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Normality through sharing of pairs of functions with derivatives

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Abstract. Let $\mathcal{F} \subset \mathcal{M}(D)$ and let a, b and c be three distinct complex numbers. If, there exist a holomorphic function h on D and a positive constant ρ such that for each $f \in \mathcal{F}$, f and f' partially share three pairs of functions (a, h), (b, c_f) and (c, d_f) on D, where c_f and d_f are some values in some punctured disk $D^*_\rho(0)$, then $\mathcal{F}$ is normal in D. This is an improvement of Schwick's result [Arch. Math. (Basel), **59** (1992), 50-54]. We also obtain several normality criteria which significantly improve the existing results and examples are given to establish the sharpness of results.

1. Introduction

Let $D \subseteq \mathbb{C}$ be a domain. For the sake of convenience we shall denote by $\mathcal{M}(D)$ the class of all meromorphic functions on D, by $\mathcal{H}(D)$ the class of all holomorphic functions on D, and by $\mathbb{D}$ the open unit disk in $\mathbb{C}$. Let $f \in \mathcal{M}(D)$ and $a \in \mathbb{C} \cup \{\infty\}$. Further, we shall denote by $E_f(a)$ the set of $a-$points of f. When $a = \infty$, $E_f(a)$ means the set of poles of f. Let $a, b \in \mathbb{C} \cup \{\infty\}$. We say that two functions f, $g \in \mathcal{M}(D)$ partially share a pair (a, b) if $z \in E_f(a) \Rightarrow z \in E_g(b)$. Further, if $E_f(a) = E_g(b)$, then f and g are said to share the pair (a, b). Clearly, f and g share the value a if they share the pair (a, a).

A family $\mathcal{F} \subset \mathcal{M}(D)$ is said to be normal if each sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly in D with respect to the spherical metric. The limit function lies in $\mathcal{M}(D) \cup \{\infty\}$.

Mues and Steinmetz [6] proved that *if f is meromorphic in the plane and if f and f' share three values, then $f' \equiv f$.* Let $\mathcal{F}$ be a subfamily of $\mathcal{M}(D)$ such that for each $f \in \mathcal{F}$, f and f' share three distinct values. In view of Bloch's principle a natural question arises: Can $\mathcal{F}$ be normal in D? Schwick [8] answered this question affirmatively:

Theorem 1.1. *Let $\mathcal{F} \subset \mathcal{M}(D)$ and let a, b and c be three distinct complex numbers. If, for each $f \in \mathcal{F}$, f and f' share three pairs of values (a, a), (b, b) and (c, c), then $\mathcal{F}$ is normal in D.*

Several extensions, improvements and related variants of Theorem 1.1 have been obtained by various authors, for example one can see [3, 4, 7, 10]. The purpose of this paper is to obtain further improvements of results of Xu [10] and Li and Yi [4].

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2. Statements of Results

Xu [10] proved that for holomorphic version of Theorem 1.1, the sharing of two distinct values is sufficient to ensure the normality:

Theorem 2.1. *Let $\mathcal{F} \subset \mathcal{H}(D)$, and let a and b be two distinct complex numbers. If for each $f \in \mathcal{F}$, f and f' share the pairs of values (a,a) and (b,b), then $\mathcal{F}$ is normal in D.*

Lü, Xu and Yi [5] proved Theorem 2.1 by using partial sharing of values:

Theorem 2.2. *Let $\mathcal{F} \subset \mathcal{H}(D)$, and let a and b be distinct complex numbers. If for each $f \in \mathcal{F}$, f and f' partially share the pairs of values (a,a) and (b,b), then $\mathcal{F}$ is normal in D.*

We prove the following result as an improvement of Theorem 1.1 in which we relax the sharing condition by partial sharing and also replace the assumed values a, b, c of f' by h, c_f and d_f respectively, where h is a holomorphic function, and c_f and d_f are complex values which depend on f:

Theorem 2.3. *Let $\mathcal{F} \subset \mathcal{M}(D)$ and let a, b and c be three distinct complex numbers. If, there exist a holomorphic function h on D and a positive constant ρ such that for each $f \in \mathcal{F}$, f and f' partially share three pairs of functions (a,h), (b,c_f) and (c,d_f) on D, where c_f and d_f are some values in a punctured disk $D^*_\rho(0)$, then $\mathcal{F}$ is normal in D.*

