Filomat 37:20 (2023), 6813–6822
https://doi.org/10.2298/FIL2320813K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Topological equicontinuity and topological uniform rigidity for dynamical system

Devender Kumar[a], **Ruchi Das**[a]

[a]*Department of Mathematics, University of Delhi, Delhi-110007, India*

Abstract. In this paper, we study topological equicontinuity, topological uniform rigidity and their properties. For a dynamical system, on a compact, T_3 space, we study relations among the set of recurrent points of the map, the set of non-wandering points of the map and the intersection of the range sets of all iterations of the map. We define topological version of uniform rigidity and show that on a compact and T_3 space any dynamical system is topologically uniformly rigid if it is first countable, almost topologically equicontinuous and transitive or it is second countable, topologically equicontinuous and has a dense set of periodic points. We show that a topologically uniformly rigid dynamical system, on a compact, Hausdorff space, has zero topological entropy. Moreover, we provide necessary examples and counterexamples.

1. Introduction

Sensitivity plays an important role in defining almost all types of chaos. Equicontinuous systems are supposed to have simple chaotic behaviours. Roughly speaking, sensitivity predicts that nearby points will go far away after a long time and equicontinuity says that points nearby will remain nearby at any given point of time. Many authors consider equicontinuity as an almost inverse of sensitivity. A generalization of concept of equicontinuity is the concept of even continuity, which says, in a rough way, if the image of a point in domain goes nearby to some point in codomain then the image of points nearby to the point in domain goes nearby to the point in codomain [13]. The definition of generalized topological equicontinuity, as defined by Royden (he called it topological equicontinuity) says that if some point nearby to a point in domain goes nearby to a point in codomain then the points nearby to the point in domain goes nearby to the point in codomain[21]. This definition (however called topological equicontinuity in the book [21]) does not seem to be a topological version of equicontinuity, rather it appears to be a generalization of topological version of equicontinuity. As mentioned earlier, equicontinuity can be considered as an almost inverse of sensitivity. In the same way, we define topological equicontinuity as the inverse of topological sensitivity.

Since equicontinuity is important in the study of dynamical systems, so many authors studied concepts which are close to the concept of equicontinuous families[3, 4, 11, 18, 19]. Akin et. al localized the concept of equicontinuity to a point[1]. Many authors have proved results concerning transitivity and equicontinuity[1, 7]. Akin et. al proved that a transitive system is either almost equicontinuous or sensitive

2020 *Mathematics Subject Classification*. Primary 37B40; Secondary 37B02 54A20, 54B10.
Keywords. Topological equicontinuity; Topological uniform rigidity; Topological entropy; Transitivity.
Received: 25 October 2022; Revised: 21 February 2023; Accepted: 24 February 2023
Communicated by Ljubiša D.R. Kočinac
The first author is supported by CSIR-SRF Sr. No. 09/045(1799)/2020-EMR-I for carrying out this research work.
Email addresses: kalgandevender@gmail.com (Devender Kumar), rdasmsu@gmail.com (Ruchi Das)

and a minimal system is either equicontinuous or sensitive[1]. Many analogues of this dichotomy result have been proved[5, 8, 10, 16, 19]. Results relating equicontinuity of subspaces and product spaces with the corresponding base spaces are also studied[4, 17, 20]. Royden defined the concept of generalized topological equicontinuity (he called it topological equicontinuity)[21]. Many authors have studied equicontinuity on uniform spaces[8, 13, 14]. For equicontinuous dynamical systems, Mai has studied relations among set of recurrent points, almost periodic points and non-wandering points of a map on metric spaces. He proved that for any dynamical system on a metric space the set of recurrent points is same as the set of non-wandering points and if the space is compact also then the set of recurrent points, the set of non-wandering points and the set of almost periodic points are same[20]. Since equicontinuous systems are supposed to have simple chaotic behaviours, many authors relate equicontinuity and entropy of a dynamical system[6, 9, 22]. Glasner and Maon introduced the concept of rigidity of a dynamical system and proved that any rigid dynamical system has zero entropy[6].

Inspired by above work, in this paper, we study topological equicontinuity, topological rigidity and topological entropy of a dynamical system. Our main results are Theorem 1, Theorem 2 and Theorem 3. In Theorem 1, we prove that on a first countable, compact and T_3 space any almost topologically equicontinuous and transitive dynamical system is topologically uniformly rigid. On a second countable, compact and uniform space any topologically equicontinuous dynamical system having dense set of periodic points is topologically uniformly rigid is proved in Theorem 2. Theorem 3 says that any topologically uniformly rigid dynamical system on a compact Hausdorff topological space has zero topological entropy. In Section 2, we provide necessary definitions required for the remaining sections of the paper. In Section 3, we introduce notions of almost topological equicontinuity and topological equicontinuity. We study relation between topological equicontinuity and generalized topological equicontinuity. We prove dichotomy result that a minimal dynamical system is either topologically equicontinuous or topologically sensitive. We also show that the product dynamical system is topologically equicontinuous if and only if both the base dynamical systems are topologically equicontinuous. We prove that if X is compact and T_3 space then the set of recurrent points of f is same as the set of non-wandering points of f and intersection of image sets of f^n is same as the set of recurrent points of f. In Section 4, we define topological version of uniform rigidity. We show that any almost topologically equicontinuous and transitive system over a first countable, compact and T_3 space is topologically uniformly rigid and if a dynamical system over a second countable, compact and Hausdorff space is topologically equicontinuous and has dense set of periodic points then the system is topologically uniformly rigid. We also show that any topologically uniformly rigid dynamical system over a compact Hausdorff space has zero topological entropy. We provide necessary examples and counterexamples.

2. Preliminaries

We denote the set of natural numbers by $\mathbb{N}$. A subset A of natural numbers is called *syndetic* if there exists a $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}, \{n, n+1, n+2, \ldots, n+k\} \cap A \neq \emptyset$. Let X be any topological space. If X is a metric space then for any set $A \subset X$, diameter of A is $diam(A) = \sup\{d(x, y) : x, y \in A\}$. For any set $A \subset X$ closure of A is denoted by $\overline{A}$. A topological space X is called T_3 if $\{x\}$ is closed for every $x \in X$ and for any $x \in X$ and any open set G containing x there exists an open set G_1 such that $x \in G_1 \subset \overline{G}_1 \subset G$. A space X is called *completely regular* if for any closed set $A \subset X$ and any element $x \notin A$ there exists a continuous map $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(A) = \{0\}$ where $f(A) = \{f(a) : a \in A\}$.

