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Large deviations for stochastic pantograph integrodifferential equation

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Abstract. The pantograph equation, a specific type of delay differential equation is examined in this study in its stochastic form. Our main intention is to establish the Wentzell-Freidlin type large deviation estimates for stochastic pantograph integrodifferential equation. The existence and uniqueness of solution is established by using the method of successive approximations. We then take up the weak convergence approach to obtain the main result. The established results are illustrated with examples.

1. Introduction

The Large Deviation Principle (LDP) is the study of events whose probabilities of occurrence are meager. The frequency distribution of all events, excluding the rare event, remains the same as rare event will exhibit a significantly larger deviation about the mean. It is concerned with the events that are not captured by the central limit theorem or the law of large numbers. Though these events are concerned only with the tail behaviour of probability distributions, they cannot be ignored as they may have the capability of creating a massive impact on the whole system when they exist. The theory of large deviations' main application is in the prediction of rare events, as it helps geologists forecast natural calamities, biologists infer reaction networks for enzyme kinetics and gene regulation, economists guess the best investment strategies and many more [5, 28].

Many researchers were fascinated by Varadhan's eminent monograph [31] that developed the theory of large deviations. Slightly later, Freidlin and Wentzell [11] made a fundamental work to enhance this theory and provide a set of hypothesis to illustrate the result. Since then, Wentzell-Freidlin type large deviations abundantly featured many papers. Sritharan and Sundar [25] investigated the large deviations for a perturbed Navier-Stokes equation both in bounded and unbounded domain. Mohammed and Zhang [18] discussed this theory for the multiplicative noise system with history in the diffusion term. Mo and Luo [17] improved and enhanced the results by Mohammed and Zhang. Inahama [12] established the LDP for pinned diffusion processes by employing a mild ellipticity assumption. The fundamental principle and motivation on this theory can be found in many literatures [10, 14, 20, 30].

Ordinary differential equation has a wider role in framing many real world problems in science and technology. But it is noticed that such equations could not mirror the actual scenarios in the process of modeling a problem with hereditary properties or with memory. In such a situation, system is designed a

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way better by incorporating a non-local parameter like delay in it. Generally, delay differential equation is the rate of change of a particular quantity which depends on the same quantity of previous time. Analysis of delay differential equation is more challenging when compared to ordinary differential equation but on the account of problem description it gives the substantially refined results.

The functional differential equations have a prominent kind of differential equation with proportional delay called the pantograph equation. The name was originated in 1851 from the device named pantograph that is used in the construction of current collection system for an electric locomotive. Then, pantograph got a complete mathematical framing in 1971 with the prestigious work of Ockendon [19]. In addition, pantograph equation may contain delay in both constant as well as variable aspects. An approximate solution for the pantograph equation with proportional delay is obtained in [22] using a semi-analytic numerical method. Yuan and Song [33] addressed the existence and uniqueness of an exact solution to construct the exponential integrators for semi-linear stochastic pantograph integro-differential equations and demonstrated the convergence of the exponential Euler method. For further numerical analysis on this model, one can refer [21, 23].

As the environment is occupied with lots of random circumstances in a day to day stuff, framing a system altogether with above mentioned scenarios will allow to have an even more better analysis on the setup. The inconsistency in the environment leads to randomness, which in turn leads to stochastic systems. The basic analysis on stochastic model is on the study of probabilistic nature of the considered system. To get familiar with stochastic system, one may refer [8, 16] and references therein.

This work is based on establishing the LDP for the pantograph equation in a stochastic version. Besides this analysis, there is a significant amount of literature on various other asymptotic analyses of solutions by considering the stochastic pantograph equation. Appleby [1] investigated the growth and decay rates of solutions of the Itô type stochastic delay differential equation, in which the linear drift term has an unbounded delay and the nonlinear diffusion term solely depends on the current state. In [3], the stochastic pantograph equation is considered to achieve mean-square convergence of approximations. Such kind of analysis on this system can be found in [2, 9, 13].

Even though there are many techniques to illustrate the LDP for the system, in this article, method of weak convergence is adopted. This approach builds up much interest among math enthusiasts and have some fruitful outcomes on several frameworks. Dupuis and Lipshutz [7] investigated the LDP for empirical measures of a diffusion in Euclidean space using the weak convergence approach with the techniques developed and generalised for large deviation problem in [6]. In [15, 24, 26, 27], weak convergence technique is employed to establish the large deviation for various systems.

The organisation of this work is as follows: In Section 2, a precise setting and few assumptions are given. The main results on the theory of large deviations are stated as well. Section 3 consists of existence and uniqueness of considered stochastic pantograph integrodifferential equation using Picard's iterative scheme as in [29]. The main result of this work is examined in Section 4, the most important among them is Theorem 4.1. Examples are provided in Section 5 to emphasize the theory developed.

2. Mathematical Model

Let us see the mathematical foundation needed for our study. Denote by $(\Omega, \mathscr{F}, \mathcal{P})$ a complete filtered probability space which is assumed to satisfy the right continuity and completeness in a probabilistic sense. Consider a collection of random variables denoted by $R = \{x(t,\omega) : \omega \in \Omega, t \geq 0\}$ called the stochastic process. For convenience, we suppress the dependency on $\omega \in \Omega$ and simply write $x(t)$ throughout the paper. Consider the stochastic pantograph integrodifferential equation,

$$\begin{cases} \mathrm{d}x(t) = f\left(t, x(t), x(pt), \int_0^t k(t,s,x(s),x(ps))\mathrm{d}s\right)\mathrm{d}t \\ \qquad\qquad +g\left(t, x(t), x(pt), \int_0^t k(t,s,x(s),x(ps))\mathrm{d}s\right)\mathrm{d}W(t), \quad t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1}$$

where $pt < t$ for $t \geq 0$ and $0 < p < 1$, a differential equation with time lag. The delay parameter pt satisfies $pt \to \infty$ as $t \to \infty$. Furthermore, $W(t)$ is r-dimensional Wiener process on the filtered probability space

$(\Omega, \mathscr{F}, \mathcal{P})$ and the initial condition $x_0 \in \mathbb{R}^d$. Let $\|\cdot\|$ denote the usual Euclidean norm with appropriate dimension. In addition, the Borel measurable functions $f : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ are continuous and $k : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. The drift and noise terms satisfies Lipschitz condition and linear growth condition, that is, for all $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}^d$ and $t \geq 0$, there exists positive constants L_1, L_2 such that

$$\begin{aligned}\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \quad &\vee \quad \|g(t,x_1,x_2,x_3) - g(t,y_1,y_2,y_3)\| \\ &\leq L_1(\|x_1 - y_1\| + \|x_2 - y_2\| + \|x_3 - y_3\|), \end{aligned} \tag{2}$$

$$\|f(t,x_1,x_2,x_3)\| \vee \|g(t,x_1,x_2,x_3)\| \leq L_2(1 + \|x_1\| + \|x_2\| + \|x_3\|). \tag{3}$$

Let us make one more assumption on the coefficient k: For any $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ and $0 \leq s \leq t$, there exists a non-negative constant δ such that

$$\|k(t,s,x_1,x_2) - k(t,s,y_1,y_2)\| \leq \delta(\|x_1 - y_1\| + \|x_2 - y_2\|). \tag{4}$$

Also, assume that $k(t,s,0,0) = 0$ for all $0 \leq s \leq t$.

