac{1}{n}} = \lim_{n\to+\infty} \|T^n\|_A^{\frac{1}{n}}$.
6. $r_A(TS) = r_A(ST)$.
7. *If $TS = ST$ then* $r_A(TS) \le r_A(T)r_A(S)$ *and* $r_A(T+S) \le r_A(T) + r_A(S)$.

For more results concerning $\|\cdot\|_A$, $w_A(\cdot)$ and $r_A(\cdot)$, we invite the readers to see [5, 15–17, 31, 34].

In 2018, Baklouti et al.[8] defined the A-numerical range of an operator $T \in \mathcal{B}(\mathcal{H})$, as follows:

$$W_A(T) = \{\langle Tx, x\rangle_A : x \in \mathcal{H},\ \|x\|_A = 1\}.$$

They proved that $W_A(T)$ is a non-empty convex subset of $\mathbb{C}$ which is not necessarily compact even if $\dim(\mathcal{H}) < +\infty$. More precisely, they showed that $W_A(T) = \mathbb{C}$ when $T(\mathcal{N}(A)) \not\subseteq \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of A. On the other hand, it is easy to see that $w_A(T) = \sup\{|\lambda| : \lambda \in W_A(T)\}$ for every $T \in \mathcal{B}(\mathcal{H})$. This implies that the closure of $W_A(T)$ is a non-empty compact subset of $\mathbb{C}$ when $T \in \mathcal{B}_A(\mathcal{H})$. Note that if T is A-selfadjoint then $W_A(T)$ is a real segment, see [8, Proposition 2.1].

Let $T \in \mathcal{B}_A(\mathcal{H})$, by Lemma 2.1, there exists a unique complex number ξ_T such that

$$R_A(T) = \inf_{\lambda\in\mathbb{C}} w_A\Big(T - \lambda I\Big) = w_A\Big(T - \xi_T I\Big).$$

Moreover, the disk of the center ξ_T and radius $R_A(T)$ is the smallest disk of the complex plane containing the A-numerical range $W_A(T)$. It follows from Lemma 2.2 that $\xi_{T^\sharp} = \overline{\xi_T}$ and $\xi_{\lambda T} = \lambda\xi_T$ for every $\lambda \in \mathbb{C}$. This gives that $R_A(T^\sharp) = R_A(T)$ and $R_A(\lambda T) = |\lambda|R_A(T)$ for all $\lambda \in \mathbb{C}$. In addition, $R_A(T) = 0$ if and only if $AT = \alpha A$ for some $\alpha \in \mathbb{C}$. On the other hand, we define the quantity $K_A(T)$ as follows:

$$K_A(T) = \inf_{\lambda\in\mathbb{R}} w_A\Big(T - i\lambda\xi_T I\Big).$$

It is clear that $R_A(T) \le K_A(T) \le w_A(T)$. This inequality can be strict, as shown in the following example.

Example 4.2. *Let* $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $A^{\frac{1}{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $T = \begin{bmatrix} 2+i & 1 \\ -2 & 1+i \end{bmatrix}$. *It is easy to see that* $T \in \mathcal{B}_A(\mathcal{H})$. *A simple calculation shows that* $W_A(T) = W\Big(A^{\frac{1}{2}}TA^{-\frac{1}{2}}\Big)$ *and* $A^{\frac{1}{2}}TA^{-\frac{1}{2}} = \begin{bmatrix} 1+i & 0 \\ 0 & i \end{bmatrix}$. *Hence, it follows from Example 2.3 that* $\xi_T = \dfrac{1}{2} + i, R_A(T) = \dfrac{1}{2}, K_A(T) = \dfrac{3}{\sqrt{5}}$ *and* $w_A(T) = \sqrt{2}$. *This implies that*

$$R_A(T) < K_A(T) < w_A(T).$$

It is easy to verify that $K_A(T^\sharp) = K_A(T)$ and $K_A(\lambda T) = |\lambda|K_A(T)$ for all $\lambda \in \mathbb{C}$. Moreover, all properties of $K(T)$ can be extended to $K_A(T)$. Some of them are presented in the following result, the others are left to the reader. It should be indicated that the continuity of K_A will be proved at the end of this section.

