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# On $B$-statistical uniform nonintegrability of a sequence of random variables

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**Abstract.** In this paper, we introduce a notion of $B$-statistical uniform nonintegrability with respect to $A$ (B-UNI(c) w.r.t. $A$, in short), which is weaker than UNI(c) w.r.t. $A$ (see [3]). We establish the de La Vallée Poussin criterion for B-UNI(c) w.r.t. $A$, which extends Theorem 2.1 of Chandra et al. (2021) [3]. Moreover, we also give a necessary condition for the sequence of random variables $\{X_k, k \in \mathbb{N}\}$ to be B-UNI(c) w.r.t. $A$.

## 1. Introduction

Chandra, Hu and Rosalsky (2016) [2] introduced the concept of uniform nonintegrability of a sequence of random variables and gave the sufficient and necessary conditions of uniform nonintegrability respectively. And they also gave two equivalent characterizations of uniform nonintegrability.

In this paper all random variables under consideration are defined on the probability space $(\Omega, \mathcal{F}, P)$. We shall denote $\mathbb{E}(X\mathbf{1}_A)$ by $\mathbb{E}(X : A)$. $a$ is a real number with $a > 0$, and $\mathbf{1}_A$ is the indicator function of set $A$.

**Definition 1.1.** (Chandra et al. (2016) [2]) A sequence of random variables $\{X_k, k \in \mathbb{N}\}$ is said to be uniformly nonintegrable (UNI) if

$$\lim_{a\to\infty} \inf_{k\in\mathbb{N}} \mathbb{E}(|X_k| : |X_k| \le a) = \infty.$$

The following equivalent characterizations of UNI is similar to the La Vallée Poussin criterion for uniform integrability (see [5]).

**Theorem 1.2.** (Theorem 3.3 of Chandra et al. (2016) [2]) A sequence of random variables $\{X_k, k \in \mathbb{N}\}$ is UNI if and only if there exists a continuous function $Q : [0, \infty) \to [0, \infty)$ such that $Q(0) = 0$, $Q$ is strictly increasing, $Q(x) \to \infty$ as $x \to \infty$, $x^{-1}Q$ is strictly decreasing to 0 as $0 < x \uparrow \infty$, and $\{Q(|X_k|), k \in \mathbb{N}\}$ is UNI.

Later on, Hu and Peng (2018) [4] introduced the following new concept of weakly uniform nonintegrability (W-UNI), which is weaker than UNI.

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**Definition 1.3.** A sequence of random variables $\{X_k, k \in \mathbb{N}\}$ is said to be W-uniformly nonintegrable (W-UNI), where "W" means "weak" , if

$$\lim_{a\to\infty} \inf_{k\in\mathbb{N}} \mathbb{E}(|X_k| \wedge a) = \infty.$$

Based on the work of Hu and Peng, Chandra et al. (2021) [3] defined a general notion of UNI(c) w.r.t. $A$ of the sequence of random variables $\{X_k, k \in \mathbb{N}\}$, (where $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ is an array of nonnegative real numbers with $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} < \infty$) and also pointed out that the concept of UNI and the concept of W-UNI can be obtained immediately by taking for $A$ of

$$a_{nk} = 1 \ if \ n = k \ and \ a_{nk} = 0 \ if \ n \neq k, n \in \mathbb{N}, k \in \mathbb{N}.$$

**Definition 1.4.** Let $c \in \{0, 1\}$ and $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ be an array of nonnegative real numbers with

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} < \infty.$$

We say that the sequence of random variables $\{X_k, k \in \mathbb{N}\}$ is UNI(c) w.r.t. $A$ if

$$\lim_{a\to\infty} \inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}[c\mathbb{E}(|X_k| \wedge a) + (1 - c)\mathbb{E}(|X_k| : |X_k| \leq a)] = \infty.$$

According to the Remarks 1.1 of Chandra et al. (2021) [3], we can obtain the following facts.

If $\{X_k, k \in \mathbb{N}\}$ is UNI(0) w.r.t. $A$, then we can say that $\{X_k, k \in \mathbb{N}\}$ is UNI w.r.t. $A$ and if $A$ is the identity matrix, the condition that $\{X_k, k \in \mathbb{N}\}$ is UNI w.r.t. $A$ is obviously the condition that $\{X_k, k \in \mathbb{N}\}$ is UNI. Similarly, if $\{X_k, k \in \mathbb{N}\}$ is UNI(1) w.r.t. $A$, we say that $\{X_k, k \in \mathbb{N}\}$ is W-UNI w.r.t. $A$. Moreover, if we take the identity matrix as $A$, then W-UNI w.r.t. $A$ reduces to W-UNI.