Let $\mathcal{X}$ be a polish space and consider a family of random variables X^ϵ defined on $\mathcal{X}$. Large deviation theory deals with the random events $\mathcal{A}$ with probability $\mathcal{P}(X^\epsilon \in \mathcal{A})$ that converges exponentially to 0 as $\epsilon \to 0$. The rate at which exponential decay of a probability occurs is termed as rate function and is formally defined as follows:

Definition 2.1 (Rate function). *[4] A function $\mathcal{I}$ from $\mathcal{X}$ to $[0, +\infty]$ is called*

- *a rate function if $\mathcal{I}$ is lower semi-continuous, which means that the level sets $\{f \in \mathcal{X} : \mathcal{I}(f) \leq k\}$ are closed for any $k \geq 0$.*
- *a good rate function if for each $k < \infty$, the level set is compact.*

The primary setting in this work is that the Laplace principle and the LDP (for both definitions, we refer to [17]) are equivalent in Polish space $\mathcal{X}$. We end this section with the following result on the equivalence.

Theorem 2.2. *The family $\{X^\epsilon\}$ satisfies the Laplace principle with a good rate function $\mathcal{I}(\cdot)$ on $\mathcal{X}$ if and only if $\{X^\epsilon\}$ satisfies the large deviation principle with the same rate function $\mathcal{I}(\cdot)$.*

3. Existence and Uniqueness

This section contains the existence and uniqueness of solution of stochastic pantograph integro-differential equation using Picard's successive approximation method. Let us have the following lemma to assure the existence and uniqueness of the solution in the space $C([0,T];\mathbb{R}^d)$.

Lemma 3.1. *Let assumptions (3) and (4) hold. Then the solution $x(t)$ of the system (1) on the solution space $C([0,T];\mathbb{R}^d)$ satisfies the estimate*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|x(t)\|^2\right] \leq \left(3\|x_0\|^2 + 3L_2^2(T+4)T\right) e^{(3L_2^2(T+4)(2+4T^2\delta^2)T)}. \tag{5}$$

Proof. For $m \geq 1$, define the stopping time,

$$\tau^m = T \wedge \inf\{t \in [0,T] : \|x(t)\| \geq m\}.$$

Set $x_m(t) = x(t \wedge \tau^m)$ for $t \in [0,T]$ satisfying

$$\begin{aligned} x_m(t) \quad = \quad & x_0 + \int_0^t f\left(s, x_m(s), x_m(ps), \int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right) I_{[[0,\tau^m]]}(s)\mathrm{d}s \\ & + \int_0^t g\left(s, x_m(s), x_m(ps), \int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right) I_{[[0,\tau^m]]}(s)\mathrm{d}W(s). \end{aligned} \tag{6}$$

Taking square norm and applying the elementary inequality,

$$|a_1 + a_2 + \cdots + a_q|^2 \leq q(|a_1|^2 + |a_2|^2 + \cdots + |a_q|^2), \tag{7}$$

we obtain,

$$\begin{aligned}||x_m(t)||^2 = 3||x_0||^2 + 3\left\|\int_0^t f\left(s, x_m(s), x_m(ps), \int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right) I_{[[0,\tau^m]]}(s)\mathrm{d}s\right\|^2 \\ +3\left\|\int_0^t g\left(s, x_m(s), x_m(ps), \int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right) I_{[[0,\tau^m]]}(s)\mathrm{d}W(s)\right\|^2.\end{aligned} \tag{8}$$

By using Holder's inequality and (3),

$$\begin{aligned}||x_m(t)||^2 \leq 3||x_0||^2 + 12TL_2^2\int_0^t \left(1 + ||x_m(s)||^2 + ||x_m(ps)||^2 + \left\|\int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right\|^2\right)\mathrm{d}s \\ +3\left\|\int_0^t g\left(s, x_m(s), x_m(ps), \int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right) I_{[[0,\tau^m]]}(s)\mathrm{d}W(s)\right\|^2.\end{aligned} \tag{9}$$

Note that, by using (4),

$$\begin{aligned}\left\|\int_0^s k(s,u,x_1(u),x_2(u))\mathrm{d}u\right\|^2 &\leq T\int_0^s ||k(s,u,x_1(u),x_2(u))||^2\,\mathrm{d}u \\ &\leq 2T^2\delta^2 \sup_{0\leq\eta\leq s}(||x_1(\eta)||^2 + ||x_2(\eta)||^2).\end{aligned} \tag{10}$$

Making use of (10), we obtain

$$\begin{aligned}||x_m(t)||^2 &\leq 3||x_0||^2 + 12TL_2^2\int_0^t \left(1 + ||x_m(s)||^2 + ||x_m(ps)||^2 + 2T^2\delta^2 \sup_{0\leq\eta\leq s}(||x_m(\eta)||^2 + ||x_m(p\eta)||^2)\right)\mathrm{d}s \\ &+3\left\|\int_0^t g\left(s, x_m(s), x_m(ps), \int_0^s k(s,u,x_m(u),x_m(pu))\mathrm{d}u\right) I_{[[0,\tau^m]]}(s)\mathrm{d}W(s)\right\|^2.\end{aligned}$$

Taking expectation of supremum and using Doob's Martingale inequality (refer to[16]), one gets that

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\leq\vartheta\leq t}||x_m(\vartheta)||^2\right\} &\leq 3||x_0||^2 + 12TL_2^2\int_0^t \sup_{0\leq\vartheta\leq s}\left(1 + ||x_m(\vartheta)||^2 + ||x_m(p\vartheta)||^2\right. \\ &\left. +2T^2\delta^2 \sup_{0\leq\eta\leq\vartheta}(||x_m(\eta)||^2 + ||x_m(p\eta)||^2)\right)\mathrm{d}s \\ &+48L_2^2\int_0^t \sup_{0\leq\vartheta\leq s}\left(1 + ||x_m(\vartheta)||^2 + ||x_m(p\vartheta)||^2\right. \\ &\left. +2T^2\delta^2 \sup_{0\leq\eta\leq\vartheta}(||x_m(\eta)||^2 + ||x_m(p\eta)||^2)\right)\mathrm{d}s.\end{aligned}$$

By simplification, one easily sees that

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\leq\vartheta\leq t}||x_m(\vartheta)||^2\right\} &\leq 3||x_0||^2 + 12TL_2^2\int_0^t \left(1 + 2\sup_{0\leq\vartheta\leq s}||x_m(\vartheta)||^2 + 4T^2\delta^2\sup_{0\leq\vartheta\leq s}||x_m(\vartheta)||^2\right)\mathrm{d}s \\ &+48L_2^2\int_0^t \left(1 + 2\sup_{0\leq\vartheta\leq s}||x_m(\vartheta)||^2 + 4T^2\delta^2\sup_{0\leq\vartheta\leq s}||x_m(\vartheta)||^2\right)\mathrm{d}s \\ &\leq 3||x_0||^2 + 12L_2^2(T+4)T + 12L_2^2(T+4)\int_0^t (2 + 4T^2\delta^2)\sup_{0\leq\vartheta\leq s}||x_m(\vartheta)||^2\mathrm{d}s.\end{aligned}$$

Finally, Gronwall's inequality yields

$$\mathbb{E}\left\{\sup_{0\le\vartheta\le T}\|x_m(\vartheta)\|^2\right\} \le \left(3\|x_0\|^2+12L_2^2(T+4)T\right)e^{(12L_2^2(T+4)(2+4T^2\delta^2)T)}. \tag{11}$$

Thus,

$$\mathbb{E}\left\{\sup_{0\le\vartheta\le\tau^m}\|x(\vartheta)\|^2\right\} \le \left(3\|x_0\|^2+12L_2^2(T+4)T\right)e^{(12L_2^2(T+4)(2+4T^2\delta^2)T)}. \tag{12}$$

Consequently by letting $m\to\infty$, we obtain $\tau^m\to T$ a.s. Hence, the solution of the control equation is therefore limited by a constant. □

Theorem 3.2. *Assume that (2), (3) and (4) hold. Then the system (1) has a unique solution on the space* $C([0,T];\mathbb{R}^d)$.

