Filomat 37:20 (2023), 6855–6866
https://doi.org/10.2298/FIL2320855B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Compactness of boundary value problems for impulsive integro-differential equation

Moustafa Beddani[a], **Hamid Beddani**[b]

[a] *Department of Mathematics, E. N. S. of Mostaganem, 27000 Mostaganem, Algeria*
[b] *Laboratory of Mathematics, Higher School in Electrical and Energy Engineering, 31000 Oran, Algeria*

Abstract. In this paper, we establish sufficient conditions to show the compactness of solution set of boundary value problems for impulsive integro-differential equation using ψ-Hilfer fractional operator in a appropriate Banach space. The method we use to show our result is based on fixed point theorems for Meir-Keeler condensing operators via measure of non-compactness, an example is presented to illustrate our method.

1. Introduction

The concept of fractional calculus has tremendous potential to change the way we see the model. Several theoretical studies showed that some systems in physics and medicine are governed by fractional differential equations [21, 22, 26]. The effectiveness of fractional calculus in many scientific fields has motivated researchers to the theoretical study of differential problems, see [1, 6, 12, 17, 19, 23–25].

The class of impulsive fractional differential equations is distinguished from others by the modeling of phenomena which undergo distortions, in particular in the field of medicine and physics, see [5]. In the references [2, 7, 15], the authors are interested in the study of impulsive differential equations involving the derivative of Riemann or Hilfer.

Recently, many interesting works have appeared in the study fractional differential equations on Banach spaces, which resides in the existence and uniqueness results by using fixed point theorems and some basic tools from functional analysis [13, 14]. One of the properties of solutions is the compactness of solution set, Recently, this property have been studied by many researchers for certain differential problems considered, see [10, 11, 16].

In this present work, we consider the following terminal value problem for impulsive fractional integro-differential equation

$$(\mathbf{P})\qquad \begin{cases} {}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+} y(t) = f(t, y(t), \Im^{\delta,\psi}_{t_k^+} y(t)), \;\; t \in I_k = (t_k, t_{k+1}],\ k = 0, \cdots, m, \\ y(L) = M \text{ and } \Delta_{\gamma,\psi} y|_{t_k} = J_k(y_{\gamma,\psi}(t_k^+)),\ k = 1, \cdots, m, \end{cases}$$

where ${}^{H}\mathcal{D}^{\alpha,\beta,\psi}$ denote the $\psi-$ Hilfer fractional derivative of ordr $0 < \alpha < 1$ and type $0 \le \beta \le 1$, $\gamma = \alpha + \beta(1-\alpha)$, $\delta > 1 - \gamma$. The operator $\Im^{\delta,\psi}$ denotes the left-sided $\psi-$ Riemann-Liouville fractional integral of order δ,

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 34A60, 34A08.
Keywords. Measure of non-compactness; Meir-Keeler condensing operators; Banach space; fixed point theorems.
Received: 27 November 2022; Accepted: 22 February 2023
Communicated by Maria Alessandra Ragusa
Email addresses: m.beddani@univ-chlef.dz (Moustafa Beddani), beddanihamid@gmail.com (Hamid Beddani)

$f : (c, L] \times E^2 \to E$ a function satisfying some specified conditions, t_k, $k = 0, \ldots, m$ are pre-fixed points satisfying $t_0 = c < t_1 \leq \cdots \leq t_m < t_{m+1} = L$, E is a Banach space with the norm $\|.\|$, $J_k : E \to E$, $M \in E$, $\psi \in C^1([c, L], \mathbb{R}^+)$ satisfies $\psi'(t) > 0$, for all $t \in [c, L]$ and $\Delta_{\gamma,\psi} y|_{t_k} = \frac{\Im^{1-\gamma,\psi}_{t_k^+} y(t_k^+)}{(\psi(t_{k+1})-\psi(t_k))^{1-\gamma}\Gamma(\gamma)} - y(t_k^-)$, where $\Im^{1-\gamma,\psi}_{t_k^+} y(t_k^+) = \lim_{t\to t_k^+} \Im^{1-\gamma,\psi}_{t_k^+} y(t)$, $y(t_k^-) = \lim_{t\to t_k^-} y(t)$ and $y_{\gamma,\psi}(t_k^+) = \lim_{t\to t_k^+} (\psi(t) - \psi(t_k))^{1-\gamma} y(t)$.
The present work is ordered as follows: to make the reader understand our problem, we give in section 2 some definitions, lemmas and basic results. Next, in the section 3, we present our main results by applying the fixed point theorem combined with the technique of measure of non-compactness to show the compactness of solution set of Problem (**P**). Finally an example to reinforce our work is presented in the section 4.

2. Background and basic results

We introduce in this section some notation and technical results which are used throughout this paper. Let $C([a, b], E)$ be the space of E-valued continuous functions on the interval $[a, b] \subset \mathbb{R}$ endowed with the following uniform norm topology:

$$\|u\|_\infty = \sup\{\|u(t)\|,\ t \in [a, b]\}.$$

Let $C_{1-\gamma}([a, b], E)$ be the Banach spaces of functions from the interval $(a, b]$ into E which is defined as:

$$C_{1-\gamma,\psi}((a, b], E) = \Big\{u \in C([a, b], E) : \ \ (\psi(.) - \psi(a))^{1-\gamma} u(.) \in C([a, b], E)\Big\}.$$

A norm in this space is given by

$$\|u\|_{C_{\gamma,\psi}} = \sup_{t\in(a,b]} \frac{(\psi(t) - \psi(a))^{1-\gamma}}{(\psi(b) - \psi(a))^{1-\gamma}} \|u(t)\|.$$

In the following, for all $\eta > -1$, we pose $\Psi_\eta(r, s) = (\psi(r) - \psi(s))^\eta$, for all $s, r \in [c, L]$ with $r > s$ and $\Psi^*_\eta = \max\{\Psi_\eta(t_{k+1}, t_k),\ k = 0, \ldots, m\}$.
We consider the following Banach spaces

$$\begin{aligned}\mathcal{PC}_{1-\gamma,\psi}([c, L], E) = \ \ &\Big\{y : [c, L] \to E : y_k \in C_{1-\gamma,\psi}([t_k, t_{k+1}], E), k = 0, \ldots, m \text{ with} \\ &y(t_k) = y(t_k^-), k = 1, \ldots, m\Big\},\end{aligned}$$

with the norm

$$\|y\|_{\mathcal{PC}_{\gamma,\psi}} = \max_{k=0,\ldots m.} \|y_k\|_{\gamma,\psi},$$

where y_k is the restriction of y to $(t_k, t_{k+1}]$.
For any subset N of $\mathcal{PC}_{1-\gamma,\psi}([c, L], E)$, we put $N_{\gamma,\psi} =: \{u_{\gamma,\psi},\ u \in N\}$, where

$$(u_{\gamma,\psi})_k(t) = \begin{cases} \Psi_{1-\gamma}(t, t_k)u(t), & \text{if } t \in (t_k, t_{k+1}] \\ \lim\limits_{t\to t_k} \Psi_{1-\gamma}(t, t_k)u(t), & \text{if } t = t_k, \end{cases}$$

where $(u_{\gamma,\psi})_k$ is the restriction of $u_{\gamma,\psi}$ on $[t_k, t_{k+1}]$,

Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of G.