Before giving the following definition, we first introduce the concepts of $B$-statistical supremum and $B$-statistical infimum of a sequence $\{x_k\}$.

Let $K$ be a subset of the set of positive integers $\mathbb{N}$ and $B = \{b_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ be a nonnegative regular summability matrix i.e. $b_{nk} \geq 0$ and $\sum_{k=1}^{\infty} b_{nk} = 1$. Set

$$\delta_B(K) := \lim_{n\to\infty} \sum_{k\in K} b_{nk}.$$

If the limit exists, we said it to be $B$-density of $K$.

Let $m$ be a real number and $\{x_k, k \in \mathbb{N}\}$ be a sequence. If

$$\delta_B(\{k \in \mathbb{N} : x_k > m\}) = 0 \ \ (or \ \delta_B(\{k \in \mathbb{N} : x_k \leq m\}) = 1),$$

the number $m$ is said to be a $B$-statistical upper bound of $\{x_k\}$. The infimun of the set of all $B$-statistical upper bounds is said to be the $B$-statistical supremum of $\{x_k\}$, where $\{x_k\}$ is a $B$-statistical upper bounded sequence, and it is denoted by $\sup_{st_B k\in\mathbb{N}} x_k$.

The definition of concept $B$-statistical infimum is similar to $B$-statistical supremum. Let $l$ be a real number. If

$$\delta_B(\{k \in \mathbb{N} : x_k < l\}) = 0 \ \ (or \ \delta_B(\{k \in \mathbb{N} : x_k \geq l\}) = 1),$$

then $l$ is said to be a $B$-statistical lower bound of $\{x_k\}$ and the supremun of the set of all $B$-statistical lower bounds of a $B$-statistical lower bounded sequence is said to be the $B$-statistical infimum of $\{x_k\}$. It is denoted by $\inf_{st_B k\in\mathbb{N}} x_k$.

From Altınok and Küçükaslan. (2014) [1], we know that

$$\inf_{k\in\mathbb{N}} x_k \leq \inf_{\substack{st_B \\ k\in\mathbb{N}}} x_k \leq \sup_{\substack{st_B \\ k\in\mathbb{N}}} x_k \leq \sup_{k\in\mathbb{N}} x_k.$$

The main purpose of our work is to introduce the notion of $B$-statistical uniform nonintegrability w.r.t. $A$, (B-UNI(c) w.r.t. $A$) (where $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ is an array of nonnegative real numbers with $\sup_{st_B n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} < \infty$). The notion of UNI(c) w.r.t. $A$ can be obtained immediately by taking for $B$ the identity matrix. In this paper, we established the de La Vallée Poussin criterion for B-UNI(c) w.r.t. $A$, which extends Theorem 2.1 of Chandra et al. (2021) [3]. Moreover, we also give a necessary condition for the sequence of random variables $\{X_k, k \in \mathbb{N}\}$ to be B-UNI(c) w.r.t. $A$.

**Definition 1.5.** Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of random variables and $c \in \{0, 1\}$. $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ is an array of nonnegative real numbers with

$$\sup_{\substack{st_B\\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk} < \infty.$$

We say that $\{X_k, k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$ if

$$\lim_{a\to\infty} \alpha_{c,A}(a) = \infty,$$

where

$$\alpha_{c,A}(a) = \inf_{\substack{st_B\\ k\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}[c\mathbb{E}(|X_k| \wedge a) + (1 - c)\mathbb{E}(|X_k| : |X_k| \leq a)], a > 0.$$

**Remark 1.6.** (I) If the sequence of random variables $\{X_k, k \in \mathbb{N}\}$ is B-UNI(0) w.r.t. $A$, then we say that $\{X_k, k \in \mathbb{N}\}$ is B-UNI w.r.t. $A$. If $\{X_k, k \in \mathbb{N}\}$ is B-UNI(1) w.r.t. $A$, we say that $\{X_k, k \in \mathbb{N}\}$ is B-WUNI w.r.t. $A$.