For any $x \in X$ let $\mathcal{N}_x$ denotes the collection of all open sets containing x. Let $\mathcal{F}$ be a family of continuous functions from a topological space X to a topological space Y. We say that $\mathcal{F}$ is *generalized topologically equicontinuous at* $(x, y) \in X \times Y$ if for any $O \in \mathcal{N}_y$ there exist neighbourhoods U of x and V of y such that for any $f \in \mathcal{F}$ if $f(U) \cap V \neq \emptyset$ then $f(U) \subset O$. If $\mathcal{F}$ is generalized topologically equicontinuous at $(x, y) \in X \times Y$ for every $y \in Y$ then we say that $\mathcal{F}$ is *generalized topologically equicontinuous at* x and if $\mathcal{F}$ is generalized topologically equicontinuous at x for every $x \in X$ then we say that $\mathcal{F}$ is *generalized topologically equicontinuous*[21]. Let (X, d_1) and (Y, d_2) be metric spaces then a point $x \in X$ is called an *equicontinuity point* if given any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $y \in Y$, $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$ for every

$f \in \mathcal{F}$[1]. A family $\mathcal{F}$ of continuous maps is called *almost equicontinuous* if there exists an equicontinuity point in X. A family $\mathcal{F}$ of continuous functions is called *equicontinuous* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_1(x,y) < \delta$ then $d_2(f(x), f(y)) < \epsilon$ for every $f \in \mathcal{F}$ and for all $(x,y) \in X \times Y$. A map $f : X \to X$ is called *sensitive* if there exists a $\delta > 0$ such that for any $\epsilon > 0$ and any $x \in X$ there exists a $y \in B(x,\epsilon) = \{z \in X | d_1(x,z) < \epsilon\}$ and $n \in \mathbb{N}$ such that $d_1(f^n(x), f^n(y)) > \delta$[2].

Let X be a topological space and $T : X \to X$ be a continuous map then (X,T) is called a *dynamical system*. We say that (X,T) is *transitive* if for any pair of nonempty open sets $U, V \subset X$ there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A point $x \in X$ is called a *transitive point* if the set $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X. The set containing all transitive points of (X,T) is denoted by $Trans(T) = \{x : x \in X$ and x is a transitive point$\}$. For a dynamical system (X,T), with X being a metric space, we say (X,T) is *uniformly rigid* if there exists a sequence $n_k \nearrow \infty$ such that $\lim T^{n_k} =$ Identity uniformly [6]. For two dynamical systems (X,T) and (Y,S) the *product dynamical system* is $(X \times Y, T \times S)$ where $(T \times S)(x,y) = (T(x), S(y))$ and $X \times Y$ is equipped with the product topology. For any topological space X *orbit of* $x \in X$, is $O(x,f) = \{f^n(x) : n \in \mathbb{N} \cup \{0\}\}$ and ω *limit set of* x is $\omega(x,f) = \bigcap_{n=0}^{\infty} \overline{O(f^n(x), f)}$. The set of *recurrent points of* f is $R(f) = \{x : x \in \omega(x,f)\}$, the set of *non-wandering points of* f is $\Omega(f) = \{x :$ for any open set G containing x there exists an $m \in \mathbb{N}$ such that $f^m(G) \cap G \neq \emptyset\}$ and the set of *almost periodic points of* f is $AP(f) = \{x :$ for any open set G containing x there exists an $m \in \mathbb{N}$ such that for any $i \in \mathbb{N}, G \cap \{f^{j+i}(x) : j \in \{1, 2, \ldots, m\}\} \neq \emptyset$. For any set $A \subset X$, we say that A is $f-$ *invariant* if $f(A) \subset A$. A closed and $f-$ invariant set A is called a *minimal set* if $O(a)$ is dense in A for every $a \in A$ where A is considered with the subspace topology. Let (X,T) and (Y,S) be two dynamical systems, we say that (Y,S) is a *factor* of (X,T) (or (X,T) is an extension of (Y,S)) if there exists an onto continuous map $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. If the map π is a homeomorphism then we say that (X,T) and (Y,S) are conjugate dynamical systems.

For any sets $A, B \subset X \times X$, $A \circ B = \{(x,z) :$ there exists a $y \in X$ such that $(x,y) \in A$ and $(y,z) \in B\}$. *Diagonal* of $X \times X$ is defined as $\Delta_X = \{(x,x) : x \in X\}$ and $\mathscr{D}_X = \{A \subset X \times X : \Delta_X \subset A$ and $A = A^{-1}\}$ where for any set $A \subset X \times X$, $A^{-1} = \{(y,x) : (x,y) \in A\}$

Definition 2.1. ([13]) For any set X, a *uniformity* on X is a nonempty collection $\mathcal{U}$ of subsets of $X \times X$ satisfying the following conditions:

1. $\mathcal{U} \subset \mathscr{D}_X$;
2. If $A_1 \in \mathcal{U}$ and $A_1 \subset A_2 \in \mathscr{D}_x$ then $A_2 \in \mathcal{U}$;
3. For any $A_1, A_2 \in \mathcal{U}$, $A_1 \cap A_2 \in \mathcal{U}$;
4. For any $A \in \mathcal{U}$ there exists a $B \in \mathcal{U}$ such that $B \circ B \subset A$.

The pair $(X, \mathcal{U})$ is called a *uniform space* and the elements of $\mathcal{U}$ are called *entourages*. Any uniform space $(X, \mathcal{U})$ induces a topology on X, called *uniform topology* generated by basic open sets $A[x] = \{y : (x,y) \in A$ where $A \in \mathcal{U}\}$ and we say that the topology is induced by uniformity $\mathcal{U}$.

Let X be any topological space and $\mathscr{U}$ and $\mathscr{V}$ be any open covers of X then $\mathscr{U} \vee \mathscr{V} = \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}$. For a topological space X, an open cover $\mathscr{U}$ of X and a continuous map $f : X \to X$, $f^{-1}(\mathscr{U}) = \{f^{-1}(U) : U \in \mathscr{U}\}$ and for any $n \in \mathbb{N}$, $f^{-(n+1)}(\mathscr{U}) = \{f^{-1}(f^{-n}(U)) : U \in \mathscr{U}\}$. Let X be any compact topological space and $\mathscr{U}$ be any open cover of X then the minimum cardinality of $\mathscr{U}$ that covers X is denoted by $N(\mathscr{U})$. For any compact topological space X, an open cover $\mathscr{U}$ of X and a continuous map $f : X \to X$, $h(f, \mathscr{U})$ is defined as $h(f, \mathscr{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathscr{U} \vee f^{-1}(\mathscr{U}) \vee \cdots \vee f^{-(n-1)}(\mathscr{U}))$ and *topological entropy* of f is denoted by $h(f)$ and is defined by $h(f) = \sup\{h(f, \mathscr{U}) : \mathscr{U}$ is an open cover of $X\}$.

3. Topological equicontinuity and some properties

Definition 3.1. Let X and Y be any topological spaces. For any $x \in X$, a family $\mathcal{F}$ of continuous maps from X to Y is said to be *topologically equicontinuous at* $x \in X$ if for every open cover $\mathscr{V}$ of Y there exists an open set G containing x such that for every $f \in \mathcal{F}$ there exists a $V \in \mathcal{V}$ such that $f(G) \subset V$. A point $x \in X$ is called a *topologically equicontinuous point*. We say that $\mathcal{F}$ is *almost topologically equicontinuous* if $\mathcal{F}$ is topologically equicontinuous at x for some $x \in X$. $\mathcal{F}$ is said to be *topologically equicontinuous* if $\mathcal{F}$ is topologically equicontinuous at x for every $x \in X$.

