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On some new tpes of convergence for double-indexed sequences

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Abstract. In this paper, we define and study analogous of some new types of convergence for double-indexed sequence and the inter-relation among them is investigated. Also, their certain basic properties are discussed.

1. Introduction

Much of classical probability theory and its applications to statistics concerns limit theorems, i.e., the asymptotic behavior of a sequence of random variables. In [4], for the first time, P. Hsu and Robbins proposed the new concept of complete convergence and had led to many follow-up studies. Baum and Katz [2] extended the idea to the case of fractional moments and higher-order moments. Asmussen and Katz [1] studied the complete convergence of subsequences of partial sums of i.i.d. random variables in 1980. After that, Gut [3] improved the results in [1] and simplified the proof, and obtained the equivalent conditions for the existence of first-order moments with complete convergence of a certain class of subsequence. Rosalsky [8] studied on complete convergence in mean of normed sums of independent random elements in Banach spaces. Sung [9] presented a note on the complete convergence of moving average processes. Zhou [11] investigated complete moment convergence of moving processes under ϕ-mixing assumptions. Wang et al. in [10] studied the complete moment convergence of double-indexed randomly weighted sums of NSD random variables and the almost sure convergence and mean square convergence of the state observers of linear-time-invariant systems. Motivated by the above work, in [5] Hu and Sun gave some new concepts of convergence of random variables and discussed some properties. In this paper, we extend these concepts of convergence to double-indexed sequence and prove some analogues.

Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$. Let $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ be a filed of random variables where $\mathbb{N}$ denotes the set of nature number. Firstly, recall some basic definitions and notations for double-indexed sequence. As everyone knows, there are different type definitions for limit of double-indexed series, in this paper we only consider the following three definitions.

Definition 1.1. *(See [7]) We say that a double-indexed series* $\{x_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ lim(min)-*converges to a number* $x_{(0,0)}$ *if, for all positive* ε*, there exists a positive integer* n_0 *such that* $|x_{(i,j)} - x_{(0,0)}| < \varepsilon$ *for all elements* (i,j) *whose*

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coordinates are such that

$$i \wedge j \geqslant n_0,$$

and we write $\lim(\min)x_{(i,j)} = x_{(0,0)}$.

Definition 1.2. *(See [7]) We say that a double-indexed series* $\{x_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ $\lim(\max)$*-converges to a number* $x_{(0,0)}$ *if, for all positive* ε*, there exists a positive integer* n_0 *such that* $|x_{(i,j)} - x_{(0,0)}| < \varepsilon$ *for all elements* (i,j) *whose coordinates are such that*

$$i \vee j \geqslant n_0,$$

and we write $\lim(\max)x_{(i,j)} = x_{(0,0)}$.

Both modes of convergence discussed above are particular cases of the following general concept.

Definition 1.3. *(See [7]) Let a function* f *of 2 arguments be defined for all* $(i,j) \in \mathbb{N}^2$*. We say that a field* $\{x_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ *is* f*-convergent to* $x_{(0,0)}$ *if, for all positive* ε*, there exists a positive integer* n_0 *such that* $|x_{(i,j)} - x_{(0,0)}| < \varepsilon$ *for all elements* (i,j) *whose coordinates are such that*

$$f(i,j) \geqslant n_0,$$

where f *is non-decreasing with respect to each of its arguments and we write* $\lim(f)x_{(i,j)} = x_{(0,0)}$.

Based on the Definition 1.1, 1.2 and 1.3, we define the analogues of different types of convergence for double-indexed random sequence as follows:

• $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ is said to f-almost surely converge to $X_{(0,0)}$, if there exists a set $N \in \mathcal{F}$ such that $P(N) = 0$ and $\forall\omega \in \Omega\backslash N, \lim(f)X_{(i,j)}(\omega) = X_{(0,0)}(\omega)$ (resp. $\lim(\min)X_{(i,j)}(\omega) = X_{(0,0)}(\omega)$, $\lim(\max)X_{(i,j)}(\omega) = X_{(0,0)}(\omega)$), and we write $X_{(i,j)} \stackrel{f-a.s.}{\longrightarrow} X_{(0,0)}$ (resp. $X_{(i,j)} \stackrel{\min -a.s.}{\longrightarrow} X_{(0,0)}$, $X_{(i,j)} \stackrel{\max -a.s.}{\longrightarrow} X_{(0,0)}$).

Each of the following definitions has three forms. For the sake of brevity, only one form of definition is given here, and the other two definitions are completely similar, which are omitted.

• $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ is said to min −converge to $X_{(0,0)}$ in probability, if for any $\varepsilon > 0$, $\lim(\min)P(\{|X_{(i,j)} - X_{(0,0)}| \geq \varepsilon\}) = 0$, and we write $X_{(i,j)} \stackrel{\min -P}{\longrightarrow} X_{(0,0)}$.

• $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ is said to min $-\mathbf{L}^P -$ *converge* to $X_{(0,0)}$ $(p > 0)$ if $\lim(\min)\mathbb{E}[|X_{(i,j)} - X_{(0,0)}|^p] = 0$, and we write $X_{(i,j)} \stackrel{\min -\mathbf{L}^P}{\longrightarrow} X_{(0,0)}$.

• $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ is said to min $-\mathbf{L}^\infty -$ *converge* to $X_{(0,0)}$ if $\lim(\min)\|X_{(i,j)} - X_{(0,0)}\|_\infty = 0$, and we write $X_{(i,j)} \stackrel{\min -\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)}$.

• $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ is said to min-converge to $X_{(0,0)}$ in distribution, if for any bounded continuous function f, $\lim(\min)\mathbb{E}[f(X_{(i,j)})] = \mathbb{E}[f(X_{(0,0)})]$, and we write $X(i,j) \stackrel{\min -d}{\longrightarrow} X(0,0)$.