(II) If the nonnegative regular summability matrix $B$ is the identity matrix, the condition that $\{X_k, k \in \mathbb{N}\}$ is B-UNI w.r.t. $A$ (resp., B-WUNI w.r.t. $A$) is the condition that $\{X_k, k \in \mathbb{N}\}$ is UNI w.r.t. $A$ (resp., W-UNI w.r.t. $A$). Furthermore, if both $B$ and the array of nonnegative real numbers $A$ are identity matrix, the condition that $\{X_k, k \in \mathbb{N}\}$ is B-UNI w.r.t. $A$ (resp., B-WUNI w.r.t. $A$) is the condition that $\{X_k, k \in \mathbb{N}\}$ is UNI (resp., W-UNI).

**Lemma 1.7.** By Definition 1.5, it is clear that $\alpha_{c,A}(\cdot)$ is a nondecreasing function on $(0, \infty)$ for $c \in \{0, 1\}$, thus $\lim_{a\to\infty} \alpha_{c,A}(a)$ always exists. Then for a sequence of random variables $\{X_k, k \in \mathbb{N}\}$ and $c \in \{0, 1\}$, the following three statements are equivalent.

(1) The sequence of random variables $\{X_k, k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$.

(2) There exists a function $\Psi : (0, \infty) \to (0, \infty)$ with $0 < \Psi(a) \to \infty$ as $a \to \infty$ such that $\lim_{a\to\infty} \alpha_{c,A}(\Psi(a)) = \infty$.

(3) There exists a sequence $\{\Psi_k, k \in \mathbb{N}\}$ on $(0, \infty)$ with $\lim_{k\to\infty} \Psi_k = \infty$ such that $\lim_{k\to\infty} \alpha_{c,A}(\Psi_k) = \infty$.

**Lemma 1.8.** (see Example 2.2 of Hu and Peng. (2018) [4]) Let $X$ be a random variable and $a > 0$. Then

$$\mathbb{E}(|X| \wedge a) = \mathbb{E}(|X| : |X| \leq a) + aP(|X| > a).$$

**Remark 1.9.** For a sequence of random variables $\{X_k, k \in \mathbb{N}\}$, it is obvious from Lemma 1.8 that when $c = 1$ whereas when $c = 0$,

$$\alpha_{c,A}(a) = \inf_{\substack{st_B\\ k\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a) + caP(|X_k| > a)], a > 0.$$

**Lemma 1.10.** Let $\{x_k, k \in \mathbb{N}\}$ and $\{y_k, k \in \mathbb{N}\}$ be two $B$-statistical bounded sequences of real numbers. Then

(1) $\inf_{st_B k\in\mathbb{N}}\{-x_k\} = -\sup_{st_B k\in\mathbb{N}}\{x_k\}$.

(2) $\inf_{st_B k\in\mathbb{N}}\{x_k + y_k\} \geq \inf_{st_B k\in\mathbb{N}}\{x_k\} + \inf_{st_B k\in\mathbb{N}}\{y_k\}$.

(3) $\inf_{st_B k\in\mathbb{N}}\{x_k + y_k\} \leq \sup_{st_B k\in\mathbb{N}}\{x_k\} + \inf_{st_B k\in\mathbb{N}}\{y_k\}$.

*Proof.* (1) Let $\sup_{st_B k\in\mathbb{N}}\{x_k\} = m$. By the definition of $B$-statistical super bound, we have $\delta_B(\{k \in \mathbb{N} : x_k > m\}) = 0$, hence

$$\delta_B(\{k \in \mathbb{N} : -x_k < -m\}) = \delta_B(\{k \in \mathbb{N} : x_k > m\}) = 0.$$

It is obvious that $-m$ is a $B$-statistical lower bound of $\{-x_k\}$. Since $m$ is the $B$-statistical supremum of $\{x_k\}$, for any $\varepsilon > 0$, $m - \varepsilon$ is not a $B$-statistical super bound of $\{x_k\}$, then we have

$$\delta_B(\{k \in \mathbb{N} : -x_k < -m + \varepsilon\}) = \delta_B(\{k \in \mathbb{N} : x_k > m - \varepsilon\}) > 0.$$

Similarly, for any $\varepsilon > 0$, $-m + \varepsilon$ is not a $B$-statistical lower bound of $-x_k$, so we know $-m$ is the $B$-statistical infimum of $\{-x_k\}$, i.e. $\inf_{st_B k\in\mathbb{N}}\{-x_k\} = -m = -\sup_{st_B k\in\mathbb{N}}\{x_k\}$.