The values c_f and d_f in Theorem 2.3 need to be in a finite punctured disk as shown by the following example:

Example 2.4. *Consider the family $\mathcal{F} := \{f_n(z) = \tan nz : n \in \mathbb{N}\}$ of meromorphic functions in $\mathbb{D}$. Then each f_n and f_n' partially share the pairs (i,h), $(-i,1)$ and $(1,2n)$, where h can be any holomorphic function on $\mathbb{D}$. Note that the values $d_{f_n} = 2n$ do not lie in any given finite punctured disk. But $\mathcal{F}$ fails to be normal in $\mathbb{D}$.*

The following example shows that the three pairs of functions in Theorem 2.3 can not be replaced by two pairs of functions:

Example 2.5. *Consider the family*

$$\mathcal{F} := \left\{ f_n(z) = \frac{e^{nz}}{1+e^{nz}} : n \geq 4 \right\} \subset \mathcal{M}(\mathbb{D}).$$

Note that each $f \in \mathcal{F}$ omits 0 and 1 in $\mathbb{D}$ and therefore, f and f' partially shares the pairs of functions $(0,h)$ and $(1,c_f)$, where h can be any holomorphic function and $c_f \in \mathbb{C}$. But the family $\mathcal{F}$ is not normal in $\mathbb{D}$ since $f_n(0) = 1/2$ and for each positive real number x in $\mathbb{D}$, $f_n(x) \to 1$ as $n \to \infty$.

In the following example, we show that the values b and c in Theorem 2.3 can not be made to depend on f:

Example 2.6. *Let $\mathcal{F} := \{f_n(z) = 1/nz : n \in \mathbb{N}\} \subset \mathcal{M}(\mathbb{D})$. Then, clearly, $f_n \neq 0$ and so f_n and f_n' partially share the pair $(0,0)$. Also, f_n and f_n' partially share $(1/n, -1/n)$ and $(-1/n, -1/n)$. Note the values $b = 1/n$ and $c = -1/n$ are not fixed and depend on f_n and the family $\mathcal{F}$ is not normal at $z = 0$.*

The holomorphic version of Theorem 2.3 is

Theorem 2.7. *Let $\mathcal{F} \subset \mathcal{H}(D)$ and let a and b be two distinct complex numbers. If there exist a holomorphic function h on D and positive constant ρ such that for each $f \in \mathcal{F}$, f and f' partially share the two pairs (a,h) and (b,c_f), where $c_f \in D^*_\rho(0)$, then $\mathcal{F}$ is normal in D.*

Note that Theorem 2.7 is an improvement of Theorem 2.2. The values c_f in Theorem 2.7 have to be essentially in a finite punctured disk, which is clear from the following example:

Example 2.8. *Consider the family*

$$\mathcal{F} := \{f_n(z) = e^{nz} : n \in \mathbb{N}\} \subset \mathcal{H}(\mathbb{D}).$$

Then f_n and f_n' partially share the pairs $(0,0)$ and $(1,n)$. Note that $c_{f_n} = n$ are not contained in any finite disk and the family $\mathcal{F}$ is not normal in $\mathbb{D}$ since $f_n(0) = 1$ and for each neighborhood N of 0, we can choose a positive real number $x \in N$ such that $f_n(x) \to \infty$ as $n \to \infty$.

Li and Yi [4] considered partial sharing of the pair of values (a,a) by f and f' and another pair of values (b,b) partially shared by f' and f, and obtained the following normality criterion:

Theorem 2.9. *Let $\mathcal{F} \subset \mathcal{H}(D)$ and let $a, b \in \mathbb{C}$ be distinct such that $b \neq 0$. If for each $f \in \mathcal{F}$, f and f' partially share the pair (a,a) and f' and f partially share the pair (b,b), then $\mathcal{F}$ is normal in D.*

Let $A \subset \mathbb{C}$ and $a \in \mathbb{C}$. For $f, g \in \mathcal{M}(D)$, we shall say that f and g partially share the pair (a,A), if $f(z) = a$ implies $g(z) \in A$.

