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The algebraic classification of nilpotent Novikov algebras

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Abstract. This paper is devoted to the complete algebraic classification of complex 5-dimensional nilpotent Novikov algebras.

Introduction

One of the classical problems in the theory of non-associative algebras is to classify (up to isomorphism) the algebras of dimension n from a certain variety defined by some family of polynomial identities. It is typical to focus on small dimensions, and there are two main directions for the classification: algebraic and geometric. Varieties as Jordan, Lie, Leibniz or Zinbiel algebras have been studied from these two approaches ([3, 9, 11, 19, 27–29] and [11, 19, 24], respectively). In the present paper, we give the algebraic classification of 5-dimensional nilpotent Novikov algebras.

The variety of Novikov algebras is defined by the following identities:

$$\begin{array}{rcl}(xy)z & = & (xz)y,\\ (xy)z - x(yz) & = & (yx)z - y(xz).\end{array}$$

It contains commutative associative algebras as a subvariety. On the other hand, the variety of Novikov algebras is the intersection of the variety of right commutative algebras (defined by the first Novikov identity) and the variety of left symmetric (Pre-Lie) algebras (defined by the second Novikov identity, see about it [6] and references therein). Also, a Novikov algebra with the commutator multiplication gives a Lie algebra, and Novikov algebras are related to Tortken and Novikov—Poisson algebras [5, 38]. The class of Novikov (known as Gelfand-Dorfman-Novikov algebras) appeared in papers of Gelfand — Dorfman [22] and Novikov — Balinsky [1].

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The systematic study of Novikov algebras from an algebraic view started after the paper of Zelmanov where all complex finite-dimensional simple Novikov algebras were classified [40]. The first nontrivial examples of infinite-dimensional simple Novikov algebras were constructed by Filippov in [20]. Also, simple Novikov algebras (under some special conditions) were described in the infinite-dimensional case and over fields of positive characteristic in some papers by Osborn and Xu [33, 37, 39].

Many other purely algebraic properties of Novikov algebras were studied in a series by papers of Dzhumadildaev [15–18]. So, Dzhumadildaev and Löfwall described the basis of free Novikov algebras [17]; Dzhumadildaev proved that the Novikov operad is not Koszul [15]; Dzhumadildaev and Ismailov found the S_n-module structure of the multilinear component of degree n of the n-generated free Novikov algebra over a field of characteristic 0 [16]. Makar-Limanov and Umirbaev proved The Freiheitssatz for Novikov algebras [32], and Duĭsengalieva and Umirbaev constructed a wild automorphism of the three-generated free Novikov algebra [14]. Novikov central extensions of n-dimensional restricted polynimonial algebras are studied by Kaygorodov, Lopes and Páez-Guillán in [29]. Chen, Niu and Meng gave some new realizations of two Novikov algebras [12]. Lebzioui studied pseudo-Euclidean Novikov algebras in [31].

Filippov proved that each Novikov nilalgebra is nilpotent [21]. Dzhumadildaev and Tulenbaev proved that if each left multiplication of a Novikov algebra over K (char $K = p, p = 0$ or $p > n+1$) has nil-index n, then A^2 is nilpotent with nilpotency index less than or equal to n [18]. Shestakov and Zhang proved analogues of Itô's and Kegel's theorems for Novikov algebras [34]. Another interesting direction in the algebraic study of Novikov algebras is the description of possible Novikov structures on a certain Lie algebra [7, 36]. Some fundamental results on Novikov algebras satisfying nontrivial identities were received in a paper by Dotsenko, Ismailov and Umirbaev [13].

The algebraic classification of 3-dimensional Novikov algebras was given in [2], and for some classes of 4-dimensional algebras, it was given in [8]; 4-dimensional and one-generated 6-dimesional complex nilpotent Novikov algebras are described in [10, 26], respectively. The geometric classification of 3-dimensional Novikov algebras was given in [4] and of 4-dimensional nilpotent Novikov algebras in [26].

Our method for classifying nilpotent Novikov algebras is based on the calculation of central extensions of nilpotent algebras of smaller dimensions from the same variety. The algebraic study of central extensions of algebras has been an important topic for years [23, 25, 29, 35]. First, Skjelbred and Sund used central extensions of Lie algebras to obtain a classification of nilpotent Lie algebras [35]. Note that the Skjelbred-Sund method of central extensions is an important tool in the classification of nilpotent algebras. Using the same method, small dimensional nilpotent (associative, terminal [28], Jordan, Lie, anticommutative) algebras, and some others have been described. Our main results related to the algebraic classification of the variety of Novikov algebras are summarized below.

Theorem A. *Up to isomorphism, there are infinitely many isomorphism classes of complex non-split non-one-generated* 5*-dimensional nilpotent (non-2-step nilpotent) non-commutative Novikov algebras, described explicitly in section 2 in terms of* 82 *one-parameter families,* 27 *two-parameter families,* 5 *three-parameter families and* 104 *additional isomorphism classes.*

1. The algebraic classification of nilpotent Novikov algebras

1.1. Method of classification of nilpotent algebras

Throughout this paper, we use the notations and methods well written in [23], which we have adapted for the Novikov case with some modifications. Further in this section, we give some important definitions.

Let $(\mathbf{A}, \cdot)$ be a Novikov algebra over $\mathbb{C}$ and $\mathbb{V}$ a vector space over $\mathbb{C}$. The $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ is defined as the set of all bilinear maps $\theta\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ such that

$$\begin{aligned} \theta(xy,z) &= \theta(xz,y), \\ \theta(xy,z) - \theta(x,yz) &= \theta(yx,z) - \theta(y,xz). \end{aligned}$$

These elements will be called *cocycles*. For a linear map f from $\mathbf{A}$ to $\mathbb{V}$, if we define $\delta f\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ by $\delta f(x,y) = f(xy)$, then $\delta f \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. We define $\mathrm{B}^2(\mathbf{A}, \mathbb{V}) = \{\theta = \delta f \ : f \in \mathrm{Hom}(\mathbf{A}, \mathbb{V})\}$. We define the *second cohomology space* $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ as the quotient space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V}) \big/ \mathrm{B}^2(\mathbf{A}, \mathbb{V})$.

Let Aut($\mathbf{A}$) be the automorphism group of $\mathbf{A}$ and let $\phi \in$ Aut($\mathbf{A}$). For $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ define the action of the group Aut($\mathbf{A}$) on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ by $\phi\theta(x, y) = \theta\left(\phi(x), \phi(y)\right)$. It is easy to verify that $\mathrm{B}^2(\mathbf{A}, \mathbb{V})$ is invariant under the action of Aut($\mathbf{A}$). So, we have an induced action of Aut($\mathbf{A}$) on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathbf{A}$ be a Novikov algebra of dimension m over $\mathbb{C}$ and $\mathbb{V}$ be a $\mathbb{C}$-vector space of dimension k. For $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$, define on the linear space $\mathbf{A}_\theta = \mathbf{A} \oplus \mathbb{V}$ the bilinear product " $[-,-]_{\mathbf{A}_\theta}$" by $[x + x', y + y']_{\mathbf{A}_\theta} = xy + \theta(x, y)$ for all $x, y \in \mathbf{A}, x', y' \in \mathbb{V}$. The algebra $\mathbf{A}_\theta$ is called an *k-dimensional central extension* of $\mathbf{A}$ by $\mathbb{V}$. One can easily check that $\mathbf{A}_\theta$ is a Novikov algebra if and only if $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$.

Call the set $\mathrm{Ann}(\theta) = \{x \in \mathbf{A} : \theta(x, \mathbf{A}) + \theta(\mathbf{A}, x) = 0\}$ the *annihilator* of θ. We recall that the *annihilator* of an algebra $\mathbf{A}$ is defined as the ideal $\mathrm{Ann}(\mathbf{A}) = \{x \in \mathbf{A} : x\mathbf{A} + \mathbf{A}x = 0\}$. Observe that $\mathrm{Ann}(\mathbf{A}_\theta) = (\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A})) \oplus \mathbb{V}$.

The following result shows that every algebra with a non-zero annihilator is a central extension of a smaller-dimensional algebra.

Lemma 1.1. *Let* $\mathbf{A}$ *be an n-dimensional Novikov algebra such that* $\dim(\mathrm{Ann}(\mathbf{A})) = m \neq 0$. *Then there exists, up to isomorphism, a unique* $(n - m)$*-dimensional Novikov algebra* $\mathbf{A}'$ *and a bilinear map* $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ *with* $\mathrm{Ann}(\mathbf{A}) \cap \mathrm{Ann}(\theta) = 0$, *where* $\mathbb{V}$ *is a vector space of dimension m, such that* $\mathbf{A} \cong {\mathbf{A}'}_\theta$ *and* $\mathbf{A}/\mathrm{Ann}(\mathbf{A}) \cong \mathbf{A}'$.

Proof. Let $\mathbf{A}'$ be a linear complement of Ann($\mathbf{A}$) in $\mathbf{A}$. Define a linear map $P\colon \mathbf{A} \longrightarrow \mathbf{A}'$ by $P(x + v) = x$ for $x \in \mathbf{A}'$ and $v \in \mathrm{Ann}(\mathbf{A})$, and define a multiplication on $\mathbf{A}'$ by $[x, y]_{\mathbf{A}'} = P(xy)$ for $x, y \in \mathbf{A}'$. For $x, y \in \mathbf{A}$, we have

$$P(xy) = P((x - P(x) + P(x))(y - P(y) + P(y))) = P(P(x)P(y)) = [P(x), P(y)]_{\mathbf{A}'}.$$

Since P is a homomorphism $P(\mathbf{A}) = \mathbf{A}'$ is a Novikov algebra and $\mathbf{A}/\mathrm{Ann}(\mathbf{A}) \cong \mathbf{A}'$, which gives us the uniqueness. Now, define the map $\theta\colon \mathbf{A}' \times \mathbf{A}' \longrightarrow \mathrm{Ann}(\mathbf{A})$ by $\theta(x, y) = xy - [x, y]_{\mathbf{A}'}$. Thus, $\mathbf{A}'_\theta$ is $\mathbf{A}$ and therefore $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ and $\mathrm{Ann}(\mathbf{A}) \cap \mathrm{Ann}(\theta) = 0$. □

Definition 1.2. *Let* $\mathbf{A}$ *be an algebra and I be a subspace of* Ann($\mathbf{A}$). *If* $\mathbf{A} = \mathbf{A}_0 \oplus I$ *then I is called an annihilator component of* $\mathbf{A}$.

Definition 1.3. *A central extension of an algebra* $\mathbf{A}$ *without annihilator component is called a non-split central extension.*

Our task is to find all central extensions of an algebra $\mathbf{A}$ by a space $\mathbb{V}$. In order to solve the isomorphism problem we need to study the action of Aut($\mathbf{A}$) on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$. To do that, let us fix a basis $e_1, \ldots, e_s$ of $\mathbb{V}$, and $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. Then θ can be uniquely written as $\theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i$, where $\theta_i \in \mathrm{Z}^2(\mathbf{A}, \mathbb{C})$. Moreover, $\mathrm{Ann}(\theta) = \mathrm{Ann}(\theta_1) \cap \mathrm{Ann}(\theta_2) \cap \cdots \cap \mathrm{Ann}(\theta_s)$. Furthermore, $\theta \in \mathrm{B}^2(\mathbf{A}, \mathbb{V})$ if and only if all $\theta_i \in \mathrm{B}^2(\mathbf{A}, \mathbb{C})$. It is not difficult to prove (see [23, Lemma 13]) that given a Novikov algebra $\mathbf{A}_\theta$, if we write as above $\theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ and $\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A}) = 0$, then $\mathbf{A}_\theta$ has an annihilator component if and only if $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $\mathrm{H}^2(\mathbf{A}, \mathbb{C})$.

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{C}$. The *Grassmannian* $G_k(\mathbb{V})$ is the set of all k-dimensional linear subspaces of $\mathbb{V}$. Let $G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$ be the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(\mathbf{A}, \mathbb{C})$. There is a natural action of Aut($\mathbf{A}$) on $G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$. Let $\phi \in$ Aut($\mathbf{A}$). For $W = \langle[\theta_1], [\theta_2], \ldots, [\theta_s]\rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$ define $\phi W = \left\langle\left[\phi\theta_1\right], \left[\phi\theta_2\right], \ldots, \left[\phi\theta_s\right]\right\rangle$. We denote the orbit of $W \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$ under the action of Aut($\mathbf{A}$) by Orb(W). Given

$$W_1 = \langle[\theta_1], [\theta_2], \ldots, [\theta_s]\rangle, W_2 = \langle[\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s]\rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right),$$

we easily have that if $W_1 = W_2$, then $\bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = \bigcap_{i=1}^{s} \mathrm{Ann}(\vartheta_i) \cap \mathrm{Ann}(\mathbf{A})$, and therefore we can introduce the set

$$\mathbf{T}_s(\mathbf{A}) = \left\{ W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right) : \bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = 0 \right\},$$

which is stable under the action of $\mathrm{Aut}(\mathbf{A})$.

Now, let $\mathbb{V}$ be an s-dimensional linear space and let us denote by $\mathbf{E}(\mathbf{A}, \mathbb{V})$ the set of all *non-split s-dimensional central extensions* of $\mathbf{A}$ by $\mathbb{V}$. By above, we can write

$$\mathbf{E}(\mathbf{A}, \mathbb{V}) = \left\{ \mathbf{A}_\theta : \theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i \ \text{ and } \ \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in \mathbf{T}_s(\mathbf{A}) \right\}.$$

We also have the following result, which can be proved as in [23, Lemma 17].

Lemma 1.4. *Let $\mathbf{A}_\theta, \mathbf{A}_\vartheta \in \mathbf{E}(\mathbf{A}, \mathbb{V})$. Suppose that $\theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i$ and $\vartheta(x, y) = \sum_{i=1}^{s} \vartheta_i(x, y) e_i$. Then the Novikov algebras $\mathbf{A}_\theta$ and $\mathbf{A}_\vartheta$ are isomorphic if and only if*

$$\mathrm{Orb}\, \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle = \mathrm{Orb}\, \langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle.$$

This shows that there exists a one-to-one correspondence between the set of $\mathrm{Aut}(\mathbf{A})$-orbits on $\mathbf{T}_s(\mathbf{A})$ and the set of isomorphism classes of $\mathbf{E}(\mathbf{A}, \mathbb{V})$. Consequently, we have a procedure that allows us, given a Novikov algebra $\mathbf{A}'$ of dimension $n - s$, to construct all non-split central extensions of $\mathbf{A}'$. This procedure is:

Procedure

1. For a given Novikov algebra $\mathbf{A}'$ of dimension $n - s$, determine $\mathrm{H}^2(\mathbf{A}', \mathbb{C})$, $\mathrm{Ann}(\mathbf{A}')$ and $\mathrm{Aut}(\mathbf{A}')$.
2. Determine the set of $\mathrm{Aut}(\mathbf{A}')$-orbits on $T_s(\mathbf{A}')$.
3. For each orbit, construct the Novikov algebra associated with a representative of it.

1.2. Notations

Let $\mathbf{A}$ be a Novikov algebra with a basis $e_1, e_2, \ldots, e_n$. Then by Δ_{ij} we denote the bilinear form $\Delta_{ij} \colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{C}$ with $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then the set $\left\{\Delta_{ij} : 1 \le i, j \le n\right\}$ is a basis for the space of the bilinear forms on $\mathbf{A}$. Then every $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{C})$ can be uniquely written as $\theta = \sum_{1 \le i,j \le n} c_{ij}\Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. Let us fix the following notations:

$\mathcal{N}_j^{i*}$	—	jth i-dimensional nilpotent Novikov algebra with identity $xyz = 0$
$\mathcal{N}_j^{i}$	—	jth i-dimensional nilpotent "pure" Novikov algebra (without identity $xyz = 0$)
$\mathfrak{N}_i$	—	ith 4-dimensional 2-step nilpotent algebra
N_i	—	ith non-split non-one-generated 5-dimensional nilpotent (non-2-step nilpotent) non-commutative Novikov algebra

1.3. 1-dimensional central extensions of 4-dimensional 2-step nilpotent Novikov algebras

1.3.1. The description of second cohomology space

In the following table, we give the description of the second cohomology space of 4-dimensional 2-step nilpotent Novikov algebras.

$\mathfrak{N}_{01}:\quad e_1e_1=e_2$ $\mathrm{H}^2_{com}(\mathfrak{N}_{01})=\Big\langle[\Delta_{12}+\Delta_{21}],[\Delta_{13}+\Delta_{31}],[\Delta_{14}+\Delta_{41}],[\Delta_{33}],[\Delta_{34}+\Delta_{43}],[\Delta_{44}]\Big\rangle$ $\mathrm{H}^2(\mathfrak{N}_{01})=\mathrm{H}^2_{com}(\mathfrak{N}_{01})\oplus\langle[\Delta_{21}],[\Delta_{31}],[\Delta_{41}],[\Delta_{43}]\rangle$
$\mathfrak{N}_{02}:\quad e_1e_1=e_3\quad e_2e_2=e_4$ $\mathrm{H}^2_{com}(\mathfrak{N}_{02})=\Big\langle[\Delta_{12}+\Delta_{21}],[\Delta_{13}+\Delta_{31}],[\Delta_{24}+\Delta_{42}]\Big\rangle$ $\mathrm{H}^2(\mathfrak{N}_{02})=\mathrm{H}^2_{com}(\mathfrak{N}_{02})\oplus\langle[\Delta_{21}],[\Delta_{31}],[\Delta_{42}]\rangle$
$\mathfrak{N}_{03}:\quad e_1e_2=e_3\quad e_2e_1=-e_3$ $\mathrm{H}^2(\mathfrak{N}_{03})=\Big\langle[\Delta_{11}],[\Delta_{14}],[\Delta_{21}],[\Delta_{22}],[\Delta_{24}],[\Delta_{41}],[\Delta_{42}],[\Delta_{44}]\Big\rangle$
$\mathfrak{N}^{\alpha}_{04}:\quad e_1e_1=e_3\quad e_1e_2=e_3\quad e_2e_2=\alpha e_3$ $\mathrm{H}^2(\mathfrak{N}^{\alpha\neq0}_{04})=\Big\langle[\Delta_{12}],[\Delta_{14}],[\Delta_{21}],[\Delta_{22}],[\Delta_{24}],[\Delta_{41}],[\Delta_{42}],[\Delta_{44}]\Big\rangle=\Phi_\alpha$ $\mathrm{H}^2(\mathfrak{N}^{0}_{04})=\Phi_0\oplus\Big\langle[\Delta_{13}],[\Delta_{31}+\Delta_{32}-\Delta_{23}]\Big\rangle$
$\mathfrak{N}_{05}:\quad e_1e_1=e_3\quad e_1e_2=e_3\quad e_2e_1=e_3$ $\mathrm{H}^2(\mathfrak{N}_{05})=\Big\langle[\Delta_{12}],[\Delta_{14}],[\Delta_{21}],[\Delta_{22}],[\Delta_{24}],[\Delta_{41}],[\Delta_{42}],[\Delta_{44}]\Big\rangle$
$\mathfrak{N}_{06}:\quad e_1e_2=e_4\quad e_3e_1=e_4$ $\mathrm{H}^2(\mathfrak{N}_{06})=\Big\langle[\Delta_{11}],[\Delta_{13}],[\Delta_{21}],[\Delta_{22}],[\Delta_{23}],[\Delta_{31}],[\Delta_{32}],[\Delta_{33}]\Big\rangle$
$\mathfrak{N}_{07}:\quad e_1e_2=e_3\quad e_2e_1=e_4\quad e_2e_2=-e_3$ $\mathrm{H}^2(\mathfrak{N}_{07})=\Big\langle[\Delta_{11}],[\Delta_{22}],[\Delta_{23}-\Delta_{13}],[\Delta_{24}],[\Delta_{32}-\Delta_{13}],[\Delta_{41}-\Delta_{14}]\Big\rangle$
$\mathfrak{N}^{\alpha}_{08}:\quad e_1e_1=e_3\quad e_1e_2=e_4\quad e_2e_1=-\alpha e_3\quad e_2e_2=-e_4$ $\mathrm{H}^2(\mathfrak{N}^{\alpha\neq1}_{08})=\Big\langle\begin{array}{l}[\Delta_{12}],[\Delta_{21}],[\Delta_{13}-\alpha\Delta_{23}],[\Delta_{14}-\Delta_{24}],\\ [\Delta_{31}-\Delta_{13}],[\Delta_{42}-\Delta_{24}]\end{array}\Big\rangle=\Phi_\alpha$ $\mathrm{H}^2(\mathfrak{N}^{1}_{08})=\Phi_1\oplus\Big\langle[\Delta_{32}+\Delta_{41}-\Delta_{23}-\Delta_{14}]\Big\rangle$
$\mathfrak{N}^{\alpha}_{09}:\quad e_1e_1=e_4\quad e_1e_2=\alpha e_4\quad e_2e_1=-\alpha e_4\quad e_2e_2=e_4\quad e_3e_3=e_4$ $\mathrm{H}^2(\mathfrak{N}^{\alpha}_{09})=\Big\langle[\Delta_{12}][\Delta_{12}],[\Delta_{21}],[\Delta_{22}],[\Delta_{23}],[\Delta_{31}],[\Delta_{32}],[\Delta_{33}]\Big\rangle$
$\mathfrak{N}_{10}:\quad e_1e_2=e_4\quad e_1e_3=e_4\quad e_2e_1=-e_4\quad e_2e_2=e_4\quad e_3e_1=e_4$ $\mathrm{H}^2(\mathfrak{N}_{10})=\Big\langle[\Delta_{12}],[\Delta_{12}],[\Delta_{21}],[\Delta_{22}],[\Delta_{23}],[\Delta_{31}],[\Delta_{32}],[\Delta_{33}]\Big\rangle$
$\mathfrak{N}_{11}:\quad e_1e_1=e_4\quad e_1e_2=e_4\quad e_2e_1=-e_4\quad e_3e_3=e_4$ $\mathrm{H}^2(\mathfrak{N}_{11})=\Big\langle[\Delta_{12}],[\Delta_{12}],[\Delta_{21}],[\Delta_{22}],[\Delta_{23}],[\Delta_{31}],[\Delta_{32}],[\Delta_{33}]\Big\rangle$
$\mathfrak{N}_{12}:\quad e_1e_2=e_3\quad e_2e_1=e_4$ $\mathrm{H}^2(\mathfrak{N}_{12})=\Big\langle[\Delta_{11}],[\Delta_{13}],[\Delta_{14}-\Delta_{41}],[\Delta_{22}],[\Delta_{23}-\Delta_{32}],[\Delta_{24}]\Big\rangle$
$\mathfrak{N}_{13}:\quad e_1e_1=e_4\quad e_1e_2=e_3\quad e_2e_1=-e_3\quad e_2e_2=2e_3+e_4$ $\mathrm{H}^2(\mathfrak{N}_{13})=\Big\langle\begin{array}{l}[\Delta_{21}],[\Delta_{22}],[\Delta_{14}+\Delta_{23}],[-\Delta_{13}+2\Delta_{14}+\Delta_{24}],\\ [\Delta_{31}-2\Delta_{13}-2\Delta_{32}+\Delta_{42}],[\Delta_{41}-2\Delta_{14}-\Delta_{32}]\end{array}\Big\rangle$
$\mathfrak{N}^{\alpha}_{14}:\quad e_1e_2=e_4\quad e_2e_1=\alpha e_4\quad e_2e_2=e_3$ $\mathrm{H}^2(\mathfrak{N}^{\alpha\neq0,1}_{14})=\Big\langle\begin{array}{l}[\Delta_{11}],[\Delta_{21}],[\Delta_{23}],[\Delta_{13}+\Delta_{24}],\\ [\Delta_{32}],[(\alpha-1)\Delta_{24}+\alpha\Delta_{31}+\Delta_{42}]\end{array}\Big\rangle=\Phi_\alpha$ $\mathrm{H}^2(\mathfrak{N}^{0}_{14})=\Phi_0\oplus\Big\langle[\Delta_{14}]\Big\rangle$ $\mathrm{H}^2_{com}(\mathfrak{N}^{1}_{14})=\Big\langle[\Delta_{11}],\ [\Delta_{13}+\Delta_{31}+\Delta_{24}+\Delta_{42}],[\Delta_{23}+\Delta_{32}]\Big\rangle$ $\mathrm{H}^2(\mathfrak{N}^{1}_{14})=\mathrm{H}^2_{com}(\mathfrak{N}^{1}_{14})\oplus\Big\langle[\Delta_{21}],[\Delta_{32}],[\Delta_{31}+\Delta_{42}]\Big\rangle$
$\mathfrak{N}_{15}:\quad e_1e_2=e_4\quad e_2e_1=-e_4\quad e_3e_3=e_4$ $\mathrm{H}^2(\mathfrak{N}_{15})=\Big\langle[\Delta_{11}],[\Delta_{13}],[\Delta_{21}],[\Delta_{22}],[\Delta_{23}],[\Delta_{31}],[\Delta_{32}],[\Delta_{33}]\Big\rangle$

1.3.2. Central extensions of $\mathfrak{N}_{01}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12} + \Delta_{21}], \quad \nabla_2 = [\Delta_{13} + \Delta_{31}], \quad \nabla_3 = [\Delta_{14} + \Delta_{41}], \quad \nabla_4 = [\Delta_{33}], \quad \nabla_5 = [\Delta_{34} + \Delta_{43}],$$
$$\nabla_6 = [\Delta_{44}], \quad \nabla_7 = [\Delta_{21}], \quad \nabla_8 = [\Delta_{31}], \quad \nabla_9 = [\Delta_{41}], \quad \nabla_{10} = [\Delta_{43}].$$

Take $\theta = \sum_{i=1}^{10} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{01})$. The automorphism group of $\mathfrak{N}_{01}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ q & x^2 & r & u \\ w & 0 & t & k \\ z & 0 & y & l \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 + \alpha_7 & 0 & 0 & 0 \\ \alpha_2 + \alpha_8 & 0 & \alpha_4 & \alpha_5 \\ \alpha_3 + \alpha_9 & 0 & \alpha_5 + \alpha_{10} & \alpha_6 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* & \alpha_3^* \\ \alpha_1^* + \alpha_7^* & 0 & 0 & 0 \\ \alpha_2^* + \alpha_8^* & 0 & \alpha_4^* & \alpha_5^* \\ \alpha_3^* + \alpha_9^* & 0 & \alpha_5^* + \alpha_{10}^* & \alpha_6^* \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{01})$ on the subspace $\langle \sum_{i=1}^{10} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{10} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= x^3\alpha_1, \\
\alpha_2^* &= rx\alpha_1 + y\left(x\alpha_3 + w\alpha_5 + z\alpha_6\right) + t\left(x\alpha_2 + w\alpha_4 + z\left(\alpha_5 + \alpha_{10}\right)\right), \\
\alpha_3^* &= ux\alpha_1 + l\left(x\alpha_3 + w\alpha_5 + z\alpha_6\right) + k\left(x\alpha_2 + w\alpha_4 + z\left(\alpha_5 + \alpha_{10}\right)\right), \\
\alpha_4^* &= t^2\alpha_4 + y\left(2t\alpha_5 + y\alpha_6 + t\alpha_{10}\right), \\
\alpha_5^* &= kt\alpha_4 + (lt + ky)\alpha_5 + y\left(l\alpha_6 + k\alpha_{10}\right) \\
\alpha_6^* &= k^2\alpha_4 + l\left(2k\alpha_5 + l\alpha_6 + k\alpha_{10}\right), \\
\alpha_7^* &= x^3\alpha_7, \\
\alpha_8^* &= rx\alpha_7 + tx\alpha_8 + xy\alpha_9 + wy\alpha_{10} - tz\alpha_{10}, \\
\alpha_9^* &= ux\alpha_7 + kx\alpha_8 + lx\alpha_9 + lw\alpha_{10} - kz\alpha_{10}, \\
\alpha_{10}^* &= (lt - ky)\alpha_{10}.
\end{aligned}$$

We are interested only in the cases with

$$(\alpha_1, \alpha_7) \neq (0,0), (\alpha_2, \alpha_4, \alpha_5, \alpha_8, \alpha_{10}) \neq (0,0,0,0,0),$$
$$(\alpha_3, \alpha_5, \alpha_6, \alpha_9, \alpha_{10}) \neq (0,0,0,0,0), (\alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \neq (0,0,0,0).$$

1. $\alpha_1 = 0,\ \alpha_7 \neq 0$, then choosing $r = -\frac{tx\alpha_8 + xy\alpha_9 + (wy - tz)\alpha_{10}}{x\alpha_7}$, $u = -\frac{kx\alpha_8 + lx\alpha_9 + (lw - kz)\alpha_{10}}{x\alpha_7}$, we have $\alpha_8^* = \alpha_9^* = 0$. The family of orbits $\langle \alpha_4\nabla_4 + \alpha_5\nabla_5 + \alpha_6\nabla_6 + \alpha_{10}\nabla_{10} \rangle$ gives us characterised structure of three dimensional ideal whose a one dimensional extension of two dimensional subalgebra with basis $\{e_3, e_4\}$. Let us remember the classification of algebras of this type.

$$\begin{array}{llll}
\mathcal{N}_{01}^{3*} & : & e_1e_1 = e_2 & \\
\mathcal{N}_{02}^{3*} & : & e_1e_1 = e_3 & e_2e_2 = e_3 \\
\mathcal{N}_{03}^{3*} & : & e_1e_2 = e_3 & e_2e_1 = -e_3 \\
\mathcal{N}_{04}^{3*}(\lambda) & : & e_1e_1 = \lambda e_3 & e_2e_1 = e_3 \quad e_2e_2 = e_3
\end{array}$$

Using the classification of three dimensional nilpotent algebras, we may consider following cases.

   (a) $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three dimensional ideal is abelian. Then we may suppose $\alpha_2 \neq 0$ and choosing $y = 0$, $l = \alpha_2$, $k = -\alpha_3$, we obtain that $\alpha_3^* = 0$, which implies $(\alpha_3^*, \alpha_5^*, \alpha_6^*, \alpha_9^*, \alpha_{10}^*) = (0,0,0,0,0)$. Thus, in this case we do not have new algebras.

   (b) $\alpha_4 = 1$, $\alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{01}^{3*}$. Then $\alpha_3 \neq 0$ and choosing $x = 1$, $t = 1$, $k = 0$, $y = 0$, $w = -\alpha_2$, $l = \frac{\alpha_7}{\alpha_3}$ and $t = \sqrt{\alpha_7}$, we have the representative $\langle \nabla_3 + \nabla_4 + \nabla_7 \rangle$.

(c) $\alpha_4 = \alpha_6 = 1$, $\alpha_5 = \alpha_{10} = 0$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{02}^{3*}$. Then choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}, k = y = 0, l = t = 1, w = -\frac{\alpha_2}{\sqrt[3]{\alpha_7}}, z = -\frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, we have the representative $\langle \nabla_4 + \nabla_6 + \nabla_7 \rangle$.

(d) $\alpha_4 = \alpha_6 = 0$, $\alpha_5 = 1, \alpha_{10} = -2$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{03}^{3*}$. Then choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}, k = y = 0, l = t = 1, w = -\frac{\alpha_3}{\sqrt[3]{\alpha_7}}, z = \frac{\alpha_2}{\sqrt[3]{\alpha_7}}$, we have the representative $\langle \nabla_5 + \nabla_7 - 2\nabla_{10} \rangle$.

(e) $\alpha_4 = \lambda, \alpha_5 = 0, \alpha_6 = 1, \alpha_{10} = 1$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{04}^{3*}(\lambda)$.

i. If $\lambda \neq 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}, k = 0, y = 0, l = t = 1, z = \frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, and $w = \frac{\alpha_2 - \alpha_3}{\lambda \sqrt[3]{\alpha_7}}$, we have the family of representatives $\langle \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle_{\lambda \neq 0}$.

ii. If $\lambda = 0$ and $\alpha_2 = \alpha_3$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_7}}, k = 0, y = 0, l = t = 1$ and $z = \frac{\alpha_3}{\sqrt[3]{\alpha_7}}$, we have the representative $\langle \nabla_6 + \nabla_7 + \nabla_{10} \rangle$.

iii. If $\lambda = 0$ and $\alpha_2 \neq \alpha_3$, then choosing $x = \frac{(\alpha_2-\alpha_3)^2}{\alpha_7}, k = 0, y = 0, l = t = \frac{(\alpha_2-\alpha_3)^3}{\alpha_7}$ and $z = -\frac{\alpha_3(\alpha_2-\alpha_3)^2}{\alpha_7}$, we have the representative $\langle \nabla_2 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle$.