Definition 2.1. *[8, 18] Let $D \in S_b(E)$. The Kuratowski measure of non-compactness ϑ of the subset D is defined as follows:*

$$\vartheta(D) = \inf\{e > 0 : \Omega \textit{ admits a finite cover by sets of diameter} \leq e\}.$$

Lemma 2.2. *[8, 18] Let $A, B \in S_b(E)$. The following properties hold:*

(i_1) $\vartheta(A) = 0$ *if and only if A is relatively compact,*

(i_2) $\vartheta(A) = \vartheta(\overline{A})$, *where $\overline{A}$ denotes the closure of A,*

(i_3) $\vartheta(A + B) \leq \vartheta(A) + \vartheta(B)$,

(i_4) $A \subset B$ *implies* $\vartheta(A) \leq \vartheta(B)$,

(i_5) $\vartheta(a.A) = |a|.\vartheta(A)$ *for all* $a \in \mathbb{R}$,

(i_6) $\vartheta(\{a\} \cup A) = \vartheta(A)$ *for all* $a \in E$,

(i_7) $\vartheta(A) = \vartheta(Conv(A))$, *where $Conv(A)$ is the smallest convex that contains A.*

Lemma 2.3. *[16] Let $D \in S_b(E)$ and $\varepsilon > 0$. Then, there is a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset D$, such that*

$$\vartheta(D) \leq 2\vartheta(\{\mu_n, n \in \mathbb{N}\}) + \varepsilon.$$

Lemma 2.4. *[18] If D is a equicontinuous and bounded subset of $C([a, b], E)$, then $\vartheta(D(.)) \in C([a, b], \mathbb{R}^+)$*

$$\vartheta_C(D) = \max_{t\in[a,b]} \vartheta(D(t)),\ \vartheta\left(\left\{\int_a^b w(t)dt : w \in D\right\}\right) \leq \int_a^b \vartheta(D(t))dr,$$

where $D(t) = \{w(t) : w \in D\}$ and ϑ_C is the non-compactness measure on the space $C([a, b], E)$.

We denote by $\vartheta^k_{\gamma,\psi}$ and $\vartheta_{\gamma,\psi}$ the Kuratowski measures of non-compactness defines respictively on $C_{1-\gamma,\psi}([t_k, t_{k+1}], E)$ and $\mathcal{P}C_{1-\gamma,\psi}([c, L], E), k = 0, \dots, m$.

Lemma 2.5. *[18] For all bounded subset D of $\mathcal{P}C_{1-\gamma,\psi}([c, L], E)$, we have*

$$\vartheta_{\gamma,\psi}(D) = \max_{k=0,\dots,m} \vartheta^k_{\gamma,\psi}(D_k),$$

where D_k is the restriction of D on $(t_k, t_{k+1}]$.

Definition 2.6. *[4] Let κ be an arbitrary measure of non-compactness on E and G be a nonempty subset of E . Let Λ be an operator from G to G. Λ is said Meir-Keeler condensing operator if*

$$\forall \varepsilon > 0,\ \exists k(\varepsilon) > 0, \forall D \in S_b(G) : \varepsilon \leq \kappa(D) < \varepsilon + k \Longrightarrow \kappa(\Lambda D) < \varepsilon.$$

Theorem 2.7. *[4] Let κ be an arbitrary measure of non-compactness on E and G a closed, bounded and convex subset of E. Let Λ be an operator from G to G, assume that Λ is a Meir-Keeler condensing operator and continuous, then the set $\{w \in G : \Lambda(w) = w\}$ is nonempty and compact.*

We begin with some definitions from the theory of fractional calculus.

Definition 2.8. *[19, 27] Let ℓ be an integrable function defined on $(a, b]$,*

(i) the ψ-Riemann- Liouville fractional integral of order $\alpha > 0$ of the function ℓ is defined by

$$\Im^{\alpha,\psi}_{a^+}\ell(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\Psi_{\alpha-1}(t, s)\ell(s)ds,$$

(ii) the ψ-Riemann- Liouville fractional derivative of order $\alpha > 0$ of the function ℓ is defined by

$$^{RL}\mathcal{D}_{a^+}^{\alpha,\psi}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \left(\int_a^t \psi'(s)\Psi_{n-\alpha-1}(t,s)\ell(s)ds\right),$$

where Γ is the gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad (x > 0)$ and $n = [\alpha] + 1$ ($[\alpha]$ represents the integer part of the real number α).

Definition 2.9. *[19, 27] Let $\psi \in C^1([a,b], E)$ a functions such that $\psi'(t) > 0$, for all $t \in [a,b]$. The ψ-Hilfer fractional derivative of a function ℓ of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ is given by*

$$^{H}\mathcal{D}_{a^+}^{\alpha,\beta,\psi}\ell(t) = \mathfrak{I}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\beta)(1-\alpha),\psi}\ell(t) = \mathfrak{I}^{\beta(1-\alpha),\psi\, RL}D^{\gamma,\psi}\ell(t),$$

where $\gamma = \alpha + \beta(1-\alpha)$.

Lemma 2.10. *[19] Let $\alpha, \rho \in \mathbb{R}_+^*$ and $t > a$. We have then*

(i_1) $\mathfrak{I}_{a^+}^{\alpha,\psi}\Psi_{\rho-1}(t,a) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}\Psi_{\alpha+\rho-1}(t,a)$.

(i_2) $^{H}\mathcal{D}_{a^+}^{\alpha,\rho,\psi}\Psi_{\rho-1}(t,a) = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\Psi_{\rho-\alpha-1}(t,a), 0 < \alpha < 1,\ \rho > 1$.