As an improvement of Theorem 2.9, we have obtained the following result:

Theorem 2.10. *Let $\mathcal{F} \subset \mathcal{H}(D)$, and let a and $b \neq 0$ be two distinct complex numbers. Let A be a compact set such that $b \notin A$ and $B = \{z : |z-a| \geq \epsilon\}$, for some $\epsilon > 0$. If for each $f \in \mathcal{F}$, f and f' partially share the pair (a,A) and f' and f partially share the pair (b,B), then $\mathcal{F}$ is normal in D.*

Remark 2.11. *After obtaining Theorem 2.10 as an improvement of Theorem 2.9 we came across a result of Sauer and Schweizer [9]: Let $\mathcal{F}$ be a family of holomorphic functions in a domain D. Let a and $b \neq 0$ be two complex numbers such that $b \neq a$, and let A and B be compact subsets of $\mathbb{C}$ with $b \notin A$ and $a \notin B$. If, for each $f \in \mathcal{F}$ and $z \in D$, f and f' partially share the pair (a,A) and f' and f partially share the pair (b,B), then $\mathcal{F}$ is normal in D. This result is also an improvement of Theorem 2.9. Theorem 2.10 also provides an improvement of Sauer and Schweizer's result.*

The condition 'the set B must be at a positive distance away from the point a' in Theorem 2.10 cannot be dropped as shown by the following example:

Example 2.12. *Let $\mathcal{F} := \{f_n(z) = e^{nz} : n \in \mathbb{N}\} \subset \mathcal{H}(\mathbb{D})$. Take $a = 0$ and $b = 1$. Then $f_n(z) \neq a$ and $f_n'(z) = b \Rightarrow f_n(z) = 1/n \to a$. But $\mathcal{F}$ is not normal at $z = 0$.*

In the next example, we show that the boundedness of set A in Theorem 2.10 can not be relaxed:

Example 2.13. *Let $\mathcal{F} := \{f_n(z) = e^{nz}/n : n \in \mathbb{N}\} \subset \mathcal{H}(\mathbb{D})$. Take $a = 1$ and $b = -1$. Then $f_n(z) = 1 \Rightarrow f_n'(z) = n \in \mathbb{N}$ and $f_n'(z) = -1 \Rightarrow f_n(z) = -1/n \in \{z : |z-1| \geq 1\}$. But $\mathcal{F}$ is not normal at $z = 0$ since $f_n(0) = 1/n \to 0$ as $n \to \infty$ and for any positive real number x, $f_n(x) \to \infty$ as $n \to \infty$.*

Another variant of Theorem 2.10 is obtained as:

Theorem 2.14. *Let $\mathcal{F} \subset \mathcal{H}(D)$ be such that zeros of each $f \in \mathcal{F}$ have multiplicity at least k, where $k \in \mathbb{N}$ and $b(\neq 0) \in \mathbb{C}$. Let A be a compact set and $B = \{z : |z| \geq \epsilon\}$ for some $\epsilon > 0$. If for each $f \in \mathcal{F}$, f and $f^{(k)}$ partially share the pair $(0,A)$ and $f^{(k)}$ and f partially share the pair (b,B) in D, then $\mathcal{F}$ is normal in D.*

The condition '$b \neq 0$' in Theorem 2.14 can not be dropped, as can be seen from the following example:

Example 2.15. *Let $\mathcal{F} := \{e^{nz} : n \in \mathbb{N}\} \subset \mathcal{H}(\mathbb{D})$. Then $\mathcal{F}$ satisfies all the conditions of Theorem 2.14 with $b = 0$, but $\mathcal{F}$ is not normal in $\mathbb{D}$.*

Also, the condition 'the zeros of $f \in \mathcal{F}$ have multiplicity at least k' in Theorem 2.14 can not be weakened:

Example 2.16. *Consider the family $\mathcal{F} := \{f_n(z) = n \sinh z : n \in \mathbb{N}\} \subset \mathcal{H}(\mathbb{D})$. Then, clearly, the zeros of $f_n \in \mathcal{F}$ are simple and $f_n \equiv f_n''$. But the family $\mathcal{F}$ is not normal at $z = 0$ since $f_n(0) = 0$ and for a sufficiently small positive real number x, $f_n(x) \to \infty$ as $n \to \infty$. .*

The meromorphic version of Theorem 2.10 does not hold as shown by the following example :