(2) Assume that $\inf_{st_B k\in\mathbb{N}}\{x_k\} = l$ and $\inf_{st_B k\in\mathbb{N}}\{y_k\} = p$, we have

$$\delta_B(\{k \in \mathbb{N} : x_k < l\}) = 0$$

and

$$\delta_B(\{k \in \mathbb{N} : y_k < p\}) = 0.$$

Note that

$$\begin{aligned}&\{k \in \mathbb{N} : x_k + y_k < l + p\}\\ \subset&\{k \in \mathbb{N} : x_k < l\} \cup \{k \in \mathbb{N} : y_k < p\},\end{aligned}$$

then

$$\begin{aligned}&\delta_B(\{k \in \mathbb{N} : x_k + y_k < l + p\})\\ \leq&\delta_B(\{k \in \mathbb{N} : x_k < l\}) + \delta_B(\{k \in \mathbb{N} : y_k < p\})\\ =&0 + 0 = 0.\end{aligned}$$

Hence, $l + p$ is a $B$-statistical lower bound of $\{x_k + y_k\}$, i.e. $l + p \leq \inf_{st_B k\in\mathbb{N}}\{x_k + y_k\}$, which implies

$$\inf_{\substack{st_B\\ k\in\mathbb{N}}}\{x_k + y_k\} \geq \inf_{\substack{st_B\\ k\in\mathbb{N}}}\{x_k\} + \inf_{\substack{st_B\\ k\in\mathbb{N}}}\{y_k\}.$$

(3) By (2), we have

$$\inf_{\substack{st_B\\ k\in\mathbb{N}}}\{x_k + y_k\} + \inf_{\substack{st_B\\ k\in\mathbb{N}}}\{-x_k\} \leq \inf_{\substack{st_B\\ k\in\mathbb{N}}}\{y_k\},$$

consequently, from (1) we have

$$\begin{aligned}\inf_{\substack{st_B\\ k\in\mathbb{N}}}\{x_k + y_k\} &\leq -\inf_{\substack{st_B\\ k\in\mathbb{N}}}\{-x_k\} + \inf_{\substack{st_B\\ k\in\mathbb{N}}}\{y_k\}\\ &= \sup_{\substack{st_B\\ k\in\mathbb{N}}}\{x_k\} + \inf_{\substack{st_B\\ k\in\mathbb{N}}}\{y_k\}.\end{aligned}$$

□

## 2. Main results and Proofs

In Theorem 2.1, we will give a de La Vallée Poussin-type criterion for B-UNI(c) w.r.t. $A$.

**Theorem 2.1.** *Let $c \in \{0, 1\}$ and $\{X_k, k \in \mathbb{N}\}$ be a sequence of random variables. Let $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ be an array of nonnegative real numbers with*

$$\sup_{\substack{st_B\\ k\in\mathbb{N}}}\sum_{k=1}^{\infty} a_{nk} < \infty.$$

*Then $\{X_k, k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$ if and only if there exists a continuous function $Q : [0,\infty) \to [0,\infty)$ such that $Q(0) = 0$, $Q$ is strictly increasing, $Q(x) \to \infty$ as $x \to \infty, x^{-1}Q$ is strictly decreasing to $0$ as $0 < x \uparrow \infty$, and $\{Q(|X_k|), k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$.*

*Proof.* (Sufficiency). Assume that there exists a function Q satisfying the above properties. Then, there exists $M > 0$ such that for all $x \geq M$, $Q(x) \leq x$. By Remark 1.9 and Lemma 1.10, we have

$$\begin{aligned}
\infty = & \lim_{a\to\infty} \inf_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}[\mathbb{E}(Q(|X_k|) : Q(|X_k|) \leq a) + caP(Q(|X_k|) > a)] \\
= & \lim_{a\to\infty} \inf_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}[\mathbb{E}(Q(|X_k|) : [Q(|X_k|) \leq a] \cap [|X_k| \leq M]) \\
& + \mathbb{E}(Q(|X_k|) : [Q(|X_k|) \leq a] \cap [|X_k| > M]) + caP(Q(|X_k|) > a)] \\
\leq & \lim_{a\to\infty} \inf_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}[Q(M) + \mathbb{E}(|X_k| : Q(|X_k|) \leq a) + caP(Q(|X_k|) > a)] \\
\leq & \lim_{a\to\infty} \inf_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}[Q(M) + \mathbb{E}(|X_k| : |X_k| \leq Q^{-1}(a)) + cQ^{-1}(a)P(|X_k| > Q^{-1}(a))] \\
\leq & Q(M) \sup_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} \\
& + \lim_{a\to\infty} \inf_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : |X_k| \leq Q^{-1}(a)) + cQ^{-1}(a)P(|X_k| > Q^{-1}(a))]
\end{aligned}$$