We consider the following spaces

$$C_{1-\gamma,\psi}^{\gamma}([a,b]) = \left\{u \in C_{1-\gamma,\psi}([a,b]),\ ^{RL}\mathcal{D}_{a^+}^{\gamma}u \in C_{1-\gamma,\psi}([a,b])\right\},$$

$$\mathcal{P}C_{1-\gamma,\psi}^{\gamma}([c,L]) = \left\{u \in \mathcal{P}C_{1-\gamma,\psi}([c,L]):\ ^{RL}\mathcal{D}_{t_k^+}^{\gamma,\psi}u_k \in C_{1-\gamma}([t_k,t_{k+1}]),\ k = 0,\ldots,m\right\},$$

and

$$\mathcal{P}C_{1-\gamma,\psi}^{\alpha,\beta}([c,L]) = \left\{u \in \mathcal{P}C_{1-\gamma,\psi}([c,L]):\ ^{H}\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi}u_k \in C_{1-\gamma}([t_k,t_{k+1}]),\ k = 0,\ldots,m\right\}.$$

From the definition of $^{H}\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi}$ and since $\mathfrak{I}_{t_k^+}^{\beta(1-\alpha),\psi}$ is defined from $C_{1-\gamma}([t_k,t_{k+1}])$ into $C_{1-\gamma}([t_k,t_{k+1}])$, $k = 0,\ldots,m$, we have $\mathcal{P}C_{1-\gamma,\psi}^{\gamma}([c,L]) \subset \mathcal{P}C_{1-\gamma,\psi}^{\alpha,\beta}([c,L])$.

Lemma 2.11. *[20] Let $0 < \alpha < 1,\ 0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $\omega \in C_{1-\gamma}^{\gamma}([a,b])$, then*

$$\mathfrak{I}_{a^+}^{\gamma,\psi}\mathcal{D}_{a^+}^{\gamma,\psi}\omega = \mathfrak{I}_{a^+}^{\alpha,\psi}\mathcal{D}_{a^+}^{\alpha,\beta,\psi}\omega$$

and

$$\mathcal{D}_{a^+}^{\gamma,\psi}\mathfrak{I}_{a^+}^{\alpha,\psi}\omega = \mathcal{D}_{a^+}^{\beta(1-\alpha)}\omega.$$

Lemma 2.12. *[20] Let $\omega : (a.b] \to E$ be a function such that $\omega(.) \in C_{1-\gamma,\psi}([a,b])$. Then, a function $y \in C_{1-\gamma,\psi}^{\gamma}([a,b])$ is a solution of linear fractional differential problem:*

$$\begin{cases} ^{H}\mathcal{D}_{a^+}^{\alpha,\beta,\psi}y(t) = \omega(t), & 0 < \alpha < 1,\ 0 \leq \beta \leq 1; \\ \mathfrak{I}_{a^+}^{1-\gamma,\psi}y(a^+) = \omega_0, & \gamma = \alpha + \beta - \alpha\beta. \end{cases}$$

if and only if y satisfies the following integral equation:

$$y(t) = \frac{\omega_0\Psi_{\gamma-1}(t,a)}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\Psi_{\alpha-1}(t,s)\omega(s)ds.$$

Lemma 2.13. *Let $h : (c, L]\times \to E$ be a function such that $h(.) \in C_{1-\gamma,\psi}([t_k, t_{k+1}])$ $k = 0, \ldots, m$. If $y \in \mathcal{PC}^{\gamma}_{1-\gamma,\psi}([c, L])$. Then, y is a solution of the following problem*

$$ {}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{t^+_{m-k}} y(t) = h(t), \;\; t \in (t_{m-k}, t_{m-k+1}],\; k = 0, \ldots, m, \tag{2.1}$$

$$ y(L) = M \text{ and } \Delta_{\gamma,\psi} y|_{t_{m-k}} = J_{m-k}(y_{\gamma,\psi}(t^+_{m-k})),\; k = 1, \ldots, m, \tag{2.2}$$

if and only if y satisfies the following integral equation:

$$ y(t) = \begin{cases} \frac{\Psi_{\gamma-1}(t,t_m)}{\Psi_{\gamma-1}(t_{m+1},t_m)} \left(M - \Im^{\alpha,\psi}_{t_m} h(L)\right) + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \psi'(s)\Psi_{\alpha-1}(s, t_m)h(s)ds, & \text{if } t \in I_m, \\ \frac{\Psi_{\gamma-1}(t,t_{m-2})}{\Psi_{\gamma-1}(t_{m-1},t_{m-2})} \left(M - \sum_{i=0}^{k} \Im^{\alpha,\psi}_{t_{m-i}} h(t_{m-i+1}) - \sum_{i=0}^{k-1} J_{m-i}(y_\gamma(t^+_{m-i}))\right) & \\ + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \psi'(s)\Psi_{\alpha-1}(t, s)h(s)ds, & \text{if } t \in I_{m-k}, \\ & k = 1, \ldots, m. \end{cases} \tag{2.3}$$

Proof. Assume y satisfies the problem (2.1)-(2.2). We want to prove that y verified (2.3).
If $t \in (t_m, t_{m+1}]$, we have ${}^{H}\mathcal{D}^{\alpha,\beta,\psi} y(t) = h(t)$, from Lemma 2.12, we obtain

$$ y(t) = \frac{\Im^{1-\gamma,\psi}_{t^+_m} y(t^+_m)}{\Gamma(\gamma)} \Psi_{\gamma-1}(t, t_m) + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \psi'(s)\Psi_{\alpha-1}(t, s)h(s)ds.$$

Since $y(L) = M$, we obtain

$$ \frac{\Im^{1-\gamma,\psi}_{t^+_m} y(t^+_m)}{\Psi_{1-\gamma}(t_{m+1}, t_m)\Gamma(\gamma)} = M - \Im^{\alpha,\psi}_{t_m} h(L). \tag{2.4}$$

So,

$$ y(t) = \left(M - \Im^{\alpha,\psi}_{t_m} h(L)\right) \frac{\Psi_{\gamma-1}(t, t_m)}{\Psi_{\gamma-1}(t_{m+1}, t_m)} + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \psi'(s)\Psi_{\alpha-1}(s, t_m)h(s)ds.$$

If $t \in (t_{m-1}, t_m]$, then, from Lemma 2.12, we get

$$ y(t) = \frac{\Im^{1-\gamma,\psi}_{t^+_{m-1}} y(t^+_{m-1})}{\Gamma(\gamma)} \Psi_{\gamma-1}(t, t_{m-1}) + \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} \psi'(s)\Psi_{\alpha-1}(t, s)h(s)ds.$$

By Equations (2.2) and (3.4), we have

$$ \frac{\Im^{1-\gamma,\psi}_{t^+_{m-1}} y(t^+_{m-1})}{\Psi_{1-\gamma}(t_m, t_{m-1})\Gamma(\gamma)} = M - \Im^{\alpha,\psi}_{t_m} h(L) - \Im^{\alpha,\psi}_{t_{m-1}} h(t_m) - J_m(y_\gamma(t^+_m)). \tag{2.5}$$