Example 2.17. *Let $a \in \mathbb{C} \setminus \{1\}$ and consider the family*

$$\mathcal{F} := \left\{ f_n(z) = \frac{n + (nz-1)^2}{n(nz-1)} + a : n \in \mathbb{N} \right\} \subset \mathcal{M}(\mathbb{D}).$$

One can easily verify that for each $f \in \mathcal{F}$, $f(z) = a \Rightarrow f'(z) = 2$ and $f' \neq 1$. Thus f and f' partially shares the pair (a, A) and f' and f partially shares the pair $(1, B)$, where $A = \{2\}$ and $B = \{z : |z - a| \geq \epsilon\}$ for any $\epsilon > 0$. Note that $f_n(0) \to a - 1$ as $n \to \infty$ and for any non-zero complex number z in the neighborhood of 0, $f_n(z) \to z + a$ as $n \to \infty$ and therefore, $\mathcal{F}$ is not normal at $z = 0$.

However, the following related meromorphic version holds:

Theorem 2.18. *Let $\mathcal{F} \subset \mathcal{M}(D)$ be such that zeros of each $f \in \mathcal{F}$ have multiplicity at least $k + 1$, where $k \in \mathbb{N}$. Let a and b be two distinct non-zero complex numbers, and A be a compact set and $B = \{z \in \mathbb{C} : |z| \geq \epsilon\}$ for some $\epsilon > 0$. If for each $f \in \mathcal{F}$, f and $f^{(k)}$ partially share the pair (a, A) and $f^{(k)}$ and f partially share the pair (b, B), then $\mathcal{F}$ is normal in D.*

The following example shows that the condition 'zeros of each $f \in \mathcal{F}$ have multiplicity at least $k + 1$,' in Theorem 2.18 is essential:

Example 2.19. *Consider the family*

$$\mathcal{F} := \left\{ f_n(z) = \frac{e^{nz}}{n} + 2 : n \in \mathbb{N} \right\}.$$

of entire functions. Then, clearly, $f_n(z) \neq 2$ and $f_n'(z) = 1 \Rightarrow f_n(z) = 1/n + 2 \in \{z : |z| \geq 2\}$. Since $f_n'(z) \neq 0$, the zeros of f_n are simple. But the family $\mathcal{F}$ is not normal at $z = 0$.

Also, the condition 'set B must be at a positive distance away from the origin' in Theorem 2.18 cannot be dropped:

Example 2.20. *Consider the family*

$$\mathcal{F} := \left\{ f_n(z) = \frac{1}{e^{nz} + 1} : n \in \mathbb{N} \right\} \subset \mathcal{M}(\mathbb{D}).$$

Take $a = 1,\ b = -1$. Then, clearly, $f_n(z) \neq 0, 1$. Also,

$$f_n'(z) = -1 \Rightarrow f_n(z) = \frac{2}{\left\{(n-2) \pm \sqrt{(n-2)^2 - 4}\right\} + 2}$$

which are not contained in any set of the form $\{z : |z| \geq \epsilon\}$, for any $\epsilon > 0$. But the family $\mathcal{F}$ is not normal at $z = 0$.

3. Proofs of the results

To prove the results of this paper, we require the following lemmas:

Lemma 3.1. *[7] Let $\mathcal{F} \subset \mathcal{M}(\mathbb{D})$ be such that for each $f \in \mathcal{F}$, all zeros of f are of multiplicity at least k. Suppose that there exists a number $L \geq 1$ such that $|f^{(k)}(z)| \leq L$ whenever $f \in \mathcal{F}$ and $f(z) = 0$. If $\mathcal{F}$ is not normal in $\mathbb{D}$, then for every $\alpha \in [0, k]$, there exist $r \in (0, 1)$, $\{z_n\} \subset D_r(0)$, $\{f_n\} \subset \mathcal{F}$ and $\{\rho_n\} \subset (0, 1) : \rho_n \to 0$ such that*

$$g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$ with $g^{\#}(\zeta) \leq g^{\#}(0) = kL + 1$.

Lemma 3.2. *[2] Let $g \in \mathcal{M}(\mathbb{C})$ be of finite order. If g has only finitely many critical values, then it has only finitely many asymptotic values.*

Lemma 3.3. *[1] Let $g \in \mathcal{M}(\mathbb{C})$ be transcendental having no poles at the origin and let the set of finite critical and asymptotic values of g be bounded. Then there exists $R > 0$ such that*

$$|g'(z)| \geq \frac{|g(z)|}{2\pi|z|} \log \frac{g(z)}{R},$$

for all $z \in \mathbb{C} \setminus \{0\}$ which are not poles of g.