2. $\alpha_1 \neq 0$, then choosing

$$\begin{array}{rcl} r & = & -\frac{tx\alpha_2 + xy\alpha_3 + tw\alpha_4 + wy\alpha_5 + tz\alpha_5 + yz\alpha_6 + tz\alpha_{10}}{x\alpha_1}, \\ u & = & -\frac{kx\alpha_2 + lx\alpha_3 + kw\alpha_4 + lw\alpha_5 + kz\alpha_5 + lz\alpha_6 + kz\alpha_{10}}{x\alpha_1}, \end{array}$$

we have $\alpha_2^* = \alpha_3^* = 0$.

(a) $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three dimensional ideal is abelian. Then we may suppose $\alpha_8 \neq 0$ and choosing $y = 0$, $l = \alpha_8$, $k = -\alpha_9$, we obtain that $\alpha_9^* = 0$, which implies $(\alpha_3^*, \alpha_5^*, \alpha_6^*, \alpha_9^*, \alpha_{10}^*) = (0, 0, 0, 0, 0)$. Thus, in this case we do not have new algebras.

(b) $\alpha_4 = 1$, $\alpha_5 = \alpha_6 = \alpha_{10} = 0$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{01}^{3*}$. Then $\alpha_9 \neq 0$, and choosing $x = 1, k = 0, t = \sqrt{\alpha_1}, y = -\frac{\sqrt{\alpha_1}\alpha_8}{\alpha_9}, l = \frac{\alpha_1}{\alpha_9}$ and $w = 0$, we have the family of representatives $\langle \nabla_1 + \nabla_4 + \alpha\nabla_7 + \nabla_9 \rangle$.

(c) $\alpha_4 = \alpha_6 = 1$, $\alpha_5 = \alpha_{10} = 0$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{02}^{3*}$.

i. $\alpha_7 = 0$, then $\alpha_8^2 + \alpha_9^2 \neq 0$, then choosing $x = \frac{\alpha_8^2+\alpha_9^2}{\alpha_1}$, $t = \frac{\alpha_8(\alpha_8^2+\alpha_9^2)}{\alpha_1}$, $y = \frac{\alpha_9(\alpha_8^2+\alpha_9^2)}{\alpha_1}$, $l = \frac{\alpha_8(\alpha_8^2+\alpha_9^2)}{\alpha_1}$, $k = -\frac{\alpha_9(\alpha_8^2+\alpha_9^2)}{\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_4 + \nabla_6 + \nabla_8 \rangle$.

ii. $\alpha_7 = 0$, $\alpha_8^2 + \alpha_9^2 = 0$, i.e., $\alpha_9 = \pm i\alpha_8 \neq 0$, then choosing $x = \sqrt{\alpha_8}$, $t = \frac{\alpha_1}{2}$, $y = \pm\frac{\alpha_1}{2i}$, $l = \pm ix\alpha_8$, $k = x\alpha_8$, have the representative $\langle \nabla_1 + \nabla_5 + \nabla_8 \rangle$.

iii. $\alpha_7 \neq 0$, then choosing $x = 1$, $y = k = 0$, $t = l = \sqrt{\alpha_7}$, $z = \frac{\alpha_1\alpha_9}{\alpha_7}$, $w = \frac{\alpha_1\alpha_8}{\alpha_7}$ we have the family of representatives $\langle \alpha\nabla_1 + \nabla_4 + \nabla_6 + \nabla_7 \rangle_{\alpha \neq 0}$.

(d) $\alpha_4 = \alpha_6 = 0$, $\alpha_5 = 1, \alpha_{10} = -2$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{03}^{3*}$.

i. $2\alpha_1 + \alpha_7 \neq 0$, then choosing $x = 1$, $y = k = 0$, $t = \sqrt{\alpha_1}$, $l = 1$, $z - \frac{\alpha_1\alpha_8}{2\alpha_1+\alpha_7}$ and $w = \frac{\alpha_1\alpha_9}{2\alpha_1+\alpha_7}$, we have the family of representatives $\langle \nabla_1 + \nabla_5 + \alpha\nabla_7 - 2\nabla_{10} \rangle_{\alpha \neq -2}$.

ii. $2\alpha_1 + \alpha_7 = 0$, then in case of $(\alpha_8, \alpha_9) = (0, 0)$, we have the representative $\langle \nabla_1 + \nabla_5 - 2\nabla_7 - 2\nabla_{10} \rangle$ and case of $(\alpha_8, \alpha_9) \neq (0, 0)$, without loss of generality we may assume $\alpha_8 \neq 0$ and choosing $x = 1$, $y = 0$, $l = \alpha_8$, $k = -\alpha_9$, $t = \frac{\alpha_1}{\alpha_8}$, we have the representative $\langle \nabla_1 + \nabla_5 - 2\nabla_7 + \nabla_8 - 2\nabla_{10} \rangle$.

(e) $\alpha_4 = \lambda, \alpha_5 = 0, \alpha_6 = 1, \alpha_{10} = 1$, i.e., three dimensional ideal is isomorphic to $\mathcal{N}_{04}^{3*}(\lambda)$.

i. $\alpha_1^2 + \alpha_1\alpha_7 + \lambda\alpha_7^2 \neq 0$, then choosing $x = 1$, $y = k = 0$ and

$$t = l = \sqrt{\alpha_7}, z = \frac{\alpha_1(\alpha_1\alpha_8 + \alpha_4\alpha_7\alpha_9)}{\alpha_1^2 + \alpha_1\alpha_7 + \alpha_4\alpha_7^2}, w = \frac{\alpha_1(\alpha_7(\alpha_8 - \alpha_9) + \alpha_1\alpha_9)}{\alpha_1^2 + \alpha_1\alpha_7 + \alpha_4\alpha_7^2},$$

we have the representative $\langle \alpha\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle$.

ii. $\alpha_1^2 + \alpha_1\alpha_7 + \lambda\alpha_7^2 = 0$, then choosing $y = k = 0$, $w = \frac{z\alpha_7}{\alpha_1} - x\alpha_9$, we have $\alpha_9^* = 0$, $\alpha_8^* = \frac{tx}{\alpha_1}(\alpha_1\alpha_8 - \lambda\alpha_7\alpha_9)$. Thus, in this case we have the representatives $\langle \frac{-1\pm\sqrt{1-4\lambda}}{2}\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle$ and $\langle \frac{-1\pm\sqrt{1-4\lambda}}{2}\nabla_1 + \lambda\nabla_4 + \nabla_6 + \nabla_7 + \nabla_8 + \nabla_{10} \rangle$ depending on $\alpha_1\alpha_8 = \lambda\alpha_7\alpha_9$ or not.

Summarizing all cases, we have the following distinct orbits

$$\langle\nabla_3+\nabla_4+\nabla_7\rangle, \langle\nabla_5+\nabla_7-2\nabla_{10}\rangle, \langle\nabla_2+\nabla_6+\nabla_7+\nabla_{10}\rangle\ \langle\nabla_1+\nabla_4+\alpha\nabla_7+\nabla_9\rangle, \langle\nabla_1+\nabla_4+\nabla_6+\nabla_8\rangle,$$
$$\langle\nabla_1+\nabla_5+\nabla_8\rangle, \langle\alpha\nabla_1+\nabla_4+\nabla_6+\nabla_7\rangle, \langle\nabla_1+\nabla_5+\alpha\nabla_7-2\nabla_{10}\rangle,$$
$$\langle\nabla_1+\nabla_5-2\nabla_7+\nabla_8-2\nabla_{10}\rangle, \langle\alpha\nabla_1+\lambda\nabla_4+\nabla_6+\nabla_7+\nabla_{10}\rangle,$$
$$\langle\tfrac{-1\pm\sqrt{1-4\lambda}}{2}\nabla_1+\lambda\nabla_4+\nabla_6+\nabla_7+\nabla_8+\nabla_{10}\rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{01}, \mathrm{N}_{02}, \mathrm{N}_{03}, \mathrm{N}_{04}^{\alpha}, \mathrm{N}_{05}, \mathrm{N}_{06}, \mathrm{N}_{07}^{\alpha}, \mathrm{N}_{08}^{\alpha}, \mathrm{N}_{09}, \mathrm{N}_{10}^{\alpha,\lambda}, \mathrm{N}_{11}^{\lambda}, \mathrm{N}_{12}^{\lambda\neq\frac{1}{4}}.$$

1.3.3. Central extensions of $\mathfrak{N}_{02}$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1=[\Delta_{12}+\Delta_{21}], & \nabla_2=[\Delta_{13}+\Delta_{31}], & \nabla_3=[\Delta_{24}+\Delta_{42}],\\ \nabla_4=[\Delta_{21}], & \nabla_5=[\Delta_{31}], & \nabla_6=[\Delta_{42}]. \end{array}$$

Take $\theta=\sum\limits_{i=1}^{6}\alpha_i\nabla_i\in \mathrm{H}^2(\mathfrak{N}_{02})$. The automorphism group of $\mathfrak{N}_{02}$ consists of invertible matrices of the form

$$\phi_1=\begin{pmatrix} x & 0 & 0 & 0\\ 0 & y & 0 & 0\\ z & u & x^2 & 0\\ t & v & 0 & y^2 \end{pmatrix}, \quad \phi_2=\begin{pmatrix} 0 & x & 0 & 0\\ y & 0 & 0 & 0\\ z & u & 0 & x^2\\ t & v & y^2 & 0 \end{pmatrix}.$$

Since

$$\phi_1^T\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & 0\\ \alpha_1+\alpha_4 & 0 & 0 & \alpha_3\\ \alpha_2+\alpha_5 & 0 & 0 & 0\\ 0 & \alpha_3+\alpha_6 & 0 & 0 \end{pmatrix}\phi_1=\begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* & 0\\ \alpha_1^*+\alpha_4^* & \alpha^{**} & 0 & \alpha_3^*\\ \alpha_2^*+\alpha_5^* & 0 & 0 & 0\\ 0 & \alpha_3^*+\alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{02})$ on the subspace $\langle\sum\limits_{i=1}^{10}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{10}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* &=& xy\alpha_1+ux\alpha_2+ty(\alpha_3+\alpha_6), & \alpha_2^* &=& x^3\alpha_2, & \alpha_3^* &=& y^3\alpha_3,\\ \alpha_4^* &=& xy\alpha_4+ux\alpha_5-ty\alpha_6, & \alpha_5^* &=& x^3\alpha_5, & \alpha_6^* &=& y^3\alpha_6. \end{array}$$

We are interested only in the cases with

$$(\alpha_3,\alpha_6)\neq(0,0), (\alpha_2,\alpha_5)\neq(0,0), (\alpha_4,\alpha_5,\alpha_6)\neq(0,0,0).$$

1. $(\alpha_5,\alpha_6)=(0,0)$, then $\alpha_2\alpha_3\alpha_4\neq 0$ and by choosing $x=\frac{\alpha_4}{\sqrt[3]{\alpha_2^2\alpha_3}}$, $y=\frac{\alpha_4}{\sqrt[3]{\alpha_2\alpha_3^2}}$ and $t=-\frac{x(y\alpha_1+u\alpha_2)}{y\alpha_3}$, we have the representative $\langle\nabla_2+\nabla_3+\nabla_4\rangle$;
2. $(\alpha_5,\alpha_6)\neq(0,0)$, then without loss of generality (maybe with an action of a suitable ϕ_2), we can suppose $\alpha_5\neq 0$ and choosing $u=\frac{ty\alpha_6-xy\alpha_4}{x\alpha_5}$, we have $\alpha_4^*=0$.
   (a) $\alpha_3\alpha_5+(\alpha_2+\alpha_5)\alpha_6=0$, then $\alpha_6\neq 0$.
      i. if $\alpha_1\neq 0$, then choosing $x=\frac{\alpha_1}{\sqrt[3]{\alpha_5^2\alpha_6}}$, $y=\frac{\alpha_1}{\sqrt[3]{\alpha_5\alpha_6^2}}$, we have the family of representatives $\langle\nabla_1+\alpha\nabla_2-(1+\alpha)\nabla_3+\nabla_5+\nabla_6\rangle$;
      ii. if $\alpha_1=0$, then choosing $x=y\sqrt[3]{\frac{\alpha_6}{\alpha_5}}$, we have the family if representatives $\langle\alpha\nabla_2-(1+\alpha)\nabla_3+\nabla_5+\nabla_6\rangle$.
   (b) $\alpha_3\alpha_5+(\alpha_2+\alpha_5)\alpha_6\neq 0$, then choosing $t=-\frac{x\alpha_1\alpha_5}{\alpha_3\alpha_5+(\alpha_2+\alpha_5)\alpha_6}$, we have $\alpha_1^*=0$.
      i. if $\alpha_6=0$, then choosing $x=y\sqrt[3]{\frac{\alpha_3}{\alpha_5}}$, we have the representative $\langle\alpha\nabla_2+\nabla_3+\nabla_5\rangle$;
      ii. if $\alpha_6\neq 0$, then choosing $x=y\sqrt[3]{\frac{\alpha_6}{\alpha_5}}$, we have the representative $\langle\alpha\nabla_2+\beta\nabla_3+\nabla_5+\nabla_6\rangle_{\beta\neq-(1+\alpha)}$.

Summarizing all cases, we have the following distinct orbits

$$\langle\nabla_2+\nabla_3+\nabla_4\rangle, \langle\nabla_1+\alpha\nabla_2-(1+\alpha)\nabla_3+\nabla_5+\nabla_6\rangle,$$
$$\langle\alpha\nabla_2+\nabla_3+\nabla_5\rangle, \langle\alpha\nabla_2+\beta\nabla_3+\nabla_5+\nabla_6\rangle^{O(\alpha,\beta)\simeq O(\beta,\alpha)},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{13}, \mathrm{N}_{14}^{\alpha}, \mathrm{N}_{15}^{\alpha}, \mathrm{N}_{16}^{\alpha,\beta}.$$

1.3.4. Central extensions of $\mathfrak{N}_{04}^0$

Let us use the following notations:

$$\begin{array}{lllll}\nabla_1=[\Delta_{12}], & \nabla_2=[\Delta_{13}], & \nabla_3=[\Delta_{14}], & \nabla_4=[\Delta_{21}], & \nabla_5=[\Delta_{22}],\\ \nabla_6=[\Delta_{24}], & \nabla_7=[\Delta_{41}], & \nabla_8=[\Delta_{42}], & \nabla_9=[\Delta_{44}], & \nabla_{10}=[\Delta_{31}+\Delta_{32}-\Delta_{23}].\end{array}$$

Take $\theta=\sum\limits_{i=1}^{10}\alpha_i\nabla_i\in \mathrm{H}^2(\mathfrak{N}_{04}^0)$. The automorphism group of $\mathfrak{N}_{04}^0$ consists of invertible matrices of the form

$$\phi=\begin{pmatrix} x & 0 & 0 & 0\\ y & x+y & 0 & 0\\ z & t & x(x+y) & w\\ u & v & 0 & r\end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3\\ \alpha_4 & \alpha_5 & -\alpha_{10} & \alpha_6\\ \alpha_{10} & \alpha_{10} & 0 & 0\\ \alpha_7 & \alpha_8 & 0 & \alpha_9\end{pmatrix}\phi=\begin{pmatrix} \alpha^* & \alpha_1^*+\alpha^* & \alpha_2^* & \alpha_3^*\\ \alpha_4^* & \alpha_5^* & -\alpha_{10}^* & \alpha_6^*\\ \alpha_{10}^* & \alpha_{10}^* & 0 & 0\\ \alpha_7^* & \alpha_8^* & 0 & \alpha_9^*\end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{04}^0)$ on the subspace $\langle\sum\limits_{i=1}^{10}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{10}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{lll}
\alpha_1^* &=& x^2\alpha_1+x(t+y-z)\alpha_2-xy\alpha_4-ux(\alpha_7-\alpha_8)-(u-v)(x\alpha_3+y\alpha_6+u\alpha_9)-y(t-z)\alpha_{10},\\
\alpha_2^* &=& x(x+y)(x\alpha_2-y\alpha_{10}),\\
\alpha_3^* &=& wx\alpha_2+rx\alpha_3+ry\alpha_6+ru\alpha_9-wy\alpha_{10},\\
\alpha_4^* &=& u((x+y)\alpha_6+v\alpha_9)-(x+y)z\alpha_{10}+x((x+y)\alpha_4+v\alpha_7+t\alpha_{10})+y((x+y)\alpha_2+v\alpha_8+t\alpha_{10}),\\
\alpha_5^* &=& (x+y)^2\alpha_5+v((x+y)\alpha_6+(x+y)\alpha_8+v\alpha_9),\\
\alpha_6^* &=& r(x+y)\alpha_6+rv\alpha_9-w(x+y)\alpha_{10},\\
\alpha_7^* &=& rx\alpha_7+ry\alpha_8+ru\alpha_9+w(x+y)\alpha_{10},\\
\alpha_8^* &=& rv\alpha_9+(x+y)(r\alpha_8+w\alpha_{10}),\\
\alpha_9^* &=& r^2\alpha_9,\\
\alpha_{10}^* &=& x(x+y)^2\alpha_{10}.
\end{array}$$

Since we are interested only in the cases with

$$(\alpha_2,\alpha_{10})\neq(0,0),\quad (\alpha_3,\alpha_6,\alpha_7,\alpha_8,\alpha_9)\neq(0,0,0,0,0),$$

consider the following subcases:

1. $\alpha_{10}=0$, then $\alpha_2\neq0$ and choosing $w=-\frac{r(x\alpha_3+y\alpha_6+u\alpha_9)}{x\alpha_2}$ and
$$t=\frac{xz\alpha_2-x^2\alpha_1+ux\alpha_3-vx\alpha_3+xy\alpha_4-xy\alpha_5+uy\alpha_6-vy\alpha_6+ux\alpha_7-ux\alpha_8+u^2\alpha_9-uv\alpha_9}{x\alpha_2},$$
we have $\alpha_1^*=\alpha_3^*=0$.

(a) $\alpha_9 \neq 0$, then choosing $u = -\frac{x\alpha_7+y\alpha_8}{\alpha_9}$, $v = -\frac{(x+y)\alpha_8}{\alpha_9}$, we have $\alpha_7^* = \alpha_8^* = 0$.

i. $\alpha_5 = \alpha_4 = \alpha_6 = 0$, then choosing $r = \sqrt{\frac{\alpha_2}{\alpha_9}}$, we have the representative $\langle \nabla_2 + \nabla_9 \rangle$;

ii. $\alpha_5 = \alpha_4 = 0$, $\alpha_6 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_2\alpha_9-\alpha_6^2}{\alpha_6^2}$, $r = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_6 + \nabla_9 \rangle$;

iii. $\alpha_5 = 0$, $\alpha_4 \neq 0$, $\alpha_6 = 0$ then choosing $x = \frac{\alpha_4}{\alpha_2}$, $y = 0$, $r = \frac{\sqrt{\alpha_4^3}}{\alpha_2\sqrt{\alpha_9}}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_9 \rangle$;

iv. $\alpha_5 = 0$, $\alpha_4 \neq 0$, $\alpha_6 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_2}$, $y = \frac{\alpha_4(\alpha_4\alpha_9-\alpha_6^2)}{\alpha_2\alpha_6^2}$, $r = \frac{\alpha_4^2}{\alpha_2\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_6 + \nabla_9 \rangle$;

v. $\alpha_5 \neq 0$, $\alpha_4 = \alpha_5$, then choosing $x = 1$, $y = \frac{\alpha_2-\alpha_5}{\alpha_5}$, $r = \frac{\alpha_2}{\sqrt{\alpha_5\alpha_9}}$, we have the family of representatives $\langle \nabla_2 + \nabla_4 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle$;

vi. $\alpha_5 \neq 0$, $\alpha_4 \neq \alpha_5$, then choosing $x = \frac{\alpha_5-\alpha_4}{\alpha_2}$, $y = \frac{\alpha_4(\alpha_4-\alpha_5)}{\alpha_2\alpha_5}$, $r = \frac{(\alpha_4-\alpha_5)^2}{\alpha_2\sqrt{\alpha_5\alpha_9}}$ we have the family of representatives $\langle \nabla_2 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle$;

(b) $\alpha_9 = 0$, $\alpha_8 \neq 0$, $\alpha_7 = \alpha_8$, then choosing $v = -\frac{x\alpha_4+y\alpha_5+u\alpha_6}{\alpha_8}$, we have $\alpha_4^* = 0$.

i. $\alpha_6 = -\alpha_8, \alpha_5 = 0$, then choosing $x = 1$, $y = 0$, $r = \frac{\alpha_2}{\alpha_5}$, we have the representative $\langle \nabla_2 - \nabla_6 + \nabla_7 + \nabla_8 \rangle$;

ii. $\alpha_6 = -\alpha_8, \alpha_5 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_2-\alpha_5}{\alpha_5}$, $r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_7 + \nabla_8 \rangle$;

iii. $\alpha_6 \neq -\alpha_8$, $\alpha_6 = 0$, $\alpha_5 = 0$, then choosing $x = 1$, $y = 0$, $r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_7 + \nabla_8 \rangle$;

iv. $\alpha_6 \neq -\alpha_8$, $\alpha_6 = 0$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2}$, $r = \frac{\alpha_5^2}{\alpha_2\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_5 + \nabla_7 + \nabla_8 \rangle$;

v. $\alpha_6 \neq -\alpha_8$, $\alpha_6 \neq 0$, then choosing $x = 1$, $y = 0$, $r = \frac{\alpha_2}{\alpha_8}$, we have the family of representatives $\langle \nabla_2 + \alpha\nabla_6 + \nabla_7 + \nabla_8 \rangle_{\alpha\neq 0,-1}$.

(c) $\alpha_9 = 0$, $\alpha_8 \neq 0$, $\alpha_7 \neq \alpha_8$, then choosing $y = -\frac{x\alpha_7}{\alpha_8}$, we have $\alpha_7^* = 0$. Hence,

i. $\alpha_6 = -\alpha_8$, $\alpha_5 = 0$, then choosing $x = 1$, $u = \frac{\alpha_4}{\alpha_8}$, $r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 - \nabla_6 + \nabla_8 \rangle$;

ii. $\alpha_6 = -\alpha_8$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2}$, $u = \frac{\alpha_4\alpha_5}{\alpha_2\alpha_8}$, $r = \frac{\alpha_5^2}{\alpha_2\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_8 \rangle$;

iii. $\alpha_6 \neq -\alpha_8$, $\alpha_6 = \alpha_4 = 0$, then choosing $x = 1$, $v = -\frac{\alpha_5}{\alpha_8}$, $r = \frac{\alpha_2}{\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_8 \rangle$;

iv. $\alpha_6 \neq -\alpha_8, \alpha_6 = 0$, $\alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_2}$, $v = -\frac{\alpha_4\alpha_5}{\alpha_2\alpha_8}$, $r = \frac{\alpha_4^2}{\alpha_2\alpha_8}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_8 \rangle$;

v. $\alpha_6 \neq -\alpha_8$, $\alpha_6 \neq 0$, then choosing $x = 1$, $u = -\frac{\alpha_4}{\alpha_6}$, $v = -\frac{\alpha_5}{\alpha_6+\alpha_8}$, $r = \frac{\alpha_2}{\alpha_8}$, we have the family of representatives $\langle \nabla_2 + \alpha\nabla_6 + \nabla_8 \rangle_{\alpha\neq 0,-1}$.

(d) $\alpha_9 = \alpha_8 = 0$, $\alpha_7 \neq 0$, then choosing $v = -\frac{(x+y)(x\alpha_4+y\alpha_5+u\alpha_6)}{x\alpha_7}$ we have $\alpha_4^* = 0$. Hance,

i. $\alpha_6 = \alpha_5 = 0$, then choosing $x = 1$, $y = 0$, $r = \frac{\alpha_2}{\alpha_7}$, we have the representative $\langle \nabla_2 + \nabla_7 \rangle$;

ii. $\alpha_6 = 0$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2}$, $y = 0$, $r = \frac{\alpha_5^2}{\alpha_2\alpha_7}$, we have the representative $\langle \nabla_2 + \nabla_5 + \nabla_7 \rangle$;

iii. $\alpha_6 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_7-\alpha_6}{\alpha_6}$, $u = \frac{\alpha_5}{\alpha_6}$, $r = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_6 + \nabla_7 \rangle$.

(e) $\alpha_9 = \alpha_8 = \alpha_7 = 0$, $\alpha_6 \neq 0$, then choosing $x = 1$, $y = 0$, $u = -\frac{\alpha_4}{\alpha_6}$, $v = -\frac{\alpha_5}{\alpha_6}$, $r = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_6 \rangle$.

2. $\alpha_{10} \neq 0$, then choosing $t = -\frac{x(x+y)\alpha_4+y(x+y)\alpha_5+ux\alpha_6+uy\alpha_6+vx\alpha_7+vy\alpha_8+uv\alpha_9-xz\alpha_{10}-yz\alpha_{10}}{(x+y)\alpha_{10}}$ and $w = -\frac{r((x+y)\alpha_8+v\alpha_9)}{(x+y)\alpha_{10}}$, we have $\alpha_4^* = \alpha_8^* = 0$. Now we consider following subcases:

(a) $\alpha_9 \neq 0$, then choosing $u = -\frac{(x+y)\alpha_6+2x\alpha_7}{2\alpha_9}$, $v = -\frac{(x+y)\alpha_6}{2\alpha_9}$, we have $\alpha_6^* = \alpha_7^* = 0$. Hence,

i. $\alpha_2 = -\alpha_{10},\ \alpha_5 = \alpha_1 = 0,\ \alpha_3 = 0$, then choosing $x = 1,\ y = 0,\ r = \sqrt{\frac{\alpha_{10}}{\alpha_9}}$, we have the representative $\langle -\nabla_2 + \nabla_9 + \nabla_{10}\rangle$;

ii. $\alpha_2 = -\alpha_{10},\ \alpha_5 = \alpha_1 = 0,\ \alpha_3 \neq 0$, then choosing $x = 1,\ y = -1 + \frac{\alpha_3}{\sqrt{\alpha_9\alpha_{10}}},\ r = \frac{\alpha_3}{\alpha_9}$, we have the representative $\langle -\nabla_2 + \nabla_3 + \nabla_9 + \nabla_{10}\rangle$;

iii. $\alpha_2 = -\alpha_{10},\ \alpha_5 = 0,\ \alpha_1 \neq 0$, then choosing $x = 1,\ y = -1 + \sqrt{\frac{\alpha_1}{\alpha_{10}}},\ r = \sqrt{\frac{\alpha_1}{\alpha_9}}$, we have the family of representatives $\langle \nabla_1 - \nabla_2 + \alpha\nabla_3 + \nabla_9 + \nabla_{10}\rangle$, where $O(\alpha) \simeq O(-\alpha)$;

iv. $\alpha_2 = -\alpha_{10},\ \alpha_5 \neq 0,\ \alpha_1 = \alpha_5,\ \alpha_3 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}},\ y = 0,\ r = \frac{\alpha_5\sqrt{\alpha_5}}{\alpha_{10}\sqrt{\alpha_9}}$, we have the representative $\langle \nabla_1 - \nabla_2 + \nabla_5 + \nabla_9 + \nabla_{10}\rangle$;

v. $\alpha_2 = -\alpha_{10},\ \alpha_5 \neq 0,\ \alpha_1 = \alpha_5,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}},\ y = \frac{\alpha_3\sqrt{\alpha_5} - \alpha_5\sqrt{\alpha_9}}{\alpha_{10}\sqrt{\alpha_9}},\ r = \frac{\alpha_3\alpha_5}{\alpha_9\alpha_{10}}$, we have the representative $\langle \nabla_1 - \nabla_2 + \nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10}\rangle$;

vi. $\alpha_2 = -\alpha_{10},\ \alpha_5 \neq 0,\ \alpha_1 \neq \alpha_5$, then choosing $x = \frac{\alpha_5}{\alpha_{10}},\ y = \frac{-\alpha_5 + \sqrt{\alpha_5(-\alpha_1+\alpha_5)}}{\alpha_{10}},\ r = \frac{\alpha_5\sqrt{\alpha_5-\alpha_1}}{\alpha_{10}\sqrt{\alpha_9}}$, we have the family of representatives $\langle -\nabla_2 + \alpha\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10}\rangle$, where $O(\alpha) \simeq O(-\alpha)$;

vii. $\alpha_2 \neq -\alpha_{10},\ \alpha_5 = \alpha_3 = \alpha_1 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_{10}},\ r = \frac{\alpha_2+\alpha_{10}}{\sqrt{\alpha_9\alpha_{10}}}$, we have the representative $\langle \nabla_9 + \nabla_{10}\rangle$;

viii. $\alpha_2 \neq -\alpha_{10},\ \alpha_5 = \alpha_3 = 0,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_{10}}{(\alpha_2+\alpha_{10})^2},\ y = \frac{\alpha_1\alpha_2}{(\alpha_2+\alpha_{10})^2},\ r = \frac{\alpha_1\alpha_{10}\sqrt{\alpha_1}}{\sqrt{\alpha_9}(\alpha_2+\alpha_{10})^2}$, we have the representative $\langle \nabla_1 + \nabla_9 + \nabla_{10}\rangle$;

ix. $\alpha_2 \neq -\alpha_{10},\ \alpha_5 = 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3^2\alpha_{10}}{\alpha_9(\alpha_2+\alpha_{10})^2},\ y = \frac{\alpha_2\alpha_3^2}{\alpha_9(\alpha_2+\alpha_{10})^2},\ r = \frac{\alpha_3^3\alpha_{10}}{\alpha_9^2(\alpha_2+\alpha_{10})^2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3 + \nabla_9 + \nabla_{10}\rangle$;

x. $\alpha_2 \neq -\alpha_{10},\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}},\ y = \frac{\alpha_2\alpha_5}{\alpha_{10}^2},\ r = \frac{\alpha_5\sqrt{\alpha_5}(\alpha_2+\alpha_{10})}{\sqrt{\alpha_9}\alpha_{10}^2}$, we have the family of representatives $\langle \alpha\nabla_1 + \beta\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10}\rangle$, where $O(\alpha,\beta) \simeq O(\alpha,-\beta)$.

(b) $\alpha_9 = 0,\ \alpha_6 \neq 0$, then choosing $v = -\frac{(x+y)\alpha_5}{\alpha_6}$, we have $\alpha_5^* = 0$. Hence,

i. $\alpha_2 = -\alpha_{10},\ \alpha_3 = \alpha_6,\ \alpha_7 = 0,\ \alpha_1 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_{10}}{\alpha_6}$, we have the representative $\langle -\nabla_2 + \nabla_3 + \nabla_6 + \nabla_{10}\rangle$;

ii. $\alpha_2 = -\alpha_{10},\ \alpha_3 = \alpha_6,\ \alpha_7 = 0,\ \alpha_1 \neq 0$, then choosing $x = 1,\ y = -1 + \sqrt{\frac{\alpha_1}{\alpha_{10}}},\ r = \frac{\sqrt{\alpha_1\alpha_{10}}}{\alpha_6}$, we have the representative $\langle \nabla_1 - \nabla_2 + \nabla_3 + \nabla_6 + \nabla_{10}\rangle$;

iii. $\alpha_2 = -\alpha_{10},\ \alpha_3 = \alpha_6,\ \alpha_7 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_1-\alpha_6}{\alpha_6},\ u = \frac{\alpha_1}{\alpha_7},\ r = \frac{\alpha_7\alpha_{10}}{\alpha_6^2}$, we have the representative $\langle -\nabla_2 + \nabla_3 + \nabla_6 + \nabla_7 + \nabla_{10}\rangle$;

iv. $\alpha_2 = -\alpha_{10},\ \alpha_3 \neq \alpha_6,\ \alpha_3 + \alpha_7 = \alpha_6,\ \alpha_1 = 0$, then choosing $x = 1,\ y = -\frac{\alpha_3}{\alpha_6},\ r = -\frac{(\alpha_3-\alpha_6)\alpha_{10}}{\alpha_6^2}$, we have the representative $\langle -\nabla_2 + \nabla_6 + \nabla_7 + \nabla_{10}\rangle$;

v. $\alpha_2 = -\alpha_{10},\ \alpha_3 \neq \alpha_6,\ \alpha_3 + \alpha_7 = \alpha_6,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_6^2}{(\alpha_3-\alpha_6)^2\alpha_{10}},\ y = -\frac{\alpha_1\alpha_3\alpha_6}{(\alpha_3-\alpha_6)^2\alpha_{10}},$ $r = \frac{\alpha_1^2\alpha_6^2}{(\alpha_6-\alpha_3)^3\alpha_{10}}$, we have the representative $\langle \nabla_1 - \nabla_2 + \nabla_6 + \nabla_7 + \nabla_{10}\rangle$;

vi. $\alpha_2 = -\alpha_{10},\ \alpha_3 \neq \alpha_6,\ \alpha_3 + \alpha_7 \neq \alpha_6$, then choosing $x = 1,\ y = -\frac{\alpha_3}{\alpha_6},\ u = \frac{\alpha_1}{\alpha_3-\alpha_6+\alpha_7},\ r = -\frac{(\alpha_3-\alpha_6)\alpha_{10}}{\alpha_6^2}$, we have the family of representatives $\langle -\nabla_2 + \nabla_6 + \alpha\nabla_7 + \nabla_{10}\rangle_{\alpha\neq 1}$;

vii. $\alpha_2 \neq -\alpha_{10}$, then choosing $y = \frac{x\alpha_2}{\alpha_{10}}$, we have $\alpha_2^* = 0$.