For any family $\mathcal{F}$ of continuous functions from X to Y, in case X is compact, we can easily observe that if $\mathcal{F}$ is generalized topologically equicontinuous then $\mathcal{F}$ is topologically equicontinuous. Next example justifies that the converse is not true.

Example 1. Let $X = Y = \{0, 1, 2\}$ with topology $\tau = \{\emptyset, X, \{0, 1\}, \{0, 2\}, \{0\}\}$. Define $f : X \to Y$ by $f(0) = 0, f(1) = 2, f(2) = 1$ and family $\mathcal{F} = \{f^n : n \in \mathbb{N}\} = \{e, f\}$ where e is the identity map then $\mathcal{F}$ is topologically equicontinuous. For $(1, 1) \in X \times Y$, take open set $O = \{0, 1\} \subset Y$ then for any open set $U \subset X$ containing 1 and $V \subset Y$ containing 1, $f(U) \cap V \neq \emptyset$, but $f(U) \not\subset O$. So, $\mathcal{F}$ is not generalized topologically equicontinuous at $(1, 1)$ and hence not generalized topologically equicontinuous.

Next we see relations between topological equicontinuity and equicontinuity in case X and Y are metric spaces.

Proposition 1. *Let (X, d_1) and (Y, d_2) be metric spaces and $\mathcal{F}$ be a family of continuous functions from X to Y.*

1. *If Y is compact then equicontinuity of $\mathcal{F}$ implies topological equicontinuity of $\mathcal{F}$*
2. *If X is compact then topological equicontinuity of $\mathcal{F}$ implies equicontinuity of $\mathcal{F}$.*

Proof. 1. Assume $\mathcal{F}$ is equicontinuous and $x \in X$ be any element. Let $\mathcal{V}$ be any open cover of Y then as Y is a compact metric space, so there exists a Lebesgue number $\epsilon > 0$ of $\mathcal{V}$. By definition of equicontinuity, for above $\epsilon > 0$ there exists a $\delta > 0$ such that for any $y \in Y$ if $d_1(x, y) < \delta$ then $d_2(f(x), f(y)) < \epsilon$ for every $f \in \mathcal{F}$. Take $G = B(x, \delta/2)$ then for any $x_1, x_2 \in G$, $d_1(x_1, x_2) < \delta$ and hence $d_2(f(x_1), f(x_2)) < \epsilon$ for every $f \in \mathcal{F}$ and so $\operatorname{diam}(f(G)) < \epsilon$ implying that $f(G) \subset V$ for some $V \in \mathcal{V}$. Hence, $\mathcal{F}$ is topologically equicontinuous at x for every $x \in X$. Therefore, $\mathcal{F}$ is topologically equicontinuous.

2. Assume that $\mathcal{F}$ is topologically equicontinuous and $\epsilon > 0$ be arbitrary. By definition of topological equicontinuity, for any $x \in X$ there exists a nonempty open set G_x such that for any $f \in \mathcal{F}$ there exists a $V \in \mathcal{V}$ such that $f(G_x) \subset V$ where $\mathcal{V} = \{B(y, \epsilon/2) : y \in Y\}$ is an open cover of Y. Note that $\mathcal{U} = \{G_x : x \in X\}$ is an open cover of X and as X is a compact metric space, so there exists a Lebesgue number, say δ of $\mathcal{U}$, such that for any $x_1, x_2 \in X$ if $d_1(x_1, x_2) < \delta$ then $x_1, x_2 \in G_x$ for some $x \in X$ and hence for any $f \in \mathcal{F}$; $f(x_1), f(x_2) \in f(G_x) \subset V$ for some $V \in \mathcal{V}$. Therefore, $d_2(f(x_1), f(x_2)) < \epsilon$. Thus, for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_1(x_1, x_2) < \delta$ then $d_2(f(x_1), f(x_2)) < \epsilon$ for every $f \in \mathcal{F}$ implying that $\mathcal{F}$ is equicontinuous. □

From above proposition it is clear that if X and Y are compact metric spaces then any family $\mathcal{F}$ of continuous functions from X to Y is topologically equicontinuous if and only if it is equicontinuous. However, in general, the above result need not be true. Next example justifies this.

Example 2. Let $X = \mathbb{R}$ with usual metric and define $f : X \to X$ by $f(x) = x + \alpha$ where α is a real number. Now, consider open cover $\mathcal{U} = \{B(0, 1), B(\pm(1 + \frac{1}{2} + \cdots + \frac{1}{n}), \frac{1}{n}) : n \in \mathbb{N}\}$ where $B(x, r)$ denotes the open ball of radius r centred at x. Let $x \in X$ be any element and G be any open set containing x then there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset G$. As for any $n \in \mathbb{N}$ $\operatorname{diam}(f^n(G)) = \operatorname{diam}(G)$, so there exists an $n \in \mathbb{N}$ such that $f^n(G) \not\subset U$ for any $U \in \mathcal{U}$. Hence, (X, f) cannot be topologically equicontinuous at x and since x was arbitrary, therefore (X, f) is not almost topologically equicontinuous. But (X, f) is equicontinuous as $f : X \to X$ is an isometry.

Next we provide dichotomy results for transitive and minimal systems.

Proposition 2. *Any dynamical system (X, T) is either almost topologically equicontinuous or topologically sensitive and if (X, T) is a transitive dynamical system with X being a T_3 space then almost topological equicontinuity of (X, T) implies that the set of topologically equicontinuous points of (X, T) is same as the set of transitive points of (X, T).*

Proof. From definition of almost topological equicontinuity and definition of topological sensitivity, it is clear that a dynamical system is either almost topologically equicontinuous or topologically sensitive.

Now assume that (X, T) is almost topologically equicontinuous and transitive. Let z be any transitive point of (X, T) and $\mathscr{V}$ be any open cover of X. By definition of almost topological equicontinuity, there exists a nonempty open set G, containing a point of topological equicontinuity, such that for every $n \in \mathbb{N}$ there exists a $V \in \mathscr{V}$ such that $T^n(G) \subset V$ and by definition of a transitive point, $\{T^n(z) : n \in \mathbb{N}\}$ is dense in X. So, there exists an $n_0 \in \mathbb{N}$ such that $T^{n_0}(z) \in G$. For any $i \in \mathbb{N}$, let $V_i \in \mathscr{V}$ be an open set such that $T^i(z) \in V_i$, define $U = \bigcap_{i=1}^{n_0} T^{-i}(V_i) \cap T^{-n_0}(G)$. Then U is an open set containing z and $T^n(U) \subset V_n$ for $n \in \{1, 2, \ldots, n_0\}$ and $T^{n_0}(U) \subset G$. Now, as we know that for every $k \in \mathbb{N}$ there exists a $V \in \mathscr{V}$ such that $T^k(G) \subset V$ so, for every $k \in \mathbb{N}$, $T^{n_0+k}(U) \subset T^k(G) \subset V$ for some $V \in \mathscr{V}$. Hence, for every $m \in \mathbb{N}$, there exists a $V \in \mathscr{V}$ such that $T^m(U) \subset V$. Thus, $z \in X$ is a topologically equicontinuous point.