So,

$$ y(t) = \left(M - \mathcal{I}^{\alpha,\psi}_{t_m} h(L) - \mathcal{I}^{\alpha,\psi}_{t_{m-1}} h(t_m) - J_m(y_\gamma(t^+_m))\right) \frac{\Psi_{\gamma-1}(t, t_{m-1})}{\Psi_{\gamma-1}(t_m, t_{m-1})} + \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} \psi'(s)\Psi_{\alpha-1}(s, t_m)h(s)ds.$$

If $t \in (t_{m-2}, t_{m-1}]$, then. By Lemma 2.12, we have

$$ y(t) = \left(y(t_{m-1}) - \Im^{\alpha,\psi}_{t_{m-2}} h(t_{m-1})\right) \frac{\Psi_{\gamma-1}(t, t_{m-2})}{\Psi_{\gamma-1}(t_{m-1}, t_{m-2})} + \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \psi'(s)\Psi_{\alpha-1}(s, t_{m-1})h(s)ds.$$

By Equations (2.2) and (2.5), we have

$$ y(t) = \left(M - \sum_{i=0}^{2} \Im^{\alpha,\psi}_{t_{m-i}} h(t_{m-i+1}) - \sum_{i=0}^{1} J_{m-i}(y_\gamma(t^+_{m-i}))\right) \frac{\Psi_{\gamma-1}(t, t_{m-2})}{\Psi_{\gamma-1}(t_{m-1}, t_{m-2})} + \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} \psi'(s)\Psi_{\alpha-1}(t, s)h(s)ds.$$

If $t \in (t_{m-k}, t_{m-k+1}]$, using, again, Lemma 2.12, by recurrence, we find

$$y(t) = \left(M - \sum_{i=0}^{k} \Im_{t_{m-i}}^{\alpha,\psi} h(t_{m-i+1}) - \sum_{i=0}^{k-1} J_{m-i}(y_\gamma(t_{m-i}^+))\right) \frac{\Psi_{\gamma-1}(t, t_{m-2})}{\Psi_{\gamma-1}(t_{m-1}, t_{m-2})} + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} \psi'(s)\Psi_{\alpha-1}(t,s)h(s)ds.$$

Conversely, assume that y satisfies the impulsive equation (2.3). If $t \in (t_m, t_{m+1}]$, then $y(L) = M$ and using the lemma 2.12, we get

$$^H\mathcal{D}_{t_m}^{\alpha,\beta,\psi} y(t) = h(t), \quad \text{for each } t \in (t_m, t_{m+1}].$$

By recurrence. If $t \in (t_{m-k}, t_{m-k+1}]$, $k = 1, \ldots, m$ and using the lemma 2.12 again, we get

$$^H\mathcal{D}_{t_{m-k}}^{\alpha,\beta,\psi} y(t) = h(t), \quad \text{for each } t \in (t_{m-k}, t_{m-k+1}].$$

Also, we can easily show that

$$\Delta_{\gamma,\psi} y|_{t=t_{m-k}} = J_{m-k}(y_\gamma(t_{m-k}^+)), \quad k = 0, \ldots, m-1.$$

□

3. Existence and compactness of solution set

Suppose that the function $f : (c, L] \times E^2 \to E$ verifies: $f(., u(.), v(.)) \in \mathcal{PC}_{1-\gamma,\psi}^{\gamma}([c, L])$, for all $u(.), v(.) \in \mathcal{PC}_{1-\gamma}([c, L])$, $f(., 0, 0) \in C([c, L], E)$ and there exists $A, B \in \mathbb{R}^+$ and $\lambda \geq 1 - \gamma$ such that

(H$_1$) For any $u, v, \overline{u}, \overline{v} \in E$:

$$\|f(t, u, v) - f(t, \overline{u}, \overline{v})\| \leq A\Psi_\lambda(t, t_k)\|u - \overline{u}\| + B\|v - \overline{v}\|, \text{ for all } t \in I_k,\ k = 1, \ldots, m.$$

(H$_2$) For each nonempty, bounded set $\Omega \subset \mathcal{PC}_{1-\gamma,\psi}([c, L])$, for all $t \in I_k, k = 0, \ldots, m$, we have

$$\vartheta\Big(f(t, \Omega(t), \Im_{t_k}^{\delta,\psi}\Omega(t)\Big) \leq A\Psi_\lambda(t, t_k)\vartheta\Big(\Omega(t))\Big) + B\vartheta\Big(\Im_{t_k}^{\delta,\psi}\Omega(t)\Big),$$

where $\Im_{t_k}^{\delta,\psi}\Omega(t) = \{\Im_{t_k}^{\delta,\psi} y(t), y \in \mathcal{PC}_{1-\gamma,\psi}([c, L])\}$, $k = 0, \ldots, m$.

Suppose that the functions $J_k : E \to E$, $k = 1, \ldots, m$, are continuous and there exists $C \in \mathbb{R}^+$ such that

(H$_3$) For any $u \in E$:

$$\|J_k(u)\| \leq C\|u\|,\ k = 1, \ldots, m.$$

(H$_4$) For each nonempty, bounded set $\Omega \subset \mathcal{PC}_{1-\gamma,\psi}([c, L])$, we have

$$\vartheta\Big(J_k(\Omega(t))\Big) \leq C\vartheta\Big(\Omega(t)\Big),\ k = 0, \ldots, m.$$

(H$_5$)

$$mC\Psi_{1-\gamma}^*\Gamma(\alpha + 1) + (m + 2)(A\Psi_{\alpha+\lambda}^* + BT\Psi_{\alpha+\delta}^*) < \Gamma(\alpha + 1),$$

where $T = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)}$.

Our result concerning the existence and compactness of solution set of the problem (**P**) for which we have used the fixed point theorem of Meir-Keeler is as follows

Theorem 3.1. *We assume that the hypotheses from* (**H$_1$**) *to* (**H$_5$**) *are satisfied, Then, the solution set of problem* (**P**) *is nonempty and compact. Moreover its solutions belong to* $\mathcal{PC}_{1-\gamma,\psi}^{\gamma}([c, L]) \subset \mathcal{PC}_{1-\gamma,\psi}^{\alpha,\beta}([c, L])$.