A. $\alpha_3 = -\alpha_7,\ \alpha_1 = 0$, then choosing $x = 1,\ r = \frac{\alpha_{10}}{\alpha_6}$, we have the family of representatives $\langle \alpha\nabla_3 + \nabla_6 - \alpha\nabla_7 + \nabla_{10}\rangle$;

B. $\alpha_3 = -\alpha_7,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_{10}},\ r = \frac{\alpha_1^2}{\alpha_6\alpha_{10}}$, we have the family of representatives $\langle \nabla_1 + \alpha\nabla_3 + \nabla_6 - \alpha\nabla_7 + \nabla_{10}\rangle$;

C. $\alpha_3 \neq -\alpha_7$, then choosing $x = 1,\ u = \frac{x\alpha_1}{\alpha_3+\alpha_7},\ r = \frac{\alpha_{10}}{\alpha_6}$, we have the family of representatives $\langle \alpha\nabla_3 + \nabla_6 + \beta\nabla_7 + \nabla_{10}\rangle_{\alpha+\beta\neq 0}$.

(c) $\alpha_9 = \alpha_6 = 0,\ \alpha_7 \neq 0$.

i. $\alpha_2 = -\alpha_{10},\ \alpha_3 = -\alpha_7,\ \alpha_5 = \alpha_1 = 0$, then choosing $x = 1,\ y = 0,\ r = \frac{\alpha_{10}}{\alpha_7}$, we have the representative $\langle -\nabla_2 - \nabla_3 + \nabla_7 + \nabla_{10}\rangle$;

ii. $\alpha_2 = -\alpha_{10},\ \alpha_3 = -\alpha_7,\ \alpha_5 = 0,\ \alpha_1 \neq 0$, then choosing $x = 1$, $y = \sqrt{\frac{\alpha_1}{\alpha_{10}}} - 1$, $r = \frac{\alpha_1}{\alpha_7}$, we have the representative $\langle \nabla_1 - \nabla_2 - \nabla_3 + \nabla_7 + \nabla_{10} \rangle$;

iii. $\alpha_2 = -\alpha_{10},\ \alpha_3 = -\alpha_7,\ \alpha_5 \neq 0,\ \alpha_1 - \alpha_5 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}}$, $y = 0$, $r = \frac{\alpha_5^2}{\alpha_7\alpha_{10}}$, we have the representative $\langle \nabla_1 - \nabla_2 - \nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle$;

iv. $\alpha_2 = -\alpha_{10},\ \alpha_3 = -\alpha_7,\ \alpha_5 \neq 0,\ \alpha_1 \neq \alpha_5$, then by choosing $x = \frac{\alpha_5}{\alpha_{10}}$, $y = \frac{\sqrt{\alpha_5(\alpha_5-\alpha_1)}-\alpha_5}{\alpha_{10}}$, $r = \frac{\alpha_5(\alpha_5-\alpha_1)}{\alpha_7\alpha_{10}}$, we have the representative $\langle -\nabla_2 - \nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle$;

v. $\alpha_2 = -\alpha_{10},\ \alpha_3 \neq -\alpha_7,\ \alpha_5 = 0$, then choosing $x = 1$, $y = 0$, $v = \frac{u(\alpha_3+\alpha_7)-\alpha_1}{\alpha_3+\alpha_7}$, $r = \frac{\alpha_{10}}{\alpha_7}$, we have the family of representatives $\langle -\nabla_2 + \alpha\nabla_3 + \nabla_7 + \nabla_{10} \rangle_{\alpha\neq-1}$;

vi. $\alpha_2 = -\alpha_{10},\ \alpha_3 \neq -\alpha_7,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}}$, $y = 0$, $u = 0$, $v = \frac{-\alpha_1\alpha_5}{(\alpha_3+\alpha_7)\alpha_{10}}$, $r = \frac{\alpha_5^2}{\alpha_7\alpha_{10}}$, we have the family of representatives $\langle -\nabla_2 + \alpha\nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle_{\alpha\neq-1}$;

vii. $\alpha_2 \neq -\alpha_{10}$, then choosing $y = \frac{x\alpha_2}{\alpha_{10}}$, we have $\alpha_2^* = 0$. If $\alpha_3 = -\alpha_7$, then choosing $v = \frac{\alpha_1}{\alpha_7}$, if $\alpha_3 \neq -\alpha_7$, then choosing $u = \frac{x\alpha_1+v\alpha_3}{\alpha_3+\alpha_7}$, we have $\alpha_1^* = 0$. Thus, we always can assume $\alpha_1^* = 0$, $u = v = 0$.

A. $\alpha_5 = 0$, then choosing $x = 1$, $r = \frac{\alpha_{10}}{\alpha_7}$, we have the representative $\langle \alpha\nabla_3 + \nabla_7 + \nabla_{10} \rangle$;

B. $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}}$, $r = \frac{\alpha_5^2}{\alpha_7\alpha_{10}}$, we have the representative $\langle \alpha\nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle$.

(d) $\alpha_9 = \alpha_7 = \alpha_6 = 0$, then $\alpha_3 \neq 0$, and choosing $u = \frac{(x^2\alpha_1+vx\alpha_3+y(x+y)\alpha_5)\alpha_{10}-xy\alpha_2\alpha_5}{x\alpha_3\alpha_{10}}$, we have $\alpha_1^* = 0$.

i. $\alpha_2 = -\alpha_{10},\ \alpha_5 = 0$, then choosing $x = 1$, $y = 0$, $r = \frac{\alpha_{10}}{\alpha_3}$, we have the representative $\langle -\nabla_2 + \nabla_3 + \nabla_{10} \rangle$;

ii. $\alpha_2 = -\alpha_{10},\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_{10}}{\alpha_3}$, $y = 0$, $r = \frac{\alpha_5^2}{\alpha_3\alpha_{10}}$, we have the representative $\langle -\nabla_2 + \nabla_3 + \nabla_5 + \nabla_{10} \rangle$;

iii. $\alpha_2 \neq -\alpha_{10},\ \alpha_5 = 0$, then choosing $x = 1$, $y = \frac{\alpha_2}{\alpha_{10}}$, $r = \frac{(\alpha_2+\alpha_{10})^2}{\alpha_3\alpha_{10}}$, we have the representative $\langle \nabla_3 + \nabla_{10} \rangle$;

iv. $\alpha_2 \neq -\alpha_{10},\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_{10}}$, $y = \frac{\alpha_2\alpha_5}{\alpha_{10}^2}$, $r = \frac{\alpha_5^2(\alpha_2+\alpha_{10})^2}{\alpha_3\alpha_{10}^3}$, we have the representative $\langle \nabla_3 + \nabla_5 + \nabla_{10} \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\begin{gathered}
\langle \nabla_2 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_6 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_6 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle, \\
\langle \nabla_2 + \nabla_5 + \alpha\nabla_6 + \nabla_9 \rangle, \langle \nabla_2 + \nabla_5 + \nabla_7 + \nabla_8 \rangle, \langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_7 + \nabla_8 \rangle, \langle \nabla_2 + \alpha\nabla_6 + \nabla_7 + \nabla_8 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_8 \rangle, \\
\langle \nabla_2 + \nabla_5 - \nabla_6 + \nabla_8 \rangle, \langle \nabla_2 + \alpha\nabla_6 + \nabla_8 \rangle, \langle \nabla_2 + \nabla_7 \rangle, \langle \nabla_2 + \nabla_5 + \nabla_7 \rangle, \langle \nabla_2 + \nabla_6 + \nabla_7 \rangle, \langle \nabla_2 + \nabla_6 \rangle, \langle -\nabla_2 + \nabla_9 + \nabla_{10} \rangle, \\
\langle -\nabla_2 + \nabla_3 + \nabla_9 + \nabla_{10} \rangle, \langle \nabla_1 - \nabla_2 + \alpha\nabla_3 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)}, \langle \nabla_1 - \nabla_2 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle, \\
\langle -\nabla_2 + \alpha\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha)\simeq O(-\alpha)}, \langle \nabla_1 - \nabla_2 + \nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle, \langle \nabla_9 + \nabla_{10} \rangle, \langle \nabla_1 + \nabla_9 + \nabla_{10} \rangle, \\
\langle \alpha\nabla_1 + \nabla_3 + \nabla_9 + \nabla_{10} \rangle, \langle \alpha\nabla_1 + \beta\nabla_3 + \nabla_5 + \nabla_9 + \nabla_{10} \rangle^{O(\alpha,\beta)\simeq O(\alpha,-\beta)}, \langle -\nabla_2 + \nabla_3 + \nabla_6 + \nabla_{10} \rangle, \\
\langle \nabla_1 - \nabla_2 + \nabla_3 + \nabla_6 + \nabla_{10} \rangle, \langle -\nabla_2 + \nabla_3 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle, \langle \nabla_1 - \nabla_2 + \nabla_6 + \nabla_7 + \nabla_{10} \rangle, \langle -\nabla_2 + \nabla_6 + \alpha\nabla_7 + \nabla_{10} \rangle, \\
\langle \nabla_1 + \alpha\nabla_3 + \nabla_6 - \alpha\nabla_7 + \nabla_{10} \rangle, \langle \alpha\nabla_3 + \nabla_6 + \beta\nabla_7 + \nabla_{10} \rangle, \langle \nabla_1 - \nabla_2 - \nabla_3 + \nabla_7 + \nabla_{10} \rangle, \langle -\nabla_2 + \alpha\nabla_3 + \nabla_7 + \nabla_{10} \rangle, \\
\langle \nabla_1 - \nabla_2 - \nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle, \langle -\nabla_2 + \alpha\nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle, \langle \alpha\nabla_3 + \nabla_7 + \nabla_{10} \rangle, \langle \alpha\nabla_3 + \nabla_5 + \nabla_7 + \nabla_{10} \rangle, \\
\langle -\nabla_2 + \nabla_3 + \nabla_{10} \rangle, \langle -\nabla_2 + \nabla_3 + \nabla_5 + \nabla_{10} \rangle, \langle \nabla_3 + \nabla_{10} \rangle, \langle \nabla_3 + \nabla_5 + \nabla_{10} \rangle,
\end{gathered}$$

which gives the following new algebras (see section 2):

$$\begin{gathered}
N_{17}, N_{18}, N_{19}, N_{20}, N_{21}^{\alpha}, N_{22}^{\alpha}, N_{23}, N_{24}, N_{25}^{\alpha}, N_{26}, N_{27}, N_{28}^{\alpha}, N_{29}, N_{30}, N_{31}, N_{32}, N_{33}, N_{34}, N_{35}^{\alpha}, N_{36}, N_{37}^{\alpha}, N_{38}, \\
N_{39}, N_{40}, N_{41}^{\alpha}, N_{42}^{\alpha,\beta}, N_{43}, N_{44}, N_{45}, N_{46}, N_{47}^{\alpha}, N_{48}^{\alpha}, N_{49}^{\alpha,\beta}, N_{50}, N_{51}, N_{52}, N_{53}^{\alpha}, N_{54}^{\alpha}, N_{55}^{\alpha}, N_{56}, N_{57}, N_{58}, N_{59}.
\end{gathered}$$

1.3.5. Central extensions of $\mathfrak{N}_{07}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{22}], \quad \nabla_3 = [\Delta_{23} - \Delta_{13}],$$
$$\nabla_4 = [\Delta_{24}], \quad \nabla_5 = [\Delta_{32} - \Delta_{13}], \quad \nabla_6 = [\Delta_{41} - \Delta_{14}].$$

Take $\theta = \sum_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{07})$. The automorphism group of $\mathfrak{N}_{07}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ z & u & x^2 & 0 \\ t & v & 0 & x^2 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & -\alpha_3 - \alpha_5 & -\alpha_6 \\ 0 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & \alpha_5 & 0 & 0 \\ \alpha_6 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & -\alpha_3^* - \alpha_5^* & -\alpha_6^* \\ \alpha^{**} & -\alpha^* + \alpha_2^* & \alpha_3^* & \alpha_4^* \\ 0 & \alpha_5^* & 0 & 0 \\ \alpha_6^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{07})$ on the subspace $\langle \sum_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* & = & x(x\alpha_1 - z(\alpha_3 + \alpha_5)), & \alpha_3^* & = & x^3\alpha_3, & \alpha_5^* & = & x^3\alpha_5 \\ \alpha_2^* & = & x\left(x\alpha_2 + v\alpha_4 + z\alpha_5 - v\alpha_6\right), & \alpha_4^* & = & x^3\alpha_4, & \alpha_6^* & = & x^3\alpha_6. \end{array}$$

We are interested only in the cases with $(\alpha_3, \alpha_5) \neq (0,0)$, $(\alpha_4, \alpha_6) \neq (0,0)$.

1. $\alpha_5 \neq 0$, then choosing $z = -\frac{x\alpha_2 + v\alpha_4 - v\alpha_6}{\alpha_5}$, we have $\alpha_2^* = 0$. Now we consider following subcases:
   (a) $\alpha_4 \neq 0$.
      i. if $\alpha_3 = -\alpha_5,\ \alpha_1 = 0$, then we have the family of representatives $\langle -\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\neq 0}$;
      ii. if $\alpha_3 = -\alpha_5,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, we have the family of representatives $\langle \nabla_1 - \nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\neq 0}$;
      iii. if $\alpha_3 \neq -\alpha_5$, then choosing $u = \frac{-v\alpha_3(\alpha_4+\alpha_6)+\alpha_5(x\alpha_1 - v(\alpha_4+\alpha_6))}{2(\alpha_3+\alpha_5)^2}$, we have the family of representatives $\langle \gamma\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\neq 0, \gamma\neq -1}$.
   (b) $\alpha_4 = 0, \alpha_6 \neq 0$.
      i. if $\alpha_3 = -\alpha_5,\ \alpha_1 = 0$, then we have the family of representatives $\langle -\nabla_3 + \nabla_5 + \beta\nabla_6 \rangle_{\beta\neq 0}$;
      ii. if $\alpha_3 = -\alpha_5,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, we have the family of representatives $\langle \nabla_1 - \nabla_3 + \nabla_5 + \beta\nabla_6 \rangle_{\beta\neq 0}$;
      iii. if $\alpha_3 \neq -\alpha_5$, then choosing $v = \frac{\alpha_5(x\alpha_1 - 2u\alpha_5) - 2u\alpha_3^2 - 4u\alpha_3\alpha_5}{(\alpha_3+\alpha_5)\alpha_6}$, we have the family of representatives $\langle \gamma\nabla_3 + \nabla_5 + \beta\nabla_6 \rangle_{\gamma\neq -1, \beta\neq 0}$.
2. $\alpha_5 = 0,\ \alpha_3 \neq 0$, then choosing $z = \frac{x\alpha_1}{\alpha_3}$, we have $\alpha_1 = 0$.
   (a) if $\alpha_4 = \alpha_6,\ \alpha_2 = 0$, then we have the family of representatives $\langle \nabla_3 + \alpha\nabla_4 + \alpha\nabla_6 \rangle_{\alpha\neq 0}$;
   (b) if $\alpha_4 = \alpha_6,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \alpha\nabla_4 + \alpha\nabla_6 \rangle_{\alpha\neq 0}$;
   (c) if $\alpha_4 \neq \alpha_6$, then choosing $x = 1,\ v = -\frac{\alpha_2}{\alpha_4 - \alpha_6}$, we have the family of representatives $\langle \nabla_3 + \alpha\nabla_4 + \beta\nabla_6 \rangle_{\alpha\neq\beta,\ (\alpha,\beta)\neq(0,0)}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_1 - \nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)}, \langle \gamma\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)},$$
$$\langle \nabla_3 + \alpha\nabla_4 + \beta\nabla_6 \rangle_{(\alpha,\beta)\neq(0,0)}, \langle \nabla_2 + \nabla_3 + \alpha\nabla_4 + \alpha\nabla_6 \rangle_{\alpha\neq 0},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{60}^{(\alpha,\beta)\neq(0,0)}, \mathrm{N}_{61}^{(\alpha,\beta)\neq(0,0)}, \mathrm{N}_{62}^{(\alpha,\beta)\neq(0,0)}, \mathrm{N}_{63}^{\alpha\neq 0}.$$

1.3.6. Central extensions of $\mathfrak{N}_{08}^{\alpha\neq 1}$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1 = [\Delta_{12}], & \nabla_2 = [\Delta_{21}], & \nabla_3 = [\Delta_{13} - \alpha\Delta_{23}], \\ \nabla_4 = [\Delta_{14} - \Delta_{24}], & \nabla_5 = [\Delta_{31} - \Delta_{13}], & \nabla_6 = [\Delta_{42} - \Delta_{24}]. \end{array}$$

Take $\theta = \sum_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{08}^{\alpha\neq 1})$. The automorphism group of $\mathfrak{N}_{08}^{\alpha\neq 1}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ t & v & x^2 & 0 \\ u & w & 0 & x^2 \end{pmatrix}, \quad \phi_2(\alpha \neq 0) = \begin{pmatrix} 0 & \alpha x & 0 & 0 \\ x & 0 & 0 & 0 \\ t & v & 0 & -\alpha^2 x^2 \\ u & w & -x^2 & 0 \end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} 0 & \alpha_1 & \alpha_3 - \alpha_5 & \alpha_4 \\ \alpha_2 & 0 & -\alpha\alpha_3 & -\alpha_4 - \alpha_6 \\ \alpha_5 & 0 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi_1 = \begin{pmatrix} \alpha^* & \alpha_1^* + \alpha^{**} & \alpha_3^* - \alpha_5^* & \alpha_4^* \\ \alpha_2^* - \alpha\alpha^* & -\alpha^{**} & -\alpha\alpha_3^* & -\alpha_4^* - \alpha_6^* \\ \alpha_5^* & 0 & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{08}^{\alpha\neq 1})$ on the subspace $\langle \sum_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* & = & x(x\alpha_1 + v(1-\alpha)\alpha_3 - v\alpha_5 + u\alpha_6), & \alpha_3^* & = & x^3\alpha_3, & \alpha_5^* & = & x^3\alpha_5 \\ \alpha_2^* & = & x\left(x\alpha_2 - u(1-\alpha)\alpha_4 + v\alpha_5 - u\alpha_6\right), & \alpha_4^* & = & x^3\alpha_4, & \alpha_6^* & = & x^3\alpha_6. \end{array}$$

We are interested only in the cases with $(\alpha_3, \alpha_5) \neq (0,0)$, $(\alpha_4, \alpha_6) \neq (0,0)$.

1. $\alpha_5 = \alpha_6 = 0$, then $\alpha_3\alpha_4 \neq 0$, and choosing $u = \frac{x\alpha_2}{(1-\alpha)\alpha_4}$, $v = -\frac{x\alpha_1}{(1-\alpha)\alpha_3}$, we have $\alpha_1^* = \alpha_2^* = 0$ and obtain the representative $\langle \nabla_3 + \beta\nabla_4 \rangle_{\beta\neq 0}$.
2. $(\alpha_5, \alpha_6) \neq (0,0)$, $\alpha \neq 0$, then with an action of a suitable ϕ_2, we can suppose $\alpha_5 \neq 0$ and choosing $v = \frac{u((1-\alpha)\alpha_4+\alpha_6)-x\alpha_2}{\alpha_5}$, we can suppose $\alpha_2^* = 0$. Now we consider following subcases:
   (a) $\alpha_4\alpha_5 + (\alpha-1)\alpha_3\alpha_4 \neq \alpha_3\alpha_6$, then choosing $u = \frac{x\alpha_1\alpha_5}{(1-\alpha)(\alpha_4\alpha_5 - \alpha_3((1-\alpha)\alpha_4+\alpha_6))}$, we have the family of representatives $\langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_5 + \delta\nabla_6 \rangle_{\gamma+(\alpha-1)\beta\gamma\neq\beta\delta}$,
   (b) $\alpha_4\alpha_5 + (\alpha-1)\alpha_3\alpha_4 = \alpha_3\alpha_6$, $\alpha_1 = 0$, then we have the family of representatives $\langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_5 + \delta\nabla_6 \rangle_{\gamma+(\alpha-1)\gamma=\beta\delta}$;
   (c) $\alpha_4\alpha_5 + (\alpha-1)\alpha_3\alpha_4 = \alpha_3\alpha_6$, $\alpha_1 \neq 0$, $\alpha_5 \neq (1-\alpha)\alpha_3$, we have the family of representatives $\langle \nabla_1 + \beta\nabla_3 + \frac{\beta\delta}{(\alpha-1)\beta+1}\nabla_4 + \nabla_5 + \delta\nabla_6 \rangle_{\beta\neq\frac{1}{1-\alpha}}$;
   (d) $\alpha_4\alpha_5 + (\alpha-1)\alpha_3\alpha_4 = \alpha_3\alpha_6$, $\alpha_1 \neq 0$, $\alpha_5 = (1-\alpha)\alpha_3$, then $\alpha_3 \neq 0$ and $\alpha_6 = 0$. Hence, we have the family of representatives $\langle \nabla_1 + \frac{1}{1-\alpha}\nabla_3 + \beta\nabla_4 + \nabla_5 \rangle$.
3. $(\alpha_5, \alpha_6) \neq (0,0)$, $\alpha = 0$. If $\alpha_5 \neq 0$, then we obtain the previous cases. Thus, we consider the case of $\alpha_5 = 0$. Then $\alpha_3\alpha_6 \neq 0$ and choosing $u = -\frac{x\alpha_1+v\alpha_3}{\alpha_6}$, we can suppose $\alpha_1^* = 0$. Now we consider following subcases:
   (a) $\alpha_4 \neq -\alpha_6$, then choosing $v = -\frac{x\alpha_2\alpha_6}{\alpha_3(\alpha_4+\alpha_6)}$, we obtain $\alpha_2^* = 0$ and obtain the family of representatives $\langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_6 \rangle_{\alpha=0,\gamma\neq -1}$,
   (b) $\alpha_4 = -\alpha_6$, $\alpha_2 = 0$, then we have the family of representatives $\langle \beta\nabla_3 - \nabla_4 + \nabla_6 \rangle_{\alpha=0}$,
   (c) $\alpha_4 = -\alpha_6$, $\alpha_2 \neq 0$, then we have the family of representatives $\langle \nabla_2 + \beta\nabla_3 - \nabla_4 + \nabla_6 \rangle_{\alpha=0}$.

Summarizing all cases for the algebra $\mathfrak{N}_{08}^{\alpha\neq 1}$, we have the following distinct orbits:

$$\langle \nabla_3 + \beta\nabla_4 \rangle_{\alpha\neq 1,\beta\neq 0}, \langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_5 + \delta\nabla_6 \rangle_{\alpha\neq 1}, \langle \nabla_1 + \beta\nabla_3 + \frac{\beta\delta}{(\alpha-1)\beta+1}\nabla_4 + \nabla_5 + \delta\nabla_6 \rangle_{\beta\neq\frac{1}{1-\alpha},\, \alpha\neq 1},$$
$$\langle \nabla_1 + \tfrac{1}{1-\alpha}\nabla_3 + \beta\nabla_4 + \nabla_5 \rangle_{\alpha\neq 1}, \langle \beta\nabla_3 + \gamma\nabla_4 + \nabla_6 \rangle_{\alpha=0}, \langle \nabla_2 + \beta\nabla_3 - \nabla_4 + \nabla_6 \rangle_{\alpha=0},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{64}^{\alpha\neq 1,\beta\neq 0}, \mathrm{N}_{65}^{\alpha\neq 1,\beta,\gamma,\delta}, \mathrm{N}_{66}^{\alpha\neq 1,\beta\neq\frac{1}{1-\alpha},\delta}, \mathrm{N}_{67}^{\alpha\neq 1,\beta}, \mathrm{N}_{68}^{\beta,\gamma}, \mathrm{N}_{69}^{\beta}.$$

1.3.7. Central extensions of $\mathfrak{N}_{08}^{1}$

Let us use the following notations:

$$\begin{array}{llll} \nabla_1 = [\Delta_{12}], & \nabla_2 = [\Delta_{21}], & \nabla_3 = [\Delta_{13} - \Delta_{23}], & \nabla_4 = [\Delta_{14} - \Delta_{24}], \\ \nabla_5 = [\Delta_{31} - \Delta_{13}], & \nabla_6 = [\Delta_{42} - \Delta_{24}], & \nabla_7 = [\Delta_{32} + \Delta_{41} - \Delta_{23} - \Delta_{14}]. & \end{array}$$

Take $\theta = \sum_{i=1}^{7} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{08}^{1})$. The automorphism group of $\mathfrak{N}_{08}^{1}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & y & 0 & 0 \\ x+y-z & z & 0 & 0 \\ t & v & x(z-y) & y(z-y) \\ u & w & (x+y-z)(z-y) & z(z-y) \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_3-\alpha_5 & \alpha_4-\alpha_7 \\ \alpha_2 & 0 & -\alpha_3-\alpha_7 & -\alpha_4-\alpha_6 \\ \alpha_5 & \alpha_7 & 0 & 0 \\ \alpha_7 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^*+\alpha^{**} & \alpha_3^*-\alpha_5^* & \alpha_4^*-\alpha_7^* \\ \alpha_2^*-\alpha^* & -\alpha^{**} & -\alpha_3^*-\alpha_7^* & -\alpha_4^*-\alpha_6^* \\ \alpha_5^* & \alpha_7^* & 0 & 0 \\ \alpha_7^* & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{08}^{1})$ on the subspace $\langle \sum_{i=1}^{7} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{7} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lcl} \alpha_1^* &=& (x+y)z\alpha_1 + y(x+y)\alpha_2 - (vx-ty)\alpha_5 - (w(x+y)-z(u+w))\alpha_6 \\ && \qquad -(x(v+w)-y(u-v)-z(t+v))\alpha_7, \\ \alpha_2^* &=& (x+y)(x+y-z)\alpha_1 + x(x+y)\alpha_2 + (vx-ty)\alpha_5 + (w(x+y)-z(u+w))\alpha_6 \\ && \qquad +(x(v+w)-y(u-v)-z(t+v))\alpha_7, \\ \alpha_3^* &=& (z-y)^2(x\alpha_3 + (x+y-z)\alpha_4), \\ \alpha_4^* &=& (z-y)^2(y\alpha_3 + z\alpha_4), \\ \alpha_5^* &=& (z-y)(x^2\alpha_5 + (x+y-z)((x+y-z)\alpha_6 + 2x\alpha_7)) \\ \alpha_6^* &=& (z-y)(y^2\alpha_5 + z(z\alpha_6 + 2y\alpha_7)), \\ \alpha_7^* &=& (z-y)((x+y-z)(z\alpha_6 + y\alpha_7) + x(y\alpha_5 + z\alpha_7)). \end{array}$$

We are interested only in the cases with

$$(\alpha_3, \alpha_5, \alpha_7) \neq (0,0,0), (\alpha_4, \alpha_6, \alpha_7) \neq (0,0,0).$$

1. $(\alpha_5, \alpha_6, \alpha_7) = (0,0,0)$, then $\alpha_3 \neq 0$, $\alpha_4 \neq 0$. If $\alpha_4 \neq -\alpha_3$, then choosing $z = -\frac{y\alpha_3}{\alpha_4}$, we obtain that $\alpha_4^* = 0$ which implies $(\alpha_4^*, \alpha_6^*, \alpha_7^*) = (0,0,0)$. Thus, we have that $\alpha_3 = -\alpha_4$.
   (a) $(\alpha_1, \alpha_2) = (0,0)$, then we have the representative $\langle \nabla_3 - \nabla_4 \rangle$;
   (b) $(\alpha_1, \alpha_2) \neq (0,0)$, without loss of generality, we can suppose $\alpha_1 \neq 0$.
      i. $\alpha_1 = -\alpha_2$, then choosing $x = \frac{\alpha_3}{\alpha_1}, y = 0, z = 1$, we have the representative $\langle \nabla_1 - \nabla_2 + \nabla_3 - \nabla_4 \rangle$;
      ii. $\alpha_1 \neq -\alpha_2$, then choosing $x = 0$, $y = -\frac{\alpha_1^3}{(\alpha_1+\alpha_2)^2\alpha_3}$, $z = \frac{\alpha_1^3\alpha_2}{(\alpha_1+\alpha_2)^2\alpha_1\alpha_3}$, we have the representative $\langle \nabla_2 + \nabla_3 - \nabla_4 \rangle$.
2. $(\alpha_5, \alpha_6, \alpha_7) \neq (0,0,0)$, then without loss of generality we can assume $\alpha_5 \neq 0$ and consider following subcases:
   (a) $\alpha_6\alpha_5 = \alpha_7^2, \alpha_7 = -\alpha_5$, $\alpha_4 = -\alpha_3$, $\alpha_1 = -\alpha_2$, then taking $v = u = w = 0$, $t = \frac{(x+y)\alpha_1}{\alpha_5}$, we have the family of representatives $\langle \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7 \rangle$;
   (b) $\alpha_6\alpha_5 = \alpha_7^2, \alpha_7 = -\alpha_5$, $\alpha_4 = -\alpha_3$, $\alpha_1 \neq -\alpha_2$, then taking
   $$y = v = u = w = 0, z = x = \frac{(\alpha_1+\alpha_2)}{\alpha_5}, t = \frac{\alpha_1(\alpha_1+\alpha_2)}{\alpha_5^2},$$
   we have the family of representatives $\langle \nabla_2 + \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7 \rangle$;

(c) $\alpha_6\alpha_5 = \alpha_7^2$, $\alpha_7 = -\alpha_5$, $\alpha_4 \neq -\alpha_3$, then we can choose y and z such that $\alpha_4^* = 0$, $\alpha_3^* \neq 0$. Thus we can suppose $\alpha_4 = 0$, moreover taking $v = u = w = 0$, $t = \frac{x\alpha_1}{\alpha_5}$, we can also suppose $\alpha_1 = 0$.
  i. if $\alpha_2 = 0$, then choosing $x = 1$, $z = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle\nabla_3 + \nabla_5 + \nabla_6 - \nabla_7\rangle$;
  ii. if $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_5^2}{\alpha_3^3}$, $z = \frac{\alpha_2\alpha_5}{\alpha_3^2}$, we have the representative $\langle\nabla_2 + \nabla_3 + \nabla_5 + \nabla_6 - \nabla_7\rangle$.

(d) $\alpha_6\alpha_5 = \alpha_7^2, \alpha_5 \neq -\alpha_7$, then choosing $y = -\frac{z\alpha_7}{\alpha_5}$, $w = 0$, $v = \frac{z(\alpha_1\alpha_5-\alpha_2\alpha_7)}{\alpha_5(\alpha_5+\alpha_7)}$, we can suppose $\alpha_1 = \alpha_6 = \alpha_7 = 0$. Since $(\alpha_4, \alpha_6, \alpha_7) \neq (0, 0, 0)$, we have that $\alpha_4 \neq 0$.
  i. $\alpha_2 = 0$, then choosing $x = z\sqrt{\frac{\alpha_4}{\alpha_5}}$, we have the family of representatives $\langle\beta\nabla_3 + \nabla_4 + \nabla_5\rangle$.
  ii. $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_5}\sqrt{\frac{\alpha_4}{\alpha_5}}$, $z = \frac{\alpha_2}{\alpha_5}$, we have the representative $\langle\nabla_2 + \beta\nabla_3 + \nabla_4 + \nabla_5\rangle$.