Now, assume $z \in X$ be any topologically equicontinuous point and U_1 be any nonempty open subset of X. Since X is a T_3 space, so there exists a nonempty open set U such that $U \subset \overline{U} \subset U_1$. Define open cover $\mathscr{V}' = \{U_1, X/\overline{U}\}$. For open cover $\mathscr{V}'$ there exists an open set G' containing z such that for any $n \in \mathbb{N}$ there exists a $V \in \mathscr{V}'$ such that $T^n(G') \subset V'$. Since (X, T) is transitive, so there exists an $n \in \mathbb{N}$ such that $T^n(G') \cap U \neq \emptyset$. By definition of equicontinuous point $T^n(G') \subset U_1$ or $T^n(G') \subset X/\overline{U}$ but since $T^n(G') \cap U \neq \emptyset$ that is $T^n(x) \notin X/\overline{U}$ for some $x \in G'$. Therefore, $T^n(G') \subset U_1$ and since $z \in G'$ so $T^n(z) \in U_1$. Hence, $\{T^n(z) : n \in \mathbb{N}\}$ is dense in X that is z is a transitive point. Thus, the set of equicontinuous points is same as the set of transitive points. □

Proposition 3. *Let (X, T) be a minimal dynamical system. Then either (X, T) is topologically equicontinuous or (X, T) is topologically sensitive.*

Proof. Let (X, T) be topologically sensitive then there exists an open cover $\mathscr{V}$ of X such that for any nonempty open set G there exists an $n \in \mathbb{N}$ such that $T^n(G) \not\subset V$ for any $V \in \mathscr{V}$ which contradicts the definition of topological equicontinuity. So, (X, T) is not topologically equicontinuous.

Now, assume that (X, T) is not topologically sensitive then by Proposition 2, (X, T) is almost topologically equicontinuous and the set of transitive points is same as the set of topologically equicontinuous points. Since (X, T) is minimal, so the set of transitive points is X and hence, every point is a topologically equicontinuous point implying that (X, T) is topologically equicontinuous. □

It is an obvious concern to check the behaviour of topological equicontinuity on product of two dynamical systems and on conjugate dynamical systems. Our next propositions justify the behaviour of topological equicontinuity under product dynamical systems and conjugate dynamical systems.

Lemma 3.1. ([15]) *Let X, Y be compact spaces and $\mathscr{V}$ be any open cover of X then there exist open covers $\mathscr{U}_1$ of X and $\mathscr{U}_2$ of Y such that for any $U_1 \in \mathscr{U}_1$ and $U_2 \in \mathscr{U}_2$ there exists a $V \in \mathscr{V}$ such that $U_1 \times U_2 \subset V$.*

Proposition 4. *Let (X, T) and (Y, S) be two dynamical systems, where X and Y are compact topological spaces, then the product dynamical system $(X \times Y, T \times S)$ is topologically equicontinuous if and only if (X, T) and (Y, S) both are topologically equicontinuous.*

Proof. Suppose that (X, T) and (Y, S) both are topologically equicontinuous and $\mathscr{V}$ be any open cover of $X \times Y$ then there exist open covers $\mathscr{U}_1$ of X and $\mathscr{U}_2$ of Y such that for any $U_1 \in \mathscr{U}_1$ and $U_2 \in \mathscr{U}_2$ there exists a $V \in \mathscr{V}$ satisfying $U_1 \times U_2 \subset V$. For any $(x, y) \in X \times Y$ there exist open sets $G_1 \subset X$ containing x and $G_2 \subset Y$ containing y such that for any $n \in \mathbb{N}$ there exists a $U_1 \in \mathscr{U}_1$ and a $U_2 \in \mathscr{U}_2$ such that $T^n(G_1) \subset U_1$ and $S^n(G_2) \subset U_2$. Let $V \in \mathscr{V}$ be an open set such that $U_1 \times U_2 \subset V$, then for any $n \in \mathbb{N}$ there exists a $V \in \mathscr{V}$ such that $(T \times S)^n(G_1 \times G_2) \subset V$ and hence $(X \times Y, T \times S)$ is topologically equicontinuous.

Conversely, assume that $(X \times Y, T \times S)$ is topologically equicontinuous and $\mathscr{U}$ be any open cover of X. Let $(x, y) \in X \times Y$. For open cover $\mathscr{V} = \{U \times Y : U \in \mathscr{U}\}$ of $X \times Y$, we have an open set $G_1 \times G_2$ containing (x, y) such that for any $n \in \mathbb{N}$ there exists a $U \in \mathscr{U}$ such that $(T \times S)^n(G_1 \times G_2) \subset U \times Y$ and hence for any $n \in \mathbb{N}$ there exists a $U \in \mathscr{U}$ such that $T^n(G_1) \subset U$. Therefore, (X, T) is topologically equicontinuous and similarly (Y, S) is also topologically equicontinuous. □

Consider a dynamical system (X, T) and Z a $T-$invariant closed subspace of X then it is easy to verify that topological equicontinuity of (X, T) implies the same for (Z, T). Next, we prove that for any factor (Y, S) of (X, T) topological equicontinuity of (X, T) implies the same for (Y, S) whenever factor map is open.

Proposition 5. *Let (X, T) be any topologically equicontinuous dynamical system and (Y, S) be any factor of (X, T) such that the factor map is an open map then (Y, S) is also topologically equicontinuous.*

Proof. Let $\pi : X \to Y$ be the factor map which is open and onto and $\mathscr{V}$ be any open cover of Y. Then $\mathscr{U} = \{\pi^{-1}(V) : V \in \mathscr{V}\}$ is an open cover of X. For any $y \in Y$ there exists an $x \in X$ such that $\pi(x) = y$ and as (X, T) is topologically equicontinuous, so for open cover $\mathscr{U}$ of X there exists an open set G_1 containing x such that for every $n \in \mathbb{N}$ there exists a $U \in \mathscr{U}$ such that $T^n(G_1) \subset U$ that is for some $V \in \mathscr{V}$, $T^n(G_1) \subset \pi^{-1}(V)$ and as π is a factor map, so $\pi \circ T = S \circ \pi$ and hence $\pi \circ T^n = S^n \circ \pi$ for every $n \in \mathbb{N}$. As $T^n(G_1) \subset \pi^{-1}(V)$, so $S^n(\pi(G_1)) = \pi(T^n(G_1)) \subset V$ and $y = \pi(x) \in \pi(G_1) = G_2$(say). As π is open so, $\pi(G_1)$ is an open set containing y. Hence, for any open cover $\mathscr{V}$ of Y there exists an open set G_2 containing y such that for every $n \in \mathbb{N}$ there exists a $V \in \mathscr{V}$ such that $S^n(G_2) \subset V$ implying that (Y, S) is a topologically equicontinuous dynamical system. □

If (X, T) and (Y, S) are conjugate dynamical systems then map $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$ is a homeomorphism and hence, π is an open, onto map. So, by above proposition, (X, T) is topologically equicontinuous if and only if (Y, S) is also topologically equicontinuous.