Proposition 4.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following assertions hold:*

1. *There exists a real number η_T such that* $K_A(T) = w_A\Big(T - i\eta_T\xi_T I\Big)$.
2. *If $\xi_T \neq 0$ then* $\Big\|\mathfrak{R}_A(c_T T)\Big\|_A \le K_A(T)$ *and* $\Big\|\mathfrak{I}_A(c_T T)\Big\|_A \le R_A(T)$, *where* $c_T = \dfrac{\overline{\xi_T}}{|\xi_T|}$.
3. $w_A(T) \le \sqrt{K_A(T)^2 + R_A(T)^2}$.

4. *$K_A(T) = 0$ if and only if $AT = 0$.*
5. *If T is A-adjoint then $K_A(T) = w_A(T)$.*

Proof. Only parts (2) and (5) are shown here, the others are found using the same techniques used in Proposition 2.4.

(2) Let $\eta_T \in \mathbb{R}$ such that $K(T) = w_A(T - i\eta_T\xi_T I)$. A straightforward calculation shows that

$$\left\|\Re_A(\overline{c_T}T^\sharp)\right\|_A = \left\|\Re_A\Big(\overline{c_T}(T^\sharp + i\eta_T\overline{\xi_T}P)\Big)\right\|_A = \left\|\Big[\Re_A\Big(c_T(T - i\eta_T\xi_T I)\Big)\Big]^\sharp\right\|_A,$$

and

$$\left\|\Im_A(\overline{c_T}T^\sharp)\right\|_A = \left\|\Im_A\Big(\overline{c_T}(T^\sharp - \overline{\xi_T}P)\Big)\right\|_A = \left\| - \Big[\Im_A\Big(c_T(T - \xi_T I)\Big)\Big]^\sharp\right\|_A.$$

Then, it follows from Theorem 4.1 that $\|\Re_A(c_T T)\|_A \leq K_A(T)$ and $\|\Im_A(c_T T)\|_A \leq R_A(T)$.

(5) Using the facts $c_T = \pm 1$ and $T^\sharp = \Re_A(T^\sharp)$, together with part (2), we get the desired result. □

Let us begin with the following definition before moving on to our first extension in this section. For an operator $T \in \mathcal{B}_A(\mathcal{H})$ we define the set $\mathscr{S}_A(T)$ by

$$\mathscr{S}_A(T) = \Big\{C \in \mathcal{B}_A(\mathcal{H}) \mid (C^\sharp)^\sharp = C^\sharp \text{ and } T^\sharp C^\sharp = C^\sharp T^\sharp\Big\}.$$

It is obvious that $\mathscr{S}_A(T)$ is a real linear space, $\mathscr{S}_A(T) = \mathscr{S}_A(T^\sharp)$ and $I, P \in \mathscr{S}_A(T)$. Also, when $A = I$ then $\mathscr{S}_A(T)$ is exactly $\mathscr{S}(T)$. Moreover, we have the following properties.

Proposition 4.4. *Let $T, C \in \mathcal{B}_A(\mathcal{H})$. Then*

1. *$C \in \mathscr{S}_A(T)$ if and only if $C^\sharp \in \mathscr{S}_A(T)$.*
2. *If C is A-selfadjoint such that $TC = CT$ or $CT^\sharp = T^\sharp C$ then $C \in \mathscr{S}_A(T)$.*
3. *$\mathscr{S}_A(T)$ is closed in $\Big(\mathcal{B}_A(\mathcal{H}), \|\cdot\|_A\Big)$.*

Proof.
1. This comes immediately from the facts $((R^\sharp)^\sharp)^\sharp = R^\sharp$ ($R \in \mathcal{B}_A(\mathcal{H})$) and $\mathscr{S}_A(T) = \mathscr{S}_A(T^\sharp)$.
2. Since C is A-selfadjoint, we have $(C^\sharp)^\sharp = C^\sharp$. So, by assumption we get the desired result.
3. Observe that, if $R \in \mathcal{B}_A(\mathcal{H})$ then $\|R^\sharp\|_A = 0$ if and only if $R^\sharp = 0$, because $R^\sharp = A^\dagger R^* A$. Now, let $C \in \mathcal{B}_A(\mathcal{H})$ and a sequence $\{C_n\}$ of elements of $\mathscr{S}_A(T)$ such that $\lim\limits_{n\to+\infty} \|C - C_n\|_A = 0$. Let $n \in \mathbb{N}$, by the fact that $C_n \in \mathscr{S}_A(T)$, we get