Note that $\sup_{st_B k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} < \infty$, then we have

$$\lim_{a\to\infty} \inf_{st_B \atop k\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : |X_k| \leq Q^{-1}(a)) + cQ^{-1}(a)P(|X_k| > Q^{-1}(a))] = \infty.$$

It follows from Lemma 1.7 that $\{X_k, k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$.

(Necessity). Assume that $\{X_k, k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$. Let $n_0 = 0$. Since $\lim_{a\to\infty} \alpha_{c,A}(a) = \infty$, then we can find a sequence of positive integers $\{n_j, j \in \mathbb{N}\}$ such that

$$n_j > max\{2n_{j-1}, j2^j\}$$

and

$$\alpha_{c,A}(n_j) > 8^j, j \in \mathbb{N}.$$

Define a continuous function $Q : [0, \infty) \to [0, \infty)$ with

$$Q(n_j) = 2^{-j}n_j,$$

$$Q(x) = \frac{\sqrt{n_1 x}}{2}, 0 < x < n_1,$$

$$Q(x) = 2^{-j}n_j + \frac{2^{-(j+1)}n_{j+1} - 2^{-j}n_j}{n_{j+1} - n_j}(x - n_j), n_j < x < n_{j+1}, j \in \mathbb{N}.$$

For $j \in \mathbb{N}$, since $n_j > max\{2n_{j-1}, j2^j\}$, we have

$$Q(n_j) = 2^{-j}n_j > j,$$

and

$$Q(n_j) = 2^{-j}n_j < 2^{-(j+1)}n_{j+1} = Q(n_{j+1}).$$

Thus the function Q is strictly increasing with $\lim_{x\to\infty} Q(x) = \infty$.

Next, we show that $x^{-1}Q(x)$ is strictly decreasing to 0 as $0 < x \uparrow \infty$.

When $0 < x < n_1$, $x^{-1}Q(x) = \frac{\sqrt{n_1}}{2}x^{-\frac{1}{2}}$ is strictly decreasing, and when $n_j < x < n_{j+1}$,

$$x^{-1}Q(x) = \frac{2^{-j}n_j}{x} + \frac{2^{-(j+1)}n_{j+1} - 2^{-j}n_j}{n_{j+1} - n_j} - \frac{n_j(2^{-(j+1)}n_{j+1} - 2^{-j}n_j)}{x(n_{j+1} - n_j)},$$

note that

$$\begin{aligned}\frac{d(x^{-1}Q(x))}{dx} =& \frac{n_j}{x^2}\Big(\frac{2^{-(j+1)}n_{j+1} - 2^{-j}n_j}{n_{j+1} - n_j} - \frac{1}{2^j}\Big)\\ =& \frac{(2^{-1} - 1)n_{j+1}}{2^j(n_{j+1} - n_j)} < 0,\end{aligned}$$

and $\frac{Q(n_j)}{n_j} = 2^{-j},\ j \in \mathbb{N}$ is strictly decreasing to 0 as $j \to \infty$. Therefore $x^{-1}Q(x)$ is strictly decreasing to 0 as $0 < x \uparrow \infty$.

Finally, we show that $\{Q(|X_k|), k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$. Set

$$\beta_{c,A}(a) = \inf_{n\in\mathbb{N}}{}_{st_B} \sum_{k=1}^{\infty} a_{nk}[c\mathbb{E}(Q(|X_k|) \wedge a) + (1 - c)\mathbb{E}(Q(|X_k|) : Q(|X_k|) \leq a)], a > 0.$$