Next few propositions show that for topologically equicontinuous maps on compact, T_3 space the set of recurrent points of f, the set of non-wandering points of f and intersection of image set of all iterations of f are same. These propositions are proved in [20] for metric spaces.

Proposition 6. *Let X be a compact and T_3 space and $f : X \to X$ be any topologically equicontinuous map then $R(f) = \Omega(f)$*

Proof. From definitions it is clear that $R(f) \subset \Omega(f)$. Now, let $x \in \Omega(f)$ and O be any open set containing x then as space X is T_3, so there exists an open set O_1, containing x, such that $O_1 \subset \overline{O_1} \subset O$. Therefore, $\{O, X/\overline{O_1}\}$ is an open cover of X. Now let $\mathscr{U}_1$ be an open cover of X such that for any $U_1, U_2 \in \mathscr{U}_1$ if $U_1 \cap U_2 \neq \emptyset$ then $U_1 \cup U_2 \subset U$ for some $U \in \mathscr{U}$. Since $\mathscr{U}_1$ is an open cover of X, so there exists an open set $O' \subset U_1$(for some $U_1 \in \mathscr{U}_1$) such that $x \in O'$ and for any $n \in \mathbb{N}$ there exists a $U' \in \mathscr{U}_1$ such that $f^n(O') \subset U'$. Now, as x is a non-wandering point, so there exists an $m \in \mathbb{N}$ such that $f^m(O') \cap O' \neq \emptyset$ and say $f^m(O') \subset U'' \in \mathscr{U}_1$. As $O' \subset U_1$, hence $U'' \cap U_1 \neq \emptyset$ which implies $U_1 \cup U'' \subset U$ for some $U \in \mathscr{U}$. As $x \in U'' \cup U_1$ and $x \notin X/\overline{O_1}$, so $U'' \cup U_1 \subset O$. Therefore, $x, f^m(x) \in O$. Hence, for any open set O containing x there exists an $m \in \mathbb{N}$ such that $f^m(x) \in O$ implying that $x \in \omega(x, f)$. Therefore, $x \in R(f)$. So, $R(f) = \Omega(f)$. □

Proposition 7. *Let $f : X \to X$ be a topologically equicontinuous map where X is a compact and T_3 space. Then $\omega(x, f)$ is minimal.*

Proof. Let $a \in \omega(x, f)$. Take any point $z \in \omega(x, f)$ and O' any open set containing z. As X is a T_3 space so there exists an open set O_1 containing z such that $O_1 \subset \overline{O_1} \subset O'$ then $\mathscr{U} = \{O', X/\overline{O_1}\}$ is an open cover of X. For open cover $\mathscr{U}$ there exists an open cover $\mathscr{U}_1$ such that, for any $U_1, U_2 \in \mathscr{U}_1$ if $U_1 \cap U_2 \neq \emptyset$ then $U_1 \cup U_2 \subset U$ for some $U \in \mathscr{U}$. Now, there exists an open set G containing a such that for any $n \in \mathbb{N}$ there exists a $U_1 \in \mathscr{U}_1$ such that $f^n(G) \subset U_1$. As $a \in \omega(x, f)$, so there exists a $y \in O(x, f) = \{f^n(x) : n \in \mathbb{N} \cup \{0\}\}$ such that $y \in G$. Therefore, for any $n \in \mathbb{N}$ there exists a $U_1 \in \mathscr{U}_1$ such that $(f^n(a), f^n(y)) \in U_1 \times U_1$. Since $\omega(x, f) = \omega(y, f)$, so there exists an $n \in \mathbb{N}$ such that $f^n(y) \in U'$ where $U' \in \mathscr{U}_1$ be such that $z \in U'$. Then for n; $f^n(a), f^n(y) \in U_1$ and $f^n(y), z \in U'$ implying that $U_1 \cap U' \neq \emptyset$. Therefore, there exists a $U \in \mathscr{U}$ such that $U_1 \cup U' \subset U$ and as $z \notin X/\overline{O_1}$, so $f^n(a), f^n(y), z \in O'$. Hence, $O(a, f) \cap O' \neq \emptyset$. Therefore, $\omega(x, f)$ is minimal. □

If $\omega(x, f)$ is a minimal set then for any $y \in \omega(x, f)$, $y \in \omega(y, f)$. Therefore, y is a recurrent point of f. Using this we prove our next Proposition.

Proposition 8. *Let $f : X \to X$ be a topologically equicontinuous map where X is a compact and T_3 space then* $\bigcap_{n=0}^{\infty} f^n(X) = R(f)$.

Proof. Since f is a continuous map, so $f(R(f)) \subset R(f)$ and if $x \in R(f)$ then there exists a sequence $\{n_k : k \in \mathbb{N}\}$ of natural numbers such that $f^{n_k}(x) \to x$. Now, as X is compact, so sequence $\{f^{n_k - 1}(x) : k \in \mathbb{N}\}$ has a convergent subsequence. Let y be a limit point of this sequence. Then $y \in \omega(x, f)$ and by Proposition 7 $\omega(x, f)$ is a minimal set, so $y \in R(f)$ and note that, because of our construction, $f(y) = x$ that is $x \in f(R(f))$. Hence, $f(R(f)) = R(f)$ which implies that $R(f) \subset \bigcap_{i=0}^{\infty} f^n(X)$.

Conversely, assume that $x \in \bigcap_{i=0}^{\infty} f^n(X)$. Then for any $n \in \mathbb{N}$ there exists a $y_n \in X$ such that $f^n(y_n) = x$. As X is compact, so $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(y_{n_k})_{k\in\mathbb{N}}$ converges to $y_0 \in X$. Let G be any open set containing x, as X is a T_3 space, so there exists an open set G_1 such that $G_1 \subset \overline{G_1} \subset G$ and $x \in G_1$. Therefore, $\mathscr{U} = \{G, X/\overline{G_1}\}$ is an open cover of X and by topological equicontinuity of f, there exists an open set O containing y_0 such that for every $n \in \mathbb{N}$ there exists a $U \in \mathscr{U}$ such that $f^n(O) \subset U$. As $(y_{n_k})_{k\in\mathbb{N}}$ is converging to y_0. So, there exists an $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, $y_{n_k} \in O$ and since $f^{n_k}(O) \subset U$ for some $U \in \mathscr{U}$ and $x \notin X/G_1$, so $f^{n_k}(O) \subset G$ for all $k \geq n_0$. Hence, $x \in \omega(y_0, f)$ and so, $x \in \omega(x, f)$, implying that $x \in R(f)$. Hence, $\bigcap_{i=0}^{\infty} f^n(X) = R(f)$. □

4. Topological uniform rigidity, topological entropy and their relations

We start this section with the definition of topological version of uniform rigidity, called topological uniform rigidity and then we show that any almost topologically equicontinuous and transitive dynamical system over a compact, first countable and T_3 space is topologically uniformly rigid.