$$\begin{aligned} \|T^\sharp C^\sharp - C^\sharp T^\sharp\|_A &= \|T^\sharp C^\sharp - T^\sharp C_n^\sharp + C_n^\sharp T^\sharp - C^\sharp T^\sharp\|_A \\ &\leq 2\|T^\sharp\|_A \|C^\sharp - C_n^\sharp\|_A, \end{aligned}$$

and,

$$\begin{aligned} \|(C^\sharp)^\sharp - C^\sharp\|_A &= \|(C^\sharp)^\sharp - (C_n^\sharp)^\sharp + C_n^\sharp - C^\sharp\|_A \\ &\leq 2\|C_n - C\|_A. \end{aligned}$$

Letting $n \to +\infty$ yields $\Big(C^\sharp\Big)^\sharp = C^\sharp$ and $T^\sharp C^\sharp = C^\sharp T^\sharp$. Thus, $C \in \mathscr{S}_A(T)$. □

Now we present our first extension in this section, which generalizes Theorem 3.1 and improves [19, Theorem 2.13]. The proof is similar to that of Theorem 3.1.

Theorem 4.5. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned} w_A\Big(TS + ST^\sharp\Big) &\leq 2\, dist_A\Big(T, i\mathscr{S}_A(S)\Big)\, w_A(S) \\ &\leq 2 \inf_{\mu\in\mathbb{R}} \|T - i\mu C\|_A\, w_A(S) \ \ \text{(for all } C \in \mathscr{S}_A(S)) \\ &\leq 2\|T\|_A w_A(S), \end{aligned} \tag{23}$$

with $dist_A\Big(T, i\mathscr{S}_A(S)\Big) = \inf\Big\{\|T - iC\|_A \mid C \in \mathscr{S}_A(S)\Big\}$.

Proof. We show the first inequality, the others follow. Let $C \in \mathscr{S}_A(S)$, a simple calculation shows that:

$$w_A\Big(TS + ST^\sharp\Big) = w_A\Big(S^\sharp[T^\sharp + iC^\sharp] + (S[T^\sharp + iC^\sharp])^\sharp\Big).$$

By applying [34, Lemma 3.1] we get

$$w_A\Big(TS + ST^\sharp\Big) \leq 2w_A(S)\|T - iC\|_A.$$

Hence, by taking the infimum over all $C \in \mathscr{S}_A(S)$, we obtain the desired inequality. □

Remark 4.6. 1. *Note that,* $dist_A(T, i\mathscr{S}_A(S)) = dist_A(T^\sharp, i\mathscr{S}_A(S))$.

2. *It follows from (23) that*

$$\begin{aligned} w_A\Big((ST)^\sharp + S^\sharp T\Big) &= w_A\Big(\Big([(ST)^\sharp + S^\sharp T]^\sharp\Big)^\sharp\Big) \\ &= w_A\Big(T^\sharp S^\sharp + S^\sharp(T^\sharp)^\sharp\Big) \\ &\leq 2\, dist_A\Big(T, i\mathscr{S}_A(S)\Big)\, w_A(S). \end{aligned} \tag{24}$$

This gives an improvement of [34, Lemma 3.1].

3. *By replacing T by iT in (23) and (24), we obtain*

$$\begin{aligned} \max\Big\{w_A\Big(TS - ST^\sharp\Big), w_A\Big((ST)^\sharp - S^\sharp T\Big)\Big\} &\leq 2\, dist_A\Big(T, \mathscr{S}_A(S)\Big)\, w_A(S) \\ &\leq 2 \inf_{\mu\in\mathbb{R}} \|T - \mu C\|_A\, w_A(S) \\ &\leq 2\|T\|_A\, w_A(S), \end{aligned}$$

for every $C \in \mathscr{S}_A(S)$.

Corollary 4.7. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_A\Big(TS + ST^\sharp\Big) \leq 2k_A(T)w_A(S) \ \text{ and } \ w_A\Big(TS - ST^\sharp\Big) \leq 2k'_A(T)w_A(S),$$

where $k_A(T) = \inf_{\mu\in\mathbb{R}} \|T - i\mu I\|_A$ *and* $k'_A(T) = \inf_{\mu\in\mathbb{R}} \|T - \mu I\|_A$.