Proof. **Existence:** We begin with establishing the existence of solution in suitable solution space. Set $x_0(t)=x_0$ and define the Picard iterative processes

$$\begin{aligned} x_n(t) &= x_0+\int_0^t f\left(s,x_{n-1}(s),x_{n-1}(ps),\int_0^s k(s,u,x_{n-1}(u),x_{n-1}(pu))\mathrm{d}u\right)\mathrm{d}s \\ &\quad+\int_0^t g\left(s,x_{n-1}(s),x_{n-1}(ps),\int_0^s k(s,u,x_{n-1}(u),x_{n-1}(pu))\mathrm{d}u\right)\mathrm{d}W(s) \end{aligned} \tag{13}$$

for $n=1,2,\ldots$ and $t\in[0,T]$. As $x_0(t)\in C([0,T];\mathbb{R}^d)$ is obvious, subsequently induction tends to reach $x_n(t)\in C([0,T];\mathbb{R}^d)$ for $n=1,2,\ldots$. Applying the inequality (7) and then using (3), we obtain

$$\begin{aligned} \mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x_n(\vartheta)\|^2\right\} &\le 3\|x_0\|^2+12L_2^2(T+4)T+12L_2^2(T+4)(2+4T^2\delta^2)\mathbb{E}\int_0^t\sup_{0\le\vartheta\le s}\|x_{n-1}(\vartheta)\|^2\mathrm{d}s \\ &\le c_1+12L_2^2(T+4)(2+4T^2\delta^2)\int_0^t\mathbb{E}\left(\sup_{0\le\vartheta\le s}\|x_{n-1}(\vartheta)\|^2\right)\mathrm{d}s \end{aligned} \tag{14}$$

where $c_1=3\|x_0\|^2+12L_2^2(T+4)T$. For any $k\ge 1$, from (14)

$$\begin{aligned} \max_{1\le n\le k}\mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x_n(\vartheta)\|^2\right\} &\le c_1+12L_2^2(T+4)(2+4T^2\delta^2)\int_0^t\max_{1\le n\le k}\mathbb{E}\left(\sup_{0\le\vartheta\le s}\|x_{n-1}(\vartheta)\|^2\right)\mathrm{d}s \\ &\le c_1+12L_2^2(T+4)(2+4T^2\delta^2)\int_0^t\left(\mathbb{E}\|x_0\|^2+\max_{1\le n\le k}\mathbb{E}\left(\sup_{0\le\vartheta\le s}\|x_n(\vartheta)\|^2\right)\right)\mathrm{d}s \\ &\le c_2+12L_2^2(T+4)(2+4T^2\delta^2)\int_0^t\max_{1\le n\le k}\mathbb{E}\left(\sup_{0\le\vartheta\le s}\|x_n(\vartheta)\|^2\right)\mathrm{d}s \end{aligned}$$

where $c_2=c_1+12L_2^2(T+4)(2+4T^2\delta^2)\mathbb{E}\|x_0\|^2T$. Then Gronwall's inequality implies,

$$\max_{1\le n\le k}\mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x_n(\vartheta)\|^2\right\} \le c_2e^{(12L_2^2(T+4)(2+4T^2\delta^2)T)}.$$

Since k is arbitrary

$$\mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x_n(\vartheta)\|^2\right\} \le c_2e^{(12L_2^2(T+4)(2+4T^2\delta^2)T)} \tag{15}$$

for all $0 \le t \le T, n \ge 1$. We note that,

$$\begin{aligned}
\|x_1(t) - x_0(t)\|^2 &= \|x_1(t) - x_0\|^2 \\
&\le 8TL_2^2 \int_0^t \left(1 + \|x_0(s)\|^2 + \|x_0(ps)\|^2 + \left\|\int_0^s k(s,u,x_0(u),x_0(pu))\mathrm{d}u\right\|^2\right)\mathrm{d}s \\
&\quad +32L_2^2 \int_0^t \left(1 + \|x_0(s)\|^2 + \|x_0(ps)\|^2 + \left\|\int_0^s k(s,u,x_0(u),x_0(pu))\mathrm{d}u\right\|^2\right)\mathrm{d}s.
\end{aligned}$$

Taking expectation, we get

$$\mathbb{E}\|x_1(t) - x_0(t)\|^2 \le 8L_2^2(T+4)(1 + (2 + 4T^2\delta^2)\mathbb{E}\|x_0\|^2)T = C(\text{say}). \tag{16}$$

We now claim that

$$\mathbb{E}\|x_{n+1}(t) - x_n(t)\|^2 \le \frac{C(Mt)^n}{n!}, \quad 0 \le t \le T, \quad n \ge 0, \tag{17}$$

where $M = 6L_1^2(T+4)(2+4T^2\delta^2)$. The above estimate provides an error of the approximation to the solution. Let us show this by induction. Taking (16) into consideration, we notice (17) holds for $n = 0$. By inductive assumption, (17) holds for $n-1$. Thus, we have

$$\mathbb{E}\|x_n(t) - x_{n-1}(t)\|^2 \le \frac{C(Mt)^{n-1}}{(n-1)!}. \tag{18}$$

Now consider,

$$\begin{aligned}
\|x_{n+1}(t) - x_n(t)\|^2 &\le 2\left\|\int_0^t \left(f\left(s, x_n(s), x_n(ps), \int_0^s k(s,u,x_n(u),x_n(pu))\mathrm{d}u\right)\right.\right. \\
&\quad \left.\left. - f\left(s, x_{n-1}(s), x_{n-1}(ps), \int_0^s k(s,u,x_{n-1}(u),x_{n-1}(pu))\mathrm{d}u\right)\right)\mathrm{d}s\right\|^2 \\
&\quad +2\left\|\int_0^t \left(g\left(s, x_n(s), x_n(ps), \int_0^s k(s,u,x_n(u),x_n(pu))\mathrm{d}u\right)\right.\right. \\
&\quad \left.\left. - g\left(s, x_{n-1}(s), x_{n-1}(ps), \int_0^s k(s,u,x_{n-1}(u),x_{n-1}(pu))\mathrm{d}u\right)\right)\mathrm{d}W(s)\right\|^2.
\end{aligned}$$

Taking expectation over supremum and applying the inequality (2), it is evident that,

$$\begin{aligned}
\mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x_{n+1}(\vartheta) - x_n(\vartheta)\|^2\right\} &\le 6L_1^2(T+4)\int_0^t (2 + 4T^2\delta^2)\mathbb{E}\sup_{0\le\vartheta\le s}\|x_n(\vartheta) - x_{n-1}(\vartheta)\|^2\mathrm{d}s \\
&\le M\int_0^t \frac{C(Ms)^{n-1}}{(n-1)!}\mathrm{d}s \\
&\le \frac{C(Mt)^n}{(n)!}.
\end{aligned} \tag{19}$$