Definition 1.4. *(See [7]) The limit of the field* $\{s_{(m,n)}, (m,n) \in \mathbb{N}^2\}$ *in the sense of lim*$(\min)$*-convergence (if exists) is called the sum of the limit of the double-indexed series constructed from the field* $\{x_{(i,j)}, (i,j) \in \mathbb{N}^2\}$*. The sum of a double sequence is defined by*

$$\sum_{(i,j)\in\mathbb{N}^2} x_{(i,j)} = \lim(\min)s_{(m,n)},$$

where $s_{(m,n)} = \sum\limits_{(i,j)\leqslant(m,n)} x_{(i,j)}$*,* $(m,n) \in \mathbb{N}^2$ *(*$(i,j) \leqslant (m,n)$ *means that* $i \leqslant m$ *and* $j \leqslant n$*). If the limit exists, we say that the double-indexed sequence*

$$\sum_{(i,j)\in\mathbb{N}^2} x_{(i,j)} = \sum x_{(i,j)}$$

converges. If the $\lim(\min)$*-limit for* $\{s_{(m,n)}, (m,n) \in \mathbb{N}^2\}$ *does not exist, we say that the double-indexed sequence diverges.*

We are now consider some new type concepts of convergence for double sequence of random variables.

Definition 1.5. *$\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to completely* min −*converge to $X_{(0,0)}$, if for any $\varepsilon > 0$, $\sum_{(i,j)\in\mathbb{N}^2} P(\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\}) < \infty$, and we write $X_{(i,j)} \overset{\min -c.c.}{\longrightarrow} X_{(0,0)}$.*

Definition 1.6. *Let $\alpha > 0$, $\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to strongly almost surely* min −*converge to $X_{(0,0)}$ with order α, if*

$$\sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)} - X_{(0,0)}|^\alpha < \infty \ \ a.s.$$

and we write $X_{(i,j)} \overset{\min -S_\alpha - a.s.}{\longrightarrow} X_{(0,0)}$.

Definition 1.7. *$\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to* $\min -S - \mathbf{L}^p$ −*converge to $X_{(0,0)}$ $(p > 0)$ if $\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - X_{(0,0)}|^p] < \infty$, and we write $X_{(i,j)} \overset{\min -S - \mathbf{L}^p}{\longrightarrow} X_{(0,0)}$.*

Definition 1.8. *$\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to strongly* $\min -\mathbf{L}^\infty$ − *converge to $X_{(0,0)}$ if*

$$\sum_{(i,j)\in\mathbb{N}^2} ||X_{(i,j)} - X_{(0,0)}||_\infty < \infty$$

and we write $X_{(i,j)} \overset{\min -S - \mathbf{L}^\infty}{\longrightarrow} X_{(0,0)}$.

Definition 1.9. *$\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to* $\min -S_1 - d$ *converge to $X_{(0,0)}$, if for any bounded Lipschitz continuous function f,*

$$\sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[f(X_{(i,j)}) - f(X_{(0,0)})]| < \infty$$

and we write $X_{(i,j)} \overset{\min -S_1 - d}{\longrightarrow} X_{(0,0)}$.

Definition 1.10. *Let $F_{(i,j)}$ and $F_{(0,0)}$ be the distribution functions of $X_{(i,j)}$ and $X_{(0,0)}$, respectively. $\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to* $\min -S_2 - d$ *converge to $X_{(0,0)}$, if for any continuous point x of F,*

$$\sum_{(i,j)\in\mathbb{N}^2} |F_{(i,j)}(x) - F_{(0,0)}(x)| < \infty$$

and we write $X_{(i,j)} \overset{\min -S_2 - d}{\longrightarrow} X_{(0,0)}$.

Definition 1.11. *Let $\{X_{(0,0)}, X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ be a field of random variables. If for any bounded Lipschitz continuous function f, $\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|f(X_{(i,j)}) - f(X_{(0,0)})|] < \infty$ holds, then $\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to* $\min -S_1^* - d$ *converge to $X_{(0,0)}$, and we write $X_{(i,j)} \overset{\min -S_1^* - d}{\longrightarrow} X_{(0,0)}$.*

Definition 1.12. *Let $\{X_{(0,0)}, X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ be a field of random variables. If for any real number t, it holds that $\sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[e^{itX_{(i,j)}}] - \mathbb{E}[e^{itX_{(0,0)}}]| < \infty$, then $\{X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ is said to* $\min -S_3 - d$ *converge to $X_{(0,0)}$, and we write $X_{(i,j)} \overset{\min -S_3 - d}{\longrightarrow} X_{(0,0)}$.*

The rest of this paper is organized as follows. In Section 2, we prove the relations among some types of convergence for double-indexed sequences we proposed. In section 3, we present some examples to compare the relations between any two types of convergence.

2. Main results and Proofs

With the preliminary preparation, we now state and prove some analogues for double-indexed random variables. For single sequence such results have been proved by Hu et al. [5, 6].

Proposition 2.1. *If* $X_{(i,j)} \xrightarrow{f-P} X_{(0,0)}$*, then there exists a subsequence* $\{X_{(i_k,j_k)}\}$ *of* $\{X_{(i,j)}\}$ *such that* $X_{(i_k,j_k)} \xrightarrow{f-a.s.} X_{(0,0)}$ *as* $k \to \infty$*.*

Proof. Noticing that $X_{(i,j)} \xrightarrow{f-P} X_{(0,0)}$, then we have, for any $\varepsilon > 0$, $\lim(f)P(|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon) = 0$. Denote, for each $i, j \in \mathbb{N}$, $E(i,j) = \{|X_{(i,j)} - X_{(0,0)}| > \frac{1}{f(i,j)}\}$, choose a subsequence $(i_1, j_1), (i_2, j_2), \cdots$ such that $P\{|X_{(i_k,j_k)} - X_{(0,0)}| > \frac{1}{f(i_k,j_k)}\} < \frac{1}{2^k}$. It is easy to see that $\sum_{k=1}^{\infty} P\{|X_{(i_k,j_k)} - X_{(0,0)}| > \frac{1}{i_k j_k}\} < 1$, hence we have by the Borel-Cantelli lemma that $P\{\cap_{s=1}^{\infty} \cup_{k=s}^{\infty} E(i_k, j_k)\} = 0$. Therefore, $X_{(i_k,j_k)} \xrightarrow{f-a.s.} X_{(0,0)}$ as $k \to \infty$. □

Proposition 2.2. *Let C be a constant. Then,* $X_{(i,j)} \xrightarrow{f-d} C \Leftrightarrow X_{(i,j)} \xrightarrow{f-P} C$*.*

