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Caputo fractional derivatives and inequalities via preinvex stochastic processes

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Abstract. In the present paper, we present some Hermite-Hadamard type inequalities for preinvex stochastic process using the Caputo fractional derivatives. Using preinvexity of $|X^{(n)}|^k$, $k \geq 1$ we find the estimates of the difference of the fractional differential inequality. Some special results that reduced from our findings have also been considered.

1. Introduction

The popularity of fractional calculus (calculus of derivatives and integrals with arbitrary order) and the interest for the subject have grown astoundingly during the past three decades or so [6, 9, 22, 30]. Several real life problems have been studied using the fractional derivatives, specifically with the Caputo fractional derivative which is widely applied in various areas of sciences and engineering [8, 12]. For instance, it is known that due to their non-locality, fractional differential operators give a better description of systems with memory effect even though the non-locality takes different forms [6, 12, 22, 30].

There are various ways to define convexity for stochastic processes, it has a wide usage in optimization, especially in optimal designs, and also has importance in numerical approximations when there exist probabilistic quantities [24]. Nikodem [15] mentioned convex stochastic processes and also consider further properties which can be proved for standard convex functions. Temporal and spatiotemporal stochastic convexity was defined in [25] and [26], respectively for discrete time stochastic processes with informative examples. Convexity concepts in sample path sense can also be found in [3], and the references therein. Jensen-convex, λ-convex, Wright-convex stochastic processes were presented in [29]. Time stochastic $s-$convexity was taken into account in [28] by using order preserving functions of majorizations. Kotrys [10] generalized the classical Hermite-Hadamard inequality to convex stochastic processes. Strongly convex stochastic processes that generalized convex stochastic processes was also defined by Kotrys in [11]. These results motivated us to establish our paper.

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The following double inequality holds for any convex mapping f

$$f\Big(\frac{\kappa_1+\kappa_2}{2}\Big) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} f(x)dx \leq \frac{f(\kappa_1)+f(\kappa_2)}{2}. \tag{1}$$

The inequality (1) which is called the Hermite-Hadamard has several applications in the theory of probability and optimization (see[27]). This inequality offer us upper and lower bounds of the mean value of a continuous convex function $f:[\kappa_1,\kappa_2]\to\mathbb{R}$. In the viewpoint of probabilistic, (1) provides new estimations for $E[f(X)]$ where X is uniformly distributed on the interval $[\kappa_1,\kappa_2]$ (see[2]). Recently, researchers effort to obtain considerable integral inequalities containing generalizations, improvements and refinements. Among others, invex and preinvex functions which stand out as generalizations of convexity introduced by Ben-Israel and Mond [1], Hanson [7] and Noor [16] respectively. These remarkable findings have found the opportunity to be applied in many areas such as optimization, statistics and numerical estimations with probabilistic quantities. [24].

Noor [16] has established the following useful inequality under assumptions that f is a preinvex function on $\wp = [\kappa_1, \kappa_1+\eta(\kappa_2,\kappa_1)]$

$$f\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big) \leq \frac{1}{\eta(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\eta(\kappa_2,\kappa_1)} f(x)dx \leq \frac{f(\kappa_1)+f(\kappa_2)}{2}.$$

This famous inequality took its place in the literature as the inequality of Hermite-Hadamard-Noor inequality. Then many similar inequalities were derived for different types of preinvex functions, see [16–21] and the references therein.

Our main goal in this paper is to obtain Hermite-Hadamard type inequality by using Caputo fractional derivatives, but now for preinvex stochastic processes.

2. preliminaries

In the sequel of this paper, we use the subsequent notations:
"A stochastic process $\{X(\xi):\xi\in I\}$ is a parameterized collection of random variables defined on a familier probability space (Ω, A, P). Here, we denote the time with ξ. Then $X(\xi)$, which can also be shown as $X(\xi,\omega)$ for $\omega\in\Omega$, is noted to be state or location of the process at time ξ. For any fixed outcome ω of sample space Ω, the deterministic mapping $\xi\to X(\xi,\omega)$ denotes a realization, trajectory or sample path of the process. For any particular $\xi\in I$ the mapping depends ω alone, i.e., then we obtain a random variable. It can be said that, $X(\xi,\omega)$ changes in time in a random manner. We restrict our attention to continuous time stochastic processes, i.e., index set is $I=[0,\infty)$."