Lemma 3.4. *[2] Let $f \in \mathcal{M}(\mathbb{C})$ be transcendental and of finite order. Suppose all zeros of f have multiplicity at least $k+1$, where $k \in \mathbb{N}$. Then $f^{(k)}$ assumes every non-zero complex number infinitely often.*

Proof of Theorem 2.3: Suppose that $\mathcal{F}$ is not normal. Then $\mathcal{F}_a = \{f - a : f \in \mathcal{F}\}$ is not normal and therefore, by Zalcman Lemma, there exist a sequence $\{f_n - a\} \subset \mathcal{F}_a$, sequence $\{z_n\}$ of points in D and a sequence $\{\rho_n\}$ of positive real numbers with $\rho_n \to 0$ as $n \to \infty$ such that the re-scaled sequence $\{g_n(\zeta) := f_n(z_n + \rho_n\zeta) - a\}$ converges locally uniformly to a non-constant meromorphic function g on $\mathbb{C}$.

Suppose $g(\zeta_0) = 0$. Then by Hurwitz's Theorem, there exists a sequence $\zeta_n \to \zeta_0$ as $n \to \infty$ such that for sufficiently large n, $g_n(\zeta_n) = 0$. That is, $f_n(z_n + \rho_n\zeta_n) = a$. Thus, by hypothesis, $f_n'(z_n + \rho_n\zeta_n) = h(z_n + \rho_n\zeta_n)$, and hence

$$g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_n) = \lim_{n\to\infty} \rho_n f_n'(z_n + \rho_n\zeta_n) = \lim_{n\to\infty} \rho_n h(z_n + \rho_n\zeta_n) = 0.$$

This shows that the zeros of g have multiplicity at least 2. Similarly, we can show that the zeros of $g - (b - a)$ and $g - (c - a)$ have multiplicity at least 2.

Next, we show that g omits $b - a$. Suppose that ζ_0 is a zero of $g - (b - a)$ with multiplicity k. Then

$$g^{(k)}(\zeta_0) \neq 0. \tag{1}$$

Choose $\delta > 0$ such that

$$g(\zeta) \neq b - a,\ g'(\zeta) \neq 0, \cdots,\ g^{(k)}(\zeta) \neq 0 \tag{2}$$

on $D_\delta^*(\zeta_0)$.

Since $g(\zeta_0) = b - a$, by Hurwitz's Theorem, there exists $\zeta_{n,i} \to \zeta_0, n \to \infty$ $(i = 1, \cdots, k)$ in $D_\delta(\zeta_0)$ such that $g_n(\zeta_{n,i}) = b - a$, for sufficiently large n. That is, $f_n(z_n + \rho_n\zeta_{n,i}) = b$ and thus $0 < |f_n'(z_n + \rho_n\zeta_{n,i})| \leq \rho$.

Further,

$$g_n'(\zeta_{n,i}) = \rho_n f_n'(z_n + \rho_n\zeta_{n,i}) \neq 0, \text{ for } i = 1, \cdots, k. \tag{3}$$

This implies $\zeta_{n,i}, (i = 1, 2, \cdots, k)$ are simple zeros of $g_n - (b - a)$.

Also $\zeta_{n,i} \neq \zeta_{n,j}$ $(1 \leq i < j \leq k)$ and

$$g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_{n,i}) = 0.$$

Therefore, by (3), for sufficiently large n, $g_n' - \rho_n c_{f_n}$, where $c_{f_n} = f_n'(z_n + \rho_n\zeta_{n,i})$, has at least k zeros $\zeta_{n,i}(i = 1, \cdots, k)$ in $D_\delta^*(0)$. This implies that ζ_0 is a zero of g' with multiplicity at least k and hence $g^{(k)}(\zeta_0) = 0$, which contradicts (1). Hence $g(\zeta) \neq b - a$. Similarly, we can show that g omits $c - a$ and then by second fundamental theorem of Nevanlinna, we arrive at a contradiction. □

The Proof of Theorem 2.7 is obtained exactly on the lines of the proof of Theorem 2.3, so we omit it.