Proof. First, we prove the existence of the solutions in the space $\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$. Consider the operator $N:\mathcal{PC}_{1-\gamma,\psi}([c,L],E)\to\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$ defined by

$$\begin{aligned}Ny(t)=&\left(M-\sum_{i=0}^{k}\Im^{\alpha,\psi}_{t_{m-i}}f(t_{m-i+1},y(t_{m-i+1}),\Im^{\delta,\psi}_{t^+_{m-i}}y(t_{m-i+1})))-\sum_{i=0}^{k-1}J_{m-i}(y_\alpha(t^+_{m-i}))\right)\times\frac{\Psi_{\gamma-1}(t,t_{m-k})}{\Psi_{\gamma-1}(t_{m-k+1},t_{m-k})}\\&+\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\psi'(s)\Psi_{\alpha-1}(t,s)f(s,y(s),\Im^{\delta,\psi}_{t^+_{m-k}}y(s))ds,\end{aligned}$$

with $t\in I_{m-k}=(t_{m-k},t_{m-k+1}]$ $k=0,\dots,m$. First, we prove the existence of the set of fixed points of N is included in $\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$

From the definition of the operator N and Lemma 2.13, we see that the fixed points of N are solutions of Eq. (2.3). For this reason, it suffices to verify the axioms of Theorem 2.7, it is done in four steps.

First step. We start to prove that N is continuous. Let $\varepsilon>0$ and $\{y_n\}_{n\in\mathbb{N}}\to y$ in $\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$. The hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_3})$ confirm the existence of an integer $m\in\mathbb{N}$ such that, for all $n\geq m$ and $t\in(t_{m-k},t_{m-k+1}]$, $k=1,\dots,m$, we have

$$\|f(t,y_n(t),\Im^{\delta,\psi}_{t^+_{m-k}}y(t))-f(t,y(t),\Im^{\delta,\psi}_{t^+_{m-k}}y(t))\|<\frac{\Gamma(\alpha+1)\varepsilon}{2(m+2)(A\Psi^*_\lambda+BT\Psi^*_\delta)\Psi^*_\alpha}\tag{3.1}$$

and

$$\|J_{m-k}((y_\gamma)_n(t^+_{m-k})-J_{m-k}(y_\gamma(t^+_{m-k})\|<\frac{\varepsilon}{2m}.\tag{3.2}$$

Thus, for all $t\in(t_{m-k},t_{m-k+1}]$, $k=1,\dots,m$, we have

$$\begin{aligned}\frac{\Psi_{1-\gamma}(t,t_{m-k})}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\|Ny_n(t)-Ny(t)\|\leq&\sum_{i=0}^{k-1}\|J_{m-i}(y_{\gamma,n}(t^+_{m-i}))-J_{m-i}(y_\gamma(t^+_{m-i}))\|\\&+\sum_{i=0}^{k}\Im^{\alpha,\psi}_{t_{m-i}}\|f(t_{m-i+1},y_n(t_{m-i+1}),\Im^{\delta,\psi}_{t^+_{m-i}}y_n(t_{m-i+1}))-f(t_{m-i+1},y(t_{m-i+1}),\Im^{\delta,\psi}_{t^+_{m-i}}y(t_{m-i+1}))\|\\&+\frac{\Psi_{1-\gamma}(t,t_{m-k})}{\Gamma(\alpha)\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\\&\times\int_{m-k}^{t}\psi'(s)\Psi_{1-\alpha}(t,s)\|f(s,y_n(s),\Im^{\delta,\psi}_{t^+_{m-k}}y_n(s))-f(s,y(s),\Im^{\delta,\psi}_{t^+_{m-k}}y(s))\|ds.\end{aligned}$$

By Equations (3.1) and (3.2), for all $t\in(t_{m-k},t_{m-k+1}]$, $k=1,\dots,m$, we get

$$\begin{aligned}\frac{\Psi_{1-\gamma}(t,t_{m-k})}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\|Ny_n(t)-Ny(t)\|<&\frac{\varepsilon}{2}\\&+\frac{2(m+2)(A\Psi^*_\lambda+BT\Psi^*_\delta)\Psi^*_\alpha}{\Gamma(\alpha+1)}\times\frac{\Gamma(\alpha+1)\varepsilon}{2(m+2)(A\Psi^*_\lambda+BT\Psi^*_\delta)\Psi^*_\alpha}=\varepsilon.\end{aligned}$$

So,

$$\|Ny_n-Ny\|_{\mathcal{PC}_{\gamma,\psi}}<\varepsilon.$$

Thus, N is continuous on $\mathcal{PC}_{1-\gamma,\psi}([c,L],E))$.

Second step. Now we will prove that N is bounded. Let $y\in\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$, from $(\mathbf{H_1})$ it is easy to deduce that $Ny\in\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$. Using $(\mathbf{H_1})$, for all $y\in D_\kappa=\{y\in\mathcal{PC}_{1-\gamma,\psi}([c,L],E):\|y\|_{\mathcal{PC}_{\gamma,\psi}}<\kappa\}$ and

$t \in (t_{m-k}, t_{m-k+1}]$, we get

$$\begin{aligned}\Big\|\frac{\Psi_{1-\gamma}(t,t_{m-k})Ny(t)}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\Big\| &\le \|M\| + \sum_{i=0}^{k} \Im_{t_{m-i}}^{\alpha,\psi}\|f(t_{m-i+1}, y(t_{m-i+1}), \Im_{t_{m-i}^+}^{\delta,\psi} y(t_{m-i+1}))\| \\ &+ \sum_{i=0}^{k-1} \|J_{m-i}(y_\alpha(t_{m-i}^+))\| \\ &+ \frac{\Psi_{1-\gamma}(t,t_{m-k})}{\Gamma(\alpha)\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\int_{t_{m-k}}^{t} \psi'(s)\Psi_{\alpha-1}(t,s)\|f(s,y(s),\Im_{t_{m-k}^+}^{\delta,\psi} y(s))\|ds \\ &\le \|M\| + mC\Psi_{1-\gamma}^*\kappa + \frac{(m+2)\big(f^* + (A\Psi_\lambda^* + BT\Psi_\delta^*)\kappa)\Psi_\alpha^*}{\Gamma(\alpha+1)}.\end{aligned}$$

So,

$$\|Ny\|_{\mathcal{P}C_{\gamma,\psi}} \le \|M\| + mC\Psi_{1-\gamma}^*\kappa + \frac{(m+2)\big(f^* + (A\Psi_\lambda^* + BT\Psi_\delta^*)\kappa\big)\Psi_\alpha^*}{\Gamma(\alpha+1)} = \ell,$$

where $f^* = \max\{\sup_{t\in[t_k,t_{k+1}]}\|f(t,0,0)\|,\ k=0,\dots,m\}$.