(e) $\alpha_6\alpha_5 \neq \alpha_7^2$, then choosing suitable value of z and y such that $y - z \neq 0$, and we can suppose $\alpha_6 = 0$ and $\alpha_7 \neq 0$.
  i. $\alpha_5 = -2\alpha_7$, $\alpha_4 = 0$, $\alpha_1 = -\alpha_2$, then choosing $y = u = v = w = 0$, $t = -\frac{x\alpha_1}{\alpha_7}$, we have the family of representatives $\langle\beta\nabla_3 - 2\nabla_5 + \nabla_7\rangle$;
  ii. $\alpha_5 = -2\alpha_7$, $\alpha_4 = 0$, $\alpha_1 \neq -\alpha_2$, then choosing $y = u = v = w = 0$, $t = -\frac{x\alpha_1}{\alpha_7}$, $z = \frac{x(\alpha_1+\alpha_2)}{\alpha_7}$, we have the family of representatives $\langle\nabla_2 + \beta\nabla_3 - 2\nabla_5 + \nabla_7\rangle$;
  iii. $\alpha_5 = -2\alpha_7$, $\alpha_4 \neq 0$, $\alpha_1 = -\alpha_2$, then choosing $y = u = v = w = 0$, $t = -\frac{x\alpha_1}{\alpha_7}$, $z = \frac{x\alpha_7}{\alpha_4}$, we have the family of representatives $\langle\beta\nabla_3 + \nabla_4 - 2\nabla_5 + \nabla_7\rangle$;
  iv. $\alpha_5 = -2\alpha_7$, $\alpha_4 \neq 0$, $\alpha_1 \neq -\alpha_2$, then choosing
$$y = u = v = w = 0,\ t = -\frac{x\alpha_1}{\alpha_7},\ x = \frac{(\alpha_1+\alpha_2)\alpha_4^2}{\alpha_7^3},\ z = \frac{(\alpha_1+\alpha_2)\alpha_4}{\alpha_7^2},$$
  we have the family of representatives $\langle\nabla_2 + \beta\nabla_3 + \nabla_4 - 2\nabla_5 + \nabla_7\rangle$;
  v. $\alpha_5 \neq -2\alpha_7$, $\alpha_1 = -\alpha_2$, then choosing $y = u = v = w = 0$, $z = \frac{x(2\alpha_7+\alpha_5)}{\alpha_7}$, $t = -\frac{x\alpha_1}{\alpha_7}$, we have the family of representatives $\langle\beta\nabla_3 + \gamma\nabla_4 + \nabla_7\rangle$;
  vi. $\alpha_5 \neq -2\alpha_7$, $\alpha_1 \neq -\alpha_2$, then choosing
$$y = u = v = w = 0,\ x = \frac{4(\alpha_1+\alpha_2)\alpha_7}{(\alpha_5+2\alpha_7)^2},\ z = \frac{2(\alpha_1+\alpha_2)}{\alpha_5+2\alpha_7},\ t = -\frac{x\alpha_1}{\alpha_7},$$
  we have the family of representatives $\langle\nabla_2 + \beta\nabla_3 + \gamma\nabla_4 + \nabla_7\rangle$.

Summarizing all cases for the algebra $\mathfrak{N}_{08}^{\alpha}$, we have the following distinct orbits

$$\langle\nabla_3 - \nabla_4\rangle_{\alpha=1}, \langle\nabla_1 - \nabla_2 + \nabla_3 - \nabla_4\rangle_{\alpha=1}, \langle\nabla_2 + \nabla_3 - \nabla_4\rangle_{\alpha=1}, \langle\beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7\rangle_{\alpha=1},$$
$$\langle\nabla_2 + \beta\nabla_3 - \beta\nabla_4 + \nabla_5 + \nabla_6 - \nabla_7\rangle_{\alpha=1}, \langle\nabla_3 + \nabla_5 + \nabla_6 - \nabla_7\rangle_{\alpha=1}, \langle\nabla_2 + \nabla_3 + \nabla_5 + \nabla_6 - \nabla_7\rangle_{\alpha=1},$$
$$\langle\beta\nabla_3 + \nabla_4 + \nabla_5\rangle_{\alpha=1}, \langle\nabla_2 + \beta\nabla_3 + \nabla_4 + \nabla_5\rangle_{\alpha=1}, \langle\beta\nabla_3 - 2\nabla_5 + \nabla_7\rangle_{\alpha=1}, \langle\nabla_2 + \beta\nabla_3 - 2\nabla_5 + \nabla_7\rangle_{\alpha=1},$$
$$\langle\beta\nabla_3 + \nabla_4 - 2\nabla_5 + \nabla_7\rangle_{\alpha=1}, \langle\nabla_2 + \beta\nabla_3 + \nabla_4 - 2\nabla_5 + \nabla_7\rangle_{\alpha=1}, \langle\beta\nabla_3 + \gamma\nabla_4 + \nabla_7\rangle_{\alpha=1}, \langle\nabla_2 + \beta\nabla_3 + \gamma\nabla_4 + \nabla_7\rangle_{\alpha=1},$$

which gives the following new algebras (see section 2):

$$N_{70}, N_{71}, N_{72}, N_{73}^{\beta}, N_{74}^{\beta}, N_{75}, N_{76}, N_{77}^{\beta}, N_{78}^{\beta}, N_{79}^{\beta}, N_{80}^{\beta}, N_{81}^{\beta}, N_{82}^{\beta}, N_{83}^{\beta,\gamma}, N_{84}^{\beta,\gamma}.$$

1.3.8. Central extensions of $\mathfrak{N}_{12}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}],\quad \nabla_2 = [\Delta_{13}],\quad \nabla_3 = [\Delta_{14} - \Delta_{41}],$$
$$\nabla_4 = [\Delta_{22}],\quad \nabla_5 = [\Delta_{23} - \Delta_{32}],\quad \nabla_6 = [\Delta_{24}].$$

Take $\theta = \sum_{i=1}^{6} \alpha_i\nabla_i \in H^2(\mathfrak{N}_{12})$. The automorphism group of $\mathfrak{N}_{12}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ z & v & xy & 0 \\ u & t & 0 & xy \end{pmatrix},\quad \phi_2 = \begin{pmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ z & v & 0 & xy \\ u & t & xy & 0 \end{pmatrix}.$$

Since

$$\phi_1^T\begin{pmatrix}\alpha_1 & 0 & \alpha_2 & \alpha_3\\ 0 & \alpha_4 & \alpha_5 & \alpha_6\\ 0 & -\alpha_5 & 0 & 0\\ -\alpha_3 & 0 & 0 & 0\end{pmatrix}\phi_1=\begin{pmatrix}\alpha_1^* & \alpha^* & \alpha_2^* & \alpha_3^*\\ \alpha^{**} & \alpha_4^* & \alpha_5^* & \alpha_6^*\\ 0 & -\alpha_5^* & 0 & 0\\ -\alpha_3^* & 0 & 0 & 0\end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{12})$ on the subspace $\langle\sum\limits_{i=1}^{6}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{6}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{lllllllll}\alpha_1^* & = & x(x\alpha_1+z\alpha_2) & \alpha_2^* & = & x^2y\alpha_2, & \alpha_3^* & = & x^2y\alpha_3,\\ \alpha_4^* & = & y(y\alpha_4+v\alpha_6), & \alpha_5^* & = & xy^2\alpha_5, & \alpha_6^* & = & xy^2\alpha_6.\end{array}$$

We are interested only in the cases with

$$(\alpha_2,\alpha_5)\neq(0,0),\ (\alpha_3,\alpha_6)\neq(0,0).$$

1. $(\alpha_2,\alpha_6)=(0,0)$, then $\alpha_3\neq 0,\ \alpha_5\neq 0$.
   (a) $\alpha_4=0,\ \alpha_1=0$, then choosing $x=1,\ y=\frac{\alpha_3}{\alpha_5}$, we have the representative $\langle\nabla_3+\nabla_5\rangle$;
   (b) $\alpha_4=0,\ \alpha_1\neq 0$, then choosing $x=\frac{\alpha_1\alpha_5}{\alpha_3^2},\ y=\frac{\alpha_1}{\alpha_3}$, we have the representative $\langle\nabla_1+\nabla_3+\nabla_5\rangle$;
   (c) $\alpha_4\neq 0$, then choosing $x=\frac{\alpha_4}{\alpha_5},\ y=\frac{\alpha_3\alpha_4}{\alpha_5^2}$, we have the family of representatives $\langle\alpha\nabla_1+\nabla_3+\nabla_4+\nabla_5\rangle$;
2. $(\alpha_2,\alpha_6)\neq(0,0)$, then without loss of generality, we can suppose $\alpha_6\neq 0$ and choosing $v=-\frac{y\alpha_4}{\alpha_6}$, we have $\alpha_4^*=0$.
   (a) $\alpha_2\neq 0$, then choosing $x=1,\ y=\frac{\alpha_2}{\alpha_4},\ z=-\frac{\alpha_1}{\alpha_2}$, we have the family of representatives $\langle\nabla_2+\alpha\nabla_3+\beta\nabla_5+\nabla_6\rangle$;
   (b) $\alpha_2=0,\ \alpha_3=\alpha_1=0$, then we have the family of representatives $\langle\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0}$;
   (c) $\alpha_2=0,\ \alpha_3=0,\ \alpha_1\neq 0$, then choosing $x=\frac{\alpha_6}{\alpha_1},\ y=1$, we have the family of representatives $\langle\nabla_1+\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0}$;
   (d) $\alpha_2=0,\ \alpha_3\neq 0,\ \alpha_1=0$, then choosing $x=\frac{\alpha_6}{\alpha_3},\ y=1$, we have the family of representatives $\langle\nabla_3+\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0}$;
   (e) $\alpha_2=0,\ \alpha_3\neq 0,\ \alpha_1\neq 0$, then choosing $x=\frac{\alpha_1\alpha_6}{\alpha_3^2},\ y=\frac{\alpha_1}{\alpha_3}$, we have the family of representatives $\langle\nabla_1+\nabla_3+\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0}$;

Summarizing all cases, we have the following distinct orbits

$$\langle\nabla_3+\nabla_5\rangle,\ \langle\nabla_1+\nabla_3+\nabla_5\rangle,\ \langle\alpha\nabla_1+\nabla_3+\nabla_4+\nabla_5\rangle^{O(\alpha)\simeq O(\alpha^{-1})},\ \langle\nabla_2+\alpha\nabla_3+\beta\nabla_5+\nabla_6\rangle^{O(\alpha,\beta)\simeq O(\beta,\alpha)},\ \langle\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0},$$
$$\langle\nabla_1+\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0},\ \langle\nabla_3+\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0},\ \langle\nabla_1+\nabla_3+\alpha\nabla_5+\nabla_6\rangle_{\alpha\neq 0},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{85},\ \mathrm{N}_{86},\ \mathrm{N}_{87}^{\alpha},\ \mathrm{N}_{88}^{\alpha,\beta},\ \mathrm{N}_{89}^{\alpha\neq 0},\ \mathrm{N}_{90}^{\alpha\neq 0},\ \mathrm{N}_{91}^{\alpha\neq 0},\ \mathrm{N}_{92}^{\alpha\neq 0}.$$

1.3.9. Central extensions of $\mathfrak{N}_{13}$

Let us use the following notations:

$$\begin{array}{lll}\nabla_1=[\Delta_{21}], & \nabla_2=[\Delta_{22}], & \nabla_3=[\Delta_{14}+\Delta_{23}],\\ \nabla_4=[\Delta_{24}-\Delta_{13}+2\Delta_{14}], & \nabla_5=[\Delta_{42}-2\Delta_{13}+\Delta_{31}-2\Delta_{32}], & \nabla_6=[\Delta_{41}-2\Delta_{14}-\Delta_{32}].\end{array}$$

Take $\theta=\sum\limits_{i=1}^{6}\alpha_i\nabla_i\in\mathrm{H}^2(\mathfrak{N}_{13})$. The automorphism group of $\mathfrak{N}_{13}$ consists of invertible matrices of the form

$$\phi_1=\begin{pmatrix}x & 0 & 0 & 0\\ 0 & x & 0 & 0\\ z & u & x^2 & 0\\ t & v & 0 & x^2\end{pmatrix},\quad \phi_2=\begin{pmatrix}0 & x & 0 & 0\\ x & 0 & 0 & 0\\ z & u & -x^2 & 2x^2\\ t & v & 0 & x^2\end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} 0 & 0 & -\alpha_4 - 2\alpha_5 & \alpha_3 + 2\alpha_4 - 2\alpha_6 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_5 & -2\alpha_5 - \alpha_6 & 0 & 0 \\ \alpha_6 & \alpha_5 & 0 & 0 \end{pmatrix} \phi_1 =$$

$$= \begin{pmatrix} \alpha^* & \alpha^{**} & -\alpha_4^* - 2\alpha_5^* & \alpha_3^* + 2\alpha_4^* - 2\alpha_6^* \\ \alpha_1^* - \alpha^{**} & \alpha_2^* + \alpha^* + 2\alpha^{**} & \alpha_3^* & \alpha_4^* \\ \alpha_5^* & -2\alpha_5^* - \alpha_6^* & 0 & 0 \\ \alpha_6^* & \alpha_5^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{13})$ on the subspace $\langle \sum_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where for ϕ_1:

$$\begin{aligned} \alpha_1^* &= x(x\alpha_1 + (v+z)\alpha_3 + (t-u+2v)\alpha_4 + (t-u-2z)\alpha_5 - (v+z)\alpha_6), \\ \alpha_2^* &= x(x\alpha_2 - (t-u+2v)\alpha_3 - (2t-2u+3v-z)\alpha_4 - \\ &\qquad (2t-2u-v-5z)\alpha_5 + (t-u+4v+2z)\alpha_6), \\ \alpha_3^* &= x^3\alpha_3, \\ \alpha_4^* &= x^3\alpha_4, \\ \alpha_5^* &= x^3\alpha_5, \\ \alpha_6^* &= x^3\alpha_6. \end{aligned}$$

for ϕ_2:

$$\begin{aligned} \alpha_1^* &= x(x\alpha_1 + (t+u)\alpha_3 + (2t+v-z)\alpha_4 - (2u-v+z)\alpha_5 - (t+u)\alpha_6), \\ \alpha_2^* &= -x(2x\alpha_1 + x\alpha_2 + (2u-v+z)\alpha_3 + (t+u)\alpha_4 + (t+u)\alpha_5 + (2t+v-z)\alpha_6), \\ \alpha_3^* &= x^3(\alpha_4 + 2\alpha_5), \\ \alpha_4^* &= x^3(\alpha_3 - 2(2\alpha_5 + \alpha_6)), \\ \alpha_5^* &= x^3(2\alpha_5 + \alpha_6), \\ \alpha_6^* &= -x^3(3\alpha_5 + 2\alpha_6). \end{aligned}$$

We are interested only in the cases with $(\alpha_3, \alpha_4, \alpha_5, \alpha_6) \neq (0,0,0,0)$.

1. $(\alpha_5, \alpha_6) = (0,0)$. Let us consider the following subcases for automorphisms of type ϕ_1:
   (a) $\alpha_3 = 0$, then $\alpha_4 \neq 0$, then choosing $u = v = 0$, $z = -\frac{x(2\alpha_1+\alpha_2)}{\alpha_4}$, $t = -\frac{x\alpha_1}{\alpha_4}$, we have the representative $\langle \nabla_4 \rangle$;
   (b) $\alpha_3 \neq 0$, $\alpha_3 = -\alpha_4$, $\alpha_1 = -\alpha_2$, then choosing $u = v = z = 0$, $t = \frac{x\alpha_1}{\alpha_3}$, we have the representative $\langle \nabla_3 - \nabla_4 \rangle$;
   (c) $\alpha_3 \neq 0$, $\alpha_3 = -\alpha_4$, $\alpha_1 \neq -\alpha_2$, then choosing $u = v = z = 0$, $t = -\frac{x\alpha_2}{\alpha_3}$, $x = \frac{\alpha_1+\alpha_2}{\alpha_3}$, we have the representative $\langle \nabla_1 + \nabla_3 - \nabla_4 \rangle$;
   (d) $\alpha_3 \neq 0$, $\alpha_4 \neq -\alpha_3$, then choosing
   $$u = v = 0,\ z = -\frac{x(\alpha_2\alpha_4 + \alpha_1(\alpha_3+2\alpha_4))}{(\alpha_3+\alpha_4)^2},\ t = \frac{x(\alpha_2\alpha_3 - \alpha_1\alpha_4)}{(\alpha_3+\alpha_4)^2},$$
   we have the family of representatives $\langle \nabla_3 + \alpha\nabla_4 \rangle_{\alpha \neq -1}$;
2. $(\alpha_5, \alpha_6) \neq (0,0)$, then without loss of generality (after an action of ϕ_2), we can suppose $\alpha_5 \neq 0$ and consider the following subcases for automorphisms of type ϕ_1:
   (a) $\alpha_4 = -\alpha_5$, $\alpha_3 = 2\alpha_5 + \alpha_6$, $\alpha_1 = 0$, then choosing $u = v = z = 0$, $t = \frac{x\alpha_2}{2\alpha_5}$, we have the family of representatives $\langle (2+\alpha)\nabla_3 - \nabla_4 + \nabla_5 + \alpha\nabla_6 \rangle$;
   (b) $\alpha_4 = -\alpha_5$, $\alpha_3 = 2\alpha_5 + \alpha_6$, $\alpha_1 \neq 0$, then choosing $u = v = z = 0$, $t = \frac{\alpha_1\alpha_2}{2\alpha_5^2}$, $x = \frac{\alpha_1}{\alpha_5}$, we have the family of representatives $\langle \nabla_1 + (2+\alpha)\nabla_3 - \nabla_4 + \nabla_5 + \alpha\nabla_6 \rangle$;
   (c) $\alpha_4 = -\alpha_5$, $\alpha_3 \neq 2\alpha_5 + \alpha_6$, $\alpha_3 \neq \alpha_6$, then choosing $t = u = 0$,
   $$x = (\alpha_3 - \alpha_6)(\alpha_3 - 2\alpha_5 - \alpha_6),\ z = 2\alpha_1(\alpha_5 + \alpha_6) + \tfrac{1}{2}\alpha_2(2\alpha_5 + \alpha_6 - \alpha_3) - \alpha_1\alpha_3$$
   $$\text{and } v = \tfrac{1}{2}\alpha_2(\alpha_3 - 2\alpha_5 - \alpha_6) - \alpha_1(2\alpha_5 + \alpha_6),$$

we have the family of representatives $\langle \alpha\nabla_3 - \nabla_4 + \nabla_5 + \gamma\nabla_6 \rangle_{\alpha\notin\{\gamma,\gamma+2\}}$;

(d) $\alpha_4 = -\alpha_5$, $\alpha_3 \neq 2\alpha_5 + \alpha_6$, $\alpha_3 = \alpha_6$, then choosing $z = \frac{x\alpha_1}{2\alpha_5}$ and $v = 0$, we have two families of representatives $\langle \alpha\nabla_3 - \nabla_4 + \nabla_5 + \alpha\nabla_6 \rangle$ and $\langle \nabla_2 + \alpha\nabla_3 - \nabla_4 + \nabla_5 + \alpha\nabla_6 \rangle$, depending on $\alpha_2\alpha_5 = -\alpha_1(\alpha_3 + 2\alpha_5)$ or not. The last family for $\alpha \neq -2$ under an action of ϕ_2 gives a case with $\alpha_4 \neq -\alpha_5$, which will be considered below.

(e) $\alpha_4 \neq -\alpha_5$, $\alpha_3^2 + \alpha_4^2 + 2\alpha_4\alpha_5 + 2\alpha_3\alpha_4 + (\alpha_5 + \alpha_6)^2 \neq 2\alpha_3\alpha_6$, then by choosing

$$x = \alpha_3^2 + \alpha_4^2 + 2\alpha_4\alpha_5 + 2\alpha_3(\alpha_4 - \alpha_6) + (\alpha_5 + \alpha_6)^2,\ u = \alpha_2(\alpha_6 - \alpha_3 - 2\alpha_4) + \alpha_1(\alpha_5 + 4\alpha_6 - 2\alpha_3 - 3\alpha_4),$$
$$v = -\alpha_2(\alpha_4 + \alpha_5) - \alpha_1(\alpha_3 + 2\alpha_4 + 2\alpha_5 - \alpha_6) \text{ and } z = t = 0,$$

we have the family of representatives

$$\langle \alpha\nabla_3 + \beta\nabla_4 + \nabla_5 + \gamma\nabla_6 \rangle_{\beta\neq -1,\ \alpha^2+\beta^2+2\beta+2\alpha\beta+(1+\gamma)^2\neq 2\alpha\gamma};$$

(f) $\alpha_4 \neq -\alpha_5$, $\alpha_3^2 + \alpha_4^2 + 2\alpha_4\alpha_5 + 2\alpha_3\alpha_4 + (\alpha_5 + \alpha_6)^2 = 2\alpha_3\alpha_6$, then by choosing $u = \frac{x\alpha_1}{\alpha_4+\alpha_5}$ and $z = t = 0$, we have two families of representatives

$$\langle \nabla_2 + \Xi\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\neq -1} \text{ and } \langle \Xi\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\neq -1},$$

where $\Xi = \beta - \alpha \pm \sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1}$, depending on

$$\alpha_2(\alpha_4 + \alpha_5) + \alpha_1(\alpha_4 + 2\alpha_5) \neq \alpha_1\sqrt{-2\alpha_4(\alpha_5 + \alpha_6) - \alpha_5(\alpha_5 + 2\alpha_6)}) \text{ or not.}$$

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_4 \rangle, \langle \nabla_3 + \alpha\nabla_4 \rangle^{O(\alpha)\simeq O(\alpha^{-1})}, \langle \nabla_1 + \nabla_3 - \nabla_4 \rangle, \langle \nabla_1 + (2+\alpha)\nabla_3 - \nabla_4 + \nabla_5 + \alpha\nabla_6 \rangle^{O(\alpha)\simeq O((2+\alpha)^{-1})},$$
$$\langle \nabla_2 - 2\nabla_3 - \nabla_4 + \nabla_5 - 2\nabla_6 \rangle, \langle \alpha\nabla_3 + \beta\nabla_4 + \nabla_5 + \gamma\nabla_6 \rangle^{O(\alpha,\beta,\gamma)\simeq O(\frac{2+\beta}{2+\gamma}, \frac{\alpha-2(2+\gamma)}{2+\gamma}, -\frac{3+2\gamma}{2+\gamma})},$$
$$\langle \nabla_2 + (\beta - \alpha + \sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1})\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\neq -1},$$
$$\langle \nabla_2 + (\beta - \alpha - \sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1})\nabla_3 + \alpha\nabla_4 + \nabla_5 + \beta\nabla_6 \rangle_{\alpha\notin\{-1, -\frac{1+2\beta}{2+2\beta}\}},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{93}, \mathrm{N}_{94}^{\alpha}, \mathrm{N}_{95}, \mathrm{N}_{96}^{\alpha}, \mathrm{N}_{97}, \mathrm{N}_{98}^{\alpha,\beta,\gamma}, \mathrm{N}_{99}^{\alpha,\beta}, \mathrm{N}_{100}^{\alpha,\beta}.$$

1.3.10. Central extensions of $\mathfrak{N}_{14}^0$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{23}], \quad \nabla_4 = [\Delta_{13} + \Delta_{24}],$$
$$\nabla_5 = [\Delta_{32}], \quad \nabla_6 = [\Delta_{42} - \Delta_{24}], \quad \nabla_7 = [\Delta_{14}].$$

Take $\theta = \sum_{i=1}^{7} \alpha_i\nabla_i \in \mathrm{H}^2(\mathfrak{N}_{14}^0)$. The automorphism group of $\mathfrak{N}_{14}^0$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ w & u & y^2 & 0 \\ t & v & yz & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_4 & \alpha_7 \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 - \alpha_6 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_4^* & \alpha_7^* \\ \alpha_2^* & \alpha^{**} & \alpha_3^* & \alpha_4^* - \alpha_6^* \\ 0 & \alpha_5^* & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{14}^0)$ on the subspace $\langle \sum_{i=1}^{7} \alpha_i\nabla_i \rangle$ is given by $\langle \sum_{i=1}^{7} \alpha_i^*\nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= x(x\alpha_1 + w\alpha_4 + t\alpha_7),\\
\alpha_2^* &= xz\alpha_1 + xy\alpha_2 + wy\alpha_3 + (ty + wz)\alpha_4 - ty\alpha_6 + tz\alpha_7,\\
\alpha_3^* &= y(y^2\alpha_3 + z(2y\alpha_4 - y\alpha_6 + z\alpha_7)),\\
\alpha_4^* &= xy\left(y\alpha_4 + z\alpha_7\right),\\
\alpha_5^* &= y^2(y\alpha_5 + z\alpha_6),\\
\alpha_6^* &= xy^2\alpha_6,\\
\alpha_7^* &= x^2y\alpha_7.
\end{aligned}$$

We are interested only in the cases with

$$(\alpha_3, \alpha_4, \alpha_5) \neq (0, 0, 0), (\alpha_4, \alpha_6, \alpha_7) \neq (0, 0, 0).$$

1. $\alpha_7 \neq 0$, then choosing $z = -\frac{y\alpha_4}{\alpha_7}$, $t = -\frac{x\alpha_1 + w\alpha_4}{\alpha_7}$, we have $\alpha_1^* = \alpha_4^* = 0$. Thus, we can suppose $\alpha_1 = \alpha_4 = 0$ and consider following subcases:
   (a) $\alpha_3 \neq 0$, then choosing $x = 1$, $y = \sqrt{\alpha_7\alpha_3^{-1}}$, $w = -\alpha_2\alpha_3^{-1}$, we have the family of representatives $\langle \nabla_3 + \beta\nabla_4 + \gamma\nabla_6 + \nabla_7 \rangle$;
   (b) $\alpha_3 = 0$, then $\alpha_5 \neq 0$.
      i. $\alpha_2 = 0$, then choosing $x = 1$, $y = \sqrt{\alpha_7\alpha_5^{-1}}$, we have the family of representatives $\langle \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle$;
      ii. $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_7}$, $y = \frac{\alpha_2}{\sqrt{\alpha_5\alpha_7}}$, we have the family of representatives $\langle \nabla_2 + \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle$.
2. $\alpha_7 = 0$, $\alpha_6 \neq 0$, then choosing $z = -\frac{y\alpha_5}{\alpha_6}$, we have $\alpha_5^* = 0$. Thus, we can suppose $\alpha_5 = 0$ and consider following subcases:
   (a) $\alpha_4 = 0$, then $\alpha_3 \neq 0$ and choosing $t = \frac{x\alpha_2}{\alpha_6}$, $w = 0$, we can suppose $\alpha_2 = 0$. Consider following subcases:
      i. $\alpha_1 = 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the representative $\langle \nabla_3 + \nabla_6 \rangle$;
      ii. $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_6 \rangle$.
   (b) $\alpha_4 = \alpha_6$, then choosing $w = -\frac{x\alpha_1}{\alpha_6}$, we can suppose $\alpha_1 = 0$ and consider following subcases:
      i. $\alpha_2 = 0$, $\alpha_3 = 0$, then we have the representative $\langle \nabla_4 + \nabla_6 \rangle$;
      ii. $\alpha_2 = 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the representative $\langle \nabla_3 + \nabla_4 + \nabla_6 \rangle$;
      iii. $\alpha_2 \neq 0$, $\alpha_3 = 0$, then choosing $x = 1$, $y = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_6 \rangle$;
      iv. $\alpha_2 \neq 0, \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_3}{\alpha_6^2}$, $y = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 + \nabla_6 \rangle$.
   (c) $\alpha_4 \neq \alpha_6, \alpha_4 \neq 0$, then choosing $t = \frac{x(\alpha_1\alpha_3 - \alpha_2\alpha_4)}{\alpha_4(\alpha_4 - \alpha_6)}$, $w = -\frac{x\alpha_1}{\alpha_4}$, we can suppose $\alpha_1 = \alpha_2 = 0$ and consider following subcases:
      i. $\alpha_3 = 0$, then we have the family of representatives $\langle \beta\nabla_4 + \nabla_6 \rangle_{\beta\neq 0;1}$;
      ii. $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the family of representatives $\langle \nabla_3 + \beta\nabla_4 + \nabla_6 \rangle_{\beta\neq 0;1}$;
3. $\alpha_7 = 0$, $\alpha_6 = 0$, then $\alpha_4 \neq 0$ and choosing $z = -\frac{y\alpha_3}{2\alpha_4}$, $t = \frac{x(\alpha_1\alpha_3 - \alpha_2\alpha_4)}{\alpha_4^2}$, $w = -\frac{x\alpha_1}{\alpha_4}$, we have $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$ and consider following subcases:
   (a) $\alpha_5 = 0$, then we have the representative $\langle \nabla_4 \rangle$;
   (b) $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_4}$, $y = 1$, we have the representative $\langle \nabla_4 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_3 + \beta\nabla_4 + \gamma\nabla_6 + \nabla_7 \rangle^{O(\beta,\gamma)\simeq O(-\beta,-\gamma)}, \langle \nabla_2 + \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle^{O(\beta)\simeq O(-\beta)}, \langle \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle^{O(\beta)\simeq O(-\beta)},$$
$$\langle \nabla_1 + \nabla_3 + \nabla_6 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_4 + \nabla_6 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_6 \rangle, \langle \nabla_3 + \beta\nabla_4 + \nabla_6 \rangle, \langle \beta\nabla_4 + \nabla_6 \rangle_{\beta\neq 0}, \langle \nabla_4 + \nabla_5 \rangle, \langle \nabla_4 \rangle.$$

1.3.11. Central extensions of $\mathfrak{N}_{14}^1$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1 = [\Delta_{11}], & \nabla_2 = [\Delta_{21}], & \nabla_3 = [\Delta_{23}], \\ \nabla_4 = [\Delta_{13} + \Delta_{24}], & \nabla_5 = [\Delta_{32}], & \nabla_6 = [\Delta_{31} + \Delta_{42}]. \end{array}$$

Take $\theta = \sum_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{14}^1)$. The automorphism group of $\mathfrak{N}_{14}^1$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ w & u & y^2 & 0 \\ t & v & 2yz & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_4 & 0 \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 \\ \alpha_6 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_4^* & 0 \\ \alpha_2^* + \alpha^* & \alpha^{**} & \alpha_3^* & \alpha_4^* \\ \alpha_6^* & \alpha_5^* & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{14}^1)$ on the subspace $\langle \sum_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lll} \alpha_1^* &=& x\left(x\alpha_1 + w\left(\alpha_4 + \alpha_6\right)\right), \\ \alpha_2^* &=& xy\alpha_2 + wy\alpha_3 - (ux - ty - wz)\alpha_4 - wy\alpha_5 + (ux - ty - wz)\alpha_6, \\ \alpha_3^* &=& y^2(y\alpha_3 + 3z\alpha_4), \\ \alpha_4^* &=& xy^2\alpha_4, \\ \alpha_5^* &=& y^2\left(y\alpha_5 + 3z\alpha_6\right), \\ \alpha_6^* &=& xy^2\alpha_6. \end{array}$$

We are interested only in the cases with

$$(\alpha_4, \alpha_6) \neq (0, 0), (\alpha_2, \alpha_3 - \alpha_5, \alpha_4 - \alpha_6) \neq (0, 0, 0).$$

1. $\alpha_6 \neq 0$, then choosing $z = -\frac{y\alpha_5}{3\alpha_6}$, we have $\alpha_5^* = 0$. Thus, we can suppose $\alpha_5 = 0$ and consider following subcases:
   (a) $\alpha_4 \neq \alpha_6$, then choosing $t = -\frac{x\alpha_2 + w\alpha_3}{\alpha_4 - \alpha_6}$, we can suppose $\alpha_2 = 0$ and consider following subcases:
      i. $\alpha_4 = -\alpha_6$, $\alpha_1 = 0$, $\alpha_3 = 0$, then we have the representative $\langle -\nabla_4 + \nabla_6 \rangle$;
      ii. $\alpha_4 = -\alpha_6$, $\alpha_1 = 0$, $\alpha_3 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_3}$, we have the representative $\langle \nabla_3 - \nabla_4 + \nabla_6 \rangle$;
      iii. $\alpha_4 = -\alpha_6$, $\alpha_1 \neq 0$, $\alpha_3 = 0$, then choosing $x = \frac{\alpha_6}{\alpha_1}$, $y = 1$, we have the representative $\langle \nabla_1 - \nabla_4 + \nabla_6 \rangle$;
      iv. $\alpha_4 = -\alpha_6$, $\alpha_1 \neq 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, we have the representative $\langle \nabla_1 + \nabla_3 - \nabla_4 + \nabla_6 \rangle$;
      v. $\alpha_4 \neq -\alpha_6$, $\alpha_3 = 0$, then choosing $x = 1$, $w = -\frac{\alpha_1}{\alpha_4 + \alpha_6}$, we have the family of representatives $\langle \beta\nabla_4 + \nabla_6 \rangle_{\beta \notin \{-1,1\}}$;
      vi. $\alpha_4 \neq -\alpha_6$, $\alpha_3 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_3}$, $w = -\frac{\alpha_1}{\alpha_4 + \alpha_6}$, we have the family of representatives $\langle \nabla_3 + \beta\nabla_4 + \nabla_6 \rangle_{\beta \notin \{-1,1\}}$.
   (b) $\alpha_4 = \alpha_6$, then $\alpha_3 \neq 0$ and choosing $w = -\frac{x\alpha_2}{\alpha_3}$, we can suppose $\alpha_2 = 0$ and consider following subcases:
      i. $\alpha_1 = 0$, then choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_3}$, we have the representative $\langle \nabla_3 + \nabla_4 + \nabla_6 \rangle$;
      ii. $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_4 + \nabla_6 \rangle$.
2. $\alpha_6 = 0$, then $\alpha_4 \neq 0$, and choosing $z = -\frac{y\alpha_3}{3\alpha_4}$, $t = \frac{x\left(3\alpha_4\left(u\alpha_4 - y\alpha_2\right) + y\alpha_1(2\alpha_3 - 3\alpha_5)\right)}{3y\alpha_4^2}$, $w = -\frac{x\alpha_1}{\alpha_4}$, we have $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$ and consider following subcases:

(a) $\alpha_5 = 0$, then we have the representative $\langle \nabla_4 \rangle$;

(b) $\alpha_5 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_4}{\alpha_5}$, have the representative $\langle \nabla_4 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_1 + \nabla_3 + \nabla_4 + \nabla_6 \rangle, \langle \nabla_1 + \nabla_3 - \nabla_4 + \nabla_6 \rangle, \langle \nabla_1 - \nabla_4 + \nabla_6 \rangle, \langle \nabla_3 + \beta\nabla_4 + \nabla_6 \rangle, \langle \beta\nabla_4 + \nabla_6 \rangle_{\beta\neq 1}, \langle \nabla_4 + \nabla_5 \rangle, \langle \nabla_4 \rangle.$$

1.3.12. Central extensions of $\mathfrak{N}_{14}^{\alpha\notin\{0,1\}}$

Let us use the following notations:

$$\begin{array}{lll} \nabla_1 = [\Delta_{11}], & \nabla_2 = [\Delta_{21}], & \nabla_3 = [\Delta_{23}], \\ \nabla_4 = [\Delta_{13} + \Delta_{24}], & \nabla_5 = [\Delta_{32}], & \nabla_6 = [(\alpha-1)\Delta_{24} + \alpha\Delta_{31} + \Delta_{42}]. \end{array}$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2(\mathfrak{N}_{14}^{\alpha\notin\{0,1\}})$. The automorphism group of $\mathfrak{N}_{14}^{\alpha\notin\{0,1\}}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ w & u & y^2 & 0 \\ t & v & (1+\alpha)yz & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_4 & 0 \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 + (\alpha-1)\alpha_6 \\ \alpha\alpha_6 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_4^* & 0 \\ \alpha_2^* + \alpha\alpha^* & \alpha^{**} & \alpha_3^* & \alpha_4^* + (\alpha-1)\alpha_6^* \\ \alpha\alpha_6^* & \alpha_5^* & 0 & 0 \\ 0 & \alpha_6^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathfrak{N}_{14}^{\alpha\notin\{0,1\}})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lll} \alpha_1^* & = & x(x\alpha_1 + w(\alpha_4 + \alpha\alpha_6)), \\ \alpha_2^* & = & xz(1-\alpha)\alpha_1 + xy\alpha_2 + wy\alpha_3 + (ty + wz - ux\alpha)\alpha_4 - wy\alpha\alpha_5 - (ty - ux\alpha + wz\alpha^2)\alpha_6, \\ \alpha_3^* & = & y^2(y\alpha_3 + z((2+\alpha)\alpha_4 - (1-\alpha^2)\alpha_6)), \\ \alpha_4^* & = & xy^2\alpha_4, \\ \alpha_5^* & = & y^2(y\alpha_5 + z(1+2\alpha)\alpha_6), \\ \alpha_6^* & = & xy^2\alpha_6. \end{array}$$

We are interested only in the cases with $(\alpha_4, \alpha_6) \neq (0,0)$.