Proof of Theorem 2.10: We may assume that D is the open unit disk $\mathbb{D}$. Suppose that $\mathcal{F}$ is not normal in $\mathbb{D}$. Then $\mathcal{F}_a = \{f - a : f \in \mathcal{F}\}$ is not normal in $\mathbb{D}$. For any $h \in \mathcal{F}_a$, $|h'(z)| \leq M + 1$ whenever $h(z) = 0$, where

$M = \sup\{|z| : z \in A\}$. By Lemma 3.1, there exist a sequence $\{f_n - a\} \subset \mathcal{F}_a$, sequence $\{z_n\}$ of points in D and a sequence $\{\rho_n\}$ of positive real numbers with $\rho_n \to 0$ as $n \to \infty$ such that

$$g_n(\zeta) = \rho_n^{-1}\left(f_n(z_n + \rho_n\zeta) - a\right) \to g(\zeta) \tag{4}$$

as $n \to \infty$, locally uniformly on $\mathbb{C}$, where g is a non-constant entire function satisfying

$$g^{\#}(\zeta) \leq g^{\#}(0) = M + 2$$

implying that the order of g is at most 1.

Assertion 1: If $g(z) = 0$, then $g'(z) \in A$.
Suppose that $g(\zeta_0) = 0$. Then by Hurwitz's Theorem, there exists $\zeta_n \to \zeta_0$ as $n \to \infty$ such that for sufficiently large n, $g_n(\zeta_n) = 0$. This implies that $f_n(z_n + \rho_n\zeta_n) = a$. Since f and f' partially share the pair (a, A),

$$g_n'(\zeta_n) = f_n'(z_n + \rho_n\zeta_n) \in A.$$

Since A is compact,

$$g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_n) \in A$$

and this proves Assertion 1.

Assertion 2: $g'(\zeta) \neq b, \forall\, \zeta \in \mathbb{C}$.
Suppose that $g'(\zeta_0) = b$ for some $\zeta_0 \in \mathbb{C}$. If $g'(\zeta) \equiv b$, then $g(\zeta) = b\zeta + c$, so by Assertion 1, $b \in A$, a contradiction. Thus $g'(\zeta) \not\equiv b$.

Now by Hurwitz's Theorem, there exists $\zeta_n \to \zeta_0$ as $n \to \infty$, such that for sufficiently large n,

$$g_n'(\zeta_n) = f_n'(z_n + \rho_n\zeta_n) = b.$$

Since $f^{'}$ and f partially share the pair (b, B),

$$|g_n(\zeta_n)| = \rho_n^{-1}|\left(f_n(z_n + \rho_n\zeta_n) - a\right)| \geq \frac{\epsilon}{\rho_n} \to \infty \text{ as } n \to \infty.$$

That is, $g(\zeta_0) = \infty$, a contradiction since $g'(\zeta_0) = b$. This proves Assertion 2.

Since g is of order at most 1, so is g' and then by Assertion 2, we have

$$g'(\zeta) = b + e^{l+m\zeta}.$$

where $l, m \in \mathbb{C}$.

Now we have the following two cases:

Case-1. When $m \neq 0$. In this case, g is a transcendental entire function of order one. Since g' omits $b(\neq 0)$, by Hayman's alternative g has infinitely many zeros $\{z_i\}$: $|z_i| \to \infty$ as $i \to \infty$.

Define $G(z) = g(z) - bz$, then $G^{'}(z) = g'(z) - b \neq 0$, G has no critical values. Thus by Lemma 3.2, G has only finitely many asymptotic values. Applying Lemma 3.3 to G, we have

$$\frac{|z_i G^{'}(z_i)|}{|G(z_i)|} \geq \frac{1}{2\pi}\log\frac{|G(z_i)|}{R} = \frac{1}{2\pi}\log\frac{|bz_i|}{R}.$$

This implies

$$\frac{|z_i G^{'}(z_i)|}{|G(z_i)|} \to \infty \text{ as } i \to \infty. \tag{5}$$

Since $g = 0 \Rightarrow |g'| \leq M$, which further implies that $|z_i G'(z_i)|/|G(z_i)|$ is bounded. Thus (5) yields a contradiction.