Thereupon, (25) holds for all $n \ge 0$. By Chebyshev's inequality,

$$\mathcal{P}\left[\sup_{0\le\vartheta\le t}\|x_{n+1}(\vartheta) - x_n(\vartheta)\|^2 > \frac{1}{n^2}\right] \le \frac{1}{(1/n^2)^2}\mathbb{E}\left[\sup_{0\le\vartheta\le t}\|x_{n+1}(\vartheta) - x_n(\vartheta)\|^2\right].$$

Using (19), and summing up the resultant inequalities,

$$\sum_{n=0}^{\infty}\mathcal{P}\left[\sup_{0\le\vartheta\le t}\|x_{n+1}(\vartheta) - x_n(\vartheta)\|^2 > \frac{1}{n^2}\right] \le \sum_{n=0}^{\infty}\frac{CM^nT^nn^4}{n!}$$

where the series on the right side converges by ratio test. By Borel-Cantelli lemma, we conclude that $\sup_{0\le\vartheta\le t}\|x_{n+1}(\vartheta)-x_n(\vartheta)\|^2$ converges to 0, almost surely, i.e., successive approximations $x_n(t)$ converge almost surely uniformly on $t\in[0,T]$ to a limit $x(t)$ defined by

$$\begin{aligned}\lim_{n\to\infty}\left[x_0(t)+\sum_{i=0}^{n-1}(x_{i+1}(t)-x_i(t))\right] &= \lim_{n\to\infty}x_n(t)\\ &= x(t).\end{aligned}$$

From (13), we get

$$\begin{aligned}x(t) &= x_0+\int_0^t f\left(s,x(s),x(ps),\int_0^s k(s,u,x(u),x(pu))\mathrm{d}u\right)\mathrm{d}s\\ &\quad+\int_0^t g\left(s,x(s),x(ps),\int_0^s k(s,u,x(u),x(pu))\mathrm{d}u\right)\mathrm{d}W(s)\end{aligned}\tag{20}$$

for $t\in[0,T]$. This completes the proof of the existence of solution.
Uniqueness: Let $x(t)$ and $\bar{x}(t)$ be the two solutions of (1). Here,

$$\begin{aligned}\|x(t)-\bar{x}(t)\|^2 &\le 2\left\|\int_0^t\left(f\left(s,x(s),x(ps),\int_0^s k(s,u,x(u),x(pu))\mathrm{d}u\right)\right.\right.\\ &\quad\left.\left.-\ f\left(s,\bar{x}(s),\bar{x}(ps),\int_0^s k(s,u,\bar{x}(u),\bar{x}(pu))\mathrm{d}u\right)\right)\mathrm{d}s\right\|^2\\ &\quad+2\left\|\int_0^t\left(g\left(s,x(s),x(ps),\int_0^s k(s,u,x(u),x(pu))\mathrm{d}u\right)\right.\right.\\ &\quad\left.\left.-\ g\left(s,\bar{x}(s),\bar{x}(ps),\int_0^s k(s,u,\bar{x}(u),\bar{x}(pu))\mathrm{d}u\right)\right)\mathrm{d}W(s)\right\|^2.\end{aligned}$$

Using (2), (4) and Doob's martingale inequality, we get

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x(\vartheta)-\bar{x}(\vartheta)\|^2\right\} &\le 6L_1^2T\int_0^t\mathbb{E}\sup_{0\le\vartheta\le s}\Big(\|x(\vartheta)-\bar{x}(\vartheta)\|^2+\|x(p\vartheta)-\bar{x}(p\vartheta)\|^2\\ &\quad+2\delta^2T^2\mathbb{E}\sup_{0\le\eta\le\vartheta}\left(\left\|x(\eta)-\bar{x}(\eta)\right\|^2+\left\|x(p\eta)-\bar{x}(p\eta)\right\|^2\right)\Big)\mathrm{d}s\\ &\quad+24L_1^2T\int_0^t\mathbb{E}\sup_{0\le\vartheta\le s}\Big(\|x(\vartheta)-\bar{x}(\vartheta)\|^2+\|x(p\vartheta)-\bar{x}(p\vartheta)\|^2\\ &\quad+2\delta^2T^2\mathbb{E}\sup_{0\le\eta\le\vartheta}\left(\left\|x(\eta)-\bar{x}(\eta)\right\|^2+\left\|x(p\eta)-\bar{x}(p\eta)\right\|^2\right)\Big)\mathrm{d}s\\ &\le 6L_1^2(T+4)\int_0^t(2+4T^2\delta^2)\mathbb{E}\sup_{0\le\vartheta\le s}\|x(\vartheta)-\bar{x}(\vartheta)\|^2\mathrm{d}s.\end{aligned}$$

By applying Gronwall's inequality, we get

$$\mathbb{E}\left\{\sup_{0\le\vartheta\le t}\|x(\vartheta)-\bar{x}(\vartheta)\|^2\right\}=0.$$

Thus $x(t)=\bar{x}(t)$ for $0\le t\le T$ almost surely. Hence there exist a unique solution on the space $C([0,T];\mathbb{R}^d)$. □

4. Large Deviation Principle

Let us now formulate the large deviation estimate which was first made known by Freidlin and Wentzell in their work [11]. Consider the stochastic pantograph integrodifferential equation with some perturbation parameter ϵ in the form,

$$\begin{cases} \mathrm{d}x^\epsilon(t) = f\left(t, x^\epsilon(t), x^\epsilon(pt), \int_0^t k(t,s,x^\epsilon(s),x^\epsilon(ps))\mathrm{d}s\right)\mathrm{d}t \\ \qquad\qquad + \sqrt{\epsilon} g\left(t, x^\epsilon(t), x^\epsilon(pt), \int_0^t k(t,s,x^\epsilon(s),x^\epsilon(ps))\mathrm{d}s\right)\mathrm{d}W(t), \quad t \in [0,T], \\ x^\epsilon(0) = x_0. \end{cases} \tag{21}$$

Since $x^\epsilon(\cdot)$ is a strong solution of (21), the Yamada-Watanabe theorem [32] states that there exists a Borel measurable map $\mathscr{G}^\epsilon : C([0,T];\mathbb{R}^r) \to C([0,T];\mathbb{R}^d)$ such that $\mathscr{G}^\epsilon(W(\cdot)) = x^\epsilon(\cdot)$ taking values in a Polish space. Let the space of all controls be

$$\mathcal{A} = \left\{ v : v \text{ is } \mathbb{R}^r\text{-valued } \mathscr{F}_t\text{-predictable process and } \int_0^T \|v(s,\omega)\|^2 \mathrm{d}s < \infty \text{ a.s.} \right\}.$$

The collection of bounded deterministic controls is

$$\mathbb{S}_N = \left\{ \psi \in L^2([0,T] : \mathbb{R}^r) : \int_0^T \|\psi(s)\|^2 \mathrm{d}s \leq N \right\}, \tag{22}$$

for $N \in \mathbb{N}$ and where $L^2([0,T];\mathbb{R}^r)$ is regarded as the space of all $\mathbb{R}^r$ valued square integrable functions on $[0,T]$. Moreover, $\mathbb{S}_N$ is a compact Polish space under the weak topology. Define

$$\mathcal{A}_N = \{ v \in \mathcal{A} : v(\omega) \in \mathbb{S}_N, \ \mathcal{P}\text{-a.s.} \}.$$

Next we frame the sufficient conditions needed to establish the Laplace principle, since the Laplace principle is equivalent to the large deviation principle in a Polish space. The sufficient conditions, labelled as follows:
(A) Let $\mathscr{G}^0 : C([0,T];\mathbb{R}^r) \to C([0,T];\mathbb{R}^d)$ be a measurable map such that the following two postulates are true:

(i) For some $N < \infty$, define the family $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_N$ such that $v^\epsilon \to v$ (as $\mathbb{S}_N$-valued random elements) in distribution. Then the solution $\mathscr{G}^\epsilon\left(W(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^\cdot v^\epsilon(s)\mathrm{d}s\right)$ converges to the solution $\mathscr{G}^0\left(\int_0^\cdot v(s)\mathrm{d}s\right)$ in distribution as $\epsilon \to 0$.