1. $\alpha_6 \neq 0$, then consider following subcases:
   (a) $\alpha_4 \neq -\alpha\alpha_6,\ \alpha_4 \neq \alpha_6$, then by choosing $u = 0$, $w = -\frac{x\alpha_1}{\alpha_4+\alpha\alpha_6}$ and $t = \frac{x(\alpha_1(y\alpha_3+\alpha(z\alpha_4-y\alpha_5-z\alpha_6))-y\alpha_2(\alpha_4+\alpha\alpha_6))}{y(\alpha_4-\alpha_6)(\alpha_4+\alpha\alpha_6)}$, we have $\alpha_1^* = \alpha_2^* = 0$.
      i. $\alpha = -\frac{1}{2},\ \alpha_5 = 0$, then choosing $z = -\frac{4y\alpha_3}{3(2\alpha_4-\alpha_6)}$, we have the family of representatives $\langle \beta\nabla_4 + \nabla_6 \rangle_{\beta\notin\{1,-\alpha\},\ \alpha=-\frac{1}{2}}$;
      ii. $\alpha = -\frac{1}{2},\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_6}$, $y = 1$, $z = -\frac{4\alpha_3}{3(2\alpha_4-\alpha_6)}$ we have the family of representatives $\langle \beta\nabla_4 + \nabla_5 + \nabla_6 \rangle_{\beta\notin\{1,-\alpha\},\ \alpha=-\frac{1}{2}}$;
      iii. $\alpha \neq -\frac{1}{2}$, then choosing $z = -\frac{y\alpha_5}{(1+2\alpha)\alpha_6}$ we have $\alpha_5^* = 0$.
         A. $\alpha_3 = 0$, then we have the family of representatives $\langle \beta\nabla_4 + \nabla_6 \rangle_{\beta\notin\{1,-\alpha\},\ \alpha\neq-\frac{1}{2}}$;
         B. $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6},\ y = 1$, we have the family of representatives $\langle \nabla_3 + \beta\nabla_4 + \nabla_6 \rangle_{\beta\notin\{1,-\alpha\},\ \alpha\neq-\frac{1}{2}}$.
   (b) $\alpha_4 \neq -\alpha\alpha_6,\ \alpha_4 = \alpha_6$, then choosing $w = -\frac{x\alpha_1}{(1+\alpha)\alpha_6}$, we have $\alpha_1^* = 0$.

i. $\alpha = -\frac{1}{2}$, $\alpha_5 = 0$, $\alpha_2 = 0$, then choosing $y = 1$, $z = -\frac{4\alpha_3}{3\alpha_6}$, we have the representative $\langle\nabla_4 + \nabla_6\rangle_{\alpha=-\frac{1}{2}}$;

ii. $\alpha = -\frac{1}{2}$, $\alpha_5 = 0$, $\alpha_2 \neq 0$, then choosing $x = 1$, $y = \frac{\alpha_2}{\alpha_6}$, $z = -\frac{4\alpha_2\alpha_3}{3\alpha_6^2}$, we have the representative $\langle\nabla_2 + \nabla_4 + \nabla_6\rangle_{\alpha=-\frac{1}{2}}$;

iii. $\alpha = -\frac{1}{2}$, $\alpha_5 \neq 0$, $\alpha_2 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_6}$, $y = 1$, $z = -\frac{4\alpha_3}{3\alpha_6}$ we have the representative $\langle\nabla_4 + \nabla_5 + \nabla_6\rangle_{\alpha=-\frac{1}{2}}$;

iv. $\alpha = -\frac{1}{2}$, $\alpha_5 \neq 0$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_5}{\alpha_6^2}$, $y = \frac{\alpha_2}{\alpha_6}$, $z = -\frac{4\alpha_2\alpha_3}{3\alpha_6^2}$ we have the representative $\langle\nabla_2 + \nabla_4 + \nabla_5 + \nabla_6\rangle_{\alpha=-\frac{1}{2}}$;

v. $\alpha \neq -\frac{1}{2}$, then choosing $z = -\frac{y\alpha_5}{(1+2\alpha)\alpha_6}$ we have $\alpha_5^* = 0$.

A. $\alpha_3 = \alpha_2 = 0$, then we have the representative $\langle\nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$;

B. $\alpha_3 = 0$, $\alpha_2 \neq 0$, then choosing $y = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle\nabla_2 + \nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$;

C. $\alpha_3 \neq 0$, $\alpha_2 = 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the representative $\langle\nabla_3 + \nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$;

D. $\alpha_3 \neq 0$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_3}{\alpha_6^2}$, $y = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle\nabla_2 + \nabla_3 + \nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$.

(c) $\alpha_4 = -\alpha\alpha_6$, $\alpha \neq -1$, then choosing $u = 0$, $w = 0$, $t = \frac{x(z(1-\alpha)\alpha_1 + y\alpha_2)}{y(1+\alpha)\alpha_6}$, we have $\alpha_2^* = 0$. Hence,

i. $\alpha = -\frac{1}{2}$, $\alpha_5 = \alpha_1 = \alpha_3 = 0$, then we have the representative $\left\langle\frac{1}{2}\nabla_4 + \nabla_6\right\rangle_{\alpha=-\frac{1}{2}}$;

ii. $\alpha = -\frac{1}{2}$, $\alpha_5 = \alpha_1 = 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the representative $\left\langle\nabla_3 + \frac{1}{2}\nabla_4 + \nabla_6\right\rangle_{\alpha=-\frac{1}{2}}$;

iii. $\alpha = -\frac{1}{2}$, $\alpha_5 = 0$, $\alpha_1 \neq 0$, $\alpha_3 = 0$ then choosing $x = \frac{\alpha_6}{\alpha_1}$, $y = 1$, we have the representative $\left\langle\nabla_1 + \frac{1}{2}\nabla_4 + \nabla_6\right\rangle_{\alpha=-\frac{1}{2}}$;

iv. $\alpha = -\frac{1}{2}$, $\alpha_5 = 0, \alpha_1 \neq 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, we have the representative $\left\langle\nabla_1 + \nabla_3 + \frac{1}{2}\nabla_4 + \nabla_6\right\rangle_{\alpha=-\frac{1}{2}}$;

v. $\alpha = -\frac{1}{2}$, $\alpha_5 \neq 0$, $\alpha_1 = 0$, then choosing $x = \frac{\alpha_5}{\alpha_6}$, $y = 1$, we have the family of representatives $\left\langle\beta\nabla_3 + \frac{1}{2}\nabla_4 + \nabla_5 + \nabla_6\right\rangle_{\alpha=-\frac{1}{2}}$;

vi. $\alpha = -\frac{1}{2}$, $\alpha_5 \neq 0$, $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_5^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_5}{\alpha_6^2}$, we have the family of representatives $\left\langle\nabla_1 + \beta\nabla_3 + \frac{1}{2}\nabla_4 + \nabla_5 + \nabla_6\right\rangle_{\alpha=-\frac{1}{2}}$;

vii. $\alpha \neq -\frac{1}{2}$, then choosing $z = -\frac{y\alpha_5}{(1+2\alpha)\alpha_6}$, we have $\alpha_5^* = 0$.

A. $\alpha_1 = 0$, $\alpha_3 = 0$, then we have the family of representatives $\langle-\alpha\nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$;

B. $\alpha_1 = 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_6}$, $y = 1$, we have the family of representatives $\langle\nabla_3 - \alpha\nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$;

C. $\alpha_1 \neq 0$, $\alpha_3 = 0$, then choosing $x = \frac{\alpha_6}{\alpha_1}$, $y = 1$, we have the family of representatives $\langle\nabla_1 - \alpha\nabla_4 + \nabla_6\rangle_{\alpha\notin\{-1,-\frac{1}{2}\}}$;

D. $\alpha_1 \neq 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, we have the representative $\langle\nabla_1 + \nabla_3 - \alpha\nabla_4 + \nabla_6\rangle_{\alpha\neq-1,-\frac{1}{2}}$.

(d) $\alpha_4 = -\alpha\alpha_6$, $\alpha = -1$, then choosing $z = \frac{y\alpha_5}{\alpha_6}$, we have $\alpha_5^* = 0$.

i. $\alpha_3 = \alpha_1 = \alpha_2 = 0$, then we have the representative $\langle\nabla_4 + \nabla_6\rangle_{\alpha=-1}$;

ii. $\alpha_3 = \alpha_1 = 0$, $\alpha_2 \neq 0$, then choosing $y = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle\nabla_2 + \nabla_4 + \nabla_6\rangle_{\alpha=-1}$;

iii. $\alpha_3 = 0$, $\alpha_1 \neq 0$, $\alpha_2 = 0$, then choosing $x = \frac{\alpha_6}{\alpha_1}$, $y = 1$, we have the representative $\langle\nabla_1 + \nabla_4 + \nabla_6\rangle_{\alpha=-1}$;

iv. $\alpha_3 = 0$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2^2}{\alpha_1\alpha_6}$, $y = \frac{\alpha_2}{\alpha_6}$, we have the representative $\langle \nabla_1 + \nabla_2 + \nabla_4 + \nabla_6 \rangle_{\alpha=-1}$;

v. $\alpha_3 \neq 0$, $\alpha_1 = 0$, then choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_3}$, $w = -\frac{\alpha_2}{\alpha_3}$, we have the representative $\langle \nabla_3 + \nabla_4 + \nabla_6 \rangle_{\alpha=-1}$;

vi. $\alpha_3 \neq 0$, $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1\alpha_3^2}{\alpha_6^3}$, $y = \frac{\alpha_1\alpha_3}{\alpha_6^2}$, $w = -\frac{\alpha_1\alpha_2\alpha_3}{\alpha_6^3}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_4 + \nabla_6 \rangle_{\alpha=-1}$.

2. $\alpha_6 = 0$, then $\alpha_4 \neq 0$ and choosing $w = -\frac{x\alpha_1}{\alpha_4}$, $t = \frac{x(\alpha_4(u\alpha\alpha_4 - y\alpha_2) + \alpha_1(y\alpha_3 + z\alpha\alpha_4 - y\alpha\alpha_5))}{y\alpha_4^2}$, we have $\alpha_1^* = \alpha_2^* = 0$. Thus, we can suppose $\alpha_1 = \alpha_2 = 0$ and consider following subcases:
   (a) $\alpha = -2$, $\alpha_3 = \alpha_5 = 0$, then we have the representative $\langle \nabla_4 \rangle$;
   (b) $\alpha = -2$, $\alpha_3 = 0$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_4}$, $y = 1$, we have the representative $\langle \nabla_4 + \nabla_5 \rangle$;
   (c) $\alpha = -2$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4}$, $y = 1$, we have the family of representatives $\langle \nabla_3 + \nabla_4 + \beta\nabla_5 \rangle$;
   (d) $\alpha \neq -2$, $\alpha_5 = 0$, then choosing $y = 1$, $z = -\frac{\alpha_3}{(2+\alpha)\alpha_4}$, we have the representative $\langle \nabla_4 \rangle_{\alpha\neq-2}$;
   (e) $\alpha \neq -2$, $\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_4}$, $y = 1$, $z = -\frac{\alpha_3}{(2+\alpha)\alpha_4}$, we have the representative $\langle \nabla_4 + \nabla_5 \rangle_{\alpha\neq-2}$.

Summarizing all cases for the family of algebras $\mathfrak{N}_{14}^{\alpha}$, we have the following distinct orbits:

$$\langle \nabla_3 + \beta\nabla_4 + \gamma\nabla_6 + \nabla_7 \rangle_{\alpha=0}^{O(\beta,\gamma)\simeq O(-\beta,-\gamma)}, \langle \nabla_2 + \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle_{\alpha=0}^{O(\beta)\simeq O(-\beta)}, \langle \nabla_5 + \beta\nabla_6 + \nabla_7 \rangle_{\alpha=0}^{O(\beta)\simeq O(-\beta)},$$
$$\langle \nabla_1 + \nabla_3 + \nabla_4 + \nabla_6 \rangle_{\alpha=1}, \langle \nabla_1 + \nabla_2 + \nabla_4 + \nabla_6 \rangle_{\alpha=-1}, \langle \nabla_3 + \nabla_4 + \beta\nabla_5 \rangle_{\alpha=-2}, \langle \nabla_1 + \beta\nabla_3 + \tfrac{1}{2}\nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha=-\frac{1}{2}},$$
$$\langle \beta\nabla_3 + \tfrac{1}{2}\nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha=-\frac{1}{2},\beta\neq0}, \langle \nabla_2 + \nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha=-\frac{1}{2}}, \langle \beta\nabla_4 + \nabla_5 + \nabla_6 \rangle_{\alpha=-\frac{1}{2}}, \langle \nabla_2 + \nabla_3 + \nabla_4 + \nabla_6 \rangle_{\alpha\neq-\frac{1}{2}},$$
$$\langle \nabla_2 + \nabla_4 + \nabla_6 \rangle, \langle \nabla_1 + \nabla_3 - \alpha\nabla_4 + \nabla_6 \rangle, \langle \nabla_1 - \alpha\nabla_4 + \nabla_6 \rangle_{\alpha\neq0}, \langle \nabla_3 + \beta\nabla_4 + \nabla_6 \rangle, \langle \beta\nabla_4 + \nabla_6 \rangle, \langle \nabla_4 + \nabla_5 \rangle, \langle \nabla_4 \rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{101}^{\beta,\gamma}, \mathrm{N}_{102}^{\beta}, \mathrm{N}_{103}^{\beta}, \mathrm{N}_{104}, \mathrm{N}_{105}, \mathrm{N}_{106}^{\beta}, \mathrm{N}_{107}^{\beta}, \mathrm{N}_{108}^{\beta}, \mathrm{N}_{109},$$
$$\mathrm{N}_{110}^{\beta}, \mathrm{N}_{111}^{\alpha(\alpha\neq-\frac{1}{2})}, \mathrm{N}_{112}^{\alpha}, \mathrm{N}_{113}^{\alpha}, \mathrm{N}_{114}^{\alpha(\alpha\neq0)}, \mathrm{N}_{115}^{\alpha,\beta}, \mathrm{N}_{116}^{\alpha,\beta}, \mathrm{N}_{117}^{\alpha}, \mathrm{N}_{118}^{\alpha}.$$

1.4. 1-*dimensional central extensions of two-generated* 4-*dimensional* 3-*step nilpotent Novikov algebras*

1.4.1. *The description of second cohomology space*

In the following table, we give the description of the second cohomology space of two-generated 4-dimensional 3-step nilpotent Novikov algebras.

$\mathcal{N}_{01}^4: \quad e_1e_1 = e_2 \quad e_2e_1 = e_3$
$\mathrm{H}^2(\mathcal{N}_{01}^4) = \langle [\Delta_{12}], [\Delta_{13} - \Delta_{31}], [\Delta_{14}], [\Delta_{41}], [\Delta_{44}] \rangle$
$\mathcal{N}_{02}^4(\lambda): \quad e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_2e_1 = \lambda e_3$
$\mathrm{H}^2(\mathcal{N}_{02}^4(\lambda \neq 1)) = \langle [\Delta_{14}], [\Delta_{21}], [(2-\lambda)\Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31}], [\Delta_{41}], [\Delta_{44}] \rangle$
$\mathrm{H}^2_{Comm}(\mathcal{N}_{02}^4(1)) = \langle [\Delta_{13} + \Delta_{22} + \Delta_{31}], [\Delta_{14} + \Delta_{41}], [\Delta_{44}] \rangle$
$\mathrm{H}^2(\mathcal{N}_{02}^4(1)) = \mathrm{H}^2_{Comm}(\mathcal{N}_{02}^4(1)) \oplus \langle [\Delta_{21}], [\Delta_{41}] \rangle$
$\mathcal{N}_{04}^4(\alpha): \quad e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_2e_1 = \alpha e_4 \quad e_3e_3 = e_4$
$\mathrm{H}^2(\mathcal{N}_{04}^4) = \langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \rangle$
$\mathcal{N}_{05}^4: \quad e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4 \quad e_3e_3 = e_4$
$\mathrm{H}^2(\mathcal{N}_{05}^4) = \langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \rangle$
$\mathcal{N}_{06}^4(\alpha \neq 0): \quad e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = e_4 \quad e_2e_1 = \alpha e_4$
$\mathrm{H}^2(\mathcal{N}_{06}^4(\alpha \neq 0)) = \langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\frac{2-\alpha}{\alpha}\Delta_{14} + \Delta_{22} + \Delta_{23} - \Delta_{32} + \Delta_{41}], [\Delta_{33}] \rangle$

$\mathcal{N}^4_{07}$: $e_1e_1 = e_2 \quad e_2e_1 = e_4 \quad e_3e_3 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{07}) = \Big\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{31}], [\Delta_{33}] \Big\rangle$
$\mathcal{N}^4_{08}$: $e_1e_1 = e_2 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{08}) = \Big\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}], [-\Delta_{14} + \Delta_{23} - \Delta_{32} + \Delta_{41}] \Big\rangle$
$\mathcal{N}^4_{09}$: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_3e_1 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{09}) = \Big\langle [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}], [\Delta_{14} + \Delta_{32}] \Big\rangle$
$\mathcal{N}^4_{10}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{10}) = \Big\langle [\Delta_{11}], [\Delta_{14}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23} - \Delta_{32}] \Big\rangle$
$\mathcal{N}^4_{11}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{11}) = \Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23} - \Delta_{32}] \Big\rangle$
$\mathcal{N}^4_{12}$: $e_1e_2 = e_3 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{12}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$
$\mathcal{N}^4_{13}$: $e_1e_2 = e_3 \quad e_1e_1 = e_4 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{13}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$
$\mathcal{N}^4_{14}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{14}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$
$\mathcal{N}^4_{15}$: $e_1e_2 = e_3 \quad e_1e_1 = e_4 \quad e_1e_3 = e_4 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{15}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$
$\mathcal{N}^4_{16}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_2 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{16}) = \Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}], [\Delta_{14} + \Delta_{23}], [-\Delta_{23} + \Delta_{32}] \Big\rangle$
$\mathcal{N}^4_{17}$: $e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_2e_1 = e_4 \quad e_2e_2 = e_4$ $\mathrm{H}^2(\mathcal{N}^4_{17}) = \Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23} - \Delta_{32}] \Big\rangle$
$\mathcal{N}^4_{18}$: $e_1e_2 = e_3 \quad e_2e_2 = e_4 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{18}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$
$\mathcal{N}^4_{19}$: $e_1e_2 = e_3 \quad e_1e_1 = e_4 \quad e_2e_2 = e_4 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{19}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$
$\mathcal{N}^4_{20}(\alpha)$: $e_1e_2 = e_3 \quad e_1e_1 = \alpha e_4 \quad e_1e_3 = e_4 \quad e_2e_2 = e_4 \quad e_2e_3 = e_4 \quad e_3e_2 = -e_4$ $\mathrm{H}^2(\mathcal{N}^4_{20}) = \Big\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$

1.4.2. Central extensions of $\mathcal{N}^4_{01}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{13} - \Delta_{31}], \quad \nabla_3 = [\Delta_{14}], \quad \nabla_4 = [\Delta_{41}], \quad \nabla_5 = [\Delta_{44}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}^4_{01})$. The automorphism group of $\mathcal{N}^4_{01}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & t \\ u & 0 & 0 & r \end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix}0&\alpha_1&\alpha_2&\alpha_3\\0&0&0&0\\-\alpha_2&0&0&0\\\alpha_4&0&0&\alpha_5\end{pmatrix}\phi=\begin{pmatrix}\alpha^*&\alpha_1^*&\alpha_2^*&\alpha_3^*\\\alpha^{**}&0&0&0\\-\alpha_2^*&0&0&0\\\alpha_4^*&0&0&\alpha_5^*\end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{01}^4)$ on the subspace $\langle\sum\limits_{i=1}^{5}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{5}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{lllllllll}\alpha_1^*&=&x^2(x\alpha_1+y\alpha_2),&\alpha_2^*&=&x^4\alpha_2,&\alpha_3^*&=&tx\alpha_2+rx\alpha_3+ru\alpha_5,\\\alpha_4^*&=&-tx\alpha_2+rx\alpha_4+ru\alpha_5,&\alpha_5^*&=&r^2\alpha_5.\end{array}$$

We are interested only in the cases with

$$(\alpha_3,\alpha_4,\alpha_5)\neq(0,0,0),\ \alpha_2\neq0.$$

Since $\alpha_2\neq0$, then choosing $y=-\frac{x\alpha_1}{\alpha_2},\ t=-\frac{r(x\alpha_3+u\alpha_5)}{x\alpha_2}$, we have $\alpha_1^*=\alpha_3^*=0$.

1. If $\alpha_5\neq0$, then choosing $x=1,\ u=-\frac{\alpha_4}{2\alpha_5},\ r=\sqrt{\frac{\alpha_2}{\alpha_5}}$, we have the representative $\langle\nabla_2+\nabla_5\rangle$.
2. If $\alpha_5=0$, then $\alpha_4\neq0$ and choosing $x=1,\ r=-\frac{\alpha_2}{\alpha_4}$, we have the representative $\langle\nabla_2+\nabla_4\rangle$.

Therefore, we have the following distinct orbits

$$\langle\nabla_2+\nabla_5\rangle,\quad\langle\nabla_2+\nabla_4\rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{119},\mathrm{N}_{120}.$$

1.4.3. Central extensions of $\mathcal{N}_{02}^4(\lambda\neq1)$

Let us use the following notations:

$$\nabla_1=[\Delta_{14}],\quad\nabla_2=[\Delta_{21}],\quad\nabla_3=[(2-\lambda)\Delta_{13}+\lambda\Delta_{22}+\lambda\Delta_{31}],\quad\nabla_4=[\Delta_{41}],\quad\nabla_5=[\Delta_{44}].$$

Take $\theta=\sum\limits_{i=1}^{5}\alpha_i\nabla_i\in\mathrm{H}^2(\mathcal{N}_{02}^4(\lambda\neq1))$. The automorphism group of $\mathcal{N}_{02}^4(\lambda\neq1)$ consists of invertible matrices of the form

$$\phi=\begin{pmatrix}x&0&0&0\\y&x^2&0&0\\z&(1+\lambda)xy&x^3&t\\u&0&0&r\end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix}0&0&(2-\lambda)\alpha_3&\alpha_1\\\alpha_2&\lambda\alpha_3&0&0\\\lambda\alpha_3&0&0&0\\\alpha_4&0&0&\alpha_5\end{pmatrix}\phi=\begin{pmatrix}\alpha^*&\alpha^{**}&(2-\lambda)\alpha_3^*&\alpha_1^*\\\alpha_2^*+\lambda\alpha^{**}&\lambda\alpha_3^*&0&0\\\lambda\alpha_3^*&0&0&0\\\alpha_4^*&0&0&\alpha_5^*\end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{02}^4(\lambda\neq1))$ on the subspace $\langle\sum\limits_{i=1}^{5}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{5}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{lllllllll}\alpha_1^*&=&rx\alpha_1+tx(2-\lambda)\alpha_3+ru\alpha_5,&\alpha_2^*&=&x^2\left(x\alpha_2-y(1-\lambda)\lambda^2\alpha_3\right),&\alpha_3^*&=&x^4\alpha_3,\\\alpha_4^*&=&tx\lambda\alpha_3+rx\alpha_4+ru\alpha_5,&\alpha_5^*&=&r^2\alpha_5.\end{array}$$

We are interested only in the cases with

$$\alpha_3 \neq 0,\ (\alpha_1, \alpha_4, \alpha_5) \neq (0,0,0).$$

1. $\lambda = 0$, then choosing $t = -\frac{r(x\alpha_1+u\alpha_5)}{2x\alpha_3}$, we have $\alpha_1^* = 0$.
   (a) $\alpha_5 = \alpha_2 = 0,\ \alpha_4 \neq 0$, then choosing $x = 1,\ r = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$;
   (b) $\alpha_5 = 0,\ \alpha_2 \neq 0,\ \alpha_4 = 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$;
   (c) $\alpha_5 = 0,\ \alpha_2 \neq 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3},\ r = \frac{\alpha_2^3}{\alpha_3^2\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$;
   (d) $\alpha_5 \neq 0,\ \alpha_2 = 0$, then choosing $x = 1,\ r = \sqrt{\frac{\alpha_3}{\alpha_4}},\ u = -\frac{\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$;
   (e) $\alpha_5 \neq 0,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3},\ r = \frac{\alpha_2^2}{\alpha_3^2}\sqrt{\frac{\alpha_3}{\alpha_4}},\ u = -\frac{\alpha_2\alpha_4}{\alpha_3\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$.
2. $\lambda \neq 0$, then choosing $y = \frac{x\alpha_2}{(1-\lambda)\lambda^2\alpha_3},\ t = -\frac{r(x\alpha_4+u\alpha_5)}{x\lambda\alpha_3}$, we have $\alpha_2^* = \alpha_4^* = 0$.
   (a) $\alpha_5 \neq 0$, then choosing $u = \frac{\lambda\alpha_1}{2(1-\lambda)\alpha_5},\ x = 1,\ r = \sqrt{\frac{\alpha_3}{\alpha_5}}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle_{\lambda\neq 0}$,
   (b) $\alpha_5 = 0,\ \alpha_1 \neq 0$, then choosing $x = 1,\ r = \frac{\alpha_3}{\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle_{\lambda\neq 0}$.

Summarizing all cases, we have the following distinct orbits:

$$\langle \nabla_3 + \nabla_4 \rangle_{\lambda=0}, \langle \nabla_2 + \nabla_3 \rangle_{\lambda=0}, \langle \nabla_2 + \nabla_3 + \nabla_4 \rangle_{\lambda=0},$$
$$\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle_{\lambda=0}, \langle \nabla_3 + \nabla_5 \rangle_{\lambda\neq 1}, \langle \nabla_1 + \nabla_3 \rangle_{\lambda\neq 0;1},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{121}, \mathrm{N}_{122}, \mathrm{N}_{123}, \mathrm{N}_{124}, \mathrm{N}_{125}^{\lambda\neq 1}, \mathrm{N}_{126}^{\lambda\neq 0;1}.$$

1.4.4. Central extensions of $\mathcal{N}_{02}^4(1)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{14} + \Delta_{41}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{13} + \Delta_{22} + \Delta_{31}], \quad \nabla_4 = [\Delta_{41}], \quad \nabla_5 = [\Delta_{44}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}_{02}^4(1))$. The automorphism group of $\mathcal{N}_{02}^4(1)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & t \\ u & 0 & 0 & r \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_3 & \alpha_1 \\ \alpha_2 & \alpha_3 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \\ \alpha_1+\alpha_4 & 0 & 0 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_3^* & \alpha_1^* \\ \alpha_2^*+\alpha^{**} & \alpha_3^* & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 \\ \alpha_1^*+\alpha_4^* & 0 & 0 & \alpha_5^* \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{02}^4(1))$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* & = & rx\alpha_1 + tx\alpha_3 + ru\alpha_5, & \alpha_2^* & = & x^3\alpha_2, & \alpha_3^* & = & x^4\alpha_3, \\ \alpha_4^* & = & rx\alpha_4, & \alpha_5^* & = & r^2\alpha_5. \end{array}$$

We are interested only in the cases with

$$\alpha_3 \neq 0,\ (\alpha_1, \alpha_4, \alpha_5) \neq (0,0,0),\ (\alpha_2, \alpha_4) \neq (0,0).$$

$\alpha_3 \neq 0$, then choosing $t = -\frac{r(x\alpha_1+u\alpha_5)}{x\alpha_3}$, we have $\alpha_1^* = 0$.