Proof. $'' \Leftarrow ''$ Obviously.
$'' \Rightarrow ''$ The distribution function of the random variable degenerated to C is

$$F_{(0,0)}(x) = \mathbf{1}_{[x>C]} = \begin{cases} 0, & x \leqslant C, \\ 1, & x > C, \end{cases}$$

where $x = C$ is the unique discontinuity point of $F_{(0,0)}$. Since $X_{(i,j)} \xrightarrow{f-d} C$, we have

$$\lim(f)F_{(i,j)}(x) = \begin{cases} 0, & x < C, \\ 1, & x > C. \end{cases}$$

Thus, for any $\varepsilon > 0$, we have

$$\begin{aligned} \lim(f)P(|X_{(i,j)} - C| \geqslant \varepsilon) &= \lim(f)[P(X_{(i,j)} \geqslant C + \varepsilon) + P(X_{(i,j)} \leqslant C - \varepsilon)] \\ &= \lim(f)[1 - F_{(i,j)}(C + \varepsilon) + F_{(i,j)}(C - \varepsilon + 0)] \\ &= 0. \end{aligned}$$

Hence, $X_{(i,j)} \xrightarrow{f-P} C$. □

Proposition 2.3. *Let* $\{X_{(0,0)}, X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ *be a double-indexed sequence of random variables. Then,* $X_{(i,j)} \xrightarrow{f-\mathbf{L}^p} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{f-P} X_{(0,0)}$*.*

Proof. By the Chebyshev's inequality

$$P(|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon) \leqslant \frac{\mathbb{E}|X_{(i,j)} - X_{(0,0)}|^p}{\varepsilon^p},$$

$X_{(i,j)} \xrightarrow{f-P} X_{(0,0)}$ follows immediately. □

Proposition 2.4. $X_{(i,j)} \xrightarrow{f-P} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{f-d} X_{(0,0)}$*.*

Proof. Assume that g is a bounded Lipschitz continuous function, such that $|g| \leqslant M$, and exists a positive constant C such that $|g(x) - g(y)| \leqslant C|x - y|, \forall x, y \in \mathbb{R}$. Note that

$$\begin{aligned}|\mathbb{E}[g(X_{(i,j)}) - g(X_{(0,0)})]|(\mathbf{1}_{(|X_{(i,j)}-X_{(0,0)}|\geqslant\varepsilon)} + \mathbf{1}_{(|X_{(i,j)}-X_{(0,0)}|<\varepsilon)}) &\leqslant \mathbb{E}|g(X_{(i,j)}) - g(X_{(0,0)})|(\mathbf{1}_{(|X_{(i,j)}-X_{(0,0)}|\geqslant\varepsilon)} + \mathbf{1}_{(|X_{(i,j)}-X_{(0,0)}|<\varepsilon)}) \\ &\leqslant 2M\mathbb{E}\mathbf{1}_{(|X_{(i,j)}-X_{(0,0)}|\geqslant\varepsilon)} + C\mathbb{E}\mathbf{1}_{(|X_{(i,j)}-X_{(0,0)}|<\varepsilon)} \\ &\leqslant 2MP(|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon) + C\varepsilon P(|X_{(i,j)} - X_{(0,0)}| < \varepsilon).\end{aligned}$$

Thus, $X_{(i,j)} \xrightarrow{f-d} X_{(0,0)}$. □

Proposition 2.5. $X_{(i,j)} \xrightarrow{f-\mathbf{L}^\infty} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{f-\mathbf{L}^p} X_{(0,0)}$.

Proof. Since $\mathbb{E}|X_{(i,j)} - X_{(0,0)}|^p \leqslant \|X_{(i,j)} - X_{(0,0)}\|_\infty^p$, it follows directly. □

Proposition 2.6. $X_{(i,j)} \xrightarrow{f-a.s.} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{f-P} X_{(0,0)}$.

Proof. Since $X_{(i,j)} \xrightarrow{f-a.s.} X_{(0,0)}$, we have $P\{\bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \bigcap_{f(i,j)\geqslant n}\{\omega : |X_{(i,j)}(\omega) - X_{(0,0)}(\omega)| \geqslant \frac{1}{k}\} \Longrightarrow P\{\omega : \bigcup_{f(i,j)\geqslant n}(|X_{(i,j)}(\omega) - X_{(0,0)}(\omega)| \geqslant \varepsilon)\}$, therefore, $X_{(i,j)} \xrightarrow{f-P} X_{(0,0)}$. □

Proposition 2.7. $X_{(i,j)} \xrightarrow{f-\mathbf{L}^\infty} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{f-a.s.} X_{(0,0)}$.

Proof. Note that $|X_{(i,j)} - X_{(0,0)}| \leqslant \|X_{(i,j)} - X_{(0,0)}\|_\infty$, $X_{(i,j)} \xrightarrow{f-a.s.} X_{(0,0)}$ follows immediately. □

Remark 2.8. *Put $f = i \wedge j$ or $f = i \vee j (i, j \in \mathbb{N})$, the above results also true for* min*-convergence and* max*-convergence.*

Remark 2.9. *f-convergence is often considered for $f(i, j) = \sqrt{i^2 + j^2}$. For this particular function, we use a special notation for the f-convergence, namely $\|\cdot\|$-convergence. In what follows, we also deal with vol-convergence, which corresponds to the function $f(i, j) = ij$. Note that the three types of convergence, namely* max −, $\|\cdot\|$−, *vol-convergence are equivalent in the space $\mathbb{N}^2$, since for $(i, j) \in \mathbb{N}^2$*

$$\max\{i, j\} \leqslant \sqrt{i^2 + j^2} \leqslant \sqrt{d}\max\{i, j\}, \quad \max\{i, j\} \leqslant ij \leqslant (\max\{i, j\})^d.$$

Theorem 2.10. *Let $\{X_{(0,0)}, X_{(i,j)}, (i, j) \in \mathbb{N}^2\}$ be a double-indexed sequence of random variables. Then,*

(i) For $p \geqslant 1$, we have $X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^\infty} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^p} X_{(0,0)}$.

(ii) For $\alpha > 0$, we have $X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^\alpha} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{\min-S_\alpha-a.s.} X_{(0,0)}$.