(ii) For each $N < \infty$, the set

$$\mathbb{K}_N = \left\{ \mathscr{G}^0\left(\int_0^\cdot \psi(s)\mathrm{d}s\right) : \psi \in \mathbb{S}_N \right\},$$

is a compact subset of $C([0,T];\mathbb{R}^d)$.

Then the family of solutions $\{x^\epsilon : \epsilon > 0\} = \mathscr{G}^\epsilon(W(\cdot))$ satisfies the Laplace principle in $C([0,T];\mathbb{R}^d)$ with the rate function $\mathcal{I}$ defined by

$$\mathcal{I}(h) = \inf_{\left\{v \in L^2([0,T]:\mathbb{R}^r); h = \mathscr{G}^0\left(\int_0^\cdot v(s)\mathrm{d}s\right)\right\}} \left\{ \frac{1}{2}\int_0^T \|v(s)\|^2 \mathrm{d}s \right\}, \tag{23}$$

for each $h \in C([0,T];\mathbb{R}^d)$.
To proceed with the main result, the following controlled equation correlated with the system (1) is developed.

$$\begin{cases} \mathrm{d}z^\psi(t) = f\left(t, z^\psi(t), z^\psi(pt), \int_0^t k(t,s,z^\psi(s),z^\psi(ps))\mathrm{d}s\right)\mathrm{d}t \\ \qquad\qquad + g\left(t, z^\psi(t), z^\psi(pt), \int_0^t k(t,s,z^\psi(s),z^\psi(ps))\mathrm{d}s\right)\psi(t)\mathrm{d}t, \quad t \in [0,T], \\ z^\psi(0) = x_0, \end{cases} \tag{24}$$

where $z^\psi(t)$ denotes the solution.

Theorem 4.1. *Assume (2)-(4) hold. The family $\{x^\epsilon(t)\}$ which is the solution to (21) satisfies the LDP on $C([0,T];\mathbb{R}^d)$ with good rate function*

$$\mathcal{I}(f) = \inf\left\{\frac{1}{2}\int_0^T \|\psi(t)\|^2 \mathrm{d}t; z^\psi = f\right\}$$

where $\psi \in L^2([0,T];\mathbb{R}^r)$, otherwise $\mathcal{I}(f) = \infty$.

This work's primary result is the theorem stated above. Proving this theorem to illustrate the LDP is the same as establishing the earlier-mentioned condition (A). Next, we formulate the following controlled stochastic equation with perturbation in order to verify the condition (A).

$$\begin{cases} \mathrm{d}x^\epsilon_{v^\epsilon}(t) = f\left(t, x^\epsilon_{v^\epsilon}(t), x^\epsilon_{v^\epsilon}(pt), \int_0^t k(t,s,x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps))\mathrm{d}s\right)\mathrm{d}t \\ \qquad +g\left(t, x^\epsilon_{v^\epsilon}(t), x^\epsilon_{v^\epsilon}(pt), \int_0^t k(t,s,x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps))\mathrm{d}s\right)v^\epsilon(t)\mathrm{d}t \\ \qquad +\sqrt{\epsilon}g\left(t, x^\epsilon_{v^\epsilon}(t), x^\epsilon_{v^\epsilon}(pt), \int_0^t k(t,s,x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps))\mathrm{d}s\right)\mathrm{d}W(t), \quad t \in [0,T], \\ x^\epsilon_{v^\epsilon}(0) = x_0. \end{cases} \tag{25}$$

Then there exists a unique solution represented by

$$\begin{aligned} x^\epsilon_{v^\epsilon}(t) &= x_0 + \int_0^t f\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u), x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\mathrm{d}s \\ &\quad + \int_0^t g\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u), x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)v^\epsilon(s)\mathrm{d}s \\ &\quad + \sqrt{\epsilon}\int_0^t g\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u), x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\mathrm{d}W(s). \end{aligned} \tag{26}$$

The following lemma states the boundedness of solution and it is required to show the desired result.

Lemma 4.2. *Let assumptions (2)-(4) hold. Then the solution $x^\epsilon_{v^\epsilon}(t)$ of the system (25) on the solution space $C([0,T];\mathbb{R}^d)$ satisfies the following estimate:*

$$\mathbb{E}\left[\sup_{0\le t\le T}\|x^\epsilon_{v^\epsilon}(t)\|^2\right] \le (4\|x_0\|^2 + 16L_2^2(T+4\epsilon))e^{(16L_2^2(T+4\epsilon)(1+N)T)}. \tag{27}$$

We omit the proof here, since the estimates for the solution can be obtained as done earlier by applying the assumptions made on the drift and noise coefficients.

4.1. Compactness

First we prove (ii) of condition (A).

Lemma 4.3. *Define $\mathscr{G}^0 : C([0,T];\mathbb{R}^r) \to C([0,T];\mathbb{R}^d)$ by*

$$\mathscr{G}^0(\chi) = \begin{cases} z^\psi, \ \text{if } \chi = \int_0^\cdot \psi(s)\mathrm{d}s \ \text{for some } \psi \in L^2([0,T];\mathbb{R}^r), \\ 0, \ \text{otherwise}. \end{cases} \tag{28}$$

Then for each $N < \infty$, the set

$$\mathbb{K}_N = \left\{\mathscr{G}^0\left(\int_0^\cdot \psi(s)\mathrm{d}s\right) : \psi \in \mathbb{S}_N\right\},$$

is a compact subset of $C([0,T];\mathbb{R}^d)$.

Proof. The solution of (24) is represented as

$$\begin{aligned} z^{\psi}(t) &= x_0 + \int_0^t f\left(s, z^{\psi}(s), z^{\psi}(ps), \int_0^s k(s,u,z^{\psi}(u), z^{\psi}(pu))\mathrm{d}u\right)\mathrm{d}s \\ &\quad + \int_0^t g\left(s, z^{\psi}(s), z^{\psi}(ps), \int_0^s k(s,u,z^{\psi}(u), z^{\psi}(pu))\mathrm{d}u\right)\psi(s)\mathrm{d}s. \end{aligned} \tag{29}$$

Let us assume ψ^n converges weakly to ψ as $n \to \infty$ in $\mathbb{S}_N$. We consider the following to obtain the continuity,

$$\begin{aligned} z^{\psi^n}(t) - z^{\psi}(t) &= \int_0^t \left\{ f\left(s, z^{\psi^n}(s), z^{\psi^n}(ps), \int_0^s k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu))\mathrm{d}u\right)\right. \\ &\qquad \left. - f\left(s, z^{\psi}(s), z^{\psi}(ps), \int_0^s k(s,u,z^{\psi}(u), z^{\psi}(pu))\mathrm{d}u\right)\right\}\mathrm{d}s \\ &\quad + \int_0^t g\left(s, z^{\psi^n}(s), z^{\psi^n}(ps), \int_0^s k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu))\mathrm{d}u\right)[\psi^n(s) - \psi(s)]\mathrm{d}s \\ &\quad + \int_0^t \left\{ g\left(s, z^{\psi^n}(s), z^{\psi^n}(ps), \int_0^s k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu))\mathrm{d}u\right)\right. \\ &\qquad \left. - g\left(s, z^{\psi}(s), z^{\psi}(ps), \int_0^s k(s,u,z^{\psi}(u), z^{\psi}(pu))\mathrm{d}u\right)\right\}\psi(s)\mathrm{d}s. \end{aligned} \tag{30}$$