1. $\alpha_4 = 0$, then $\alpha_5 \neq 0,\ \alpha_2 \neq 0$. Choosing $x = \frac{\alpha_2}{\alpha_3},\ r = \frac{\alpha_2^2}{\alpha_3\sqrt{\alpha_3\alpha_5}}$, we have the representative $\langle\nabla_2 + \nabla_3 + \nabla_5\rangle$;
2. $\alpha_4 \neq 0,\ \alpha_2 = 0,\ \alpha_5 = 0$, then choosing $x = 1,\ r = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle\nabla_3 + \nabla_4\rangle$;
3. $\alpha_4 \neq 0,\ \alpha_2 = 0,\ \alpha_5 \neq 0$ then choosing $x = \frac{\alpha_4}{\sqrt{\alpha_3\alpha_5}},\ r = \frac{\alpha_4^2}{\sqrt{\alpha_3\alpha_5^3}}$, we have the representative $\langle\nabla_3 + \nabla_4 + \nabla_5\rangle$;
4. $\alpha_4 \neq 0,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3},\ r = \frac{\alpha_2^3}{\alpha_3^2\alpha_4}$, we have the representative $\langle\nabla_2 + \nabla_3 + \nabla_4 + \alpha\nabla_5\rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle\nabla_2 + \nabla_3 + \nabla_4\rangle,\quad \langle\nabla_3 + \nabla_4\rangle,\quad \langle\nabla_3 + \nabla_4 + \nabla_5\rangle,\quad \langle\nabla_2 + \nabla_3 + \nabla_4 + \alpha\nabla_5\rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{127}, \mathrm{N}_{128}, \mathrm{N}_{129}, \mathrm{N}_{130}^{\alpha}.$$

1.4.5. Central extensions of $\mathcal{N}_{06}^4(\alpha \neq 0)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{13}],\quad \nabla_2 = [\Delta_{21}],\quad \nabla_3 = [\Delta_{31}],\quad \nabla_4 = [\tfrac{2-\alpha}{\alpha}\Delta_{14} + \Delta_{22} + \Delta_{23} - \Delta_{32} + \Delta_{41}],\quad \nabla_5 = [\Delta_{33}].$$

Take $\theta = \sum_{i=1}^{5}\alpha_i\nabla_i \in \mathrm{H}^2(\mathcal{N}_{06}^4(\alpha \neq 0))$. The automorphism group of $\mathcal{N}_{06}^4(\alpha \neq 0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0\\ y & x^2 & 0 & 0\\ z & 0 & x^2 & 0\\ u & x((1+\alpha)y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \frac{2-\alpha}{\alpha}\alpha_4\\ \alpha_2 & \alpha_4 & \alpha_4 & 0\\ \alpha_3 & -\alpha_4 & \alpha_5 & 0\\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}\phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_1^* + \alpha^{**} & \frac{2-\alpha}{\alpha}\alpha_4^*\\ \alpha_2^* + \alpha\alpha^{**} & \alpha_4^* & \alpha_4^* & 0\\ \alpha_3^* & -\alpha_4^* & \alpha_5^* & 0\\ \alpha_4^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{06}^4(\alpha \neq 0))$ on the subspace $\langle\sum_{i=1}^{5}\alpha_i\nabla_i\rangle$ is given by $\langle\sum_{i=1}^{5}\alpha_i^*\nabla_i\rangle$, where

$$\begin{aligned}
\alpha_1^* &= x\Big(x^2\alpha_1 - \Big(v(\alpha-2) + x\Big(2z(1-\alpha) + y\Big(2+\alpha-\alpha^2\Big)\Big)\Big)\alpha^{-1}\alpha_4 + xz\alpha_5\Big),\\
\alpha_2^* &= x^2\left(x\alpha_2 + (2z - y(1-\alpha))\alpha\alpha_4\right),\\
\alpha_3^* &= x\Big(x^2\alpha_3 + (v - xy)\alpha_4 + xz\alpha_5\Big),\\
\alpha_4^* &= x^4\alpha_4,\\
\alpha_5^* &= x^4\alpha_5.
\end{aligned}$$

We are interested only in the cases with $\alpha_4 \neq 0$. Choosing

$$z = \tfrac{1}{2}((1-\alpha)y - \tfrac{x\alpha_2}{\alpha\alpha_4}) \text{ and } v = -\frac{x(2x\alpha\alpha_3\alpha_4 - 2y\alpha\alpha_4^2 - x\alpha_2\alpha_5 - y(\alpha-1)\alpha\alpha_4\alpha_5)}{2\alpha\alpha_4^2},$$

we have $\alpha_2^* = \alpha_3^* = 0$.

1. $\alpha = 1$, then choosing $y = \frac{x\alpha_1}{\alpha_4}$ we have the family of representatives $\langle\nabla_4 + \beta\nabla_5\rangle$;
2. $\alpha \neq 1,\ (\alpha-1)^2\alpha_5 = -\alpha_4,\ \alpha_1 = 0$, then we have the representative $\left\langle\nabla_4 + \frac{1}{(\alpha-1)^2}\nabla_5\right\rangle$;
3. $\alpha \neq 1,\ (\alpha-1)^2\alpha_5 = -\alpha_4,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_5}$, we have the representative $\left\langle\nabla_1 + \nabla_4 + \frac{1}{(\alpha-1)^2}\nabla_5\right\rangle$;

4. $\alpha \neq 1$, $(\alpha-1)^2\alpha_5 \neq -\alpha_4$ then choosing $y = \frac{x\alpha\alpha_1}{\alpha_4+(\alpha-1)^2\alpha_5}$, we have the family of representatives $\langle \nabla_4 + \beta\nabla_5 \rangle_{\beta \neq \frac{1}{(\alpha-1)^2}}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_4 + \beta\nabla_5 \rangle, \langle \nabla_1 + \nabla_4 + \tfrac{1}{(\alpha-1)^2}\nabla_5 \rangle_{\alpha\neq 1},$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{131}^{\alpha\neq 0,\beta}, \mathrm{N}_{132}^{\alpha\neq 0,1}.$$

1.4.6. Central extensions of $\mathcal{N}_{08}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{31}], \quad \nabla_4 = [\Delta_{33}], \quad \nabla_5 = [\Delta_{23} - \Delta_{32} - \Delta_{14} + \Delta_{41}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}_{08}^4)$. The automorphism group of $\mathcal{N}_{08}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 0 & x^2 & 0 \\ u & x(y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & -\alpha_5 \\ \alpha_2 & 0 & \alpha_5 & 0 \\ \alpha_3 & -\alpha_5 & \alpha_4 & 0 \\ \alpha_5 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha^{**} & -\alpha_5^* \\ \alpha_2^* + \alpha^{**} & 0 & \alpha_5^* & 0 \\ \alpha_3^* & -\alpha_5^* & \alpha_4^* & 0 \\ \alpha_5^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{08}^4)$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* & = & x^2(x\alpha_1 - (y+2z)\alpha_5), & \alpha_3^* & = & x(x^2\alpha_3 + xz\alpha_4 + (v - xy)\alpha_5), & \alpha_5^* & = & x^4\alpha_5. \\ \alpha_2^* & = & x(x^2\alpha_2 - xz\alpha_4 + (v+2xz)\alpha_5), & \alpha_4^* & = & x^4\alpha_4, & & & \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing

$$v = \frac{x\big(x\alpha_1(\alpha_4 - 2\alpha_5) + \alpha_5(2y\alpha_5 - 2x\alpha_2 - y\alpha_4)\big)}{2\alpha_5^2}, \text{ and } z = \frac{x\alpha_1 - y\alpha_5}{2\alpha_5},$$

we have $\alpha_1^* = \alpha_2^* = 0$.

1. $\alpha_4 = \alpha_3 = 0$, then we have the representative $\langle \nabla_5 \rangle$;
2. $\alpha_4 = 0$, $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$;
3. $\alpha_4 \neq 0$, then choosing $y = \frac{x\alpha_3}{\alpha_4}$, we have the family of representatives $\langle \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\neq 0}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_3 + \nabla_5 \rangle, \langle \alpha\nabla_4 + \nabla_5 \rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{133}^{\alpha\neq 0}, \mathrm{N}_{134}^{\alpha\neq 0,1}.$$

1.4.7. Central extensions of $\mathcal{N}_{09}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{31}], \quad \nabla_4 = [\Delta_{33}], \quad \nabla_5 = [\Delta_{14} + \Delta_{32}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}_{09}^4)$. The automorphism group of $\mathcal{N}_{09}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 0 & x^2 & 0 \\ u & x(y+z) & v & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_5 \\ \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & \alpha_5 & \alpha_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_1^* & \alpha_5^* \\ \alpha_2^* & 0 & 0 & 0 \\ \alpha_3^* + \alpha^{**} & \alpha_5^* & \alpha_4^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{09}^4)$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllll} \alpha_1^* & = & x(x^2\alpha_1 + xz\alpha_4 + v\alpha_5), & \alpha_2^* & = & x^3\alpha_2, & \\ \alpha_3^* & = & x^2(x\alpha_3 + z(\alpha_4 - 2\alpha_5)), & \alpha_4^* & = & x^4\alpha_4, & \alpha_5^* \; = \; x^4\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_5 \neq 0$. Choosing $v = -\frac{x(x\alpha_1 + z\alpha_4)}{\alpha_5}$, we have $\alpha_1^* = 0$.

1. $\alpha_4 = 2\alpha_5,\ \alpha_3 = \alpha_2 = 0$, then we have the representative $\langle 2\nabla_4 + \nabla_5 \rangle$;
2. $\alpha_4 = 2\alpha_5,\ \alpha_3 = 0,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_5}$, we have the representative $\langle \nabla_2 + 2\nabla_4 + \nabla_5 \rangle$;
3. $\alpha_4 = 2\alpha_5,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_3 + 2\nabla_4 + \nabla_5 \rangle$;
4. $\alpha_4 \neq 2\alpha_5,\ \alpha_2 = 0$, then choosing $x = 1,\ z = -\frac{\alpha_3}{\alpha_4 - 2\alpha_5}$, we have the representative $\langle \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\neq 2}$;
5. $\alpha_4 \neq 2\alpha_5,\ \alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_5},\ z = -\frac{\alpha_2\alpha_3}{(\alpha_4 - 2\alpha_5)\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \alpha\nabla_4 + \nabla_5 \rangle_{\alpha\neq 2}$.

Summarizing all cases, we have the following distinct orbits

$$\langle \alpha\nabla_2 + \nabla_3 + 2\nabla_4 + \nabla_5 \rangle, \quad \langle \alpha\nabla_4 + \nabla_5 \rangle, \quad \langle \nabla_2 + \alpha\nabla_4 + \nabla_5 \rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{135}^{\alpha}, \mathrm{N}_{136}^{\alpha}, \mathrm{N}_{137}^{\alpha}.$$

1.4.8. Central extensions of $\mathcal{N}_{10}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{14}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{22}], \quad \nabla_5 = [\Delta_{23} - \Delta_{32}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}_{10}^4)$. The automorphism group of $\mathcal{N}_{10}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & z & xy & 0 \\ u & v & xz & x^2y \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_2 \\ \alpha_3 & \alpha_4 & \alpha_5 & 0 \\ 0 & -\alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha^{**} & \alpha_2^* \\ \alpha_3^* & \alpha_4^* & \alpha_5^* & 0 \\ 0 & -\alpha_5^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{10}^4)$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{llllll} \alpha_1^* = x(x\alpha_1 + u\alpha_2), & \alpha_2^* = x^3 y \alpha_2, & \alpha_3^* = xy\alpha_3, \\ \alpha_4^* = y^2\alpha_4, & \alpha_5^* = xy^2\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_2 \neq 0$. Choosing $v = -\frac{x\alpha_1}{\alpha_2}$, we have $\alpha_1^* = 0$.

1. $\alpha_5 = \alpha_3 = \alpha_4 = 0$, then we have the representative $\langle \nabla_2 \rangle$;
2. $\alpha_5 = \alpha_3 = 0,\ \alpha_4 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$;
3. $\alpha_5 = 0,\ \alpha_3 \neq 0,\ \alpha_4 = 0$, then choosing $x = \sqrt{\frac{\alpha_3}{\alpha_2}}$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$;
4. $\alpha_5 = 0,\ \alpha_3 \neq 0,\ \alpha_4 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_3}{\alpha_2}},\ y = \frac{\alpha_3}{\alpha_4}\sqrt{\frac{\alpha_3}{\alpha_2}}$ we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$;
5. $\alpha_5 \neq 0,\ \alpha_3 = 0,\ \alpha_4 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_5}$ we have the representative $\langle \nabla_2 + \nabla_5 \rangle$;
6. $\alpha_5 \neq 0,\ \alpha_3 = 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5},\ y = \frac{\alpha_2\alpha_4^2}{\alpha_5^3}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_5 \rangle$;
7. $\alpha_5 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_3}{\alpha_2}},\ y = \frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \alpha\nabla_4 + \nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_2 \rangle, \langle \nabla_2 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_3 \rangle, \langle \nabla_2 + \nabla_3 + \nabla_4 \rangle, \langle \nabla_2 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_4 + \nabla_5 \rangle, \langle \nabla_2 + \nabla_3 + \alpha\nabla_4 + \nabla_5 \rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{138}, \mathrm{N}_{139}, \mathrm{N}_{140}, \mathrm{N}_{141}, \mathrm{N}_{142}, \mathrm{N}_{143}, \mathrm{N}_{144}^{\alpha}.$$

1.4.9. Central extensions of $\mathcal{N}_{16}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{22}], \quad \nabla_4 = [\Delta_{14} + \Delta_{23}], \quad \nabla_5 = [-\Delta_{23} + \Delta_{32}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2(\mathcal{N}_{16}^4)$. The automorphism group of $\mathcal{N}_{16}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ 0 & y & x^3 & 0 \\ u & v & xy & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 - \alpha_5 & 0 \\ 0 & \alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha^{**} & \alpha_4^* \\ \alpha_2^* & \alpha_3^* + \alpha^{**} & \alpha_4^* - \alpha_5^* & 0 \\ 0 & \alpha_5^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathcal{N}_{16}^4)$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{lllllllll} \alpha_1^* & = & x(x\alpha_1 + u\alpha_4), & \alpha_2^* & = & x^3\alpha_2, & \alpha_3^* & = & x^4\alpha_3, \\ \alpha_4^* & = & x^5\alpha_4, & \alpha_5^* & = & x^5\alpha_5. \end{array}$$

We are interested only in the cases with $\alpha_4 \neq 0$. Choosing $u = -\frac{x\alpha_1}{\alpha_4}$, we have $\alpha_1^* = 0$.

1. $\alpha_3 = \alpha_2 = 0$, then we have the family of representatives $\langle \nabla_4 + \alpha\nabla_5 \rangle$;
2. $\alpha_3 = 0,\ \alpha_2 \neq 0$, then choosing $x = \sqrt{\frac{\alpha_2}{\alpha_4}}$, we have the representative $\langle \nabla_2 + \nabla_4 + \alpha\nabla_5 \rangle$;
3. $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle \alpha\nabla_2 + \nabla_3 + \nabla_4 + \beta\nabla_5 \rangle$.

Summarizing all cases, we have the following distinct orbits

$$\langle \nabla_4 + \alpha\nabla_5 \rangle, \quad \langle \nabla_2 + \nabla_4 + \alpha\nabla_5 \rangle, \quad \langle \alpha\nabla_2 + \nabla_3 + \nabla_4 + \beta\nabla_5 \rangle,$$

which gives the following new algebras (see section 2):

$$\mathrm{N}_{145}^{\alpha}, \mathrm{N}_{146}^{\alpha}, \mathrm{N}_{147}^{\alpha,\beta}.$$

1.5. 2-dimensional central extensions of two-generated 3-dimensional nilpotent Novikov algebras

1.5.1. The description of second cohomology spaces.

In the following table, we give the description of the second cohomology space of two-generated 3-dimensional nilpotent Novikov algebras

$\mathcal{N}_{01}^{3*}$	: $e_1e_1 = e_2$
$\mathrm{H}^2(\mathcal{N}_{01}^{3*}) = \langle [\Delta_{12} + \Delta_{21}], [\Delta_{13} + \Delta_{31}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \rangle$	
$\mathcal{N}_{02}^{3*}$	: $e_1e_1 = e_3 \quad e_2e_2 = e_3$
$\mathrm{H}^2(\mathcal{N}_{02}^{3*}) = \langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \rangle$	
$\mathcal{N}_{03}^{3*}$	: $e_1e_2 = e_3 \quad e_2e_1 = -e_3$
$\mathrm{H}^2(\mathcal{N}_{03}^{3*}) = \langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}] \rangle$	
$\mathcal{N}_{04}^{3*}(\lambda \neq 0)$	: $e_1e_1 = \lambda e_3 \quad e_2e_1 = e_3 \quad e_2e_2 = e_3$
$\mathrm{H}^2(\mathcal{N}_{04}^{3*}(\lambda \neq 0)) = \langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \rangle$	
$\mathcal{N}_{04}^{3*}(0)$	: $e_1e_2 = e_3$
$\mathrm{H}^2(\mathcal{N}_{04}^{3*}(0)) = \langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{23}] - [\Delta_{32}] \rangle$	

1.5.2. Central extensions of $\mathcal{N}_{01}^{3*}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12} + \Delta_{21}], \quad \nabla_2 = [\Delta_{13} + \Delta_{31}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{31}], \quad \nabla_5 = [\Delta_{33}].$$

The automorphism group of $\mathcal{N}_{01}^{3*}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ u & x^2 & w \\ z & 0 & y \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 + \alpha_3 & 0 & 0 \\ \alpha_2 + \alpha_4 & 0 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* \\ \alpha_1^* + \alpha_3^* & 0 & 0 \\ \alpha_2^* + \alpha_4^* & 0 & \alpha_5^* \end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{N}_{01}^{3*})$ on subspace $\left\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \right\rangle$, where

$$\begin{array}{lcl} \alpha_1^* &=& x^3\alpha_1, \\ \alpha_2^* &=& wx\alpha_1 + xy\alpha_2 + yz\alpha_5, \\ \alpha_3^* &=& x^3\alpha_3, \\ \alpha_4^* &=& x(w\alpha_3 + y\alpha_4), \\ \alpha_5^* &=& y^2\alpha_5. \end{array}$$

We are interested only in 2-dimensional central extensions and consider the vector space generated by the following two cocycles:

$$\theta_1 = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3 + \alpha_4\nabla_4 + \alpha_5\nabla_5 \text{ and } \theta_2 = \beta_1\nabla_1 + \beta_2\nabla_2 + \beta_4\nabla_4 + \beta_5\nabla_5.$$

Our aim is to find only central extensions with $(\alpha_3, \alpha_4, \beta_3, \beta_4) \neq 0$. Hence, we have the following cases.

1. $\alpha_3 \neq 0$, then we have

$$\begin{array}{lcllcl} \alpha_1^* &=& x^3\alpha_1, & \beta_1^* &=& x^3\beta_1, \\ \alpha_2^* &=& wx\alpha_1 + xy\alpha_2 + yz\alpha_5, & \beta_2^* &=& wx\beta_1 + xy\beta_2 + yz\beta_5, \\ \alpha_3^* &=& x^3\alpha_3, & \beta_3^* &=& 0, \\ \alpha_4^* &=& x(w\alpha_3 + y\alpha_4), & \beta_4^* &=& xy\beta_4, \\ \alpha_5^* &=& y^2\alpha_5, & \beta_5^* &=& y^2\beta_5. \end{array}$$

(a) $\beta_5 \neq 0$, then we can suppose $\alpha_5 = 0$ and choosing $w = -\frac{y\alpha_4}{\alpha_3}, z = -\frac{x(\alpha_4\beta_1 - \alpha_3\beta_2)}{\alpha_3\beta_5}$, we have $\alpha_4^* = \beta_2^* = 0$. Thus, we can assume $\alpha_4 = \beta_2 = 0$ and consider following subcases:

i. $\alpha_2 = \beta_4 = \beta_1 = 0$, then we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_5 \rangle$;

ii. $\alpha_2 = \beta_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \sqrt[3]{\beta_5\beta_1^{-1}},\ y = 1$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \nabla_5 \rangle$;

iii. $\alpha_2 = 0,\ \beta_4 \neq 0,\ \beta_1 = 0$ then choosing $x = \beta_5\beta_4^{-1},\ y = 1$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_4 + \nabla_5 \rangle$;

iv. $\alpha_2 = 0,\ \beta_4 \neq 0,\ \beta_1 \neq 0$, then choosing $x = \beta_4^2\beta_1^{-1}\beta_5^{-1},\ y = \beta_4^3\beta_1^{-1}\beta_5^{-2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \nabla_4 + \nabla_5 \rangle$;

v. $\alpha_2 \neq 0,\ \beta_4 = \beta_1 = 0$, then choosing $x = 1,\ y = \alpha_3\alpha_2^{-1}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \nabla_5 \rangle$;

vi. $\alpha_2 \neq 0,\ \beta_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \alpha_2^2\beta_1\alpha_3^{-2}\beta_5^{-1}$, $y = \alpha_2^3\beta_1^2\alpha_3^{-3}\beta_5^{-2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \nabla_1 + \nabla_5 \rangle$;

vii. $\alpha_2 \neq 0,\ \beta_4 \neq 0$, then choosing $x = \alpha_2\beta_4\alpha_3^{-1}\beta_5^{-1},\ y = \alpha_2\beta_4^2\alpha_3^{-1}\beta_5^{-2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \beta\nabla_1 + \nabla_4 + \nabla_5 \rangle$;

(b) $\beta_5 = 0, \beta_4 \neq 0$.

i. $\alpha_5 = \beta_1 = 0,\ \alpha_1\beta_4 \neq \alpha_3\beta_2$, then choosing $y = 1,\ w = \frac{\alpha_4\beta_2 - \alpha_2\beta_4}{\alpha_1\beta_4 - \alpha_3\beta_2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \beta\nabla_2 + \nabla_4 \rangle_{\alpha\neq\beta}$;

ii. $\alpha_5 = \beta_1 = 0$, $\alpha_1\beta_4 = \alpha_3\beta_2$, $\alpha_2\alpha_3 = \alpha_1\alpha_4$, then choosing $y = 1,\ w = -\alpha_4\alpha_3^{-1}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \alpha\nabla_2 + \nabla_4 \rangle$;

iii. $\alpha_5 = \beta_1 = 0$, $\alpha_1\beta_4 = \alpha_3\beta_2$, $\alpha_2\alpha_3 \neq \alpha_1\alpha_4$ then choosing $x = \alpha_4\beta_2 - \alpha_2\beta_4$, $y = -\alpha_3\beta_4(\alpha_4\beta_2 - \alpha_2\beta_4)$, and $w = \alpha_4\beta_4(\alpha_4\beta_2 - \alpha_2\beta_4)$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3, \alpha\nabla_2 + \nabla_4 \rangle$;

iv. $\alpha_5 = 0, \beta_1 \neq 0$, then choosing

$$x = 1,\ y = \frac{\alpha_3}{\beta_4},\ w = \frac{(\alpha_3\beta_2 + \alpha_4\beta_1 - \alpha_1\beta_4) - \sqrt{(\alpha_3\beta_2 + \alpha_4\beta_1 - \alpha_1\beta_4)^2 - 4\alpha_3\beta_1(\alpha_4\beta_2 - \alpha_2\beta_4)}}{\alpha_3\beta_4},$$

we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3, \nabla_1 + \beta\nabla_2 + \nabla_4 \rangle$;

v. $\alpha_5 \neq 0, \beta_1 = 0$, then choosing

$$x = \alpha_5,\ y = -\sqrt{\alpha_3}\alpha_5,\ z = 0 \text{ and } w = -\frac{\sqrt{\alpha_3}\alpha_5(\alpha_4\beta_2 - \alpha_2\beta_4)}{\alpha_3\beta_2 - \alpha_1\beta_4},$$

we have the family of representatives $\langle \alpha\nabla_1 + \nabla_3 + \nabla_5, \beta\nabla_2 + \nabla_4 \rangle$;

vi. $\alpha_5 \neq 0, \beta_1 \neq 0$, then choosing

$$x=\frac{\alpha_3\beta_4^2}{\alpha_5\beta_1^2},\ y=\frac{\alpha_3^2\beta_4^3}{\alpha_5^2\beta_1^3},\ w=-\frac{\alpha_3^2\beta_2\beta_4^3}{\alpha_5^2\beta_1^4}\text{ and }z=\frac{\alpha_3(\alpha_1\beta_2-\alpha_2\beta_1)\beta_4^2}{\alpha_5^2\beta_1^3},$$

we have the family of representatives $\langle\alpha\nabla_1+\nabla_3+\nabla_5,\nabla_1+\nabla_4\rangle$;

(c) $\beta_5=0,\beta_4=0,\ \beta_1\neq0$, then we can suppose $\alpha_1=0$ and consider following subcases:

i. $\alpha_5=0$, then choosing $w=-\frac{y\beta_2}{\beta_1}$, we have $\beta_2^*=0$.

A. if $\alpha_2=\alpha_4=0$, then we have a split algebra;

B. if $\alpha_2=0,\alpha_4\neq0$, then choosing $x=1,\ y=\frac{\alpha_3}{\alpha_4}$, we have the representative $\langle\nabla_3+\nabla_4,\nabla_1\rangle$;

C. if $\alpha_2\neq0$, then choosing $x=1,\ y=\frac{\alpha_3}{\alpha_2}$, we have the family of representatives $\langle\nabla_2+\nabla_3+\alpha\nabla_4,\nabla_1\rangle$;

ii. $\alpha_5\neq0,\ \beta_2=0$, then choosing $x=\alpha_5,\ y=\sqrt{\alpha_3}\alpha_5,\ z=-\alpha_2$ and $w=0$, we have the representative $\langle\nabla_3+\nabla_5,\nabla_1\rangle$;

iii. $\alpha_5\neq0,\beta_2\neq0$, then choosing

$$x=\frac{\alpha_3\beta_2^2}{\alpha_5\beta_1^2},\ y=\frac{\alpha_3^2\beta_2^3}{\alpha_5^2\beta_1^3},\ z=\frac{\alpha_3\beta_2^2(\alpha_2\beta_1-\alpha_1\beta_2)}{\alpha_5^2\beta_1^3}\text{ and }w=0,$$

we have the representative $\langle\nabla_3+\nabla_5,\nabla_1+\nabla_2\rangle$.

(d) $\beta_5=\beta_4=\beta_1=0,\beta_2\neq0$, then we can suppose $\alpha_2=0$ and choosing $w=-\frac{y\alpha_4}{\alpha_3}$, we have $\alpha_4^*=0$. Thus, we have following subcases:

i. if $\alpha_5=0$, then we have the family of representatives $\langle\alpha\nabla_1+\nabla_3,\nabla_2\rangle$;

ii. if $\alpha_5\neq0$, then choosing $x=1,\ y=\sqrt{\alpha_3\alpha_5^{-1}}$, we have the family of representatives $\langle\alpha\nabla_1+\nabla_3+\nabla_5,\nabla_2\rangle$.

2. $\alpha_3=0,\alpha_4\neq0$, then we can suppose $\beta_4=0$.

$$\begin{array}{rclrcl}
\alpha_1^* &=& x^3\alpha_1, & \beta_1^* &=& x^3\beta_1,\\
\alpha_2^* &=& wx\alpha_1+xy\alpha_2+yz\alpha_5, & \beta_2^* &=& wx\beta_1+xy\beta_2+yz\beta_5,\\
\alpha_3^* &=& 0, & \beta_3^* &=& 0,\\
\alpha_4^* &=& xy\alpha_4, & \beta_4^* &=& 0,\\
\alpha_5^* &=& y^2\alpha_5, & \beta_5^* &=& y^2\beta_5.
\end{array}$$

(a) $\beta_5\neq0$, then we can suppose $\alpha_5=0$ and choosing $z=-\frac{x(w\beta_1+y\beta_2)}{y\beta_5}$, we have $\beta_2^*=0$. Thus, we have following subcases:

i. if $\beta_1=0$, then $\alpha_1\neq0$ and choosing $x=1,\ y=\frac{\alpha_1}{\alpha_4},\ w=-\frac{\alpha_2}{\alpha_4}$ we have the representative $\langle\nabla_1+\nabla_4,\nabla_5\rangle$;

ii. if $\beta_1\neq0,\ \alpha_1=0$ then choosing $x=1,\ y=\sqrt{\frac{\beta_1}{\beta_5}}$, we have the family of representatives $\langle\alpha\nabla_2+\nabla_4,\nabla_1+\nabla_5\rangle$;

iii. if $\beta_1\neq0,\ \alpha_1\neq0$ then choosing $x=\frac{\alpha_4^2\beta_1}{\alpha_1^2\beta_5},\ y=\frac{\alpha_2^3\beta_1^2}{\alpha_1^3\beta_5^2},\ w=-\frac{\alpha_2^4\beta_1^2}{\alpha_1^4\beta_5^2}$, we have the representative $\langle\nabla_1+\nabla_4,\nabla_1+\nabla_5\rangle$.

(b) $\beta_5=0,\ \beta_1\neq0$, then we can suppose $\alpha_1=0$ and choosing $w=-\frac{y\beta_2}{\beta_1}$, we have $\beta_2^*=0$. Thus, we have following subcases:

i. if $\alpha_5=0$, then we have the representative $\langle\alpha\nabla_2+\nabla_4,\nabla_1\rangle$;

ii. if $\alpha_5\neq0$, then choosing $x=1,\ y=\frac{\alpha_4}{\alpha_5},\ z=-\frac{\alpha_2}{\alpha_5}$ we have the representative $\langle\nabla_4+\nabla_5,\nabla_1\rangle$.

(c) $\beta_5=\beta_1=0,\beta_2\neq0$, then we can suppose $\alpha_2=0$. Since in case of $\alpha_1=0$, we have a split extension, we can assume $\alpha_1\neq0$, Thus, we have following subcases:

i. if $\alpha_5=0$, then choosing $x=1,\ y=\frac{\alpha_1}{\alpha_4}$, we have the representative $\langle\nabla_1+\nabla_4,\nabla_2\rangle$;

ii. if $\alpha_5\neq0$, then choosing $x=\frac{\alpha_4^2}{\alpha_1\alpha_5},\ y=\frac{\alpha_4^3}{\alpha_1\alpha_5}$, we have the representative $\langle\nabla_1+\nabla_4+\nabla_5,\nabla_2\rangle$.

Now we have the following distinct orbits:

$\langle\alpha\nabla_1+\nabla_3,\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_3,\nabla_1+\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_3,\nabla_4+\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_3,\nabla_1+\nabla_4+\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_2+\nabla_3,\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_2+\nabla_3,\nabla_1+\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_2+\nabla_3,\beta\nabla_1+\nabla_4+\nabla_5\rangle$, $\langle\alpha\nabla_1+\nabla_3,\beta\nabla_2+\nabla_4\rangle$, $\langle\alpha\nabla_1+\nabla_2+\nabla_3,\alpha\nabla_2+\nabla_4\rangle$, $\langle\alpha\nabla_1+\nabla_3,\nabla_1+\beta\nabla_2+\nabla_4\rangle$, $\langle\alpha\nabla_1+\nabla_3+\nabla_5,\beta\nabla_2+\nabla_4\rangle$, $\langle\alpha\nabla_1+\nabla_3+\nabla_5,\nabla_1+\nabla_4\rangle$, $\langle\nabla_1,\nabla_3+\nabla_4\rangle$, $\langle\nabla_1,\nabla_2+\nabla_3+\alpha\nabla_4\rangle$, $\langle\nabla_1,\nabla_3+\nabla_5,\rangle$, $\langle\nabla_1+\nabla_2,\nabla_3+\nabla_5,\rangle$, $\langle\alpha\nabla_1+\nabla_3,\nabla_2\rangle$, $\langle\alpha\nabla_1+\nabla_3+\nabla_5,\nabla_2\rangle$, $\langle\nabla_1+\nabla_4,\nabla_5\rangle$, $\langle\nabla_1+\nabla_5,\alpha\nabla_2+\nabla_4\rangle$, $\langle\nabla_1+\nabla_4,\nabla_1+\nabla_5\rangle$, $\langle\nabla_1,\alpha\nabla_2+\nabla_4\rangle$, $\langle\nabla_1,\nabla_4+\nabla_5\rangle$, $\langle\nabla_1+\nabla_4,\nabla_2\rangle$, $\langle\nabla_1+\nabla_4+\nabla_5,\nabla_2\rangle$.