Definition 4.1. For some sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers, a dynamical system (X, T) is called *topologically uniformly rigid with respect to* $(n_k)_{k\in\mathbb{N}}$ if for any open cover $\mathscr{U}$ of X there exists an n_{k_0} in our sequence such that for every $n_k \geq n_{k_0}$ and for every $x \in X$ there exists an open set $U \in \mathscr{U}$ such that $(T^{n_k}(x), x) \in U \times U$ and the dynamical system is called *topologically uniformly rigid* if there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that (X, T) is topologically uniformly rigid with respect to $(n_k)_{k\in\mathbb{N}}$.

First we provide some lemmas which will be used in next propositions and theorems.

Lemma 4.1. *Let $\mathscr{U}$ be any open cover of X where X is a compact uniform space then there exists an entourage W such that for any $y \in X$ there exists a $U \in \mathscr{U}$ such that $W[y] \subset U$.*

Proof. Let $\mathscr{U}$ be any open cover of X then for any $x \in X$ there exists an entourage A_x such that $A_x[x] \subset U$ for some $U \in \mathscr{U}$. Now, by definition of uniform spaces there exists an entourage B_x such that $B_x \circ B_x \subset A_x$. As $\{B_x[x] : x \in X\}$ is an open cover of X. So, there exists a finite subcover, say, $\{B_1[x_1], B_2[x_2], \ldots, B_n[x_n]\}$. Define $W = \bigcap_{i=1}^{n} B_i$ then as for any $y \in X$, $y \in B_i[x_i]$ for some $i \in \{1, 2, \ldots, n\}$, so $W[y] \subset W \circ B_i[x_i] \subset B_i \circ B_i[x_i] \subset U$. Hence, for any $y \in X$ there exists a $U \in \mathscr{U}$ such that $W[y] \subset U$. □

Lemma 4.2. *Let X be any uniform space and W be any entourage then there exists an entourage E such that if $E[x] \cap E[y] \neq \emptyset$ then $E[x] \cup E[y] \subset W[z]$ for some $z \in X$.*

Proof. By definition of uniform spaces, for entourage W there exists an entourage E such that $E \circ E \subset W$. Now, let for some $x, y \in X$, $E[x] \cap E[y] \neq \emptyset$ and say, $z \in E[x] \cap E[y]$ for some $z \in$ X. For any $z' \in E[x] \cup E[y]$, without loss of generality, we can assume that $z' \in E[x]$. So, $(z', x) \in E$ and as $(x, z) \in E$, so $(z', z) \in E \circ E \subset W$. Hence, $z' \in W[z]$. Therefore, if $E[x] \cap E[y] \neq \emptyset$ then $E[x] \cup E[y] \subset W[z]$ for some $z \in X$. □

Remark 4.1. From above two lemmas we can deduce that for a compact uniform space X and any open cover $\mathscr{U}$ of X there exists an open cover $\{E[x] : x \in X\}$ such that if $E[x] \cap E[y] \neq \emptyset$ then $E[x] \cup E[y] \subset U$ for some $U \in \mathscr{U}$.

In next two theorems, we obtain conditions under which a topologically equicontinuous dynamical system becomes topologically uniformly rigid.

Theorem 1. *Let (X, T) be an almost topologically equicontinuous and transitive dynamical system with X being a first countable, compact and T_3 space then (X, T) is topologically uniformly rigid.*

Proof. Since (X, T) is almost topologically equicontinuous, so let $x \in X$ be an equicontinuous point of (X, T). Then there exists a neighbourhood basis $\{G_j : j \in \mathbb{N}$ and $G_{j+1} \subset G_j$ for every $j \in \mathbb{N}\}$ of x. Since x is a topologically equicontinuous point of (X, T), so by Proposition 2, x ia a transitive point also and hence $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X. Therefore, there exists a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ of natural numbers such that $T^{k_j}(x) \in G_j$ for every $j \in \mathbb{N}$. Note that $T^{k_t}(x) \in G_j$ for every $t \geq j$ because $G_t \subset G_j$.

Now, let $\mathscr{U}$ be any open cover of X. As X is a T_3 space, so for every $y \in X$ and any $U \in \mathscr{U}$ containing y there exists an open set U_y such that $y \in U_y \subset \overline{U_y} \subset U$ then $\mathscr{V} = \{U_y : y \in X, U \in \mathscr{U}\}$ is an open cover of X and let $\mathscr{V}_1 = \{U_{y_1}, U_{y_2}, \dots, U_{y_k}\}$ be a finite subcover. As x is a topologically equicontinuous point of (X, T), so there exists a $j \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $T^n(G_j) \subset U_{y_i}$ for some $i \in \{1, 2, \dots, k\}$. Hence, for every $t \geq j$, $(T^{n+k_t}(x), T^n(x)) \in U_{y_i} \times U_{y_i}$ for some $i \in \{1, 2, \dots, k\}$. Since $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X and T^{k_t} is continuous, so for any $z \in X$, there exists an $i \in \{1, 2, \dots, k\}$ such that $(T^{k_t}(z), z) \in \overline{U_{y_i}} \times \overline{U_{y_i}} \subset U \times U$ for some $U \in \mathscr{U}$ that is for any open cover $\mathscr{U}$ and any $z \in X$ there exists a $j \in \mathbb{N}$ such that for any $t \geq j$ there exists a $U \in \mathscr{U}$ such that $(T^{k_t}(z), z) \in U \times U$. Therefore, (X, T) is topologically uniformly rigid. □

Theorem 2. *Let (X, T) be a second countable, compact uniform dynamical system. If (X, T) is topologically equicontinuous and has a dense set of periodic points then (X, T) is topologically uniformly rigid.*

Proof. Let $\mathscr{B} = \{B_i : i \in \mathbb{N}\}$ be a countable basis for X. As periodic points are dense in X, so for every $i \in \mathbb{N}$ there exists a $p_i \in B_i$ and a $k_i \in \mathbb{N}$ such that $T^{k_i}(p_j) = p_j$ for every $j \leq i$, $j \in \mathbb{N}$ and $k_i < k_{i+1}$ for every $i \in \mathbb{N}$.

Now, let $\mathscr{U}$ be any open cover of X. Then there exists an open cover $\mathscr{V}$ such that for any $V_1, V_2 \in \mathscr{V}$, if $V_1 \cap V_2 \neq \emptyset$ then $V_1 \cup V_2 \subset U$ for some $U \in \mathscr{U}$. Since, (X, T) is topologically equicontinuous, so for every $x \in X$ there exists a $B_x \in \mathscr{B}$ containing x such that for every $n \in \mathbb{N} \cup \{0\}$ there exists a $V \in \mathscr{V}$ such that $T^n(B_x) \subset V$. Let $\{B_t : t \in F\}$ where F is a finite subset of $\mathbb{N}$ be a finite subcover of $\{B_x : x \in X\}$ and say t' be the largest element of F. Now, for any $i \geq t'$, $T^{k_i}(p_t) = p_t$ for every $t \in F$. Since, for every $x \in X$ $x \in B_t$ for some $t \in F$ and for every $n \in \mathbb{N}$ there exists a $V \in \mathscr{V}$ such that $T^n(B_t) \subset V$, so for every $i \geq t'$ and every $x \in X$, there exists some $t \in F$ and $V \in \mathscr{V}$ such that $(T^{k_i}(x), T^{k_i}(p_t)) = (T^{k_i}(x), p_t) \in V \times V$ and $(x, p_t) \in V \times V$ for some $V \in \mathscr{V}$. Hence, for every $i \geq t'$ and any $x \in X$ there exists a $U \in \mathscr{U}$ such that $(T^{k_i}(x), x) \in U \times U$ implying that (X, T) is topologically uniformly rigid. □

From [6] we know that topological entropy of a rigid dynamical system is zero. Next result shows that the same is true for a topologically uniformly rigid dynamical system on compact T_3 spaces.