Take

$$\zeta^n(t) = \int_0^t g\left(s, z^{\psi^n}(s), z^{\psi^n}(ps), \int_0^s k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu))\mathrm{d}u\right)[\psi^n(s) - \psi(s)]\mathrm{d}s. \tag{31}$$

By Holder's inequality and linear growth condition, we have

$$\begin{aligned} \|\zeta^n(t)\| &= \left\| \int_0^t g\left(s, z^{\psi^n}(s), z^{\psi^n}(ps), \int_0^s k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu))\mathrm{d}u\right)[\psi^n(s) - \psi(s)]\mathrm{d}s \right\| \\ &\leq \left(\int_0^t \left\| g\left(s, z^{\psi^n}(s), z^{\psi^n}(ps), \int_0^s k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu))\mathrm{d}u\right)\right\|^2 \mathrm{d}s\right)^{1/2} \left(\int_0^t \|\psi^n(s) - \psi(s)\|^2 \mathrm{d}s\right)^{1/2} \\ &\leq \text{Constant} < \infty. \end{aligned} \tag{32}$$

Since $\psi^n \to \psi$ weakly in $L^2([0,T] : \mathbb{R}^r)$, Arzela-Ascoli theorem implies that $\{\zeta^n(t)\}$ converges to zero for each t, thus we have

$$\lim_{n\to\infty} \|\zeta^n(t)\| = 0. \tag{33}$$

By applying the Lipschitz condition (2) on (30),

$$\begin{aligned} \|z^{\psi^n}(t) - z^{\psi}(t)\| &\leq \|\zeta^n(t)\| + L_1 \int_0^t \left(\|z^{\psi^n}(s) - z^{\psi}(s)\| + \|z^{\psi^n}(ps) - z^{\psi}(ps)\| \right. \\ &\qquad \left. + \left\| \int_0^s (k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu)) - k(s,u,z^{\psi}(u), z^{\psi}(pu)))\mathrm{d}u \right\| \right)\mathrm{d}s \\ &\quad + L_1 \int_0^t \left(\|z^{\psi^n}(s) - z^{\psi}(s)\| + \|z^{\psi^n}(ps) - z^{\psi}(ps)\| \right. \\ &\qquad \left. + \left\| \int_0^s (k(s,u,z^{\psi^n}(u), z^{\psi^n}(pu)) - k(s,u,z^{\psi}(u), z^{\psi}(pu)))\mathrm{d}u \right\| \right)\|\psi(s)\|\mathrm{d}s. \end{aligned}$$

Using (4), we get

$$\begin{aligned}\|z^{\psi^n}(t)-z^{\psi}(t)\| \quad \leq \quad & \|\zeta^n(t)\| + L_1\int_0^t \Big(\|z^{\psi^n}(s)-z^{\psi}(s)\| + \|z^{\psi^n}(ps)-z^{\psi}(ps)\| \\ & + \sup_{0\leq\eta\leq s}\delta T(\|z^{\psi^n}(\eta)-z^{\psi}(\eta)\| + \|z^{\psi^n}(p\eta)-z^{\psi}(p\eta)\|)\Big)\mathrm{d}s \\ & + L_1\int_0^t \Big(\|z^{\psi^n}(s)-z^{\psi}(s)\| + \|z^{\psi^n}(ps)-z^{\psi}(ps)\| \\ & + \sup_{0\leq\eta\leq s}\delta T(\|z^{\psi^n}(\eta)-z^{\psi}(\eta)\| + \|z^{\psi^n}(p\eta)-z^{\psi}(p\eta)\|)\Big)\|\psi(s)\|\mathrm{d}s.\end{aligned}$$

Taking supremum on both sides, we obtain

$$\begin{aligned}\sup_{0\leq\vartheta\leq t}\|z^{\psi^n}(\vartheta)-z^{\psi}(\vartheta)\| \leq & \sup_{0\leq\vartheta\leq t}\|\zeta^n(\vartheta)\| + L_1\int_0^t \sup_{0\leq\vartheta\leq s}\Big(\|z^{\psi^n}(\vartheta)-z^{\psi}(\vartheta)\| + \|z^{\psi^n}(p\vartheta)-z^{\psi}(p\vartheta)\| \\ & + \sup_{0\leq\eta\leq\vartheta}\delta T(\|z^{\psi^n}(\eta)-z^{\psi}(\eta)\| + \|z^{\psi^n}(p\eta)-z^{\psi}(p\eta)\|)\Big)(1+\|\psi(s)\|)\mathrm{d}s \\ \leq & \sup_{0\leq\vartheta\leq t}\|\zeta^n(\vartheta)\| + L_1\int_0^t (2+2\delta T)\sup_{0\leq\vartheta\leq s}\|z^{\psi^n}(\vartheta)-z^{\psi}(\vartheta)\|(1+\|\psi(s)\|)\mathrm{d}s.\end{aligned}$$

Finally by means of Gronwall's inequality, one sees that

$$\sup_{0\leq\vartheta\leq t}\|z^{\psi^n}(\vartheta)-z^{\psi}(\vartheta)\| \quad\leq\quad \left(\sup_{0\leq\vartheta\leq t}\|\zeta^n(\vartheta)\|\right)e^{(L_1(2+2\delta)(1+\sqrt{N})T)}. \tag{34}$$

Combining the above estimate with (33), compactness of K_N is obtained. □

4.2. Weak Convergence

Lemma 4.4. *For some $N<\infty$, define the family $\{v^\epsilon:\epsilon>0\}\subset\mathcal{A}_N$ such that $v^\epsilon\to v$ (as $\mathbb{S}_N$-valued random elements) in distribution. Then the solution $\mathcal{G}^\epsilon\left(W(\cdot)+\frac{1}{\sqrt{\epsilon}}\int_0^\cdot v^\epsilon(s)\mathrm{d}s\right)$ converges to the solution $\mathcal{G}^0\left(\int_0^\cdot v(s)\mathrm{d}s\right)$ in distribution as $\epsilon\to 0$.*

Proof. Consider $v^\epsilon\to v$ in distribution in $\mathbb{S}_N$. Now, we try to prove that the weak convergence of solutions, i.e., the solution of (25), $x^\epsilon_{v^\epsilon}$ converges to the solution of (24), z^v in distribution as $\epsilon\to 0$. Take $\kappa^\epsilon(t)=x^\epsilon_{v^\epsilon}(t)-z^v(t)$,

$$\begin{aligned}\kappa^\epsilon(t) \quad = \quad & \int_0^t \Big\{f\Big(s,x^\epsilon_{v^\epsilon}(s),x^\epsilon_{v^\epsilon}(ps),\int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\Big) \\ & - f\Big(s,z^v(s),z^v(ps),\int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\Big)\Big\}\,\mathrm{d}s \\ & + \int_0^t \Big\{g\Big(s,x^\epsilon_{v^\epsilon}(s),x^\epsilon_{v^\epsilon}(ps),\int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\Big) \\ & - g\Big(s,z^v(s),z^v(ps),\int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\Big)\Big\}\,v^\epsilon(s)\mathrm{d}s \\ & + \int_0^t g\Big(s,z^v(s),z^v(ps),\int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\Big)[v^\epsilon(s)-v(s)]\mathrm{d}s \\ & + \sqrt{\epsilon}\int_0^t g\Big(s,x^\epsilon_{v^\epsilon}(s),x^\epsilon_{v^\epsilon}(ps),\int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\Big)\mathrm{d}W(s).\end{aligned} \tag{35}$$