Definition 2.1. *"Suppose that $I_\eta\neq\emptyset$ be subset of $\mathbb{R}^n$. I_η is an invex set with respect to the given vector function $\eta: I_\eta\times I_\eta\to\mathbb{R}^n$ (or η-invex, or η-connected set) if*

$$\kappa_1+\xi\eta(\kappa_2,\kappa_1)\in I_\eta$$

for all $\kappa_1,\kappa_2\in I_\eta$ and $\xi\in[0,1]$".

Remark 2.2. *"Clearly, any convex set is an invex set with respect to $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$. When interpreted geometrically, the endpoints of the cluster and line segment joining the endpoints are located in a convex set. Despite convex sets cannot be disconnected, invex sets can be disconnected. Definition 2.1 essentially says that there is a path starting from a point κ_1 which is contained in I_η. We do not require that the point κ_2 should be the one of endpoints of the path [16]".*

Definition 2.3. *"Suppose that $I_\eta \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : I_\eta \times I_\eta \to \mathbb{R}^n$. Then the mapping (not necessarily differentiable) $f : I_\eta \to \mathbb{R}$ is said to be preinvex with respect to η if*

$$f(\kappa_1 + \xi\eta(\kappa_2, \kappa_1)) \le (1-\xi)f(\kappa_1) + \xi f(\kappa_2), \forall \kappa_1, \kappa_2 \in I_\eta,\ \xi \in [0,1]."$$

Any convex function is preinvex with respect to $\eta(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$, but the converse is not necessarily true. $f(x) = \exp(x)$ is a counterexample, it is preinvex with respect to $\eta(x, \kappa_1) = -1$.

The following Condition C was introduced by Mohan and Neogy [14], which has played a key role in the papers related to variational inequalities and optimization theory.

Condition C. Suppose I_η is an open invex subset of $\mathbb{R}^n$ with respect to $\eta : I_\eta \times I_\eta \to \mathbb{R}$ and η satisfies

$$\begin{aligned} \eta\Big(\kappa_2, \kappa_2 + \xi\eta(\kappa_1, \kappa_2)\Big) &= -\xi\eta(\kappa_1, \kappa_2), \\ \eta\Big(\kappa_1, \kappa_2 + \xi\eta(\kappa_1, \kappa_2)\Big) &= (1-\xi)\eta(\kappa_1, \kappa_2), \end{aligned} \tag{2}$$

for any $\kappa_1, \kappa_2 \in I_\eta$ and $\xi \in [0,1]$.

Definition 2.4. *"Let (Ω, A, P) be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is A-measurable. Let $I \subset \mathbb{R}$ be an interval indicating time. A function $X : I \times \Omega \to \mathbb{R}$ is called a stochastic process if for every $t \in I$ the function $X(t, .)$ is a random variable.*

1. *If $X(t, w)$ takes values in $\mathbb{R}^n$ it is called vector-valued stochastic process.*
2. *If the time I can be a discrete subset of $\mathbb{R}$, then $X(t, w)$ is called a discrete time stochastic process.*
3. *If the time I is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a stochastic process with continuous time."*

Definition 2.5. *"Let (Ω, A, P) be a probability space and $I \in \mathbb{R}$ be an interval. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called:*

1. Increasing (decreasing) *if for all $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$,*

$$X(\kappa_1, .) = X(\kappa_2, .), \quad (X(\kappa_1, .) = X(\kappa_2, .)), \quad (a.e.);$$

2. *Monotonic, if it's increasing or decreasing;*

3. Continuous *in probability in the interval I, if for all $t_0 \in I$ the following limit holds:*

$$P - \lim_{t \to t_0} X(t, .) = X(t_0, .),$$

where $P - \lim$ denotes the limit in probability;

4. *Mean square continuous in the interval I, if the limit for all $t_0 \in I$*

$$\lim_{t \to t_0} E[X(t, .) - X(t_0, .)]^2 = 0,$$

where $E[X(t, .)]$ denotes the expectation value of the random variable $X(t, .)$;

5. Mean square differentiable *in I, if there exist a stochastic process $X'(t,)$ (the derivative of X) such that for all $t_0 \in I$ we have*

$$\lim_{t\to t_0} E\Big[\frac{X(t,.)-X(t_0,.)}{t-t_0} - X'(t_0,.)\Big]^2 = 0.''$$