Case-2. When $m = 0$. In this case $g(\zeta) = \left(b + e^l\right)\zeta + t$, where t is a constant. By Assertion 1, we get $b + e^l \in A$. Thus $g^\#(0) < M + 2$, a contradiction. □

Proof of Theorem 2.14: We may assume that D is the open unit disk $\mathbb{D}$. Suppose that $\mathcal{F}$ is not normal in $\mathbb{D}$. Then, by Lemma 3.1, (with $\alpha = k$ and $L = M + 1$, where $M = \sup\{|z| : z \in A\}$), there exist $f_n \in \mathcal{F}$, $z_n \in \mathbb{D}$ and $\rho_n \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \to g(\zeta)$$

locally uniformly on $\mathbb{C}$, where g is a non-constant entire function such that $g^\#(\zeta) \leq g^\#(0) = k(M + 1) + 1$ and the order of g is at most one.

Next we show that zeros of g are of multiplicity at least k and $g(z) = 0$ implies that $g^{(k)}(z) \in A$. Let $g(\zeta_0) = 0$. Then by Hurwitz's Theorem, there exists a sequence $\zeta_n \to \zeta_0$ as $n \to \infty$ such that for sufficiently large n, $g_n(\zeta_n) = 0$. That is $f_n(z_n + \rho_n\zeta_n) = 0$ and by assumption, we have, $f_n^{(i)}(z_n + \rho_n\zeta_n) = 0$ $(i = 1, \cdots, k-1)$ and $f_n^{(k)}(z_n + \rho_n\zeta_n) \in A$. Thus

$$g^{(i)}(\zeta_0) = \lim_{n\to\infty} g_n^{(i)}(\zeta_n) = \lim_{n\to\infty} \rho_n^{i-k} f_n^{(i)}(z_n + \rho_n\zeta_n) = 0 \ (i = 1, \cdots, k-1)$$

and

$$g^{(k)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k)}(\zeta_n) = \lim_{n\to\infty} f_n^{(k)}(z_n + \rho_n\zeta_n) \in A.$$

Therefore, all zeros of g are of multiplicity at least k and $g(z) = 0$ implies that $g^{(k)}(z) \in A$.

Assertion: $g^{(k)}(z) \neq b$ in $\mathbb{C}$.
Suppose that $g^{(k)}(\zeta_0) = b$. If $g^{(k)}(\zeta) \equiv b$, then g is a polynomial of degree k. Since all zeros of g are of multiplicity at least k, g has only one zero, say ζ'. Thus

$$g(\zeta) = \frac{b(\zeta - \zeta')^k}{k!}.$$

Since $g(\zeta) = 0 \Rightarrow g^{(k)}(\zeta) \in A$, $|b| \leq M$. By a simple calculation, we have

$$g^\#(0) \leq \begin{cases} k/2 & ; |\zeta'| \geq 1 \\ M & ; |\zeta'| < 1 \end{cases}$$

That is, $g^\#(0) < k(M + 1) + 1$, a contradiction. Thus $g^{(k)}(\zeta) \not\equiv b$.

Thus, we choose a sequence $\zeta_n \to \zeta_0$ as $n \to \infty$ such that $g_n^{(k)}(\zeta_n) = b$. This implies that $f_n^{(k)}(z_n + \rho_n\zeta_n) = b$ and by hypothesis, we find that $|f_n(z_n + \rho_n\zeta_n)| \geq \epsilon$.

Therefore ,

$$\left|g(\zeta_0)\right| = \lim_{n\to\infty} |g_n(\zeta_n)| = \lim_{n\to\infty} \left|\frac{f_n(z_n + \rho_n\zeta_n)}{\rho_n^k}\right| \geq \lim_{n\to\infty} \frac{\epsilon}{\rho_n^k} = \infty.$$

That is, $g(\zeta_0) = \infty$, a contradiction since $g^{(k)}(\zeta_0) = b$ and this proves the Assertion.
Since g is of order at most one, so is $g^{(k)}$ and by Assertion, we find that

$$g^{(k)}(\zeta) = b + e^{l+m\zeta},$$

where l and m are constants. Now we have the following two cases:

Case-I. If $m = 0$, then g is a polynomial of degree k. Since all zeros of g are of multiplicity at least k, g has only one zero, say ζ'. Thus

$$g(\zeta) = \frac{(b + e^l)(\zeta - \zeta')^k}{k!}.$$

By second part of Assertion, we have $|b + e^l| \leq M$ and as obtained above, we have that $g^{\#}(0) < k(M + 1) + 1$, a contradiction.