Hence, we have the following new 5-dimensional nilpotent Novikov algebras (see section 2):

$$\mathrm{N}^{\alpha}_{148}, \mathrm{N}^{\alpha}_{149}\ \mathrm{N}^{\alpha}_{150}, \mathrm{N}^{\alpha}_{151}, \mathrm{N}^{\alpha}_{152}, \mathrm{N}^{\alpha}_{153}, \mathrm{N}^{\alpha,\beta}_{154}, \mathrm{N}^{\alpha,\beta}_{155}, \mathrm{N}^{\alpha}_{156}, \mathrm{N}^{\alpha,\beta}_{157}, \mathrm{N}^{\alpha,\beta}_{158}, \mathrm{N}^{\alpha}_{159}\ \mathrm{N}_{160},$$
$$\mathrm{N}^{\alpha}_{161}\ \mathrm{N}_{162}, \mathrm{N}_{163}, \mathrm{N}^{\alpha}_{164}, \mathrm{N}^{\alpha}_{165}, \mathrm{N}_{166}, \mathrm{N}^{\alpha}_{167}, \mathrm{N}_{168}, \mathrm{N}^{\alpha}_{169}, \mathrm{N}_{170}, \mathrm{N}_{171}, \mathrm{N}_{172}.$$

1.5.3. Central extensions of $\mathcal{N}^{3*}_{04}(0)$

Let us use the following notations:

$$\nabla_1=[\Delta_{11}],\quad \nabla_2=[\Delta_{13}],\quad \nabla_3=[\Delta_{21}],\quad \nabla_4=[\Delta_{22}],\quad \nabla_5=[\Delta_{23}-\Delta_{32}].$$

The automorphism group of $\mathcal{N}^{3*}_{04}(0)$ consists of invertible matrices of the form

$$\phi=\begin{pmatrix} x & 0 & 0\\ 0 & y & 0\\ z & t & xy\end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} \alpha_1 & 0 & \alpha_2\\ \alpha_3 & \alpha_4 & \alpha_5\\ 0 & -\alpha_5 & 0\end{pmatrix}\phi=\begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_2^*\\ \alpha_3^* & \alpha_4^* & \alpha_5^*\\ 0 & -\alpha_5^* & 0\end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{N}^{3*}_{04}(0))$ on the subspace $\langle\sum\limits_{i=1}^{5}\alpha_i\nabla_i\rangle$ is given by $\langle\sum\limits_{i=1}^{5}\alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{rcl} \alpha_1^* &=& x(x\alpha_1+z\alpha_2);\\ \alpha_2^* &=& x^2y\alpha_2;\\ \alpha_3^* &=& y(x\alpha_3+z\alpha_5);\\ \alpha_4^* &=& y^2\alpha_4;\\ \alpha_5^* &=& xy^2\alpha_5. \end{array}$$

We consider the vector space generated by the following two cocycles:

$$\theta_1=\alpha_1\nabla_1+\alpha_2\nabla_2+\alpha_3\nabla_3+\alpha_4\nabla_4+\alpha_5\nabla_5 \quad\text{and}\quad \theta_2=\beta_1\nabla_1+\beta_3\nabla_3+\beta_4\nabla_4+\beta_5\nabla_5.$$

We are interested only in $(\alpha_2,\alpha_5,\beta_2,\beta_5)\neq 0$. Hence, we have the following cases.

1. $\alpha_2\neq 0$, then we can suppose $\beta_2=0$ and have

$$\begin{array}{rclrcl} \alpha_1^* &=& x(x\alpha_1+z\alpha_2), & \beta_1^* &=& x^2\beta_1,\\ \alpha_2^* &=& x^2y\alpha_2, & \beta_2^* &=& 0,\\ \alpha_3^* &=& y(x\alpha_3+z\alpha_5), & \beta_3^* &=& y(x\beta_3+z\beta_5),\\ \alpha_4^* &=& y^2\alpha_4, & \beta_4^* &=& y^2\beta_4,\\ \alpha_5^* &=& xy^2\alpha_5, & \beta_5^* &=& xy^2\beta_5. \end{array}$$

   (a) $\beta_5\neq 0$, then we can suppose $\alpha_5^*=0$ and choosing $z=-\frac{x\beta_3}{\beta_5}$, we have $\beta_3^*=0$ and we will suppose that $\beta_3=0$. Thus, we have following subcases:
      i. $\alpha_1=\alpha_3=\alpha_4=\beta_4=\beta_1=0$, then we have the representative $\langle\nabla_2,\nabla_5\rangle$;

ii. $\alpha_1 = \alpha_3 = \alpha_4 = \beta_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\beta_5}{\beta_1},\ y = 1$, we have the representative $\langle \nabla_2, \nabla_1 + \nabla_5 \rangle$;

iii. $\alpha_1 = \alpha_3 = \alpha_4 = 0, \beta_4 \neq 0,\ \beta_1 = 0$, then choosing $x = \frac{\beta_4}{\beta_5}$, we have the representative $\langle \nabla_2, \nabla_4 + \nabla_5 \rangle$;

iv. $\alpha_1 = \alpha_3 = \alpha_4 = 0,\ \beta_4 \neq 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\beta_4}{\beta_5},\ y = \frac{\sqrt{\beta_1\beta_4}}{\beta_5}$, we have the representative $\langle \nabla_2, \nabla_1 + \nabla_4 + \nabla_5 \rangle$;

v. $\alpha_1 = \alpha_3 = 0,\ \alpha_4 \neq 0,\ \beta_4 = \beta_1 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_5 \rangle$;

vi. $\alpha_1 = \alpha_3 = 0,\ \alpha_4 \neq 0,\ \beta_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \sqrt[3]{\frac{\alpha_4^2\beta_1}{\alpha_2^2\beta_5}},\ y = \sqrt[3]{\frac{\alpha_4\beta_1^2}{\alpha_2\beta_5^2}}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_1 + \nabla_5 \rangle$;

vii. $\alpha_1 = \alpha_3 = 0,\ \alpha_4 \neq 0,\ \beta_4 \neq 0$, then choosing $x = \frac{\beta_4}{\beta_5},\ y = \frac{\alpha_2\beta_4^2}{\alpha_4\beta_5^2}$, we have the family of representatives $\langle \nabla_2 + \nabla_4, \alpha\nabla_1 + \nabla_4 + \nabla_5 \rangle$;

viii. $\alpha_1 = 0,\ \alpha_3 \neq 0,\ \alpha_4 = \beta_1 = 0$, then choosing $x = \frac{\alpha_3}{\alpha_2}$, we have the family of representatives $\langle \nabla_2 + \nabla_3, \alpha\nabla_4 + \nabla_5 \rangle$;

ix. $\alpha_1 = 0,\ \alpha_3 \neq 0,\ \alpha_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \sqrt{\frac{\alpha_3\beta_1}{\alpha_2\beta_5}}$, we have the family of representatives $\langle \nabla_2 + \nabla_3, \nabla_1 + \alpha\nabla_4 + \nabla_5 \rangle$;

x. $\alpha_1 = 0,\ \alpha_3 \neq 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3^2}{\alpha_2\alpha_4}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \nabla_4, \alpha\nabla_1 + \beta\nabla_4 + \nabla_5 \rangle$;

xi. $\alpha_1 \neq 0,\ \alpha_3 = \alpha_4 = \beta_1 = \beta_4 = 0$, then choosing $y = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2, \nabla_5 \rangle$;

xii. $\alpha_1 \neq 0,\ \alpha_3 = \alpha_4 = \beta_1 = 0,\ \beta_4 \neq 0$, then choosing $x = \frac{\beta_4}{\beta_5},\ y = \frac{\alpha_1}{\alpha_2}$ we have the family of representatives $\langle \nabla_1 + \nabla_2, \nabla_4 + \nabla_5 \rangle$;

xiii. $\alpha_1 \neq 0,\ \alpha_3 = \alpha_4 = 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\alpha_1^2\beta_5}{\alpha_2^2\beta_1},\ y = \frac{\alpha_1}{\alpha_2}$, we have the family of representatives $\langle \nabla_1 + \nabla_2, \nabla_1 + \alpha\nabla_4 + \nabla_5 \rangle$;

xiv. $\alpha_1 \neq 0,\ \alpha_3 = 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\sqrt{\alpha_1\alpha_4}}{\alpha_2},\ y = \frac{\alpha_1}{\alpha_2}$, we have the family of representatives $\langle \nabla_1 + \nabla_2 + \nabla_4, \alpha\nabla_1 + \beta\nabla_4 + \nabla_5 \rangle$;

xv. $\alpha_1 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_1}{\alpha_2}$, we have the family of representatives $\langle \nabla_1 + \nabla_2 + \nabla_3 + \alpha\nabla_4, \beta\nabla_1 + \gamma\nabla_4 + \nabla_5 \rangle$.

(b) $\beta_5 = 0,\ \beta_4 \neq 0$, then choosing $z = \frac{x(\alpha_4\beta_1 - \alpha_1\beta_4)}{\alpha_2\beta_4}$, we can suppose $\alpha_1^* = \alpha_4^* = 0$ and have following subcases:

i. $\alpha_3 = \alpha_5 = \beta_3 = \beta_1 = 0$, then we have the representative $\langle \nabla_2, \nabla_4 \rangle$;

ii. $\alpha_3 = \alpha_5 = \beta_3 = 0,\ \beta_1 \neq 0$, then choosing $x = 1,\ y = \sqrt{\frac{\beta_1}{\beta_4}}$, we have the representative $\langle \nabla_2, \nabla_1 + \nabla_4 \rangle$;

iii. $\alpha_3 = \alpha_5 = 0,\ \beta_3 \neq 0$, then choosing $x = 1,\ y = \frac{\beta_3}{\beta_4}$, we have the family of representatives $\langle \nabla_2, \alpha\nabla_1 + \nabla_3 + \nabla_4 \rangle$;

iv. $\alpha_3 = 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_2}$, we have the family of representatives $\langle \nabla_2 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4 \rangle$;

v. $\alpha_3 \neq 0,\ \alpha_5 = \beta_3 = \beta_1 = 0$, then choosing $x = \frac{\alpha_3}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_4 \rangle$;

vi. $\alpha_3 \neq 0,\ \alpha_5 = \beta_3 = 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3\sqrt{\beta_1}}{\alpha_2\sqrt{\beta_4}}$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_1 + \nabla_4 \rangle$;

vii. $\alpha_3 \neq 0,\ \alpha_5 = 0,\ \beta_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3\beta_3}{\alpha_2\beta_4}$, we have the family of representatives $\langle \nabla_2 + \nabla_3, \alpha\nabla_1 + \nabla_3 + \nabla_4 \rangle$;

viii. $\alpha_3 \neq 0,\ \alpha_5 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4 \rangle$.

(c) $\beta_5 = 0,\ \beta_4 = 0,\ \beta_3 \neq 0$.

i. $\alpha_5 = \beta_1 = \alpha_4 = 0$, then we can suppose $\alpha_3^* = 0$ and choosing $x = 1,\ z = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2, \nabla_3 \rangle$;

ii. $\alpha_5 = \beta_1 = 0,\ \alpha_4 \neq 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4},\ z = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_3 \rangle$;

iii. $\alpha_5 = 0,\ \beta_1 \neq 0,\ \alpha_4 = 0$ then choosing $x = 1,\ y = \frac{\beta_1}{\beta_3},\ z = \frac{\alpha_3\beta_1 - \alpha_1\beta_3}{\alpha_2\beta_3}$, we have the representative $\langle \nabla_2, \nabla_1 + \nabla_3 \rangle$;

iv. $\alpha_5 = 0,\ \beta_1 \neq 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4\beta_1}{\alpha_2\beta_3},\ y = \frac{\alpha_4\beta_1^2}{\alpha_2\beta_3^2},\ z = \frac{\alpha_4\beta_1(\alpha_3\beta_1 - \alpha_1\beta_3)}{\alpha_2^2\beta_3^2}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_1 + \nabla_3 \rangle$;

v. $\alpha_5 \neq 0,\ \alpha_2\beta_3 \neq \alpha_5\beta_1,\ \alpha_4 = 0$, then choosing $x = 1, y = \frac{\alpha_2}{\alpha_5}, z = \frac{\alpha_3\beta_1 - \alpha_1\beta_3}{\alpha_2\beta_3 - \alpha_5\beta_1}$, we have the family of representatives$\langle \nabla_2 + \nabla_5, \alpha\nabla_1 + \nabla_3 \rangle_{\alpha \neq 1}$;

vi. $\alpha_5 \neq 0,\ \alpha_2\beta_3 \neq \alpha_5\beta_1,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5},\ y = \frac{\alpha_2\alpha_4}{\alpha_5^2},\ z = \frac{\alpha_4(\alpha_3\beta_1 - \alpha_1\beta_3)}{\alpha_5(\alpha_2\beta_3 - \alpha_5\beta_1)}$, we have the family of representatives $\langle \nabla_2 + \nabla_4 + \nabla_5, \alpha\nabla_1 + \nabla_3 \rangle_{\alpha \neq 1}$;

vii. $\alpha_5 \neq 0,\ \alpha_2\beta_3 = \alpha_5\beta_1,\ \alpha_2\alpha_3 = \alpha_1\alpha_5,\ \alpha_4 = 0$, then choosing $x = 1, y = \frac{\alpha_2}{\alpha_5}, z = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_5, \nabla_1 + \nabla_3 \rangle$;

viii. $\alpha_5 \neq 0,\ \alpha_2\beta_3 = \alpha_5\beta_1,\ \alpha_2\alpha_3 = \alpha_1\alpha_5,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5}, y = \frac{\alpha_2\alpha_4}{\alpha_5^2}, z = -\frac{\alpha_1\alpha_4}{\alpha_2\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_5, \nabla_1 + \nabla_3 \rangle$;

ix. $\alpha_5 \neq 0,\ \alpha_2\beta_3 = \alpha_5\beta_1,\ \alpha_2\alpha_3 \neq \alpha_1\alpha_5$, then choosing $x = \frac{\alpha_2\alpha_3 - \alpha_1\alpha_5}{\alpha_2^2}, y = \frac{\alpha_2\alpha_3 - \alpha_1\alpha_5}{\alpha_2\alpha_5}, z = -\frac{\alpha_1(\alpha_2\alpha_3 - \alpha_1\alpha_5)}{\alpha_2^3}$, we have the family of representatives $\langle \nabla_2 + \nabla_3 + \alpha\nabla_4 + \nabla_5, \nabla_1 + \nabla_3 \rangle$.

(d) $\beta_5 = 0, \beta_4 = 0,\ \beta_3 = 0,\ \beta_1 \neq 0$, then we can suppose $\alpha_1^* = 0$ and consider following subcases:

i. $\alpha_5 = \alpha_4 = \alpha_3 = 0$, then we have the representative $\langle \nabla_2, \nabla_1 \rangle$;

ii. $\alpha_5 = \alpha_4 = 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = 1$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_1 \rangle$;

iii. $\alpha_5 = 0,\ \alpha_4 \neq 0,\ \alpha_3 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4, \nabla_1 \rangle$;

iv. $\alpha_5 = 0,\ \alpha_4 \neq 0,\ \alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2},\ y = \frac{\alpha_3^2}{\alpha_2\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4, \nabla_1 \rangle$;

v. $\alpha_5 \neq 0,\ \alpha_4 = 0$, then choosing $x = 1,\ y = \frac{\alpha_2}{\alpha_5},\ z = -\frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_5, \nabla_1 \rangle$;

vi. $\alpha_5 \neq 0,\ \alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5},\ y = \frac{\alpha_2\alpha_4}{\alpha_5^2},\ z = -\frac{\alpha_3\alpha_4}{\alpha_5^2}$, we have the representative $\langle \nabla_2 + \nabla_4 + \nabla_5, \nabla_1 \rangle$.

2. $\alpha_2 = 0$, then $\alpha_5 \neq 0$ and we have

$$
\begin{array}{rclrcl}
\alpha_1^* &=& x^2\alpha_1, & \beta_1^* &=& x^2\beta_1,\\
\alpha_2^* &=& 0, & \beta_2^* &=& 0,\\
\alpha_3^* &=& y(x\alpha_3 + z\alpha_5), & \beta_3^* &=& xy\beta_3,\\
\alpha_4^* &=& y^2\alpha_4, & \beta_4^* &=& y^2\beta_4,\\
\alpha_5^* &=& xy^2\alpha_5, & \beta_5^* &=& 0.
\end{array}
$$

(a) $\beta_4 \neq 0$, then we can suppose $\alpha_4^* = 0$ and consider following subcases:

i. $\alpha_1 = \beta_3 = \beta_1 = 0$, then choosing $x = 1,\ z = -\frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_5, \nabla_4 \rangle$;

ii. $\alpha_1 = \beta_3 = 0,\ \beta_1 \neq 0$, then choosing $x = 1,\ y = \sqrt{\frac{\beta_1}{\beta_4}},\ z = -\frac{\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_5, \nabla_1 + \nabla_4 \rangle$;

iii. $\alpha_1 = 0,\ \beta_3 \neq 0$, then choosing $x = 1,\ y = \frac{\beta_3}{\beta_4},\ z = -\frac{\alpha_3}{\alpha_5}$, we have the family of representatives $\langle \nabla_5, \alpha\nabla_1 + \nabla_3 + \nabla_4 \rangle$;

iv. $\alpha_1 \neq 0,\ \beta_1 = \beta_3 = 0$, then choosing $x = \alpha_1\alpha_5,\ y = \alpha_1,\ z = -\alpha_1\alpha_3$, we have the family of representatives $\langle \nabla_1 + \nabla_5, \nabla_4 \rangle$;

v. $\alpha_1 \neq 0,\ \beta_3 = 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\alpha_1\beta_4}{\alpha_5\beta_1},\ y = \frac{\alpha_1\sqrt{\beta_4}}{\alpha_5\sqrt{\beta_1}},\ z = -\frac{\alpha_1\alpha_3\beta_4}{\alpha_5^2\beta_1}$, we have the representative $\langle \nabla_1 + \nabla_5, \nabla_1 + \nabla_4 \rangle$;

vi. $\alpha_1 \neq 0,\ \beta_3 \neq 0$, then choosing $x = \frac{\alpha_1\beta_4^2}{\alpha_5\beta_3^2},\ y = \frac{\alpha_1\beta_4}{\alpha_5\beta_3},\ z = -\frac{\alpha_1\alpha_3\beta_4^2}{\alpha_5^2\beta_3}$, we have the family of representatives $\langle\nabla_1 + \nabla_5, \alpha\nabla_1 + \nabla_3 + \nabla_4\rangle$.

(b) $\beta_4 = 0,\ \beta_3 \neq 0$.

i. $\alpha_4 = \beta_1 = \alpha_1 = 0$, then choosing $x = 1,\ z = -\frac{\alpha_3}{\alpha_5}$, we have the representative $\langle\nabla_5, \nabla_3\rangle$;

ii. $\alpha_4 = \beta_1 = 0,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_5}{\alpha_1},\ y = 1,\ z = -\frac{\alpha_3}{\alpha_1}$, we have the representative $\langle\nabla_5 + \nabla_1, \nabla_3\rangle$;

iii. $\alpha_4 = 0,\ \beta_1 \neq 0$, then choosing $x = 1,\ y = \frac{\beta_1}{\beta_3},\ z = \frac{\alpha_1\beta_3 - \alpha_3\beta_1}{\alpha_5\beta_1}$, we have the representative $\langle\nabla_5, \nabla_1 + \nabla_3\rangle$;

iv. $\alpha_4 \neq 0,\ \beta_1 = \alpha_1 = 0$, then choosing $x = \frac{\alpha_4}{\alpha_5},\ z = -\frac{\alpha_3\alpha_4}{\alpha_5^2}$, we have the representative $\langle\nabla_4 + \nabla_5, \nabla_3\rangle$;

v. $\alpha_4 \neq 0,\ \beta_1 = 0,\ \alpha_1 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5},\ y = \frac{\sqrt{\alpha_1\alpha_4}}{\alpha_5},\ z = -\frac{\alpha_3\alpha_4}{\alpha_5^2}$, we have the representative $\langle\nabla_1 + \nabla_4 + \nabla_5, \nabla_3\rangle$;

vi. $\alpha_4 \neq 0,\ \beta_1 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5},\ y = \frac{\alpha_4\beta_1}{\alpha_5\beta_3},\ z = \frac{\alpha_4(\alpha_1\beta_3 - \alpha_3\beta_1)}{\alpha_5^2\beta_1}$, we have the representative $\langle\nabla_4 + \nabla_5, \nabla_1 + \nabla_3\rangle$.

(c) $\beta_4 = \beta_3 = 0,\ \beta_1 \neq 0$, then we can suppose $\alpha_1^* = 0$ and choosing $z = -\frac{x\alpha_3}{\alpha_5}$, obtain $\alpha_3^* = 0$.

i. $\alpha_4 = 0$, then we have the representative $\langle\nabla_5, \nabla_1\rangle$;

ii. $\alpha_4 \neq 0$, then choosing $x = \frac{\alpha_4}{\alpha_5}$, we have the representative $\langle\nabla_4 + \nabla_5, \nabla_1\rangle$.

Now we have the following distinct orbits:

$$\begin{gathered}
\langle\nabla_2, \nabla_5\rangle, \langle\nabla_2, \nabla_1 + \nabla_5\rangle, \langle\nabla_2, \nabla_4 + \nabla_5\rangle, \langle\nabla_2, \nabla_1 + \nabla_4 + \nabla_5\rangle, \langle\nabla_2 + \nabla_4, \nabla_5\rangle, \langle\nabla_2 + \nabla_4, \nabla_1 + \nabla_5\rangle, \\
\langle\nabla_2 + \nabla_4, \alpha\nabla_1 + \nabla_4 + \nabla_5\rangle, \langle\nabla_2 + \nabla_3, \alpha\nabla_4 + \nabla_5\rangle, \langle\nabla_2 + \nabla_3, \nabla_1 + \alpha\nabla_4 + \nabla_5\rangle, \langle\nabla_2 + \nabla_3 + \nabla_4, \alpha\nabla_1 + \beta\nabla_4 + \nabla_5\rangle, \\
\langle\nabla_1 + \nabla_2, \nabla_5\rangle, \langle\nabla_1 + \nabla_2, \nabla_4 + \nabla_5\rangle, \langle\nabla_1 + \nabla_2, \nabla_1 + \alpha\nabla_4 + \nabla_5\rangle, \langle\nabla_1 + \nabla_2 + \nabla_4, \alpha\nabla_1 + \beta\nabla_4 + \nabla_5\rangle, \\
\langle\nabla_1 + \nabla_2 + \nabla_3 + \alpha\nabla_4, \beta\nabla_1 + \gamma\nabla_4 + \nabla_5\rangle, \langle\nabla_2, \nabla_4\rangle, \langle\nabla_2, \nabla_1 + \nabla_4\rangle, \langle\nabla_2, \alpha\nabla_1 + \nabla_3 + \nabla_4\rangle, \\
\langle\nabla_2 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4\rangle, \langle\nabla_2 + \nabla_3, \nabla_4\rangle, \langle\nabla_2 + \nabla_3, \nabla_1 + \nabla_4\rangle, \langle\nabla_2 + \nabla_3, \alpha\nabla_1 + \nabla_3 + \nabla_4\rangle, \\
\langle\nabla_2 + \nabla_3 + \nabla_5, \alpha\nabla_1 + \beta\nabla_3 + \nabla_4\rangle, \langle\nabla_2, \nabla_3\rangle, \langle\nabla_2 + \nabla_4, \nabla_3\rangle, \langle\nabla_2, \nabla_1 + \nabla_3\rangle, \langle\nabla_2 + \nabla_4, \nabla_1 + \nabla_3\rangle \\
\langle\nabla_2 + \nabla_5, \alpha\nabla_1 + \nabla_3\rangle, \langle\nabla_2 + \nabla_4 + \nabla_5, \alpha\nabla_1 + \nabla_3\rangle, \langle\nabla_2 + \nabla_3 + \alpha\nabla_4 + \nabla_5, \nabla_1 + \nabla_3\rangle, \langle\nabla_2, \nabla_1\rangle, \langle\nabla_2 + \nabla_3, \nabla_1\rangle, \\
\langle\nabla_2 + \nabla_4, \nabla_1\rangle, \langle\nabla_2 + \nabla_3 + \nabla_4, \nabla_1\rangle, \langle\nabla_2 + \nabla_5, \nabla_1\rangle, \langle\nabla_2 + \nabla_4 + \nabla_5, \nabla_1\rangle, \langle\nabla_5, \nabla_4\rangle, \langle\nabla_5, \nabla_1 + \nabla_4\rangle, \\
\langle\nabla_5, \alpha\nabla_1 + \nabla_3 + \nabla_4\rangle, \langle\nabla_1 + \nabla_5, \nabla_4\rangle, \langle\nabla_1 + \nabla_5, \nabla_1 + \nabla_4\rangle, \langle\nabla_1 + \nabla_5, \alpha\nabla_1 + \nabla_3 + \nabla_4\rangle, \langle\nabla_5, \nabla_3\rangle, \langle\nabla_5 + \nabla_1, \nabla_3\rangle, \\
\langle\nabla_5, \nabla_1 + \nabla_3\rangle, \langle\nabla_4 + \nabla_5, \nabla_3\rangle, \langle\nabla_1 + \nabla_4 + \nabla_5, \nabla_3\rangle, \langle\nabla_4 + \nabla_5, \nabla_1 + \nabla_3\rangle, \langle\nabla_5, \nabla_1\rangle, \langle\nabla_4 + \nabla_5, \nabla_1\rangle.
\end{gathered}$$

Hence, we have the following new 5-dimensional nilpotent Novikov algebras (see section 2):

$$\begin{gathered}
\mathrm{N}_{173}, \mathrm{N}_{174}, \mathrm{N}_{175}, \mathrm{N}_{176}, \mathrm{N}_{177}, \mathrm{N}_{178}, \mathrm{N}_{179}^{\alpha}, \mathrm{N}_{180}^{\alpha}, \mathrm{N}_{181}^{\alpha}, \mathrm{N}_{182}^{\alpha,\beta}, \mathrm{N}_{183}, \mathrm{N}_{184}, \mathrm{N}_{185}^{\alpha}, \mathrm{N}_{186}^{\alpha,\beta}, \mathrm{N}_{187}^{\alpha,\beta,\gamma}, \mathrm{N}_{188}, \mathrm{N}_{189}, \mathrm{N}_{62}^{0,0}, \\
\mathrm{N}_{64}^{\alpha\notin\{0,1\},0}, \mathrm{N}_{190}, \mathrm{N}_{191}^{\alpha,\beta}, \mathrm{N}_{192}, \mathrm{N}_{193}, \mathrm{N}_{194}^{\alpha}, \mathrm{N}_{195}^{\alpha,\beta}, \mathrm{N}_{89}^{0}\ \mathrm{N}_{90}^{0}\ \mathrm{N}_{196}, \mathrm{N}_{197}^{\alpha}, \mathrm{N}_{198}^{\alpha}, \mathrm{N}_{199}^{\alpha}, \mathrm{N}_{200}, \mathrm{N}_{201}, \mathrm{N}_{202}, \mathrm{N}_{203}, \mathrm{N}_{204}, \mathrm{N}_{205}, \\
\mathrm{N}_{206}, \mathrm{N}_{207}, \mathrm{N}_{208}^{\alpha}, \mathrm{N}_{114}^{0}, \mathrm{N}_{209}, \mathrm{N}_{210}^{\alpha}, \mathrm{N}_{211}, \mathrm{N}_{212}, \mathrm{N}_{213}, \mathrm{N}_{214}, \mathrm{N}_{215}, \mathrm{N}_{216}, \mathrm{N}_{217}, \mathrm{N}_{218}.
\end{gathered}$$

It should be noted that the family of orbits $\langle\nabla_2, \alpha\nabla_1 + \nabla_3 + \nabla_4\rangle$, gives the algebra $\mathrm{N}_{62}^{0,0}$ in case of $\alpha = 0$, the parametric family $\mathrm{N}_{64}^{\frac{\alpha-1}{\alpha},0}$ in case of $\alpha \notin \{0, 1\}$ and the algebra N_{190} in case of $\alpha = 1$.

2. Classification theorem for 5-dimensional nilpotent Novikov algebras

The algebraic classification of complex 5-dimensional nilpotent Novikov algebras consists of two parts:

1. 5-dimensional algebras with identity $xyz = 0$ (also known as 2-step nilpotent algebras) are the intersection of all varieties of algebras defined by a family of polynomial identities of degree three or more; for example, it is in the intersection of associative, Zinbiel, Leibniz, etc, algebras. All these algebras can be obtained as central extensions of zero-product algebras. The geometric classification of 2-step nilpotent algebras is given in [24]. It is the reason why we are not interested in it.
2. 5-dimensional nilpotent (non-2-step nilpotent) Novikov algebras, which are central extensions of nilpotent Novikov algebras with nonzero products of a smaller dimension. These algebras are classified by several steps:

(a) complex split 5-dimensional nilpotent Novikov algebras are classified in [26];
(b) complex non-split 5-dimensional nilpotent commutative associative algebras are listed in [30];
(c) complex one-generated 5-dimensional nilpotent Novikov algebras are classified in [10];
(d) complex non-split non-one-generated 5-dimensional nilpotent non-commutative Novikov algebras are classified in Theorem A (see below).