Third step. We prove that $(ND)_k$ is equicontinuous for all bounded subset D of $\mathcal{P}C_{1-\gamma,\psi}([c,L],E)$, $k=0,\dots,m$, where $(ND)_k$ the restriction of ND on the interval $(t_k,t_{k+1}]$, let D_κ be the subset which was previously defined. It suffices to prove that $(ND_\kappa)_k$ is equicontinuous in $\mathcal{C}_{\gamma,\psi}([t_{m-k},t_{m-k+1}],E)$. Let $y \in (D_\kappa)_{m-k}$ and $t_1,t_2 \in (t_{m-k},t_{m-k+1}]$ with $t_1 < t_2$, we have

$$\begin{aligned}&\Big\|\frac{\Psi_{1-\gamma}(t_2,t_{m-k})Ny(t_2)}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})} - \frac{\Psi_{1-\gamma}(t_1,t_{m-k})Ny(t_1)}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\Big\| \\ &\le \frac{\Psi_{1-\gamma}(t_1,t_{m-k})}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})\Gamma(\alpha)}\int_{t_{m-k}}^{t_1}\psi'(s)[\Psi_{\alpha-1}(t_1,s)-\Psi_{\alpha-1}(t_2,s)]\|f(s,y(s),\Im_{t_{m-k}^+}^{\delta,\psi}y(s))\|ds \\ &+ \frac{[\Psi_{1-\gamma}(t_2,t_{m-k})-\Psi_{1-\gamma}(t_1,t_{m-k})]}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})\Gamma(\alpha)}\int_{t_{m-k}}^{t_1}\psi'(s)\Psi_{\alpha-1}(t_2,s)\|f(s,y(s),\Im_{t_{m-k}^+}^{\delta,\psi}y(s))\|ds \\ &+ \frac{\Psi_{1-\gamma}(t_2,t_{m-k})}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\Psi_{\alpha-1}(t_2,s)\|f(s,y(s),\Im_{t_{m-k}^+}^{\delta,\psi}y(s))\|ds \\ &\le \frac{f^*+(A\Psi_\lambda^*+BT\Psi_\delta^*)\kappa}{\Gamma(\alpha)}\int_{t_{m-k}}^{t_1}\psi'(s)[\Psi_{\alpha-1}(t_1,s)-\Psi_{\alpha-1}(t_2,s)]ds \\ &+ \frac{f^*[\Psi_{1-\gamma}(t_2,t_{m-k})-\Psi_{1-\gamma}(t_1,t_{m-k})]}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})\Gamma(\alpha)}\int_{t_{m-k}}^{t_1}\psi'(s)\Psi_{\alpha-1}(t_2,s)ds \\ &+ \frac{A\kappa[\Psi_{1-\gamma}(t_2,t_{m-k})-\Psi_{1-\gamma}(t_1,t_{m-k})]}{\Gamma(\alpha)}\int_{t_{m-k}}^{t_1}\psi'(s)\Psi_{\alpha-1}(t_2,s)\Psi_{\lambda+\gamma-1}(s,t_{m-k})ds \\ &+ \frac{BT\kappa[\Psi_{1-\gamma}(t_2,t_{m-k})-\Psi_{1-\gamma}(t_1,t_{m-k})]}{\Gamma(\alpha)}\int_{t_{m-k}}^{t_1}\psi'(s)\Psi_{\alpha-1}(t_2,s)\Psi_{\delta+\gamma-1}(s,t_{m-k})ds \\ &+ \frac{f^*+(A\Psi_\lambda^*+BT\Psi_\delta^*)\kappa}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\Psi_{\alpha-1}(t_2,s)ds \\ &\le \frac{f^*+(A\Psi_\lambda^*+BT\Psi_\delta^*)\kappa}{\Gamma(\alpha+1)}[\Psi_\alpha(t_2,t_1)+\Psi_\alpha(t_1,t_{m-k})-\Psi_\alpha(t_2,t_{m-k})] \\ &+ \frac{f^*\Psi_{\alpha+\gamma-1}^* + A\kappa\Psi_{\alpha+\gamma+\lambda-1}^* + BT\kappa\Psi_{\alpha+\gamma+\delta-1}^*}{\Gamma(\alpha+1)}[\Psi_{1-\alpha}(t_2,t_{m-k})-\Psi_{1-\alpha}(t_1,t_{m-k})]\end{aligned}$$

$$+\frac{f^*+(A\Psi^*_{\lambda}+BT\Psi^*_{\delta})\kappa}{\Gamma(\alpha+1)}\Psi_{\alpha}(t_2,t_1).$$

Taking t_2 tends towards t_1, we get that, the last inequality tends to zero. Then $(ND_{\kappa})_k$ is equicontinuous in $C_{1-\gamma,\psi}([t_{m-k},t_{m-k+1}],E),\ k=0,\ldots,m$.

Final step. We verify that N satisfies the assumptions of theorem 2.7. We pose

$$D=\{y\in\mathcal{PC}_{1-\gamma,\psi}([c,L],E):\|y\|_{\mathcal{PC}_{\gamma,\psi}}\le R\},$$

where R is a real number verifies the following equality

$$R>\frac{\|M\|\Gamma(\alpha+1)+(m+2)f^*\Psi^*_{\alpha}}{\Gamma(\alpha+1)-mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)+(m+2)(A\Psi^*_{\alpha+\lambda}+BT\Psi^*_{\alpha+\delta})}.\tag{3.3}$$

First, we now show that N is defined from D to D, Indeed, for any $y\in D$, by above conditions $(\mathbf{H_2})$, $(\mathbf{H_5})$ and by according to a little calculation, for all $t\in(t_{m-k},t_{m-k+1}]$, we have

$$\begin{aligned}\left\|\frac{\Psi_{1-\gamma}(t,t_{m-k})Ny(t)}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\right\|&\le\|M\|+\frac{(m+2)f^*\Psi^*_{\alpha}}{\Gamma(\alpha+1)}+\Big(mC\Psi^*_{1-\gamma}+\frac{(m+2)(A\Psi^*_{\alpha+\lambda}+BT\Psi^*_{\alpha+\delta})}{\Gamma(\alpha+1)}\Big)R\\&\le\frac{\|M\|\Gamma(\alpha+1)+(m+2)f^*\Psi^*_{\alpha}}{\Gamma(\alpha+1)}\\&\quad+\left(\frac{mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)+(m+2)(A\Psi^*_{\alpha+\lambda}+BT\Psi^*_{\alpha+\delta})}{\Gamma(\alpha+1)}\right)R\end{aligned}$$

From the inequality (3.3), we obtain

$$\forall y\in D:\|Ny\|_{\mathcal{PC}_{\gamma,\psi}}<R.$$

Then N remains defined from D to D. Note that D is bounded, convex and closed subset of $\mathcal{PC}_{1-\gamma,\psi}([c,L],E)$ and N is continuous on D, we can easily show the following equalitie

$$\vartheta^k_{\gamma,\psi}((NV)_{m-k})=\sup\left\{\vartheta\left(\frac{\Psi_{1-\gamma}(t,t_{m-k})NV(t)}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\right),\ t\in(t_{m-k},t_{m-k+1}]\right\},$$

for all $V\subset D,\ k=0,\ldots,m$. Next, we need to prove the following implication

$$\forall\epsilon>0,\exists\varrho(\epsilon)>0:\epsilon\le\vartheta_{\gamma,\psi}(V)<\epsilon+\varrho\Longrightarrow\vartheta_{\gamma,\psi}(NV)<\epsilon,\ \text{for any } V\subset D.\tag{3.4}$$