(iii) For $\alpha \geqslant 1$, we have $X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^\infty} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{\min-S_\alpha-a.s.} X_{(0,0)}$.

Proof. (*i*) If $\|X_{(i,j)} - X_{(0,0)}\|_\infty < 1$ and $p \geqslant 1$, we have

$$\mathbb{E}[|X_{(i,j)} - X_{(0,0)}|^p] \leqslant \|X_{(i,j)} - X_{(0,0)}\|_\infty^p \leqslant \|X_{(i,j)} - X_{(0,0)}\|_\infty. \tag{2.1}$$

According to Definition 1.7 and 1.8 and Eq. (2.1), we have $X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^\infty} X_{(0,0)} \Rightarrow X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^p} X_{(0,0)}$.

(*ii*) Let $\alpha > 0$, if $X_{(i,j)} \xrightarrow{\min-S-\mathbf{L}^\alpha} X_{(0,0)}$, then $\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - X_{(0,0)}|^\alpha] < \infty$. By the monotone convergence theorem, we have

$$\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - X_{(0,0)}|^\alpha] = \mathbb{E}[\sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)} - X_{(0,0)}|^\alpha].$$

It follows that $\mathbb{E}[\sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)} - X_{(0,0)}|^\alpha] < \infty$ and thus $\sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)} - X_{(0,0)}|^\alpha < \infty \quad$ *a.s., i.e.,*

$$X_{(i,j)} \xrightarrow{\min-S_\alpha-a.s.} X_{(0,0)}.$$

(*iii*) It is a direct consequence of (*i*) and (*ii*). □

Theorem 2.11. $X_{(i,j)} \overset{\min-S-\mathbf{L}^1}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$.

Proof. Suppose that f is a bounded Lipschitz continuous function. Then, there exists a positive constant C such that

$$|f(x) - f(y)| \leqslant C|x - y|, \quad \forall x, y \in \mathbb{R}.$$

It follows that

$$|\mathbb{E}[f(X_{(i,j)}) - f(X_{(0,0)})]| \leqslant \mathbb{E}[|f(X_{(i,j)}) - f(X_{(0,0)})|] \leqslant C\mathbb{E}[|X_{(i,j)} - X_{(0,0)}|]. \tag{2.2}$$

Based on the Definition 1.7 and 1.9, Eq. (2.2) implies that $X_{(i,j)} \overset{\min-S-\mathbf{L}^1}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$. □

Theorem 2.12. *Let C be a constant. Then,* $X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} C \Leftrightarrow X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} C$.

Proof. $''\Rightarrow''$ For any $\varepsilon > 0$, we have

$$\begin{aligned} P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} &= 1 - P\{X_{(i,j)} < C + \varepsilon\} + P\{X_{(i,j)} \leqslant C - \varepsilon\} \\ &\leqslant 1 - F_{(i,j)}(C + \frac{\varepsilon}{2}) + F_{(i,j)}(C - \varepsilon), \end{aligned} \tag{2.3}$$

where $F_{(i,j)}(\cdot)$ denotes the distribution function of $X_{(i,j)}$.

If $X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} C$, then for any $\varepsilon > 0$,

$$\sum_{(i,j)\in\mathbb{N}^2} |F_{(i,j)}(C + \frac{\varepsilon}{2}) - 1| < \infty \quad \textit{and} \sum_{(i,j)\in\mathbb{N}^2} |F_{(i,j)}(C - \varepsilon) - 0| < \infty$$

these together with Eq. (2.3) imply that for any $\varepsilon > 0$,

$$\sum_{(i,j)\in\mathbb{N}^2} P\{|X_{(i,j)} - C| \geqslant \varepsilon\} < \infty.$$

That is, $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)}$.

$''\Leftarrow''$ Let $X_{(0,0)} \equiv C$ and $F_{(0,0)}$ be the distribution of $X_{(0,0)}$. For any $\varepsilon > 0$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} F_{(0,0)}(x - \varepsilon) - P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} &\leqslant F_{(i,j)}(x) \\ &\leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} + F_{(0,0)}(x + \varepsilon). \end{aligned} \tag{2.4}$$

If $x > C$, set $\varepsilon = \frac{(x-C)}{2}$ in Eq. (2.4), we have

$$1 - P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} \leqslant F_{(i,j)}(x) \leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} + 1,$$

i.e.,

$$-P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} \leqslant F_{(i,j)}(x) - 1 \leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\}.$$

If $x < C$, set $\varepsilon = \frac{(C-x)}{2}$ in Eq. (2.4), we have

$$0 - P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} \leqslant F_{(i,j)}(x) \leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} + 0,$$

i.e.,

$$-P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} \leqslant F_{(i,j)}(x) - 0 \leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\}.$$

Note that $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)}$, we have for any $x \neq C$,

$$\sum_{(i,j)\in N^2} |F_{(i,j)}(x) - F_{(0,0)}(x)| < \infty.$$

Thus, $X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} C$. □

Theorem 2.13. *Assume that* $X_{(i,j)} \stackrel{\min-S_1-d}{\longrightarrow} X_{(0,0)}$ *and* $Y_{(i,j)} \stackrel{\min-S-\mathbf{L}^1}{\longrightarrow} C$. *Then,*

(i). $X_{(i,j)} + Y_{(i,j)} \stackrel{\min-S_1-d}{\longrightarrow} X_{(0,0)} + C$.

(ii). *If* $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ *is a double-indexed sequence of bounded random variables, then* $X_{(i,j)}Y_{(i,j)} \stackrel{S_1-d}{\longrightarrow} CX_{(0,0)}$.

(iii). *If* $\{X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ *and* $\{\frac{1}{Y_{(i,j)}}\}$ *are two double-indexed sequence of bounded random variables and* $C \neq 0$, *then* $\frac{X_{(i,j)}}{Y_{(i,j)}} \stackrel{S_1-d}{\longrightarrow} \frac{X_{(0,0)}}{C}$.