From Remark 1.9, we have

$$\begin{aligned}&\beta_{c,A}(Q(n_j))\\ =& \inf_{n\in\mathbb{N}}{}_{st_B} \sum_{k=1}^{\infty} a_{nk}[\mathbb{E}(Q(|X_k|) : Q(|X_k|) \leq Q(n_j)) + cQ(n_j)P(Q(|X_k|) > Q(n_j))]\\ =& \inf_{n\in\mathbb{N}}{}_{st_B} \sum_{k=1}^{\infty} a_{nk}[\mathbb{E}(Q(|X_k|) : |X_k| \leq n_j) + c2^{-j}n_jP(|X_k| > n_j)]\\ =& 2^{-j} \inf_{n\in\mathbb{N}}{}_{st_B} \sum_{k=1}^{\infty} a_{nk}[2^j\mathbb{E}(Q(|X_k|) : |X_k| \leq n_j) + cn_jP(|X_k| > n_j)]\end{aligned} \tag{2.1}$$

Note that for all $r \in \mathbb{N}$,

$$\frac{Q(x)}{x} \geq \frac{Q(n_r)}{n_r} = 2^{-r} \textit{ when } n_{r-1} < x < n_r.$$

Thus

$$\begin{aligned}&\mathbb{E}(Q(|X_k|) : |X_k| \leq n_j)\\ =& \sum_{r=1}^{j} \mathbb{E}(Q(|X_k|) : n_{r-1} < |X_k| \leq n_r)\\ \geq& \sum_{r=1}^{j} 2^{-r}\mathbb{E}(|X_k| : n_{r-1} < |X_k| \leq n_r)\\ \geq& 2^{-j} \sum_{r=1}^{j} \mathbb{E}(|X_k| : n_{r-1} < |X_k| \leq n_r)\\ =& 2^{-j}\mathbb{E}(|X_k| : |X_k| \leq n_j).\end{aligned} \tag{2.2}$$

Since $\alpha_{c,A}(n_j) > 8^j$, $j \in \mathbb{N}$, then by Eqs. (2.1), (2.2) and Remark 1.9, we have

$$\begin{aligned}&\beta_{c,A}(Q(n_j))\\ \geq&2^{-j}\inf_{n\in\mathbb{N}}{}_{st_B}\sum_{k=1}^{\infty}a_{nk}[2^j 2^{-j}\mathbb{E}(|X_k|:|X_k|\leq n_j)+cn_jP(|X_k|>n_j)]\\ =&2^{-j}\alpha_{c,A}(n_j)\\ >&2^{-j}8^j\\ =&4^j.\end{aligned}$$

Thus

$$\lim_{j\to\infty}\beta_{c,A}(Q(n_j)) = \infty,$$

which implies that $\lim_{a\to\infty}\beta_{c,A}(a) = \infty$. By Lemma 1.7, we get that $\{Q(|X_k|), k \in \mathbb{N}\}$ is B-UNI(c) w.r.t. $A$. □

In the following, we will give a necessary condition for a sequence of random variables $\{X_k, k \in \mathbb{N}\}$ to be B-WUNI w.r.t. $A$.

**Theorem 2.2.** *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of random variables and $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ be an array of nonnegative real numbers with $\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}a_{nk} < \infty$. If $\{X_k, k \in \mathbb{N}\}$ is B-WUNI w.r.t. $A$, then for all $M > 0$, there exists $\alpha > 0$ such that for every sequence of events $\{A_k, k \in \mathbb{N}\}$,*

$$\inf_{n\in\mathbb{N}}\sum_{k=1}^{\infty}a_{nk}P(A_k) \geq \alpha \Rightarrow \inf_{n\in\mathbb{N}}{}_{st_B}\sum_{k=1}^{\infty}a_{nk}\mathbb{E}(|X_k| : A_k) \geq M.$$

*Proof.* From Remark 1.6, we know that $\{X_k, k \in \mathbb{N}\}$ is B-WUNI w.r.t. $A$, i.e.

$$\lim_{a\to\infty}\inf_{n\in\mathbb{N}}{}_{st_B}\sum_{k=1}^{\infty}a_{nk}\mathbb{E}(|X_k| \wedge a) = \infty.$$

By Lemma 1.8, we have

$$\lim_{a\to\infty}\inf_{n\in\mathbb{N}}{}_{st_B}\sum_{k=1}^{\infty}a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a) + aP(|X_k| > a)] = \infty.$$

Then for all $M > 0$, there exists $a_0 > M$ such that

$$\inf_{n\in\mathbb{N}}{}_{st_B}\sum_{k=1}^{\infty}a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a_0) + a_0P(|X_k| > a_0)] > 4M,$$

therefore, $4M$ is a $B$-statistical lower bound of $\sum_{k=1}^{\infty}a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a_0) + a_0P(|X_k| > a_0)]$. By the definition of $B$-statistical lower bound, we have

$$\delta_B(\{n \in \mathbb{N} : \sum_{k=1}^{\infty}a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a_0) + a_0P(|X_k| > a_0)] \geq 4M\}) = 1.$$