Definition 2.6. *"Let (Ω, A, P) be a probability space, $I \subset \mathbb{R}$ be an interval with $E[X(t)]^2 < \infty$ for all $t \in I$. Let $[\kappa_1, \kappa_2] \in I, \kappa_1 = t_0 < t_1 < ... < t_n = \kappa_2$ be a partition of $[a,b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1,2,...,n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,.)$ on $[\kappa_1, \kappa_2]$, if the following identity holds:*

$$\lim_{n\to\infty} E\Big[\sum_{k=1}^{n} X(\theta_k,.)(t_k - t_{k-1}) - Y(.)\Big]^2 = 0;$$

in such a way, it can be written

$$\int_a^b X(t,.)dt = Y(.) \quad (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,.)dt = \int_a^b Z(t,.)dt \quad (a.e.),$$

where $X(t,.) = Z(t,.)$ in $[\kappa_1, \kappa_2]$."

Throughout this paper, we assume that $I \subseteq [0,\infty)$ is a η-invex interval and the function η satisfies Condition C.

Definition 2.7. *"Let (Ω, A, P) be a probability space and $I \subset \mathbb{R}$ be an interval. The stochastic process $X : I\times\Omega \to \mathbb{R}$ is said to be a convex stochastic process if*

$$X(\xi\kappa_1 + (1-\xi)\kappa_2,.) = \xi X(\kappa_1,.) + (1-\xi)X(\kappa_2,.)$$

holds almost everywhere for all $\kappa_1, \kappa_2 \in I$ and $\xi \in [0,1]$."

One of the results of interest for the present work is the following.

Theorem 2.8. *Every Jensen-convex stochastic process and continuous in probability is convex.*

Using Definition 2.7, Kotrys [10], the Hermite-Hadamard integral inequality version for Stochastic Processes.

Theorem 2.9. *If $X : I\times\Omega \to \mathbb{R}$ is convex and mean square continuous in the interval $T\times\Omega$, then for any $\kappa_1, \kappa_2 \in T$, the inequality*

$$X\Big(\frac{\kappa_1+\kappa_2}{2},.\Big) \leq \frac{1}{\kappa_1-\kappa_2}\int_{\kappa_1}^{\kappa_2} X(t,.) \leq \frac{X(\kappa_1.,) + X(\kappa_2.,)}{2}$$

holds almost everywhere.

Before establishing the main results, it will be given some necessary notions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 2.10. *"Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}, n = [\alpha] + 1, f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous . The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:*

$$\left({}^{c}D^{\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt \quad (x > a) \tag{3}$$

and

$$\left({}^{c}D^{\alpha}_{b^-} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt \qquad (x < b). \tag{4}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $\left({}^{c}D^{\alpha}_{a^+} f\right)(x)$ coincides with $f^{(n)}(x)$ whereas $\left({}^{c}D^{\alpha}_{b^-} f\right)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$\left({}^{c}D^{0}_{a^+} f\right)(x) = \left({}^{c}D^{0}_{b^-} f\right)(x) = f(x),$$

where $n = 1$ and $\alpha = 0$. For more details see [9]."

3. main results

In this section of our paper, we focus to define the important generalizations of convexity for stochastic processes which are called preinvex and invex stochastic processes. Furthermore, we are in a position to establish a new variant of Hermite-Hadamard inequality for preinvex stochastic processes. Starting with motivation of the definition of convexity for random processes [15], we universalize the idea of preinvexity to processes. We expand the concept of convexity for random variables in [24].

Definition 3.1. *"Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process (not necessarily mean-square differentiable) on η-invex index set I. $X(\xi,)$ is called preinvex with respect to η if*

$$\begin{aligned} & X(\kappa_1 + \xi\eta(\kappa_2, \kappa_1), .) \\ \leq \quad & (1-\xi)X(\kappa_1, .) + \xi X(\kappa_2, .) \quad (a.e.) \quad \forall \kappa_1, \kappa_2 \in I, \xi \in [0, 1]. \end{aligned} \tag{5}$$

For a preinvex stochastic process, the inequality (5) holds almost everywhere on Ω, i.e., almost every sample path of X will be a preinvex function. For instance, the convex stochastic process $X : (0, 1)\times(0, 1) \to \mathbb{R}$ defined by

$$X(\xi, \omega) = \begin{cases} \xi^2, & \xi \neq \omega \\ \\ 0, & \xi = \omega. \end{cases}$$

is a preinvex stochastic process with respect to $\eta(\kappa_2, \kappa_1) = -\kappa_1$.
In (5), if ξ is fixed number in $(0, 1)$, then X is called
(i) ξ-preinvex stochastic process.
(ii) Jensen type preinvex stochastic process for $\xi = \frac{1}{2}$.
If we choose $\eta(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$, then preinvex $X(\xi, .)$ is also a convex stochastic process, that is, class of convex stochastic processes is contained by the class of preinvex stochastic processes."