Proof. Suppose that f is a bounded Lipschitz continuous function. Then, there exits a positive constant K such that

$$|f(x) - f(y)| \leqslant K|x-y|, \quad \forall x, y \in \mathbb{R}.$$

(i) We have

$$\begin{aligned}|\mathbb{E}[f(X_{(i,j)} + Y_{(i,j)})] - \mathbb{E}[f(X_{(0,0)} + C)]| &\leqslant |\mathbb{E}[f(X_{(i,j)} + Y_{(i,j)})] - \mathbb{E}[f(X_{(i,j)} + C)] \\ &\quad + \mathbb{E}[f(X_{(i,j)} + C)] - \mathbb{E}[f(X_{(0,0)} + C)]| \\ &\leqslant K\mathbb{E}[|Y_{(i,j)} - C|] + |\mathbb{E}[f(X_{(i,j)} + C)] - \mathbb{E}[f(X_{(0,0)} + C)]|.\end{aligned}$$

Noticing that $Y_{(i,j)} \stackrel{\min-S-\mathbf{L}^1}{\longrightarrow} C$, i.e.,

$$\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[Y_{(i,j)} - C] < \infty.$$

Define $g(x) = f(x+C)$. Then, g is a bounded Lipschitz continuous function and thus by the assumption that $X_{(i,j)} \stackrel{\min-S_1-d}{\longrightarrow} X_{(0,0)}$, we have

$$\sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[f(X_{(i,j)} + C)] - \mathbb{E}[f(X_{(0,0)} + C)]| < \infty.$$

According the definition of $S_1 - d$ convergence, we have $X_{(i,j)} + Y_{(i,j)} \stackrel{\min-S_1-d}{\longrightarrow} X_{(0,0)} + C$.
The proofs for (ii) and (iii) are quite similar to that of (i) and so are omitted.
□

Theorem 2.14. *Let* $\{X_{(0,0)}, X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ *be a double-indexed sequence of random variables and* $\{F_{(0,0)}, F_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ *be the corresponding distribution functions. Then,* $X_{(i,j)} \stackrel{\min-S_2-d}{\longrightarrow} X_{(0,0)}$ *if one of the following conditions is fulfilled.*

(1). $X_{(0,0)}$ *is a discrete random variable such that* $\{x \in \mathbb{R} : P(X_{(0,0)} = x) = 0\}$ *is an open subset of* $\mathbb{R}$ *and* $X_{(i,j)} \stackrel{\min-c.c.}{\longrightarrow} X_{(0,0)}$.

(2). $X_{(0,0)}$ *has a bounded density function and* $\sum_{(i,j)\in\mathbb{N}^2} P\{ij(\log ij)^{1+\beta}|X_{(i,j)} - X_{(0,0)}| \geqslant \delta\} < \infty$ *for two positive constants* β *and* δ.

Proof. (1) Assume that $x \in \mathbb{R}$ with $P(X_{(0,0)} = x) = 0$. Noticing that $\{x \in \mathbb{R} : P(X_{(0,0)} = x) = 0\}$ is an open subset of $\mathbb{R}$, there exists $\varepsilon > 0$ such that

$$F_{(0,0)}(x) = F_{(0,0)}(x+\varepsilon) = F_{(0,0)}(x-\varepsilon),$$

which implies that

$$\begin{aligned}|F_{(i,j)}(x) - F_{(0,0)}(x)| &\leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} + |F_{(0,0)}(x+\varepsilon) - F_{(0,0)}(x)| + |F_{(0,0)}(x-\varepsilon) - F_{(0,0)}(x)| \\ &= P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\}.\end{aligned}$$

It follows that

$$\sum_{(i,j)\in N^2} |F_{(i,j)}(x) - F_{(0,0)}(x)| \leqslant \sum_{(i,j)\in \mathbb{N}^2} P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\}. \tag{2.5}$$

Eq. (2.5) together with the definition of complete convergence in the sense of lim(min)-convergence and $\min -S_2 - d$ imply that $X_{(i,j)} \overset{\min -S_2-d}{\longrightarrow} X_{(0,0)}$.

(2) By the assumption, we know that there exists a positive constant C such that $|f(x)| \leqslant C$, $\forall x \in \mathbb{R}$. It follows that for any $x, y \in \mathbb{R}$,

$$|F_{(0,0)}(x) - F_{(0,0)}(y)| = |\int_x^y f(u)du| \leqslant C|y - x|.$$

So we have

$$\begin{aligned}|F_{(i,j)}(x) - F_{(0,0)}(x)| &\leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} + |F_{(0,0)}(x+\varepsilon) - F_{(0,0)}(x)| + |F_{(0,0)}(x-\varepsilon) - F_{(0,0)}(x)| \\ &\leqslant P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon\} + 2C\varepsilon. \end{aligned} \tag{2.6}$$

Take $\varepsilon = \frac{\delta}{ij(\log ij)^{1+\beta}}$ in Eq. (2.6), we have

$$\begin{aligned}\sum_{(i,j)\in \mathbb{N}^2} |F_{(i,j)}(x) - F_{(0,0)}(x)| &\leqslant \sum_{(i,j)\in \mathbb{N}^2} [P\{|X_{(i,j)} - X_{(0,0)}| \geqslant \frac{\delta}{ij(\log ij)^{1+\beta}}\} + 2C\frac{\delta}{ij\log(ij)^{1+\beta}}] \\ &= \sum_{(i,j)\in \mathbb{N}^2} P\{ij(\log ij)^{1+\beta}|X_{(i,j)} - X_{(0,0)}| \geqslant \delta\} + 2C\delta \sum_{(i,j)\in \mathbb{N}^2} \frac{1}{ij\log(ij)^{1+\beta}} \\ &< \infty\end{aligned}$$

and thus, $X_{(i,j)} \overset{\min -S_2-d}{\longrightarrow} X_{(0,0)}$. □

Theorem 2.15. *Let $\{X_{(0,0)}, X_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ be a double-indexed sequence of random variables and $\{F_{(0,0)}, F_{(i,j)}, (i,j) \in \mathbb{N}^2\}$ be the corresponding distribution functions. If $F_{(0,0)}$ is locally Lipschitz continuous at each continuous point x of $F_{(0,0)}$ and $X_{(i,j)} \overset{\min -S-L^\infty}{\longrightarrow} X_{(0,0)}$, then $X_{(i,j)} \overset{\min -S_2-d}{\longrightarrow} X_{(0,0)}$.*

Proof. Assume that $X_{(i,j)} \overset{\min -S-\mathbf{L}^\infty}{\longrightarrow} X(0,0)$. Put $\alpha_{(i,j)} = \|X_{(i,j)} - X_{(0,0)}\|_\infty$. Then, $\alpha_{(i,j)} \geqslant 0$ and $\sum_{(i,j)\in \mathbb{N}^2} \alpha_{(i,j)} < \infty$.