In all what follows, T and S are elements of $\mathcal{B}_A(\mathcal{H})$ such that $\xi_T\xi_S \neq 0$. We denote by $c_T = \dfrac{\overline{\xi_T}}{|\xi_T|}$ and $c_S = \dfrac{\overline{\xi_S}}{|\xi_S|}$.

In the following result, we present an extension of Theorem 3.5.

Theorem 4.8. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned} w_A(TS + ST) &\leq 2\Big(dist_A\Big(\mathfrak{R}_A(c_S S), i\mathscr{S}_A(T)\Big) + dist_A\Big(\mathfrak{I}_A(c_S S), i\mathscr{S}_A(T)\Big)\Big)\, w_A(T) \\ &\leq 2\Big(k_A\Big(\mathfrak{R}_A(c_S S)\Big) + k_A\Big(\mathfrak{I}_A(c_S S)\Big)\Big)\, w_A(T) \\ &\leq 2\Big(\|\mathfrak{R}_A(c_S S)\|_A + \|\mathfrak{I}_A(c_S S)\|_A\Big)\, w_A(T) \\ &\leq 2\Big(K_A(S) + R_A(S)\Big)\, w_A(T) \\ &\leq 2\Big(w_A(S) + R_A(S)\Big)\, w_A(T). \end{aligned}$$

In particular, if $TS = ST$ then

$$\begin{aligned} w_A(TS) &\leq \Big(dist_A\Big(\Re_A(c_S S), i\mathscr{S}_A(T)\Big) + dist_A\Big(\Im_A(c_S S), i\mathscr{S}_A(T)\Big)\Big)\, w_A(T) \\ &\leq \Big(k_A\Big(\Re_A(c_S S)\Big) + k_A\Big(\Im_A(c_S S)\Big)\Big)\, w_A(T) \\ &\leq \Big(\|\Re_A(c_S S)\|_A + \|\Im_A(c_S S)\|_A\Big)\, w_A(T) \\ &\leq \Big(K_A(S) + R_A(S)\Big)\, w_A(T) \\ &\leq \Big(w_A(S) + R_A(S)\Big)\, w_A(T). \end{aligned}$$

Proof. Since $(\Re_A(c_S S)^\sharp)^\sharp = \Re_A(c_S S)^\sharp$ and $(\Im_A(c_S S)^\sharp)^\sharp = \Im_A(c_S S)^\sharp$, we have

$$w_A(ST + TS) = w_A\Big(\Re_A(c_S S)^\sharp T^\sharp + T^\sharp(\Re_A(c_S S)^\sharp)^\sharp - i[\Im_A(c_S S)^\sharp T^\sharp + T^\sharp(\Im_A(c_S S)^\sharp)^\sharp]\Big).$$

So, by using Theorem 4.5 and replicating the proof of Theorem 3.5, we reach the desired result. □

Next, we present the following result which is a natural generalization of Theorem 3.7 and considerably improves the inequalities in [34, Theorem 3.4] and [34, Theorem 3.5.].

Theorem 4.9. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned} w_A(TS) \leq \|TS\|_A &\leq \|T\|_A\Big[K_A(S) + R_A(S)\Big] \\ &\leq \|T\|_A\Big[w_A(S) + R_A(S)\Big], \end{aligned}$$

and,

$$\begin{aligned} w_A(TS) \leq \|TS\|_A &\leq \Big[K_A(T) + R_A(T)\Big]\Big[K_A(S) + R_A(S)\Big] \\ &\leq \Big[w_A(T) + R_A(T)\Big]\Big[K_A(S) + R_A(S)\Big] \\ &\leq \Big[w_A(T) + R_A(T)\Big]\Big[w_A(S) + R_A(S)\Big]. \end{aligned}$$

Proof. We achieve the required inequalities by using Proposition 4.3 (2) and following a similar reasoning as the one used in the proof of Theorem 3.7. □

The next result follows directly from the previous theorem.