Lemma 4.3. ([12]) *A space is uniform if and only if it is completely regular*

Note that any compact and Hausdorff space is completely regular, and hence uniform. We will use this fact in our next lemma.

Lemma 4.4. *Let X be any compact and Hausdorff space and $\mathscr{U}$ be any open cover of X then for any $t \in \mathbb{N}, t \geq 2$ there exists an open cover $\mathscr{V}^t$ such that if for any $V_1^t, V_2^t, \dots, V_t^t \in \mathscr{V}^t$, $V_i^t \cap V_{i+1}^t \neq \emptyset$ for every $i \in \{1, 2, \dots, t-1\}$ then $V_1^t \cup V_2^t \cup \cdots \cup V_t^t \subset U$ for some $U \in \mathscr{U}$.*

Proof. We will prove it by using induction on t. For $t = 2$, by using Lemma 4.2 and Lemma 4.1 we get there exists an entourage E such that if $E[x] \cap E[y] \neq \emptyset$ then $E[x] \cup E[y] \subset U$ for some $U \in \mathscr{U}$. Hence, taking $\mathscr{V}^2 = \{E[x] : x \in X\}$, we get an open cover $\mathscr{V}^2$ such that if $V_1^2 \cap V_2^2 \neq \emptyset$ for any $V_1^2, V_2^2 \in \mathscr{V}^2$ then $V_1^2 \cup V_2^2 \subset U$ for some $U \in \mathscr{U}$.

Now, assume for $t \in \mathbb{N}$ there exists an open cover $\mathscr{V}^t$ such that if for any $V_1^t, V_2^t, \ldots, V_t^t \in \mathscr{V}^t$, $V_i^t \cap V_{i+1}^t \neq \emptyset$ for every $i \in \{1, 2, \ldots, t-1\}$ then $V_1^t \cup V_2^t \cup \cdots \cup V_t^t \subset U$ for some $U \in \mathscr{U}$.

Now, since $\mathscr{V}^t$ is an open cover of X, so there exists an open cover $\mathscr{V}^{t+1}$ such that if for any $V', V'' \in \mathscr{V}^{t+1}$, $V' \cap V'' \neq \emptyset$ then $V' \cup V'' \subset V^t$ for some $V^t \in \mathscr{V}^t$. Now, let $V_1^{t+1}, V_2^{t+1}, \ldots, V_{t+1}^{t+1} \in \mathscr{V}^{t+1}$ such that $V_i^{t+1} \cap V_{i+1}^{t+1} \neq \emptyset$ for every $i \in \{1, 2, \ldots, t\}$. Then there exist $V_i^t \in \mathscr{V}^t$ such that $V_i^{t+1} \cup V_{i+1}^{t+1} \subset V_i^t$ for any $i \in \{1, 2, \ldots, t\}$. Note that for any $i \in \{1, 2, \ldots, t\}$, $V_i^t \in \mathscr{V}^t$ and $V_i^t \cap V_{i+1}^t \neq \emptyset$ for every $i \in \{1, 2, \ldots, t-1\}$. Hence, by our assumption there exists a $U \in \mathscr{U}$ such that $V_1^t \cup V_2^t \cup \cdots \cup V_t^t \subset U$ and as $V_i^{t+1} \cup V_{i+1}^{t+1} \subset V_i^t$ for all $i \in \{1, 2, \ldots, t\}$, so $V_1^{t+1} \cup V_2^{t+1} \cup \cdots \cup V_{t+1}^{t+1} \subset U$ for some $U \in \mathscr{U}$. Therefore, there exists an open cover $\mathscr{V}^{t+1}$ such that if for any $V_1^{t+1}, V_2^{t+1}, \ldots, V_{t+1}^{t+1} \in \mathscr{V}^{t+1}$, $V_i^{t+1} \cap V_{i+1}^{t+1} \neq \emptyset$ for every $i \in \{1, 2, \ldots, t\}$ then $V_1^{t+1} \cup V_2^{t+1} \cup \cdots \cup V_{t+1}^{t+1} \subset U$ for some $U \in \mathscr{U}$.

Hence, by using principle of mathematical induction, we can say that for any open cover $\mathscr{U}$ and for every $t \in \mathbb{N}$ there exists an open cover $\mathscr{V}^t$ such that for any $V_1^t, V_2^t, \ldots, V_t^t \in \mathscr{V}^t$ if $V_i^t \cap V_{i+1}^t \neq \emptyset$ for every $i \in \{1, 2, \ldots, t-1\}$ then $V_1^t \cup V_2^t \cup \cdots \cup V_t^t \subset U$ for some $U \in \mathscr{U}$. □

Theorem 3. *Let (X, T) be a topologically uniformly rigid dynamical system where X is a compact and Hausdorff space, then (X, T) has zero topological entropy.*

Proof. Let $\mathscr{U}$ be any open cover of X and k be cardinality of $\mathscr{U}$. For any $t \in \mathbb{N}$ by Lemma 4.4 there exists an open cover $\mathscr{V}^t$ such that if for any $V_1^t, V_2^t, \ldots, V_t^t \in \mathscr{V}^t$, $V_i^t \cap V_{i+1}^t \neq \emptyset$ for all $i \in \{1, 2, \ldots, t-1\}$ then $V_1^t \cup V_2^t \cup \cdots \cup V_t^t \subset U$ for some $U \in \mathscr{U}$.