Put

$$K := \{n \in \mathbb{N} : \sum_{k=1}^{\infty}a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a_0) + a_0P(|X_k| > a_0)] \geq 4M\},$$

then

$$\delta_B(K) = 1.$$

For any $n \in K$, inequality $\sum_{k=1}^{\infty} a_{nk}[\mathbb{E}(|X_k| : |X_k| \leq a_0) + a_0 P(|X_k| > a_0)] \geq 4M$ holds, which implies either $\sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : |X_k| \leq a_0) \geq 2M$ or $\sum_{k=1}^{\infty} a_{nk} a_0 P(|X_k| > a_0) \geq 2M$. Note that $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} < \infty$, we have

$$\sum_{k=1}^{\infty} a_{nk} \leq \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} < \infty,$$

then there exists a constant $A_1 > \frac{M}{a_0}$ such that

$$\sum_{k=1}^{\infty} a_{nk} \leq A_1.$$

Let $\alpha = A_1 - \frac{M}{a_0}$ and a sequence of events $\{A_k, k \in \mathbb{N}\}$ satisfy $\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk} P(A_k) \geq \alpha$. When $\sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : |X_k| \leq a_0) \geq 2M$, we can get

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq & \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : [|X_k| \leq a_0] \cap A_k) \\
= & \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : |X_k| \leq a_0) - \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : [|X_k| \leq a_0] \cap A_k^c) \\
\geq & 2M - \sum_{k=1}^{\infty} a_{nk} a_0 P(A_k^c) \\
= & 2M - \sum_{k=1}^{\infty} a_{nk} a_0 (1 - P(A_k)) \\
= & 2M + a_0 \sum_{k=1}^{\infty} a_{nk}(P(A_k) - 1) \\
\geq & 2M + a_0(\alpha - \sum_{k=1}^{\infty} a_{nk}) \\
\geq & 2M + a_0(\alpha - A_1) \\
= & M.
\end{aligned}$$

When $\sum_{k=1}^{\infty} a_{nk} a_0 P(|X_k| > a_0) > 2M$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq & \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : [|X_k| > a_0] \cap A_k) \\
\geq & \sum_{k=1}^{\infty} a_{nk} a_0 P([|X_k| > a_0] \cap A_k) \\
= & \sum_{k=1}^{\infty} a_{nk} a_0 [P(|X_k| > a_0) - P([|X_k| > a_0] \cap A_k^c)] \\
\geq & \sum_{k=1}^{\infty} a_{nk} a_0 [P(|X_k| > a_0) - P(A_k^c)] \\
= & \sum_{k=1}^{\infty} a_{nk} a_0 P(|X_k| > a_0) + a_0 \sum_{k=1}^{\infty} a_{nk}(P(A_k) - 1)
\end{aligned}$$

$$\begin{aligned}&\geq 2M + a_0(\sum_{k=1}^{\infty} a_{nk}P(A_k) - \sum_{k=1}^{\infty} a_{nk})\\&\geq 2M + a_0(\alpha - A_1)\\&= M.\end{aligned}$$

Hence, for any $n \in K$, $\sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq M$ holds, then

$$K \subseteq \{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq M\}.$$

Therefore

$$1 \geq \delta_B(\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq M\}) \geq \delta_B(K) = 1.$$

So $M$ is a $B$-statistical lower bound of $\sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k)$, it follows that $\inf_{st_B\, n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq M$. □

**Corollary 2.3.** *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of random variables and $A = \{a_{nk}, n \in \mathbb{N}, k \in \mathbb{N}\}$ be an array of nonnegative real numbers with $\sum_{k=1}^{\infty} a_{nk} = 1$ for all $n \in \mathbb{N}$. Then $\{X_k, k \in \mathbb{N}\}$ is B-WUNI w.r.t. A if and only if for all $M > 0$, there exists $\alpha \in (0, 1)$ such that for every sequence of events $\{A_k, k \in \mathbb{N}\}$,*

$$\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(A_k) \geq \alpha \Rightarrow \inf_{\substack{st_B\\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq M.$$

*Proof.* The proof for necessity is similar to that of Theorem 2.2. We omit the details. In the following, we give the proof for sufficiency.