Let ϵ be a strictly positive real, $V\subset D$. From Lemmas 2.3,2.4,2.5, $(\mathbf{H_3})$ and the previous steps, we have, that there exists a sequence $\{\mu_n\}_{n=0}^{\infty}\subset V$ such that, for all $t\in(t_{m-k},t_{m-k+1}]$:

$$\begin{aligned}\vartheta\left(\frac{\Psi_{1-\gamma}(t,t_{m-k})(NV)(t)}{\Psi_{1-\gamma}(t_{m-k+1},t_{m-k})}\right)&\le\frac{\epsilon}{2}+\frac{2(A\Psi^*_{\lambda}+BT\Psi^*_{\delta})}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_{m-i}}^{t_{m-i+1}}\psi'(s)\Psi_{\alpha-1}(t_{m-i+1},s)\vartheta^{m-i}_{\gamma,\psi}(V_{m-i})ds\\&\quad+2C\Psi^*_{1-\gamma}\sum_{i=0}^{k-1}\vartheta^{m-i}_{\gamma,\psi}(V_{m-i})+\frac{2(A\Psi^*_{\lambda}+BT\Psi^*_{\delta})}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}\psi'(s)\Psi_{\alpha-1}(t,s)\vartheta^{m-k}_{\gamma,\psi}(V_{m-k})ds.\end{aligned}$$

We know that

$$\vartheta_{\gamma,\psi}(NV)\le\frac{\epsilon}{2}+\frac{2\Big[(m+2)(A\Psi^*_{\lambda+\alpha}+BT\Psi^*_{\delta+\alpha})+mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)\Big]}{\Gamma(\alpha+1)}\vartheta_{\gamma,\psi}(V).$$

If

$$\vartheta_{(\alpha,\phi)}(NV)\le\frac{\epsilon}{2}+\frac{2\Big[(m+2)(A\Psi^*_{\lambda+\alpha}+BT\Psi^*_{\delta+\alpha})+mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)\Big]}{\Gamma(\alpha+1)}\vartheta_{\gamma,\psi}(V)<\epsilon,$$

this implies that

$$\vartheta_{(\alpha,\phi)}(V) < \frac{\Gamma(\alpha+1)}{4\Big[(m+2)(A\Psi^*_{\lambda+\alpha} + BT\Psi^*_{\delta+\alpha}) + mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)\Big]}\epsilon,$$

so that implication (3.4) is fulfilled, we take

$$\varrho = \frac{\Gamma(\alpha+1) - 4\Big[(m+2)(A\Psi^*_{\lambda+\alpha} + BT\Psi^*_{\delta+\alpha}) + mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)\Big]}{4\Big[(m+2)(A\Psi^*_{\lambda+\alpha} + BT\Psi^*_{\delta+\alpha}) + mC\Psi^*_{1-\gamma}\Gamma(\alpha+1)\Big]}\epsilon.$$

So, N is a Meir-Keeler condensing operator via $\vartheta_{\gamma,\psi}$, finally all the hypotheses of the theorem 2.7 are fulfilled. Then, the solution set of Eq. (2.3) is nonempty and compact. Let us now show that the fixed point of N is included in $\mathcal{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$, Let $w \in \{u \in \mathcal{PC}_{1-\gamma,\psi}([c,L]) : Nu = u\}$, for all $t \in (t_{m-k}, t_{m-k+1]}$, $k = 0,\dots,m$, we have

$$\begin{aligned} w(t) &= \left(M - \sum_{i=0}^{k} \Im^{\alpha,\psi}_{t_{m-i}} f(t_{m-i+1}, w(t_{m-i+1}), \Im^{\delta,\psi}_{t^+_{m-i}} w(t_{m-i+1}))) - \sum_{i=0}^{k-1} J_{m-i}(w_\gamma(t^+_{m-i}))\right) \frac{\Psi_{\gamma-1}(t,t_{m-k})}{\Psi_{\gamma-1}(t_{m-k+1},t_{m-k})} \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t} \psi'(s)\Psi_{\alpha-1}(t,s) f(s, w(s), \Im^{\delta,\psi}_{t^+_{m-k}} w(s))ds, \end{aligned}$$

By entering ${}^{RL}\mathcal{D}^{\gamma}_{t^+_k}$ on both sides, it follows from Lemma 2.10 and Lemma 2.11 that

$$\begin{aligned} {}^{RL}\mathcal{D}^{\gamma}_{t^+_{m-k}} w(t) &= {}^{RL}\mathcal{D}^{\gamma}_{t^+_{m-k}} \Im^{\alpha,\psi}_{t^+_{m-k}} f\Big(t, w(t), \Im^{\delta,\psi}_{t_{m-k}} w(t)\Big) \\ &= {}^{RL}\mathcal{D}^{\beta(1-\alpha)}_{t^+_{m-k}} f\Big(t, w(t), \Im^{\delta,\psi}_{t_{m-k}} w(t)\Big). \end{aligned}$$

Thus, according to the hypotheses on f, we deduce that ${}^{RL}\mathcal{D}^{\gamma}_{t^+_k} w(t) \in C^{\gamma}_{1-\gamma}([t_k, t_{k+1}])$, $k = 0,\dots,m$, from the definition of $\mathcal{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$, we conclude that the fixed point w of N is an element of such space. Finally, the solution set of Problem (**P**) is nonempty and compact. □