Case-II. If $m \neq 0$. then g is a transcendental entire function. Since $g^{(k)}(\zeta) \neq b(\neq 0)$, by Hayman's alternative, g has infinitely many zeros $\{z_i\}$ and $|z_i| \to \infty$ as $n \to \infty$. Define $G(z) = g^{(k-1)}(z) - bz$, then $G^{'}(z) = g^{(k)}(z) - b \neq 0$, G has no critical value. Thus by Lemma 3.2, G has only finitely many asymptotic values. Applying Lemma 3.3 to G, we have

$$\frac{|z_i G^{'}(z_i)|}{|G(z_i)|} \geq \frac{1}{2\pi} \log \frac{|G(z_i)|}{R} = \frac{1}{2\pi} \log \frac{|bz_i|}{R}.$$

This implies that

$$\frac{|z_i G^{'}(z_i)|}{|G(z_i)|} \to \infty$$

as $i \to \infty$, which leads to a contradiction, since $g = 0$ implies $g^{(k)} \in A$ and $|z_i G^{'}(z_i)|/|G(z_i)|$ is bounded. □

Proof of Theorem 2.18: We may take D to be $\mathbb{D}$, the open unit disk. Suppose that $\mathcal{F}$ is not normal on $\mathbb{D}$. Then, by Lemma 3.1, there exist $z_n \in \mathbb{D}, f_n \in \mathcal{F}$ and $\rho_n \to 0^+$ such that $\left\{g_n(\zeta) = \rho_n^{-k}\left(f_n(z_n + \rho_n\zeta)\right)\right\}$ converges spherically locally uniformly on $\mathbb{C}$ to a non-constant meromorphic function g, all of whose zeros have multiplicity at least $k + 1$ and the order of g is finite.

Assertion 1: $g^{(k)} \neq b$ on $\mathbb{C}$.

Suppose that $g^{(k)}(\zeta_0) = b$, for some $\zeta_0 \in \mathbb{C}$. If $g^{(k)} \equiv b$, then g is a polynomial of degree k, a contradiction since all zeros of g are of multiplicity at least $k + 1$. Thus by Hurwitz's Theorem, there exists $\zeta_n \to \zeta_0$ such that for sufficiently large n ,

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) = b.$$

By assumption, $|f_n(z_n + \rho_n\zeta_n)| \geq \epsilon$ and so

$$|g(\zeta_0)| = \lim_{n\to\infty} |g_n(\zeta_n)| = \lim_{n\to\infty} \frac{|f_n(z_n + \rho_n\zeta_n)|}{\rho_n^k} \geq \lim_{n\to\infty} \frac{\epsilon}{\rho_n^k} = \infty.$$

That is, $g(\zeta_0) = \infty$, a contradiction since $g^{(k)}(\zeta_0) = b$.

Assertion 2: g is an entire function.

Suppose that $g(\zeta_1) = \infty$, for some $\zeta_1 \in \mathbb{C}$. For sufficiently large n, we can choose a closed disk $\overline{D}_r(\zeta_1)$ such that $g_n(\zeta) \neq 0$ and $g(\zeta) \neq 0$, and $1/g_n(\zeta) \to 1/g(\zeta)$ uniformly on $\overline{D}_r(\zeta_1)$. Thus

$$\frac{1}{g_n(\zeta)} - \frac{\rho_n^k}{a} \to \frac{1}{g(\zeta)},$$

uniformly on $\overline{D}_r(\zeta_1)$. Since $1/g(\zeta_1) = 0$, there exits $\zeta_n \to \zeta_1$ such that for sufficiently large n,

$$\frac{1}{g_n(\zeta_n)} - \frac{\rho_n^k}{a} = 0.$$

That is, $f_n(z_n + \rho_n\zeta_n) = a$. By assumption, we have $|f_n^{(k)}(z_n + \rho_n\zeta_n)| \leq M$, where $M = \sup\{|z| : z \in A\}$ and hence $|g^{(k)}(\zeta_1)| \leq M$, a contradiction since $g(\zeta_1) = \infty$.

Since g is entire and $g^{(k)} \neq b$ on $\mathbb{C}$, by Lemma 3.4, g is a polynomial of degree at most k, a contradiction. □

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