For any $x \in \mathbb{R}$, we have

$$\begin{aligned}F_{(i,j)}(x) - F_{(0,0)}(x) &= P(X_{(i,j)} \leqslant x) - F_{(0,0)}(x) = P(X_{(0,0)} + X_{(i,j)} - X_{(0,0)} \leqslant x) - F_{(0,0)}(x) \\ &\leqslant P(X_{(0,0)} \leqslant x + \alpha_{(i,j)}) - F_{(0,0)}(x) = F_{(0,0)}(x + \alpha_{(i,j)}) - F_{(0,0)}(x)\end{aligned}$$

and

$$\begin{aligned}F_{(i,j)}(x) - F_{(0,0)}(x) &= 1 - P(X_{(i,j)} > x) - F_{(0,0)}(x) = 1 - P(X_{(0,0)} + X_{(i,j)} - X_{(0,0)} > x) - F_{(0,0)}(x) \\ &\geqslant P(X_{(0,0)} \leqslant x - \alpha_{(i,j)}) - F_{(0,0)}(x) = F_{(0,0)}(x - \alpha_{(i,j)}) - F_{(0,0)}(x) \\ &= -[F_{(0,0)}(x) - F_{(0,0)}(x + \alpha_{(i,j)})].\end{aligned}$$

It follows that

$$|F_{(i,j)}(x) - F_{(0,0)}(x)| \leqslant [F_{(0,0)}(x + \alpha_{(i,j)}) - F_{(0,0)}(x)] + [F_{(0,0)}(x) - F_{(0,0)}(x - \alpha_{(i,j)})].$$

Let x be a continuous point of $F_{(0,0)}$. By the assumption, there exists two constants K and δ such that for any $u, v \in (x-\delta, x+\delta)$,

$$|F_{(0,0)}(u) - F_{(0,0)}(v)| \leqslant K|u-v|.$$

Since $\lim(\min)\alpha_{(i,j)} = 0$, there exists $N_0 \in \mathbb{N}$ such that $\alpha_{(i,j)} < \delta$ for any $i, j > N_0$. Then, we have for any $i, j > N_0$,

$$F_{(0,0)}(x + \alpha_{(i,j)}) - F_{(0,0)}(x) \leqslant K\alpha_{(i,j)}.$$

So we have

$$\sum_{(i,j)\in\mathbb{N}^2} [F_{(0,0)}(x+\alpha_{(i,j)}) - F_{(0,0)}(x)] \leqslant \sum_{(i,j)}^{(N_0,N_0)} [F_{(0,0)}(x+\alpha_{(i,j)}) - F_{(0,0)}(x)] + K \sum_{(i,j)=(N_0+1,N_0+1)}^{\infty} \alpha_{(i,j)} < \infty.$$

Similarly, we have

$$\sum_{(i,j)\in\mathbb{N}^2} [F_{(0,0)}(x) - F_{(0,0)}(x+\alpha_{(i,j)})] < \infty$$

and so,

$$\sum_{(i,j)\in\mathbb{N}^2} |F_{(i,j)}(x) - F_{(0,0)}(x)| < \infty$$

which follows that $X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} X_{(0,0)}$. □

Theorem 2.16. *If* $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)}$ *and* $\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - X_{(0,0)}|\mathbf{1}_{\{|X_{(i,j)}-X_{(0,0)}|<\varepsilon\}}] < \infty$ *for some positive number* ε, *then* $X_{(i,j)} \overset{\min-S_1^*-d}{\longrightarrow} X_{(0,0)}$, *and thus* $X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$. *Where* $\mathbf{1}_A$ *denotes the indicator function of set A.*

Proof. Assume that f is a bounded Lipschitz continuous function, then there exist two constants K, M such that for any $x, y \in \mathbb{R}$ such that $|f(x)| \leqslant M$ and $|f(x) - f(y)| \leqslant K|x-y|$. Note that

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|f(X_{(i,j)}) - f(X_{(0,0)})|] &= \sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|f(X_{(i,j)}) - f(X_{(0,0)})|\mathbf{1}_{\{|X_{(i,j)}-X_{(0,0)}|<\varepsilon\}}] \\
&\quad + \sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|f(X_{(i,j)}) - f(X_{(0,0)})|\mathbf{1}_{\{|X_{(i,j)}-X_{(0,0)}|\geqslant\varepsilon\}}] \\
&\leqslant K \sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - X_{(0,0)}|\mathbf{1}_{\{|X_{(i,j)}-X_{(0,0)}|<\varepsilon\}}] + 2M \sum_{(i,j)\in\mathbb{N}^2} P(|X_{(i,j)} - X_{(0,0)}| \geqslant \varepsilon) \\
&< \infty.
\end{aligned}$$

Thus, $X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$. □

3. Examples and Remarks

In this section we always assume that $\Omega = (0,1] \times (0,1]$, $\mathcal{F} = \mathcal{B}(\Omega)$ (the Borel field of Ω) and P be the Lebesgue measure on Ω.

The following Example 3.1 shows that, in general, $X_{(i,j)} \overset{\min-S-\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)}$ is stronger than $X_{(i,j)} \overset{\min-S-\mathbf{L}^p}{\longrightarrow} X_{(0,0)}$.