Suppose that for all $M > 0$, there exists $\alpha \in (0, 1)$ such that for every sequence of events $\{A_k, k \in \mathbb{N}\}$,

$$\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(A_k) \geq \alpha \Rightarrow \inf_{\substack{st_B\\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : A_k) \geq M. \tag{2.3}$$

Note that $\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(|X_k| \leq a)$ is an increasing function of $a$. Denote

$$\beta := \lim_{a\to\infty} \inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(|X_k| \leq a) \geq 0.$$

Now we need only consider two cases:

(I) $\alpha \in (0, \beta)$, $\beta > 0$. By the definition of $\beta$, there exists an $a_0$ such that for all $a > a_0$, $\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(|X_k| \leq a) \geq \alpha$, then by Eq.(2.3) and Lemma 1.8, we have

$$\inf_{\substack{st_B\\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| \wedge a) \geq \inf_{\substack{st_B\\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| : |X_k| \leq a) \geq M.$$

(II) $\alpha \in [\beta, \infty) \cap (0, 1)$ ($\beta$ may be zero). Since $\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(|X_k| \leq a)$ is an increasing function of $a$, for all $a > 0$, we have

$$\inf_{n\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}P(|X_k| \leq a) \leq \beta \leq \alpha.$$

Let $a > \frac{M}{1-\alpha}$. Decompose the positive integers set $\mathbb{N}$ into two subsets $\mathbb{N}_1$ and $\mathbb{N}_2$ such that $\mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$ and for all $n_j \in \mathbb{N}_1$, $\sum_{k=1}^{\infty} a_{n_jk}P(|X_k| \leq a) \leq \alpha$ and, for all $m_r \in \mathbb{N}_2$, $\sum_{k=1}^{\infty} a_{m_rk}P(|X_k| \leq a) > \alpha$.

(i) For any $n_j \in \mathbb{N}_1$, we have

$$\begin{aligned}
\inf_{\substack{st_B \\ n_j\in\mathbb{N}_1}} \sum_{k=1}^{\infty} a_{n_jk}\mathbb{E}(|X_k| \wedge a) &= \inf_{\substack{st_B \\ n_j\in\mathbb{N}_1}} \sum_{k=1}^{\infty} a_{n_jk}[\mathbb{E}(|X_k| : |X_k| \le a) + aP(|X_k| > a)] \\
&\ge \inf_{\substack{st_B \\ n_j\in\mathbb{N}_1}} \sum_{k=1}^{\infty} a_{n_jk} aP(|X_k| > a) \\
&= \inf_{\substack{st_B \\ n_j\in\mathbb{N}_1}} a \sum_{k=1}^{\infty} a_{n_jk}(1 - P(|X_k| \le a)) \\
&\ge \inf_{\substack{st_B \\ n_j\in\mathbb{N}_1}} (a \sum_{k=1}^{\infty} a_{n_jk} - a\alpha) \\
&= a(1-\alpha) \\
&> M.
\end{aligned}$$

(ii) For any $m_r \in \mathbb{N}_2$, $\sum_{k=1}^{\infty} a_{m_rk}P(|X_k| \le a) > \alpha$ holds, hence $\inf_{m_r\in\mathbb{N}_2} \sum_{k=1}^{\infty} a_{m_rk}P(|X_k| \le a) \ge \alpha$. By Eq.(2.3) and Lemma 1.8, we have

$$\inf_{\substack{st_B \\ m_r\in\mathbb{N}_2}} \sum_{k=1}^{\infty} a_{m_rk}\mathbb{E}(|X_k| \wedge a) \ge \inf_{\substack{st_B \\ m_r\in\mathbb{N}_2}} \sum_{k=1}^{\infty} a_{m_rk}\mathbb{E}(|X_k| : |X_k| \le a) \ge M.$$

Hence from (i) and (ii), when $a > \frac{M}{1-\alpha}$, for all $n \in \mathbb{N}$ and $M > 0$,

$$\inf_{\substack{st_B \\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| \wedge a) \ge M.$$

Both in the case (I) and (II) can imply that

$$\lim_{a\to\infty} \inf_{\substack{st_B \\ n\in\mathbb{N}}} \sum_{k=1}^{\infty} a_{nk}\mathbb{E}(|X_k| \wedge a) = \infty,$$

therefore, $\{X_k, k \in \mathbb{N}\}$ is B-WUNI w.r.t. $A$.

The proof is completed. □

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