Put,

$$\zeta^\epsilon(t) = \int_0^t g\left(s, z^v(s), z^v(ps), \int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\right)[v^\epsilon(s) - v(s)]\mathrm{d}s. \tag{36}$$

Taking square norm on both sides and using (3), we get

$$\begin{aligned} \|\zeta^\epsilon(t)\|^2 &\le \left(\int_0^t \left\|g\left(s, z^v(s), z^v(ps), \int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\right)\right\|^2 \mathrm{d}s\right)\left(\int_0^t \|v^\epsilon(s) - v(s)\|^2\mathrm{d}s\right) \\ &< \infty. \end{aligned} \tag{37}$$

As a consequence, $\zeta^\epsilon(\cdot)$ converges to zero in distribution as $\epsilon \to 0$. Further, by means of (7) and Holder's inequality, (35) becomes,

$$\begin{aligned} \|\kappa^\epsilon(t)\|^2 &\le 4\|\zeta^\epsilon(t)\|^2 + 4T\int_0^t \left\|f\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\right. \\ &\quad \left. - f\left(s, z^v(s), z^v(ps), \int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\right)\right\|^2 \mathrm{d}s \\ &\quad +4T\int_0^t \left\|g\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\right. \\ &\quad \left. - g\left(s, z^v(s), z^v(ps), \int_0^s k(s,u,z^v(u),z^v(pu))\mathrm{d}u\right)\right\|^2 \|v^\epsilon(s)\|^2\mathrm{d}s \\ &\quad +4\epsilon\left\|\int_0^t g\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\mathrm{d}W(s)\right\|^2. \end{aligned}$$

Using (2), we get

$$\begin{aligned} \|\kappa^\epsilon(t)\|^2 \le{}& 4\|\zeta^\epsilon(t)\|^2 + 12TL_1^2\int_0^t \Big(\|x^\epsilon_{v^\epsilon}(s) - z^v(s)\|^2 + \|x^\epsilon_{v^\epsilon}(ps) - z^v(ps)\|^2 \\ &+ \left\|\int_0^s (k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu)) - k(s,u,z^v(u),z^v(pu)))\mathrm{d}u\right\|^2\Big)(1 + \|v^\epsilon(s)\|^2)\mathrm{d}s \\ &+4\epsilon\left\|\int_0^t g\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\mathrm{d}W(s)\right\|^2. \end{aligned} \tag{38}$$

We know that,

$$\begin{aligned} \left\|\int_0^t (k(t,s,x_1(s),x_2(s)) - k(t,s,y_1(s),y_2(s)))\mathrm{d}s\right\|^2 &\le T\int_0^t \left\|k(t,s,x_1(s),x_2(s)) - k(t,s,y_1(s),y_2(s))\right\|^2\mathrm{d}s \\ &\le 2T^2\delta^2 \sup_{0\le u\le t}\left(\|x_1(u) - y_1(u)\|^2 + \|x_2(u) - y_2(u)\|^2\right). \end{aligned}$$

Equation (38) becomes,

$$\begin{aligned} \|\kappa^\epsilon(t)\|^2 &\le 4\|\zeta^\epsilon(t)\|^2 + 12TL_1^2\int_0^t \Big(\|x^\epsilon_{v^\epsilon}(s) - z^v(s)\|^2 + \|x^\epsilon_{v^\epsilon}(ps) - z^v(ps)\|^2 \\ &\quad +2T^2\delta^2 \sup_{0\le u\le s}(\|x^\epsilon_{v^\epsilon}(u) - z^v(u)\|^2 + \|x^\epsilon_{v^\epsilon}(pu) - z^v(pu)\|^2)\Big)(1 + \|v^\epsilon(s)\|^2)\mathrm{d}s \\ &\quad +4\epsilon\left\|\int_0^t g\left(s, x^\epsilon_{v^\epsilon}(s), x^\epsilon_{v^\epsilon}(ps), \int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\mathrm{d}W(s)\right\|^2. \end{aligned} \tag{39}$$

The Doob's martingale inequality allows us to bound the stochastic integral on the right side of the above inequality, described in the following:

$$\begin{aligned}
\mathbb{E}\left\{\sup_{0\leq\vartheta\leq t}\|\kappa^\epsilon(\vartheta)\|^2\right\} &\leq 4\mathbb{E}\left\{\sup_{0\leq\vartheta\leq t}\|\zeta^\epsilon(\vartheta)\|^2\right\} + 12TL_1^2\int_0^t\Bigg(2\sup_{0\leq\vartheta\leq s}\|x^\epsilon_{v^\epsilon}(\vartheta)-z^v(\vartheta)\|^2 \\
&\quad + 4T^2\delta^2\sup_{0\leq\vartheta\leq s}\|x^\epsilon_{v^\epsilon}(\vartheta)-z^v(\vartheta)\|^2\Bigg)(1+\|v^\epsilon(s)\|^2)\mathrm{d}s \\
&\quad +4\epsilon\sup_{0\leq\vartheta\leq t}\int_0^\vartheta\left\|g\left(s,x^\epsilon_{v^\epsilon}(s),x^\epsilon_{v^\epsilon}(ps),\int_0^s k(s,u,x^\epsilon_{v^\epsilon}(u),x^\epsilon_{v^\epsilon}(pu))\mathrm{d}u\right)\right\|^2\mathrm{d}s \\
&\leq 4\mathbb{E}\left\{\sup_{0\leq\vartheta\leq t}\|\zeta^\epsilon(\vartheta)\|^2\right\} + 12TL_1^2\int_0^t(2+4T^2\delta^2)\sup_{0\leq\vartheta\leq s}\|\kappa^\epsilon(\vartheta)\|^2(1+\|v^\epsilon(s)\|^2)\mathrm{d}s \\
&\quad +16\epsilon L_2^2\int_0^t\sup_{0\leq\vartheta\leq s}4\Bigg(1+\|x^\epsilon_{v^\epsilon}(\vartheta)\|^2+\|x^\epsilon_{v^\epsilon}(p\vartheta)\|^2+2T^2\delta^2\sup_{0\leq\eta\leq\vartheta}\left(\|x^\epsilon_{v^\epsilon}(\eta)\|^2+\|x^\epsilon_{v^\epsilon}(p\eta)\|^2\right)\Bigg)\mathrm{d}s.
\end{aligned}$$

Then, Gronwall's inequality yields,

$$\begin{aligned}
\mathbb{E}\left\{\sup_{0\leq\vartheta\leq t}\|\kappa^\epsilon(\vartheta)\|^2\right\} &\leq \left\{4\mathbb{E}\left(\sup_{0\leq\vartheta\leq t}\|\zeta^\epsilon(\vartheta)\|^2\right)+16\epsilon L_2^2\int_0^t\left(1+(2+4T^2\delta^2)\sup_{0\leq\vartheta\leq s}\|x^\epsilon_{v^\epsilon}(\vartheta)\|^2\right)\mathrm{d}s\right\} \\
&\quad \times\exp\left(12TL_1^2(2+4T^2\delta^2)\int_0^t(1+\|v^\epsilon(s)\|^2)\mathrm{d}s\right). \qquad (40)
\end{aligned}$$