Now, by using the definition of topologically uniformly rigid dynamical system there exists an $n_t \in \mathbb{N}$ such that for any $x \in X$, there exists a $V_1^t \in \mathscr{V}^t$ such that $(x, T^{n_t}(x)) \in V_1^t \times V_1^t$ and similarly for every $i \in \{1, 2, \ldots, t\}$ there exists a $V_i^t \in \mathscr{V}^t$ such that $(T^{(i-1)n_t}(x), T^{(i)n_t}(x)) \in V_i^t \times V_i^t$. Then $V_i^t \cap V_{i+1}^t \neq \emptyset$ for any $i \in \{1, 2, \ldots, t-1\}$ and so there exists a $U \in \mathscr{U}$ such that $V_1^t \cup V_2^t \cup \cdots \cup V_t^t \subset U$ for some $U \in \mathscr{U}$. So, $(x, T^{n_t}(x), T^{2n_t}(x), \ldots, T^{tn_t}(x)) \in U \times U \times \cdots \times U$. Therefore, $x \in U \cap T^{-n_t}(U) \cap \cdots \cap T^{-tn_t}(U)$ and hence, $\mathscr{V}_0^t = \{U \cap T^{-n_t}(U) \cap \cdots \cap T^{-tn_t}(U) : U \in \mathscr{U}\}$ is an open cover of X with cardinality k. Now, define $\mathscr{V}_i^t = \{T^{-i}(U \cap T^{-n_t}(U) \cap \cdots \cap T^{-tn_t}(U)) : U \in \mathscr{U}\}$ for every $i \in \{1, 2, \ldots, n_t - 1\}$, then $\mathscr{V}_0^t \vee \mathscr{V}_1^t \vee \cdots \vee \mathscr{V}_{n_t-1}^t$ is a subcover of $\mathscr{U} \vee T^{-1}(\mathscr{U}) \vee \cdots \vee T^{-((t+1)n_t-1)}(\mathscr{U})$ and hence, $N(\mathscr{U} \vee T^{-1}(\mathscr{U}) \vee \cdots \vee T^{-((t+1)n_t-1)}(\mathscr{U})) \leq N(\mathscr{V}_0^t \vee \mathscr{V}_1^t \vee \cdots \vee \mathscr{V}_{n_t-1}^t) \leq k^{n_t}$. Therefore, $h(T, \mathscr{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathscr{U} \vee T^{-1}(\mathscr{U}) \vee \cdots \vee T^{-(n-1)}(\mathscr{U})) = \lim_{n_t\to\infty} \frac{1}{(t+1)n_t} \log N(\mathscr{U} \vee T^{-1}(\mathscr{U}) \vee \cdots \vee T^{-((t+1)n_t-1)}(\mathscr{U})) \leq \lim_{t\to\infty} \frac{\log(k^{n_t})}{(t+1)n_t} = 0$.

Hence, $h(T, \mathscr{U}) = 0$ for any open cover $\mathscr{U}$ of X implying $h(T) = \sup\{h(T, \mathscr{U}) : \mathscr{U}$ is an open cover of $X\} = 0$. □

Remark 4.2. By using Theorem 1 and above theorem we can say that any transitive and almost topologically equicontinuous dynamical system on a first countable, compact uniform space has zero topological entropy and since any transitive but not topologically sensitive system is almost topologically equicontinuous, so any transitive but not topologically sensitive system on a first countable, compact uniform space has zero topological entropy.

Next we provide an example of a topologically uniformly rigid dynamical system with the base space a compact, Hausdorff space and hence, the system will have zero topological entropy.

Example 3. Let $X = \{(r, \theta) : r \in [0, 1], \theta \in [0, 2\pi)\}$ with usual metric and $f : X \to X$ by $f(r, \theta) = (r, (\theta + \alpha) \bmod 2\pi)$ where α is an irrational multiple of π. Then given any $i \in \mathbb{N}$ there exists an $n_i \in \mathbb{N}$ such that $d(f^{n_i}(x), x) < \frac{1}{i}$ for every $x \in X$. Now, let $\mathscr{U}$ be any open cover of X then there exists a Lebesgue number for $\mathscr{U}$, say ϵ. Take $i_0 > \frac{1}{\epsilon}$, as for any $i > i_0$, $d(f^{n_i}(x), x) < \frac{1}{i} < \epsilon$, so for any $i > i_0$ there exists a $U \in \mathscr{U}$ such that $(f^{n_i}(x), x) \in U \times U$ and hence the system (X, f) is topologically uniformly rigid. Therefore, f has zero topological entropy.

Acknowledgement

The authors express their sincere gratitude to the reviewer for useful comments and suggestions.

References

[1] E. Akin, J. Auslander, K. Berg, *When is a transitive map chaotic?*, Converg. Ergodic Theory Prob. **5** (1996), 25–40.
[2] J. Auslander, J. A. Yorke, *Interval maps, factor of maps and chaos*, Tohoku Math. J. **32** (1980), 177–188.
[3] S. Fomin, *On dynamical systems with a purely point spectrum*, Doklady Akad. Nauk SSSR **77** (1951), 22–32.
[4] G. Fuhrmann, M. Gröger, D. Lenz, *The structure of mean equicontinuous group actions*, Israel J. Math. **247** (2022), 75–123.
[5] F. Garcia-Ramos, J. Li, R. Zhang, *When is a dynamical system mean sensitive?*, Ergodic Theory Dynam. Systems **39** (2019), 1608–1636.
[6] S. Glasner, D. Maon, *Rigidity in topological dynamics*, Ergodic Theory Dynam. Systems **9** (1989), 309–320.
[7] E. Glasner, B. Weiss, *Locally equicontinuous dynamical systems*, Colloq. Math. **84**(85) (2000), 345–361.
[8] C. Good, R. Leek, J. Mitchell, *Equicontinuity, transitivity and sensitivity: the Auslander-Yorke dichotomy revisited*, Discrete Contin. Dyn. Syst. **40** (2020), 2441–2474.
[9] B. M. Hood, *Topological entropy and uniform spaces*, J. Lond. Math. Soc. **2** (1974), 633–641.
[10] W. Huang, S. Kolyada, G. Zhang, *Analogues of Auslander–Yorke theorems for multi-sensitivity*, Ergodic Theory Dynam. Systems **38** (2018), 651–655.
[11] W. Huang, J. Li, J. P. Thouvenot, L. Xu, X. Ye, *Bounded complexity, mean equicontinuity and discrete spectrum*, Ergodic Theory Dynam. Systems **41** (2021), 494–533.
[12] I. M. James, *Introduction to Uniform Spaces, London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge **144** (1990).
[13] J. L. Kelley, *General Topology*, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics No. 27. Springer-Verlag, New York-Berlin, 1975.
[14] D. Kumar, M. Salman, R. Das, *Topological sensitivity on hyperspaces*, Bull. Belg. Math. Soc. Simon Stevin **29** (2022), 19–36.
[15] D. Kumar, M. Salman, R. Das, *Topological sensitivity and its stronger forms on semiflows*, submitted.
[16] R. Li, *A note on shadowing with chain transitivity*, Commun Nonlinear Sci. Numer. Simulat. **17** (2012), 2815–2823.
[17] R. Li, *A note on chaos and the shadowing property*, Int. J. Gen. Syst. **45** (2016), 675–688.
[18] J. Li, S. Tu, X. Ye, *Mean Equicontinuity and Mean Sensitivity*, Ergodic Theory Dynam. Systems **35** (2015), 2587–2612.
[19] J. Li, X. Ye, T. Yao, *Equicontinuity and sensitivity in mean forms*, J. Dynam. Differential Equations **34** (2022), 133–154.
[20] J. Mai, *The structure of equicontinuous maps*, Trans. Amer. Math. Soc. **355** (2003), 4125–4136.
[21] H. L. Royden, *Real Analysis*, (3rd edition), Macmillan Publishing Company, New York, 1988.
[22] X. Wu, Y. Luo, X. Ma, T. Lu, *Rigidity and sensitivity on uniform spaces*, Topology Appl. **252** (2019), 145–157.