Corollary 4.10. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_A(TS) \leq 2\|T\|_A K_A(S) \quad \text{and} \quad w_A(TS) \leq 4K_A(T)K_A(S).$$

From the following results, we can immediately obtain the extensions of Theorems 3.13 and 3.15. We leave the proofs to the reader. The first result presents the relationship between $\mathcal{B}_A(\mathcal{H})$ and the algebra of all operators acting on the Hilbert space $\mathbf{R}(A^{\frac{1}{2}}) := \Big(\mathcal{R}(A^{\frac{1}{2}}), \langle\cdot,\cdot\rangle_{\mathbf{R}(A^{\frac{1}{2}})}\Big)$, where

$$\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\rangle_{\mathbf{R}(A^{\frac{1}{2}})} := \langle Px, Py\rangle \ \text{(for all } x, y \in \mathcal{H}).$$

For more information concerning $\mathbf{R}(A^{\frac{1}{2}})$ and $\mathcal{B}\Big(\mathbf{R}(A^{\frac{1}{2}})\Big)$, we invite the readers to check [7, 17, 27].

Lemma 4.11 ([7, 17, 27]). *Let $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a unique $\widetilde{T} \in \mathcal{B}\Big(\mathbf{R}(A^{\frac{1}{2}})\Big)$ such that $\widetilde{T}W_A = W_A\widetilde{T}$, where $W_A : \mathcal{H} \longrightarrow \mathbf{R}(A^{\frac{1}{2}})$ defined by $W_Ax = Ax$ for all $x \in \mathcal{H}$. Moreover, we have the following properities:*

1. $\|T\|_A = \|\widetilde{T}\|_{\mathcal{B}\left(\mathbf{R}(A^{\frac{1}{2}})\right)}$, $w_A(T) = w\left(\widetilde{T}\right)$ *and* $r_A(T) = r\left(\widetilde{T}\right)$.
2. $\widetilde{T^\sharp} = \left(\widetilde{T}\right)^*$ *and* $\widetilde{(T^\sharp)^\sharp} = \widetilde{T}$.
3. *If* $S \in \mathcal{B}_A(\mathcal{H})$ *and* $\alpha \in \mathbb{C}$ *then*

$$\widetilde{T + \alpha S} = \widetilde{T} + \alpha \widetilde{S} \text{ and } \widetilde{TS} = \widetilde{T}\widetilde{S}.$$

4. $\widetilde{I} = I_{\mathcal{B}\left(\mathbf{R}(A^{\frac{1}{2}})\right)}$ *the identity element of* $\mathcal{B}\left(\mathbf{R}(A^{\frac{1}{2}})\right)$.

The last result is the following.

Lemma 4.12. *Let* $T \in \mathcal{B}_A(\mathcal{H})$, *the following properties hold:*

1. $R_A(T) = R\left(\widetilde{T}\right)$.
2. $\xi_T = z_{\widetilde{T}}$.
3. $K_A(T) = K\left(\widetilde{T}\right)$.
4. $\widetilde{\mathfrak{R}_A(T)} = \mathfrak{R}\left(\widetilde{T}\right)$ *and* $\widetilde{\mathfrak{I}_A(T)} = \mathfrak{I}\left(\widetilde{T}\right)$.

Proof. The parts (1), (3) and (4) come directly from Lemma 4.11. Now, we give the proof of part (2). From Lemma 2.2 we have

$$w_A\left(T - \xi_T I\right) < w_A\left(T - (\xi_T + \lambda)I\right) \text{ (for all } \lambda \in \mathbb{C} \setminus \{0\}).$$

So, by applying Lemma 4.11, we get

$$w\left(\widetilde{T} - \xi_T \widetilde{I}\right) < w\left(\widetilde{T} - (\xi_T + \lambda)\widetilde{I}\right) \text{ (for all } \lambda \in \mathbb{C} \setminus \{0\}).$$

By Lemma 2.2, we conclude that $\xi_T = z_{\widetilde{T}}$. □

Combining lemmas 4.11, 4.12 and Proposition 2.7, we get the following result, showing the continuity of K_A.

Proposition 4.13. *Let* $T \in \mathcal{B}_A(\mathcal{H})$ *and* $\{T_n\}$ *be a sequence of elements of* $\mathcal{B}_A(\mathcal{H})$ *such that* $\lim\limits_{n \mapsto +\infty} \|T_n - T\|_A$. *Then, the sequence* $\{K_A(T_n)\}$ *converges to* $K_A(T)$.

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