By the virtue of (37) and the Lemma 4.2, we conclude that the solution $x^\epsilon_{v^\epsilon}$ weakly converges to the solution z^v in distribution as $\epsilon \to 0$ and hence the lemma is proved. □

5. Example

Example 5.1. Consider the following stochastic pantograph integrodifferential equation with multiplicative noise,

$$\begin{cases}
\mathrm{d}x^\epsilon(t) = f\left(t,x^\epsilon(t),x^\epsilon(pt),\int_0^t k(t,s,x^\epsilon(s),x^\epsilon(ps))\mathrm{d}s\right)\mathrm{d}t \\
\qquad\qquad + \sqrt{\epsilon}g\left(t,x^\epsilon(t),x^\epsilon(pt),\int_0^t k(t,s,x^\epsilon(s),x^\epsilon(ps))\mathrm{d}s\right)\mathrm{d}W(t), \quad t\in[0,6], \\
x^\epsilon(0)=x_0,
\end{cases} \qquad (41)$$

where

$$\begin{aligned}
f &= -\begin{bmatrix}1&0\\0&1\end{bmatrix}(1+308x^\epsilon(t))+\begin{bmatrix}2&0\\0&0\end{bmatrix}x^\epsilon(pt)+\begin{bmatrix}1&0\\0&1\end{bmatrix}\int_0^{pt}x^\epsilon(s)\mathrm{d}s, \\
g &= \begin{bmatrix}0.8&0\\0&1\end{bmatrix}x^\epsilon(pt)+\begin{bmatrix}1&0\\0&1\end{bmatrix}\int_0^{pt}x^\epsilon(s)\mathrm{d}s, \\
x_0 &= \begin{bmatrix}0\\2\end{bmatrix}.
\end{aligned}$$

Select $p = 1/2$. The following figures demonstrate that as the perturbation parameter $\epsilon \to 0$, the state variable exhibit a smoother graph. Let us consider the control $v \in L^2([0,6];\mathbb{R})$ and the corresponding

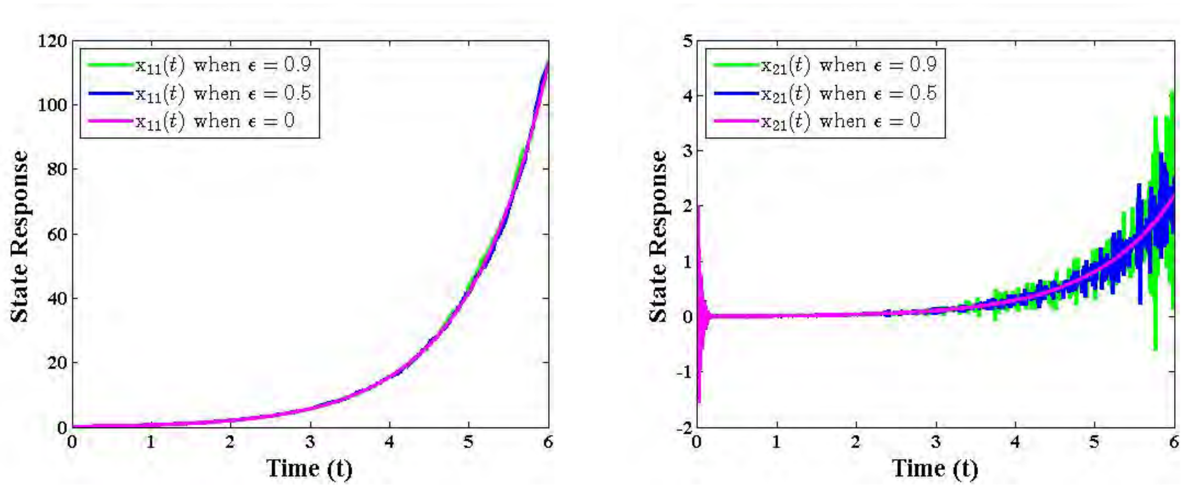

Figure 1: State trajectories for a different perturbation

controlled equation is

$$\begin{cases} dz^v(t) = \left(-\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}(1+308z^v(t)) + \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}z^v(pt) + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\int_0^{pt} z^v(s)ds\right)dt \\ \qquad\qquad + \left(\begin{bmatrix}0.8 & 0\\0 & 1\end{bmatrix}z^v(pt) + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\int_0^{pt} z^v(s)ds\right)v(t)dt, \quad t\in[0,6], \\ z^v(0) = \begin{bmatrix}0\\2\end{bmatrix}, \end{cases} \tag{42}$$

where z^v is the unique solution of the above system. Then the rate function $\mathcal{I} : C([0,6];\mathbb{R}^2) \to [0,\infty]$ is defined as

$$\mathcal{I}(\varphi) = \inf\left\{\frac{1}{2}\int_0^6 |v(t)|^2 dt\right\},$$

where the infimum is taken over $v \in L^2([0,6];\mathbb{R})$ such that $z^v = \varphi$, $\mathcal{I}(\varphi) = \infty$, otherwise.

Example 5.2. *Consider the stochastic pantograph integrodifferential equation*

$$\begin{cases} dx^\epsilon(t) = \left(\sin(x^\epsilon(t)+x^\epsilon(pt)) + \int_0^t \cos(x^\epsilon(s)+x^\epsilon(ps))ds\right)dt + \sqrt{\epsilon}(1+t^2)dW(t), \quad t\in[0,T], \\ x^\epsilon(0) = 0.2, \end{cases} \tag{43}$$

where $p = 1/2$ and $W(t)$ denotes one-dimensional Brownian motion.

Let us consider the control to be $v \in L^2([0,T];\mathbb{R})$ and the controlled system corresponding to (43) is represented as,

$$\begin{cases} dz^v(t) = \left(\sin(z^v(t)+z^v(pt)) + \int_0^t \cos(z^v(s)+z^v(ps))ds\right)dt + (1+t^2)v(t)dt, \quad t\in[0,T], \\ z^v(0) = 0.2. \end{cases} \tag{44}$$

The coefficients of the system (43) satisfy the assumptions made earlier and so the LDP holds with the rate function $\mathcal{I} : C([0,T];\mathbb{R}) \to [0,+\infty]$ defined by

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \left|\left(\frac{d\varphi(s)}{ds} - (\sin(\varphi(s)+\varphi(ps)) + \int_0^s \cos(\varphi(u)+\varphi(pu))du\right)\frac{1}{1+t^2}\right|^2 ds, \text{ if } \varphi\in L^2([0,T];\mathbb{R}), \\ \infty, \text{ otherwise.} \end{cases} \tag{45}$$

where $z^v(t)$ is represented as

$$z^v(t) = 0.2 + \int_0^t \left(\sin(z^v(s) + z^v(ps)) + \int_0^s \cos(z^v(u) + z^v(pu))\mathrm{d}u\right)\mathrm{d}s + \int_0^t (1 + s^2)v(s)\mathrm{d}s,$$

which is the unique solution of (44).

6. Declarations

6.1. Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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6.3. Competing Interests

The authors declare that they have no competing interests.

6.4. Author Contributions

All the authors have contributed equally to this paper. All authors read and approved the final manuscript.

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