Example 3.1. *For* $(i,j) \in \mathbb{N}^2$, *we define random variables* $X_{(i,j)}$ *by*

$$X_{(i,j)}(\omega) = \begin{cases} 1, & if\, \omega \in (0, \frac{1}{i^2}] \times (0, \frac{1}{j^2}], \\ 0, & otherwise. \end{cases}$$

For any $p > 0$, we have

$$\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - 0|^p] = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \int_0^{\frac{1}{j^2}} \int_0^{\frac{1}{i^2}} 1^p dP = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{i^2 j^2} < \infty.$$

Therefore, $X_{(i,j)} \overset{\min-S-\mathbf{L}^p}{\longrightarrow} 0$. *Obviously, we have* $\|X_{(i,j)} - 0\|_\infty = 1$ *for any* $(i,j) \in \mathbb{N}^2$. *Thus,* $\|X_{(i,j)} - 0\|_\infty \overset{\min-S-\mathbf{L}^\infty}{\nrightarrow} 0$, *which implies that* $X_{(i,j)} \overset{\min-S-\mathbf{L}^\infty}{\nrightarrow} 0$.

The following example shows that, in general, $X_{(i,j)} \overset{\min-S-\mathbf{L}^\alpha}{\longrightarrow} X_{(0,0)}$ is stronger than $X_{(i,j)} \overset{\min-S_\alpha-a.s.}{\longrightarrow} X_{(0,0)}$.

Example 3.2. *Let $\alpha > 0$. For $(i,j) \in \mathbb{N}^2$, define random variables $X_{(i,j)}$ by*

$$X_{(i,j)}(\omega) = \begin{cases} (ij)^\alpha, & if \omega \in (0, \frac{1}{i^{1+\alpha}}] \times (0, \frac{1}{j^{1+\alpha}}], \\ 0, & otherwise. \end{cases}$$

Clearly,

$$\sum_{(i,j)\in\mathbb{N}^2} P\{|X_{(i,j)} - 0|^\alpha \geqslant \varepsilon\} \leqslant \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{(ij)^{1+\alpha}} < \infty.$$

By Borel-Cantelli lemma, we have $P(|X_{(i,j)} - 0|^\alpha \geqslant \varepsilon, i.o.) = 0$. *Thus,* $\sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)} - 0|^\alpha < \infty \quad a.s.$.

Noticing that

$$\sum_{(i,j)\in\mathbb{N}^2} \mathbb{E}[|X_{(i,j)} - 0|^\alpha] = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \int_0^{\frac{1}{j^{1+\alpha}}} \int_0^{\frac{1}{i^{1+\alpha}}} (ij)^\alpha dP = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{ij} = \infty.$$

The Example 3.2 shows that $X_{(i,j)} \overset{\min-S_\alpha-a.s.}{\longrightarrow} 0$ *but* $X_{(i,j)} \overset{\min-S-\mathbf{L}^\alpha}{\nrightarrow} 0$.

The next example shows that $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)}$ does not imply $X_{(i,j)} \overset{\min-S_\alpha-a.s.}{\longrightarrow} X_{(0,0)}$ in general.

Example 3.3. *Let $\alpha > 0$. For $(i,j) \in \mathbb{N}^2$, define a random variable $X_{(i,j)}$ by*

$$X_{(i,j)}(\omega) = \begin{cases} 1, & if \omega \in (0, \frac{1}{i^2}] \times (0, \frac{1}{j^2}], \\ \frac{1}{ij^{\frac{1}{\alpha}}}, & otherwise. \end{cases}$$

For any $\varepsilon > 0$, there exists N_0 such that $\frac{1}{N_0^{\frac{2}{\alpha}}} < \varepsilon$, for any $i, j > N_0$, we have $\frac{1}{ij^{\frac{2}{\alpha}}} \leqslant \frac{1}{N_0^{\frac{2}{\alpha}}} < \varepsilon$ and thus,

$$\sum_{(i,j)\in\mathbb{N}^2} P\{|X_{(i,j)} - 0| \geqslant \varepsilon\} \leqslant \sum_{(i,j)=(1,1)}^{(N_0-1,N_0-1)} P\{|X_{(i,j)} - 0| \geqslant \varepsilon\} + \sum_{(i,j)=(N_0,N_0)}^{\infty} \frac{1}{i^2 j^2} < \infty.$$

Hence, $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} 0$.

Note that for any $\omega \in (0,1] \times (0,1]$, *we have* $\sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)} - 0|^\alpha = \sum_{(i,j)\in\mathbb{N}^2} |X_{(i,j)}|^\alpha = \infty$.

This shows that $X_{(i,j)} \overset{\min-S_\alpha-a.s.}{\nrightarrow} 0$.

The following two examples give the relationship between the $\min -S_1-d$ convergence and the $\min -S_2-d$ convergence.

Example 3.4. *Let $\alpha > 0$. For $(i, j) \in \mathbb{N}^2$, we define random variables $X_{(i,j)}$ by*

$$X_{(i,j)}(\omega) = \begin{cases} 1, & if \omega \in (0, \frac{1}{i^2}] \times (0, \frac{1}{j^2}], \\ \frac{1}{ij^{\frac{1}{\alpha}}}, & otherwise. \end{cases}$$

It is easy to see that $X_{(i,j)} \overset{\min -c.c.}{\longrightarrow} 0$. Therefore, $X_{(i,j)} \overset{\min -S_2-d}{\longrightarrow} 0$.

In the following, we will show that when $\alpha > 1$, $X_{(i,j)} \overset{\min -S_1-d}{\nrightarrow} 0$. Let $f(x) = \sin x$. It is obvious that $f(x)$ is a bounded Lipschitz continuous function. Note that

$$\begin{aligned} \sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}f(X_{(i,j)} - f(0))| &= \sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[\sin X_{(i,j)} - \sin 0]| \\ &= \sum_{(i,j)\in\mathbb{N}^2} |[\frac{1}{i^2 j^2} \sin 1 + (\frac{1}{i^2} - 1)(\frac{1}{j^2} - 1) \sin \frac{1}{(ij)^{\frac{1}{\alpha}}}]| \\ &= \sin 1 \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{i^2 j^2} + \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{i^2 j^2} \sin \frac{1}{(ij)^{\frac{1}{\alpha}}} - \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{i^2} \sin \frac{1}{(ij)^{\frac{1}{\alpha}}} \\ &\quad - \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{j^2} \sin \frac{1}{(ij)^{\frac{1}{\alpha}}} + \sum_{(i,j)\in\mathbb{N}^2} \sin \frac{1}{(ij)^{\frac{1}{\alpha}}}. \end{aligned}$$

It is easy to check that the first two sums are convergent. Noticing that

$$\lim_{i,j\to\infty} \frac{\sin \frac{1}{(ij)^{\frac{1}{\alpha}}}}{\frac{1}{(ij)^{\frac{1}{\alpha}}}} = 1$$

and the fact that for $\alpha > 1$, $\sum_{(i,j)\in\mathbb{N}^2} \frac{1}{(ij)^{\frac{1}{\alpha}}} = \infty$, we know that the sum $\sum_{(i,j)\in\mathbb{N}^2} \sin \frac{1}{(ij)^{\frac{1}{\alpha}}}$ is divergent. Hence,

$$\sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[f(X_{(i,j)}) - f(0)]| = \infty.$$

It follows that $X_{(i,j)} \overset{\min -S_1-d}{\nrightarrow} 0$.