Theorem A. *Let $\mathbb{N}$ be a complex non-split non-one-generated 5-dimensional nilpotent (non-2-step nilpotent) non-commutative Novikov algebra. Then $\mathbb{N}$ is isomorphic to one algebra from the following list:*

N_{01}	:	$e_1e_1 = e_2$	$e_1e_4 = e_5$	$e_2e_1 = e_5$	
		$e_3e_3 = e_5$	$e_4e_1 = e_5$		
N_{02}	:	$e_1e_1 = e_2$	$e_2e_1 = e_5$	$e_3e_4 = e_5$	$e_4e_3 = -e_5$
N_{03}	:	$e_1e_1 = e_2$	$e_1e_3 = e_5$	$e_2e_1 = e_5$	
		$e_3e_1 = e_5$	$e_4e_3 = e_5$	$e_4e_4 = e_5$	
N_{04}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = \alpha e_5$	
		$e_3e_3 = e_5$	$e_4e_1 = e_5$		
N_{05}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_5$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$	$e_4e_4 = e_5$	
N_{06}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_5$	
		$e_3e_1 = e_5$	$e_3e_4 = e_5$	$e_4e_3 = e_5$	
N_{07}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_5$	$e_2e_1 = (\alpha+1)e_5$	
		$e_3e_3 = e_5$	$e_4e_4 = e_5$		
N_{08}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = \alpha e_5$	
		$e_3e_4 = e_5$	$e_4e_3 = -e_5$		
N_{09}	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = -e_5$	
		$e_3e_1 = e_5$	$e_3e_4 = e_5$	$e_4e_3 = -e_5$	
$N_{10}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_5$	$e_2e_1 = (\alpha+1)e_5$	
		$e_3e_3 = \beta e_5$	$e_4e_3 = e_5$	$e_4e_4 = e_5$	
N_{11}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \frac{-1+\sqrt{1-4\alpha}}{2}e_5$	$e_3e_3 = \alpha e_5$	$e_4e_3 = e_5$
		$e_4e_4 = e_5$	$e_2e_1 = \frac{1+\sqrt{1-4\alpha}}{2}e_5$	$e_3e_1 = e_5$	
$N_{12}^{\alpha\neq\frac{1}{4}}$	:	$e_1e_1 = e_2$	$e_1e_2 = \frac{-1-\sqrt{1-4\alpha}}{2}e_5$	$e_3e_3 = \alpha e_5$	$e_4e_3 = e_5$
		$e_4e_4 = e_5$	$e_2e_1 = \frac{1-\sqrt{1-4\alpha}}{2}e_5$	$e_3e_1 = e_5$	
N_{13}	:	$e_1e_1 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_5$	$e_2e_2 = e_4$
		$e_2e_4 = e_5$	$e_2e_4 = e_5$	$e_3e_1 = e_5$	
N_{14}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_5$	$e_1e_3 = \alpha e_5$	$e_2e_1 = e_5$
		$e_2e_2 = e_4$	$e_2e_4 = -(1+\alpha)e_5$	$e_3e_1 = (1+\alpha)e_5$	$e_4e_2 = -\alpha e_5$
N_{15}^{α}	:	$e_1e_1 = e_3$	$e_1e_3 = \alpha e_5$	$e_2e_2 = e_4$	
		$e_2e_4 = e_5$	$e_3e_1 = (1+\alpha)e_5$	$e_4e_2 = e_5$	
$N_{16}^{\alpha,\beta}$	:	$e_1e_1 = e_3$	$e_1e_3 = \alpha e_5$	$e_2e_2 = e_4$	
		$e_2e_4 = \beta e_5$	$e_3e_1 = (1+\alpha)e_5$	$e_4e_2 = (1+\beta)e_5$	
N_{17}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_4e_4 = e_5$
N_{18}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_4 = e_5$	$e_4e_4 = e_5$		
N_{19}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_4e_4 = e_5$		
N_{20}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_1 = e_5$	$e_2e_4 = e_5$	$e_4e_4 = e_5$	
N_{21}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_1 = e_5$
		$e_2e_2 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_4 = e_5$	
N_{22}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_2 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_4 = e_5$	
N_{23}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
		$e_2e_2 = e_5$	$e_4e_1 = e_5$	$e_4e_2 = e_5$	
N_{24}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_2e_2 = e_5$

		$e_2e_4=-e_5$	$e_4e_1=e_5$	$e_4e_2=e_5$	
N_{25}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_2e_4=\alpha e_5$	$e_4e_1=e_5$	$e_4e_2=e_5$	
N_{26}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_2e_1=e_5$	$e_4e_2=e_5$		
N_{27}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_2e_2=e_5$	$e_2e_4=-e_5$	$e_4e_2=e_5$	
N_{28}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_2e_4=\alpha e_5$	$e_4e_2=e_5$		
N_{29}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_4e_1=e_5$
N_{30}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_2e_2=e_5$	$e_4e_1=e_5$		
N_{31}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	
		$e_2e_4=e_5$	$e_4e_1=e_5$		
N_{32}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_4=e_5$
N_{33}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	$e_2e_3=-e_5$
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$	
N_{34}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	$e_1e_4=e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$
N_{35}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	$e_1e_4=\alpha e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$
N_{36}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	$e_2e_2=e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$
N_{37}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	$e_1e_4=\alpha e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$
N_{38}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	
		$e_1e_4=e_5$	$e_2e_2=e_5$	$e_2e_3=-e_5$	
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$	
N_{39}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_2e_3=-e_5$	
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$	
N_{40}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_2e_3=-e_5$	
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$	
N_{41}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3+\alpha e_5$	$e_1e_4=e_5$	$e_2e_3=-e_5$
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$	
$\mathrm{N}_{42}^{\alpha,\beta}$	:	$e_1e_1=e_3$	$e_1e_2=e_3+\alpha e_5$	$e_1e_4=\beta e_5$	$e_2e_2=e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_4=e_5$
N_{43}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	$e_1e_4=e_5$
		$e_2e_3=-e_5$	$e_2e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$
N_{44}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	$e_1e_4=e_5$
		$e_2e_3=-e_5$	$e_2e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$
N_{45}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	
		$e_1e_4=e_5$	$e_2e_3=-e_5$	$e_2e_4=e_5$	
		$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$	
N_{46}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	$e_2e_3=-e_5$
		$e_2e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$
N_{47}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	$e_2e_3=-e_5$
		$e_2e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=\alpha e_5$
N_{48}^{α}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_4=\alpha e_5$	$e_2e_3=-e_5$
		$e_2e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=\alpha e_5$
$\mathrm{N}_{49}^{\alpha,\beta}$	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_4=\alpha e_5$	$e_2e_3=-e_5$
		$e_2e_4=e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=\beta e_5$
N_{50}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	$e_1e_4=-e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$
N_{51}	:	$e_1e_1=e_3$	$e_1e_2=e_3$	$e_1e_3=-e_5$	$e_1e_4=\alpha e_5$
		$e_2e_3=-e_5$	$e_3e_1=e_5$	$e_3e_2=e_5$	$e_4e_1=e_5$
N_{52}	:	$e_1e_1=e_3$	$e_1e_2=e_3+e_5$	$e_1e_3=-e_5$	

		$e_1e_4 = -e_5$	$e_2e_2 = e_5$	$e_2e_3 = -e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
N_{53}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	
		$e_1e_4 = \alpha e_5$	$e_2e_2 = e_5$	$e_2e_3 = -e_5$	
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
N_{54}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = \alpha e_5$	$e_2e_3 = -e_5$
		$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
N_{55}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = \alpha e_5$	$e_2e_2 = e_5$
		$e_2e_3 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$
N_{56}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	$e_1e_4 = e_5$
		$e_2e_3 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	
N_{57}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	$e_1e_4 = e_5$
		$e_2e_2 = e_5$	$e_2e_3 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$
N_{58}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	
		$e_2e_3 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	
N_{59}	:	$e_1e_1 = e_3$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_2 = e_5$
		$e_2e_3 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = e_5$	
$N_{60}^{(\alpha,\beta)\neq(0,0)}$	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_4 = -\beta e_5$	
		$e_2e_1 = e_4$	$e_2e_2 = -e_3$	$e_2e_3 = -e_5$	
		$e_2e_4 = \alpha e_5$	$e_3e_2 = e_5$	$e_4e_1 = \beta e_5$	
$N_{61}^{(\alpha,\beta,\gamma)\neq(0,0,\gamma)}$	:	$e_1e_2 = e_3$	$e_1e_3 = -(\gamma+1)e_5$	$e_1e_4 = -\beta e_5$	
		$e_2e_1 = e_4$	$e_2e_2 = -e_3$	$e_2e_3 = \gamma e_5$	
		$e_2e_4 = \alpha e_5$	$e_3e_2 = e_5$	$e_4e_1 = \beta e_5$	
$N_{62}^{(\alpha,\beta)\neq(0,0)}$	:	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	$e_1e_4 = -\beta e_5$	$e_2e_1 = e_4$
		$e_2e_2 = -e_3$	$e_2e_3 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_1 = \beta e_5$
$N_{63}^{\alpha\neq0}$	:	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	$e_1e_4 = -\alpha e_5$	$e_2e_1 = e_4$
		$e_2e_2 = -e_3 + e_5$	$e_2e_3 = e_5$	$e_2e_4 = \alpha e_5$	$e_4e_1 = \alpha e_5$
$N_{64}^{\alpha\neq1,\beta\neq0}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_1e_4 = \beta e_5$	$e_2e_1 = -\alpha e_3$	$e_2e_2 = -e_4$	
		$e_2e_3 = -\alpha e_5$	$e_2e_4 = -\beta e_5$		
$N_{65}^{\alpha\neq1,\beta,\gamma,\delta}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\beta-1)e_5$	
		$e_1e_4 = \gamma e_5$	$e_2e_1 = -\alpha e_3$	$e_2e_2 = -e_4$	$e_2e_3 = -\alpha\beta e_5$
		$e_2e_4 = -(\gamma+\delta)e_5$	$e_3e_1 = e_5$	$e_4e_2 = \delta e_5$	
$N_{66}^{\alpha\neq1,\beta\neq\frac{1}{1-\alpha},\gamma}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4 + e_5$	$e_1e_3 = (\beta-1)e_5$	
		$e_1e_4 = \frac{\beta\delta}{(\alpha-1)\beta+1}e_5$	$e_2e_1 = -\alpha e_3$	$e_2e_2 = -e_4$	$e_2e_3 = -\alpha\beta e_5$
		$e_2e_4 = -\frac{\gamma(\alpha\beta+1)}{(\alpha-1)\beta+1}e_5$	$e_3e_1 = e_5$	$e_4e_2 = \gamma e_5$	
$N_{67}^{\alpha\neq1,\beta}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4 + e_5$	$e_1e_3 = \frac{\alpha}{1-\alpha}e_5$	$e_1e_4 = \beta e_5$
		$e_2e_1 = -\alpha e_3$	$e_2e_2 = -e_4$	$e_2e_3 = \frac{\alpha}{\alpha-1}e_5$	
		$e_2e_4 = -\beta e_5$	$e_3e_1 = e_5$		
$N_{68}^{\alpha,\beta}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_1e_4 = \beta e_5$
		$e_2e_2 = e_4$	$e_2e_4 = -(1+\beta)e_5$	$e_4e_2 = e_5$	
N_{69}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = e_5$	$e_2e_2 = e_4$	$e_4e_2 = e_5$	
N_{70}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = -e_4$	$e_2e_3 = -e_5$	$e_2e_4 = e_5$
N_{71}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4 + e_5$	$e_1e_3 = e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3 - e_5$	$e_2e_2 = -e_4$	$e_2e_3 = -e_5$	$e_2e_4 = e_5$
N_{72}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = -e_4$	$e_2e_3 = -e_5$	$e_2e_4 = e_5$
N_{73}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha-1)e_5$	$e_1e_4 = (1-\alpha)e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = -e_4$	$e_2e_3 = (1-\alpha)e_5$	$e_2e_4 = (\alpha-1)e_5$
		$e_3e_1 = e_5$	$e_3e_2 = -e_5$	$e_4e_1 = -e_5$	$e_4e_2 = e_5$
N_{74}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha-1)e_5$	$e_1e_4 = (1-\alpha)e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = -e_4$	$e_2e_3 = (1-\alpha)e_5$	$e_2e_4 = (\alpha-1)e_5$

		$e_3e_1 = e_5$	$e_3e_2 = -e_5$	$e_4e_1 = -e_5$	$e_4e_2 = e_5$
N_{75}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = -e_3$
		$e_2e_2 = -e_4$	$e_2e_4 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = -e_5$
		$e_4e_1 = -e_5$	$e_4e_2 = e_5$		
N_{76}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = -e_3 + e_5$
		$e_2e_2 = -e_4$	$e_2e_4 = -e_5$	$e_3e_1 = e_5$	$e_3e_2 = -e_5$
		$e_4e_1 = -e_5$	$e_4e_2 = e_5$		
N_{77}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha - 1)e_5$	
		$e_1e_4 = e_5$	$e_2e_1 = -e_3$	$e_2e_2 = -e_4$	
		$e_2e_3 = -\alpha e_5$	$e_2e_4 = -e_5$	$e_3e_1 = e_5$	
N_{78}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha - 1)e_5$	
		$e_1e_4 = e_5$	$e_2e_1 = -e_3 + e_5$	$e_2e_2 = -e_4$	
		$e_2e_3 = -\alpha e_5$	$e_2e_4 = -e_5$	$e_3e_1 = e_5$	
N_{79}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha + 2)e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = -e_4$	$e_2e_3 = -(1 + \alpha)e_5$	$e_3e_1 = -2e_5$
		$e_3e_2 = e_5$	$e_4e_1 = e_5$		
N_{80}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha + 2)e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = -e_4$	$e_2e_3 = -(1 + \alpha)e_5$	$e_3e_1 = -2e_5$
		$e_3e_2 = e_5$	$e_4e_1 = e_5$		
N_{81}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha + 2)e_5$	$e_2e_1 = -e_3$
		$e_2e_2 = -e_4$	$e_2e_3 = -(1 + \alpha)e_5$	$e_2e_4 = -e_5$	$e_3e_1 = -2e_5$
		$e_3e_2 = e_5$	$e_4e_1 = e_5$		
N_{82}^{α}	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = (\alpha + 2)e_5$	$e_2e_1 = -e_3 + e_5$
		$e_2e_2 = -e_4$	$e_2e_3 = -(1 + \alpha)e_5$	$e_2e_4 = -e_5$	$e_3e_1 = -2e_5$
		$e_3e_2 = e_5$	$e_4e_1 = e_5$		
$N_{83}^{\alpha,\beta}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_1e_4 = (\beta - 1)e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = -e_4$	$e_2e_3 = -(1 + \alpha)e_5$	$e_2e_4 = -\beta e_5$
		$e_3e_2 = e_5$	$e_4e_1 = e_5$		
$N_{84}^{\alpha,\beta}$	:	$e_1e_1 = e_3$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_1e_4 = (\beta - 1)e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = -e_4$	$e_2e_3 = -(1 + \alpha)e_5$	$e_2e_4 = -\beta e_5$
		$e_3e_2 = e_5$	$e_4e_1 = e_5$		
N_{85}	:	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_1 = e_4$	
		$e_2e_3 = e_5$	$e_3e_2 = -e_5$	$e_4e_1 = -e_5$	
N_{86}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_1 = e_4$
		$e_2e_3 = e_5$	$e_3e_2 = -e_5$	$e_4e_1 = -e_5$	
N_{87}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_1 = e_4$
		$e_2e_2 = e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	$e_4e_1 = -e_5$
$N_{88}^{\alpha,\beta}$	:	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = \alpha e_5$	$e_2e_1 = e_4$
		$e_2e_3 = \beta e_5$	$e_2e_4 = e_5$	$e_3e_2 = -\beta e_5$	$e_4e_1 = -\alpha e_5$
N_{89}^{α}	:	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_2e_3 = \alpha e_5$	
		$e_2e_4 = e_5$	$e_3e_2 = -\alpha e_5$		
N_{90}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	
		$e_2e_3 = \alpha e_5$	$e_2e_4 = e_5$	$e_3e_2 = -\alpha e_5$	
$N_{91}^{\alpha\neq 0}$	:	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_1 = e_4$	$e_2e_3 = \alpha e_5$
		$e_2e_4 = e_5$	$e_3e_2 = -\alpha e_5$	$e_4e_1 = -e_5$	
$N_{92}^{\alpha\neq 0}$	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_4 = e_5$	$e_2e_1 = e_4$
		$e_2e_3 = \alpha e_5$	$e_2e_4 = e_5$	$e_3e_2 = -\alpha e_5$	$e_4e_1 = -e_5$
N_{93}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	$e_1e_4 = 2e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4$	$e_2e_4 = e_5$	
N_{94}^{α}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = -\alpha e_5$	$e_1e_4 = (2\alpha + 1)e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = e_5$	$e_2e_4 = \alpha e_5$
N_{95}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = -e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = e_5$	$e_2e_4 = -e_5$
N_{96}^{α}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	$e_1e_4 = \alpha e_5$
		$e_2e_1 = -e_3 + e_5$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = (2 + \alpha)e_5$	$e_2e_4 = -e_5$
		$e_3e_1 = e_5$	$e_3e_2 = -(2 + \alpha)e_5$	$e_4e_1 = \alpha e_5$	$e_4e_2 = e_5$

N_{97}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$	
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4 + e_5$	$e_2e_3 = -2e_5$	$e_2e_4 = -e_5$
		$e_3e_1 = e_5$	$e_4e_1 = -2e_5$	$e_4e_2 = e_5$	
$N_{98}^{\alpha,\beta,\gamma}$	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = -(\beta + 2)e_5$	$e_1e_4 = (\alpha + \beta - 2\gamma)e_5$
		$e_2e_1 = -e_3$	$e_2e_2 = 2e_3 + e_4$	$e_2e_3 = \alpha e_5$	$e_2e_4 = \beta e_5$
		$e_3e_1 = e_5$	$e_3e_2 = -(\gamma + 2)e_5$	$e_4e_1 = \gamma e_5$	$e_4e_2 = e_5$
$N_{99}^{\alpha,\beta}$	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_2 = 2e_3 + e_4 + e_5$	
		$e_1e_3 = -(\alpha + 2)e_5$	$e_2e_1 = -e_3$	$e_1e_4 = (\sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1} - \beta)e_5$	
		$e_2e_4 = \alpha e_5$	$e_3e_1 = e_5$	$e_2e_3 = (\beta - \alpha + \sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1})e_5$	
		$e_3e_2 = -(\beta + 2)e_5$	$e_4e_1 = \beta e_5$	$e_4e_2 = e_5$	
$N_{100}^{\alpha,\beta}$	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_2 = 2e_3 + e_4 + e_5$	
		$e_1e_3 = -(\alpha + 2)e_5$	$e_2e_1 = -e_3$	$e_1e_4 = -(\sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1} + \beta)e_5$	
		$e_2e_4 = \alpha e_5$	$e_3e_1 = e_5$	$e_2e_3 = (\beta - \alpha - \sqrt{-2\alpha\beta - 2\alpha - 2\beta - 1})e_5$	
		$e_3e_2 = -(\beta + 2)e_5$	$e_4e_1 = \beta e_5$	$e_4e_2 = e_5$	
$N_{101}^{\alpha,\beta}$	:	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_1e_4 = e_5$	$e_2e_2 = e_3$
		$e_2e_3 = e_5$	$e_2e_4 = (\alpha - \beta)e_5$	$e_4e_2 = \beta e_5$	
N_{102}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = e_5$	$e_2e_1 = e_5$
		$e_2e_2 = e_3$	$e_2e_4 = (1 - \alpha)e_5$	$e_4e_2 = \alpha e_5$	
N_{103}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = e_5$	$e_2e_2 = e_3$
		$e_2e_4 = (1 - \alpha)e_5$	$e_4e_2 = \alpha e_5$		
N_{104}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4$	$e_2e_2 = e_3$		$e_2e_3 = e_5$
		$e_2e_4 = e_5$	$e_3e_1 = e_5$	$e_4e_2 = e_5$	
N_{105}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = -e_4 + e_5$
		$e_2e_2 = e_3$	$e_2e_4 = -e_5$	$e_3e_1 = -e_5$	$e_4e_2 = e_5$
N_{106}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = -2e_4$	$e_2e_2 = e_3$
		$e_2e_3 = e_5$	$e_2e_4 = e_5$	$e_3e_2 = \alpha e_5$	
N_{107}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = \frac{1}{2}e_5$	$e_2e_1 = -\frac{1}{2}e_4$
		$e_2e_2 = e_3$	$e_2e_3 = \alpha e_5$	$e_2e_4 = -e_5$	
		$e_3e_1 = -\frac{1}{2}e_5$	$e_3e_2 = e_5$	$e_4e_2 = e_5$	
N_{108}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = \frac{1}{2}e_5$	$e_2e_1 = -\frac{1}{2}e_4$	
		$e_2e_2 = e_3$	$e_2e_3 = \alpha e_5$	$e_2e_4 = -e_5$	
		$e_3e_1 = -\frac{1}{2}e_5$	$e_3e_2 = e_5$	$e_4e_2 = e_5$	
N_{109}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = -\frac{1}{2}e_4 + e_5$	$e_2e_2 = e_3$
		$e_2e_4 = -\frac{1}{2}e_5$	$e_3e_2 = e_5$	$e_3e_1 = -\frac{1}{2}e_5$	$e_4e_2 = e_5$
N_{110}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	$e_2e_1 = -\frac{1}{2}e_4$	$e_2e_2 = e_3$
		$e_2e_4 = (\alpha - \frac{3}{2})e_5$	$e_3e_2 = e_5$	$e_3e_1 = -\frac{1}{2}e_5$	$e_4e_2 = e_5$
$N_{111}^{\alpha\neq-\frac{1}{2}}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4 + e_5$	$e_2e_2 = e_3$
		$e_2e_3 = e_5$	$e_2e_4 = \alpha e_5$	$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$
N_{112}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4 + e_5$	$e_2e_2 = e_3$
		$e_2e_4 = \alpha e_5$	$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$	
N_{113}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = -\alpha e_5$	
		$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$		$e_2e_3 = e_5$
		$e_2e_4 = -e_5$	$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$	
N_{114}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_4$	$e_1e_3 = -\alpha e_5$	$e_2e_1 = \alpha e_4$
		$e_2e_2 = e_3$	$e_2e_4 = -e_5$	$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$
$N_{115}^{\alpha,\beta}$	:	$e_1e_2 = e_4$	$e_1e_3 = \beta e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$
		$e_2e_3 = e_5$	$e_2e_4 = (\beta + \alpha - 1)e_5$	$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$
$N_{116}^{\alpha,\beta}$	:	$e_1e_2 = e_4$	$e_1e_3 = \beta e_5$	$e_2e_1 = \alpha e_4$	$e_2e_2 = e_3$
		$e_2e_4 = (\beta + \alpha - 1)e_5$	$e_3e_1 = \alpha e_5$	$e_4e_2 = e_5$	
N_{117}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4$	
		$e_2e_2 = e_3$	$e_2e_4 = e_5$	$e_3e_2 = e_5$	
N_{118}^{α}	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = \alpha e_4$	
		$e_2e_2 = e_3$	$e_2e_4 = e_5$		
N_{119}	:	$e_1e_1 = e_2$	$e_1e_3 = e_5$	$e_2e_1 = e_3$	$e_4e_4 = e_5$

N_{120}	:	$e_1e_1=e_2$	$e_1e_3=e_5$	$e_1e_4=e_5$	$e_2e_1=e_3$
N_{121}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=2e_5$	$e_4e_1=e_5$
N_{122}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=2e_5$	$e_2e_1=e_5$
N_{123}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=2e_5$	
		$e_2e_1=e_5$	$e_4e_1=e_5$		
N_{124}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=2e_5$	
		$e_2e_1=e_5$	$e_4e_4=e_5$		
$N_{125}^{\alpha\neq1}$	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=(2-\alpha)e_5$	$e_2e_1=\alpha e_3$
		$e_2e_2=\alpha e_5$	$e_3e_1=\alpha e_5$	$e_4e_4=e_5$	
$N_{126}^{\alpha\neq0,1}$	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=(2-\alpha)e_5$	$e_1e_4=e_5$
		$e_2e_1=\alpha e_3$	$e_2e_2=\alpha e_5$	$e_3e_1=\alpha e_5$	
N_{127}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_1=e_3+e_5$
		$e_2e_2=e_5$	$e_3e_1=e_5$	$e_4e_1=e_5$	
N_{128}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_1=e_3$
		$e_2e_2=e_5$	$e_3e_1=e_5$	$e_4e_1=e_5$	
N_{129}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_1=e_3$
		$e_2e_2=e_5$	$e_3e_1=e_5$	$e_4e_1=e_5$	$e_4e_4=e_5$
N_{130}^{α}	:	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=e_5$	$e_2e_1=e_3+e_5$
		$e_2e_2=e_5$	$e_3e_1=e_5$	$e_4e_1=e_5$	$e_4e_4=\alpha e_5$
$N_{131}^{\alpha\neq0,\beta}$	:	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_3=e_4$	$e_1e_4=\frac{2-\alpha}{\alpha}e_5$
		$e_2e_1=\alpha e_4$	$e_2e_2=e_5$	$e_2e_3=e_5$	
		$e_3e_2=-e_5$	$e_3e_3=\beta e_5$	$e_4e_1=e_5$	
$N_{132}^{\alpha\neq0,1}$	:	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_3=e_4+e_5$	$e_1e_4=\frac{2-\alpha}{\alpha}e_5$
		$e_2e_1=\alpha e_4$	$e_2e_2=e_5$	$e_2e_3=e_5$	
		$e_3e_2=-e_5$	$e_3e_3=\frac{1}{(\alpha-1)^2}e_5$	$e_4e_1=e_5$	
N_{133}	:	$e_1e_1=e_2$	$e_1e_3=e_5$	$e_1e_4=-e_5$	$e_2e_1=e_4$
		$e_2e_3=e_4$	$e_3e_1=e_5$	$e_3e_2=-e_5$	$e_4e_1=e_5$
N_{134}^{α}	:	$e_1e_1=e_2$	$e_1e_3=e_5$	$e_1e_4=-e_5$	$e_2e_1=e_4$
		$e_2e_3=e_4$	$e_3e_2=-e_5$	$e_3e_3=\alpha e_5$	$e_4e_1=e_5.$
N_{135}^{α}	:	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_4=e_5$	$e_2e_1=\alpha e_5$
		$e_3e_1=e_4+e_5$	$e_3e_2=e_5$	$e_3e_3=2e_5$	
N_{136}^{α}	:	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_4=e_5$	
		$e_3e_1=e_4$	$e_3e_2=e_5$	$e_3e_3=\alpha e_5$	
N_{137}^{α}	:	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_4=e_5$	$e_2e_1=e_5$
		$e_3e_1=e_4$	$e_3e_2=e_5$	$e_3e_3=\alpha e_5$	
N_{138}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	
N_{139}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	$e_2e_2=e_5$
N_{140}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	$e_2e_1=e_5$
N_{141}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	
		$e_2e_1=e_5$	$e_2e_2=e_5$		
N_{142}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	
		$e_2e_3=e_5$	$e_3e_2=-e_5$		
N_{143}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	$e_2e_2=e_5$
		$e_2e_3=e_5$	$e_3e_2=-e_5$		
N_{144}^{α}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	$e_2e_1=e_5$
		$e_2e_2=\alpha e_5$	$e_2e_3=e_5$	$e_3e_2=-e_5$	
N_{145}^{α}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	
		$e_2e_2=e_4$	$e_2e_3=(1-\alpha)e_5$	$e_3e_2=\alpha e_5$	
N_{146}^{α}	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	$e_2e_1=e_5$
		$e_2e_2=e_4$	$e_2e_3=(1-\alpha)e_5$	$e_3e_2=\alpha e_5$	
$N_{147}^{\alpha,\beta}$	:	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$	$e_2e_1=\alpha e_5$
		$e_2e_2=e_4+e_5$	$e_2e_3=(1-\beta)e_5$	$e_3e_2=\beta e_5$	
N_{148}^{α}	:	$e_1e_1=e_2$	$e_1e_2=\alpha e_4$	$e_2e_1=(\alpha+1)e_4$	$e_3e_3=e_5$
N_{149}^{α}	:	$e_1e_1=e_2$	$e_1e_2=\alpha e_4+e_5$	$e_2e_1=(\alpha+1)e_4+e_5$	$e_3e_3=e_5$
N_{150}^{α}	:	$e_1e_1=e_2$	$e_1e_2=\alpha e_4$	$e_2e_1=(\alpha+1)e_4$	
		$e_3e_1=e_5$	$e_3e_3=e_5$		

N_{151}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_2e_1 = (\alpha+1)e_4 + e_5$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$		
N_{152}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_4$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_4$	$e_3e_3 = e_5$	
N_{153}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_1e_3 = e_4$	
		$e_2e_1 = (\alpha+1)e_4 + e_5$	$e_3e_1 = e_4$	$e_3e_3 = e_5$	
$N_{154}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + \beta e_5$	$e_1e_3 = e_4$	
		$e_2e_1 = (\alpha+1)e_4 + \beta e_5$	$e_3e_1 = e_4 + e_5$	$e_3e_3 = e_5$	
$N_{155}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = \beta e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = (\beta+1)e_5$		
N_{156}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_4 + \alpha e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_4 + (\alpha+1)e_5$		
$N_{157}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_1e_3 = \beta e_5$	
		$e_2e_1 = (\alpha+1)e_4 + e_5$	$e_3e_1 = (\beta+1)e_5$		
$N_{158}^{\alpha,\beta}$	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = \beta e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = (\beta+1)e_5$	$e_3e_3 = e_4$	
N_{159}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4 + e_5$	$e_2e_1 = (\alpha+1)e_4 + e_5$	
		$e_3e_1 = e_5$	$e_3e_3 = e_4$		
N_{160}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4 + e_5$	$e_3e_1 = e_5$
N_{161}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4 + e_5$	$e_3e_1 = \alpha e_5$		
N_{162}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4 + e_5$	$e_3e_3 = e_5$
N_{163}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4 + e_5$	$e_3e_1 = e_4$	$e_3e_3 = e_5$	
N_{164}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_5$		
N_{165}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = \alpha e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = (\alpha+1)e_4$	$e_3e_1 = e_5$	$e_3e_3 = e_4$	
N_{166}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4$	
		$e_3e_1 = e_4$	$e_3e_3 = e_5$		
N_{167}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = (\alpha+1)e_5$	$e_3e_3 = e_4$	
N_{168}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4 + e_5$	$e_2e_1 = e_4 + e_5$	
		$e_3e_1 = e_4$	$e_3e_3 = e_5$		
N_{169}^{α}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = \alpha e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = (\alpha+1)e_5$		
N_{170}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_4$	
		$e_3e_1 = e_5$	$e_3e_3 = e_5$		
N_{171}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = e_4 + e_5$		
N_{172}	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
		$e_2e_1 = e_4$	$e_3e_1 = e_4 + e_5$	$e_3e_3 = e_4$	
N_{173}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$
N_{174}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_3 = e_5$	$e_3e_2 = -e_5$		
N_{175}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_5$	
		$e_2e_3 = e_5$	$e_3e_2 = -e_5$		
N_{176}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{177}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$	
		$e_2e_3 = e_5$	$e_3e_2 = -e_5$		
N_{178}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{179}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_4 + e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{180}^{α}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	

		$e_2e_2 = \alpha e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{181}^{α}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
		$e_2e_2 = \alpha e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
$N_{182}^{\alpha,\beta}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
		$e_2e_2 = e_4 + \beta e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{183}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_3 = e_5$	$e_3e_2 = -e_5$		
N_{184}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{185}^{α}	:	$e_1e_1 = e_4 + e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = \alpha e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
$N_{186}^{\alpha,\beta}$	:	$e_1e_1 = e_4 + \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_4 + \beta e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
$N_{187}^{\alpha,\beta,\gamma}$	:	$e_1e_1 = e_4 + \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
		$e_2e_2 = \beta e_4 + \gamma e_5$	$e_2e_3 = e_5$	$e_3e_2 = -e_5$	
N_{188}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_5$	
N_{189}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_5$
N_{190}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_2e_2 = e_5$		
$N_{191}^{\alpha,\beta}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = \beta e_5$
		$e_2e_2 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{192}	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_5$
N_{193}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4$	$e_2e_2 = e_5$		
N_{194}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4 + e_5$	$e_2e_2 = e_5$		
$N_{195}^{\alpha,\beta}$	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4 + \beta e_5$
		$e_2e_2 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{196}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_5$
N_{197}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{198}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_5$
		$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{199}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4 + e_5$
		$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{200}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
N_{201}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
N_{202}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$
N_{203}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_1 = e_4$	$e_2e_2 = e_4$		
N_{204}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		
N_{205}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
		$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{206}	:	$e_1e_2 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$
N_{207}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_2 = e_5$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		
N_{208}^{α}	:	$e_1e_1 = \alpha e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_2e_2 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{209}	:	$e_1e_1 = e_4 + e_5$	$e_1e_2 = e_3$	$e_2e_2 = e_5$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		
N_{210}^{α}	:	$e_1e_1 = e_4 + \alpha e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_2e_2 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{211}	:	$e_1e_2 = e_3$	$e_2e_1 = e_5$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$
N_{212}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		

N_{213}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		
N_{214}	:	$e_1e_2 = e_3$	$e_2e_1 = e_5$	$e_2e_2 = e_4$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		
N_{215}	:	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{216}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_1 = e_5$	
		$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$	
N_{217}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$
N_{218}	:	$e_1e_1 = e_5$	$e_1e_2 = e_3$	$e_2e_2 = e_4$	
		$e_2e_3 = e_4$	$e_3e_2 = -e_4$		

Note that

$$\mathrm{N}_{12}^{\frac{1}{4}} \simeq \mathrm{N}_{11}^{\frac{1}{4}}, \mathrm{N}_{16}^{\alpha,\beta} \simeq \mathrm{N}_{16}^{\beta,\alpha}, \mathrm{N}_{35}^{\alpha} \simeq \mathrm{N}_{35}^{-\alpha}, \mathrm{N}_{37}^{\alpha} \simeq \mathrm{N}_{37}^{-\alpha}, \mathrm{N}_{42}^{\alpha,\beta} \simeq \mathrm{N}_{42}^{\alpha,-\beta}, \mathrm{N}_{60}^{0,0} \simeq \mathrm{N}_{213}^{0}, \mathrm{N}_{61}^{0,0,-1} \simeq \mathrm{N}_{210}^{0}, \mathrm{N}_{61}^{0,0,\gamma\neq-1} \simeq \mathrm{N}_{191}^{0,-\frac{1}{\gamma+1}},$$
$$\mathrm{N}_{63}^{0} \simeq \mathrm{N}_{194}^{0}, \mathrm{N}_{64}^{0,0} \simeq \mathrm{N}_{150}^{-1}, \mathrm{N}_{87}^{\alpha} \simeq \mathrm{N}_{87}^{\frac{1}{\alpha}}, \mathrm{N}_{88}^{\alpha,\beta} \simeq \mathrm{N}_{88}^{\beta,\alpha}, \mathrm{N}_{91}^{0} \simeq \mathrm{N}_{197}^{0}, \mathrm{N}_{92}^{0} \simeq {}_{198}^{0}, \mathrm{N}_{94}^{\alpha} \simeq \mathrm{N}_{94}^{\frac{1}{\alpha}}, \mathrm{N}_{96}^{\alpha} \simeq \mathrm{N}_{96}^{\frac{1}{2+\alpha}},$$
$$\mathrm{N}_{98}^{\alpha,\beta,\gamma} \simeq \mathrm{N}_{98}^{\frac{2+\beta}{2+\gamma}, \frac{\alpha-2(2+\gamma)}{2+\gamma}, -\frac{3+2\gamma}{2+\gamma}}, \mathrm{N}_{111}^{-\frac{1}{2}} \simeq \mathrm{N}_{112}^{-\frac{1}{2}},$$

N_{125}^{1} *and* N_{126}^{1} *are commutative-associative algebras,*

N_{126}^{0} *is a split algebra.*

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