4. Example

We take the following problem

$$ {}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{k} y(t) = \left(\frac{\Im^{\delta,\psi}_{t_k} y(t)}{20+nt^2} + \frac{\sqrt{\psi(t)-\psi(t_i)}}{20+t+t^2} y_n(t)\right)_{n=1}^{\infty}, \; t \in (t_k, t_{k+1}] \subset (0,1], \; k = 0,1 \tag{4.1}$$

$$y(1) = (1,0,\dots,0,\dots). \tag{4.2}$$

$$\Delta_{\gamma,\psi} y|_{t=\frac{1}{2}} = \frac{1}{20} y_{\gamma,\psi}(\frac{1}{2}^+), \tag{4.3}$$

where $\psi(t) = t$, $t_0 = 0, t_1 = 0.5, t_2 = 1, \alpha = \beta = \delta = \lambda = 0.5$ and

$$E = \{(y_1, y_2, \dots, y_n, \dots) : \sup_n |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $(E, \|.\|)$ consists a Banach space, by comparing with the (**P**), we notice that

$$f(t, y(t)) = (f_1(t, y_1(t), \Im^{\delta,\psi}_{t_k} y_1(t)), \dots, f_n(t, y_n(t), \Im^{\delta,\psi}_{t_k} y_n(t)), \dots),$$

where

$$f_n(t, y_n(t), \Im^{\delta,\psi}_{t_k} y_n(t)) = \frac{\Im^{\delta,\psi}_{t_k} y(t)}{20+nt^2} + \frac{\sqrt{t-t_k}}{20+t+t^2} y_n(t), t \in (t_k, t_{k+1}], k = 0,1, n \in \mathbb{N}^* \text{ and}$$

$$J_1(u) = \frac{1}{20}u, \text{ for all } u \in E.$$

We can easily see that $f : (t_k, t_{k+1}] \times E \to E$, $k = 0, 1$ and $J_1 : E \to E$ are continuous and there exists $A = B = C = \frac{1}{20}$ such that

$$\|f(t,u,v) - f(t,\overline{u},\overline{v})\| \leq A\sqrt{t-t_k}\|u-\overline{u}\| + B\|v-\overline{v}\|, \text{ for all } t \in I_k \text{ and } u, v, \overline{u}, \overline{v} \in E \text{ and}$$

$$\|J_1(u)\| = C\|u\|, \text{ for all } u \in E.$$

So, ($\mathbf{H_2}$) and ($\mathbf{H_4}$) are valid. Next, let Ω be a bounded subset of $\mathcal{PC}_{1\gamma,\psi}([0,1])$, we have

$$\vartheta\Big(f\Big(t, \Omega(t), \mathfrak{I}_{t_k}^{\delta,\psi}\Omega(t)\Big)\Big) \leq \frac{1}{10}\Big(\sqrt{t-t_k}\,\vartheta\Big(\Omega(t)\Big) + \vartheta\Big(\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(t)\Big),\ t \in I_k, k = 0, 1 \text{ and}$$

$$\vartheta\Big(J_1(\Omega(t)\Big) \leq \frac{1}{10}\Big(\Omega(t)\Big).$$

Thus, ($\mathbf{H_1}$) and ($\mathbf{H_3}$) are satisfied. A quick calculation gives us

$$C\Psi^*_{1-\gamma}\Gamma(\alpha+1) + 3(A\Psi^*_{\alpha+\lambda} + BT\Psi^*_{\alpha+\delta}) < \Gamma(\alpha+1).$$

So, ($\mathbf{H_5}$) holds. Therefore, Theorem 3.1 ensures that the solution set of Problem (4.1)-(4.3) is nonempty and compact.

References

[1] M. I. Abbas, *Erdélyi-Kober Fractional Integral Boundary Conditions*, Filomat, **34** (9) (2020), 3003-3014.
[2] R. P. Agarwal, B. Hedia and M. Beddani, *Structure of solutions sets for impulsive fractional differential equation*, J. Fractional Cal. Appl. **9**(1) (2018), 15–34.
[3] R. P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge 2001.
[4] A. Aghajani, M. Mursaleen and A. Shole Haghighi, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci. Ser. **35** (2015), 552–556.
[5] Z. Agur, L. Cojocaru, G. Mazaur, R. M. Anderson, Y. L. Danon, *Pulse mass measles vaccination across age cohorts,* Proc. Nat. Acad. Sci. USA **90** (1993), 11698–11702.
[6] H. M. Ahmed, M. A. Ragusa, *Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential*, Bull. Malays. Math. Sci. Soc., **45** (6) (2022), 3239–3253.
[7] K. Balachandran, S. Kiruthika, J. J. Trujillo, *Existence results for fractional impulsive integro-differential equations in Banach spaces*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011) , 1970–1977.
[8] J. Banaŝ, K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Note in Pure App.Math, **60**, Dekker, New York 1980.
[9] J. Banaŝ, M. Mursaleen, *Sequence spaces and measures of noncompactness with applications to differential and integral equations*, Springer, New Delhi 2014.
[10] M. Beddani, *Solution set for impulsive fractional differential inclusions*, Kragujevac J. Math. **46** (1) (2022), 49–64.
[11] M. Beddani and B. Hedia, *An existence result for a fractional differential equation with ϕ-fractional derivative*, Filomat **36** (3) (2022), 753-–762.
[12] M. Beddani and B. Hedia, *Existence result for fractional differential equations on unbounded domain*, Kragujevac J. Math. **48** (5) (2024), 755-766.
[13] M. Beddani and B. Hedia, *Existence result for a fractional differential equation involving a sequential derivative*, Moroccan J. of Pure and Appl. Anal. **8** (1) (2022), 67–77.
[14] M. Benchohra, S. Bouriah and J. J. Nieto, *Terminal value problem for differential equations with Hilfer-Katugampola fractional derivative*, Symmetry, **11** (2019), 675–389.
[15] Q. Chen, A. Debbouche, Z. Luo and J. Wang, *Impulsive fractional differential equations with Riemann-Liouville derivative and iterative learning control*, Chaos Solitons Fractals, **102** (2017), 111–118.
[16] C. Derbazi, Z. Baitiche, M. Benchohra, *Cauchy problem with ψ-Caputo fractional derivative in Banach spaces*, Advances in the Theory of Nonlinear Analysis and its Applications, **4** (2020), 349–361.
[17] M. Fečkan, Y. Zhou and J. Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 3050–3060.
[18] D. J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht 1996.
[19] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V. Amsterdam 2006.

[20] K.D. Kucche, A. D. Mali, *On the Nonlinear ψ-Hilfer Hybrid Fractional Differential Equations*, Mathematics 2020.
[21] D. Luo, J. R. Wang, M. Feckan, *Applying fractional calculus to analyze economic growth modelling*, J. Appl. Math. Stat. Inform., **14** (2018), 25–36.
[22] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, vol. **198**, Academic Press, New York, London, Toronto 1999.
[23] S. Rashid, A. O. Akdemir, M. A. Noor, K. I. Noor, *Integral inequalities for exponentially harmonically convex functions via fractional integral operators*, Miskolc Math. Notes, **22** (2) (2021), 875–888.
[24] M. A. Ragusa, A. Razani, *Weak solutions for a system of quasilinear elliptic equations*, Contrib. Math. (Shahin Digital Publisher), **1** (2020), 11–16.
[25] M. A. Ragusa, V. B. Shakhmurov, *A Navier-Stokes-type problem with high-order elliptic operator and applications*, Mathematics, **8**(12) (2020), 2256.
[26] S.G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon 1993.
[27] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul., **60** (2018), 72–91.