Example 3.5. *Let α, β be two constants satisfying $0 < \alpha < 1$, $\beta > 1$. Let $X_{(0,0)}$ be a random variable defined on $(\Omega, \mathcal{F}, P)$ with the density function $f(u, v) = (1 - \alpha)(1 - u)^{-\alpha}$, $(u, v) \in (0, 1] \times (0, 1]$. For any $i, j \in \mathbb{N}$, define random variables*

$$X_{(i,j)} := X_{(0,0)} + \frac{1}{(ij)^\beta} \;\; i, j \in \mathbb{N}.$$

Then, we have

$$\sum_{(i,j)\in\mathbb{N}^2} ||X_{(i,j)} - X_{(0,0)}||_\infty = \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{(ij)^\beta} < \infty,$$

which implies that $X_{(i,j)} \overset{\min -S-\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)}$.

Denote by $F_{(i,j)}$ and $F_{(0,0)}$ the distribution functions of $X_{(i,j)}$ and $X_{(0,0)}$, respectively. Suppose that $(1-\alpha)\beta \leqslant 1$. Then,

$$\begin{aligned}\sum_{(i,j)\in\mathbb{N}^2} |F_{(i,j)}(1) - F_{(0,0)}(1)| &= \sum_{(i,j)\in N^2} |F_{(i,j)}(1 - \frac{1}{(ij)^\beta}) - F_{(0,0)}(1)| \\ &= \sum_{(i,j)\in\mathbb{N}^2} |F_{(0,0)}(1) - F_{(i,j)}(1 - \frac{1}{(ij)^\beta})| \\ &= \sum_{(i,j)\in\mathbb{N}^2} \int_{1-\frac{1}{(ij)^\beta}}^{1} (1-\alpha)(1-u)^{-\alpha} du \\ &= (1-\alpha) \sum_{(i,j)\in N^2} \int_0^{\frac{1}{(ij)^\beta}} t^{-\alpha} dt \\ &= \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{(ij)^{\beta(1-\alpha)}} \\ &= \infty.\end{aligned}$$

Obviously, $X_{(i,j)} \overset{\min-S_2-d}{\nrightarrow} 0$.

Remark 3.6. *(i) By Example 3.4, we get that $X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} X_{(0,0)} \nRightarrow X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$. By Example 3.5 and the fact that $X_{(i,j)} \overset{\min-S-\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-S-\mathbf{L}^1}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$, we obtain that $X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)} \nRightarrow X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} X_{(0,0)}$.*

(ii) By Example 3.5, we know that $X_{(i,j)} \overset{\min-S-\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)} \nRightarrow X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} X_{(0,0)}$.

(iii) By Example 3.5 and the fact that $X_{(i,j)} \overset{\min-S-\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-S-\mathbf{L}^1}{\longrightarrow} X_{(0,0)}$, we have $X_{(i,j)} \overset{\min-S-\mathbf{L}^1}{\longrightarrow} X_{(0,0)} \nRightarrow X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} X_{(0,0)}$.

(iv) By Example 3.4, we get that $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)} \nRightarrow X_{(i,j)} \overset{\min-S_1-d}{\longrightarrow} X_{(0,0)}$. By Example 3.5 and the fact that $X_{(i,j)} \overset{\min-S-\mathbf{L}^\infty}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-S-\mathbf{L}^1}{\longrightarrow} X_{(0,0)} \Rightarrow X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)}$, we obtain that $X_{(i,j)} \overset{\min-c.c.}{\longrightarrow} X_{(0,0)} \nRightarrow X_{(i,j)} \overset{\min-S_2-d}{\longrightarrow} X_{(0,0)}$.

Example 3.7. *Let $\alpha > 0$. For $(i,j) \in \mathbb{N}^2$, define random variables $X_{(i,j)}$ by*

$$X_{(i,j)}(\omega) = \begin{cases} (ij)^\alpha, & if \omega \in (0, \frac{1}{i^{1+\alpha}}] \times (0, \frac{1}{j^{1+\alpha}}], \\ 0, & otherwise. \end{cases}$$

From [Example 3.2.], we have that $X_{(i,j)} \overset{\min-S_\alpha-a.s.}{\longrightarrow} 0$.

Let $f(x) = x$. It is easy to see that f is a bounded Lipschitz continuous function. We have

$$\begin{aligned}\sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[f(X_{(i,j)}) - f(0)]| &= \sum_{(i,j)\in\mathbb{N}^2} |\mathbb{E}[X_{(i,j)}]| \\ &= \sum_{(i,j)\in\mathbb{N}^2} |\int_0^{\frac{1}{j^{1+\alpha}}} \int_0^{\frac{1}{i^{1+\alpha}}} (ij)^\alpha dP| \\ &= \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{i^{1+\alpha}} \cdot \frac{1}{j^{1+\alpha}} (ij)^\alpha \\ &= \sum_{(i,j)\in\mathbb{N}^2} \frac{1}{i} \cdot \frac{1}{j} \\ &= \infty,\end{aligned}$$

and thus, $X_{(i,j)} \overset{\min -S_1-d}{\nrightarrow} 0.$

Remark 3.8. *By Example 3.7, we have* $X_{(i,j)} \overset{\min -S_\alpha-a.s.}{\longrightarrow} X_{(0,0)}(\alpha > 0) \nRightarrow X_{(i,j)} \overset{\min -S_1-d}{\nrightarrow} X_{(0,0)}$.

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