Theorem 3.2. *Let $\alpha > 0$ and $X : [\kappa_1, \kappa_1 + \eta(\kappa_2, \kappa_1)] \times \Omega \to \mathbb{R}$ be a positive stochastic process with $\eta(\kappa_2, \kappa_1)$ and $X(\xi, .) \in AC^n[\kappa_1, \kappa_1 + \eta(\kappa_2, \kappa_1)]$. If $X^{(n)}(\xi, .)$ is preinvex and Condition C holds, then the following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned} & X^{(n)}\Big(\frac{2\kappa_1 + \eta(\kappa_2, \kappa_1)}{2}, .\Big) \\ \leq\ & \frac{\Gamma(n-\alpha+1)}{2(\eta(\kappa_2, \kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\kappa_1^+}X^{(n)}\Big)((\kappa_1 + \eta(\kappa_2, \kappa_1)), .) \\ & +(-1)^n\Big({}^{c}D^{\alpha}_{(\kappa_1+\eta(\kappa_2,\kappa_1))^-}X^{(n)}\Big)(\kappa_1, .)\Big] \\ \leq\ & \frac{X^{(n)}(\kappa_1, .) + X^{(n)}(\kappa_2, .)}{2}. \end{aligned} \tag{6}$$

Proof. Since $X^{(n)}$ is preinvex stochastic process, therefore for $u, v \in [\kappa_1, \kappa_1 + \eta(\kappa_2, \kappa_1)]$ we have

$$X^{(n)}\Big(\frac{x + \eta(y, x)}{2}, .\Big) \leq \frac{X^{(n)}(x, .) + X^{(n)}(y, .)}{2}, \tag{7}$$

Using Condition C and substituting $x = \kappa_1 + (1-\xi)\eta(\kappa_2, \kappa_1)$ and $y = \kappa_1 + \xi\eta(\kappa_2, \kappa_1)$ for $\xi \in [0, 1]$. Then $x, y \in [\kappa_1, \kappa_1 + \eta(\kappa_2, \kappa_1)]$ and (7) gives

$$\begin{aligned} & X^{(n)}\Big(\frac{\kappa_1 + \eta(\kappa_2, \kappa_1)}{2}, .\Big) \\ \leq\ & \frac{X^{(n)}(\kappa_1 + (1-\xi)\eta(\kappa_2, \kappa_1), .) + X^{(n)}(\kappa_1 + \xi\eta(\kappa_2, \kappa_1), .)}{2}, \end{aligned} \tag{8}$$

multiplying both sides of (8) by $\xi^{n-\alpha-1}$, and integrating over $[0, 1]$ we get

$$\begin{aligned} & \frac{2}{n-\alpha}X^{(n)}\Big(\frac{\kappa_1 + \eta(\kappa_2, \kappa_1)}{2}, .\Big) \\ \leq\ & \int_0^1 \xi^{n-\alpha-1}X^{(n)}(\kappa_1 + (1-\xi)\eta(\kappa_2, \kappa_1), .)d\xi \\ & + \int_0^1 \xi^{n-\alpha-1}X^{(n)}(\kappa_1 + \xi\eta(\kappa_2, \kappa_1), .)d\xi. \end{aligned}$$

By the change of variable we get

$$\begin{aligned} & \frac{2}{n-\alpha}X^{(n)}\Big(\frac{\kappa_1 + \eta(\kappa_2, \kappa_1)}{2}, .\Big) \\ \leq\ & \int_{\kappa_1+\eta(\kappa_2,\kappa_1)}^{\kappa_1} \Big(\frac{\kappa_1 + \eta(\kappa_2, \kappa_1) - u}{\eta(\kappa_2, \kappa_1)}\Big)^{n-\alpha-1}\frac{X^{(n)}(u, .)}{\eta(\kappa_2, \kappa_1)}du \\ & + \int_{\kappa_1}^{\kappa_1+\eta(\kappa_2,\kappa_1)} \Big(\frac{v - \kappa_1}{\eta(\kappa_2, \kappa_1)}\Big)^{n-\alpha-1}\frac{X^{(n)}(v, .)}{\eta(\kappa_2, \kappa_1)}du \\ =\ & \frac{\Gamma(n-\alpha)}{(\eta(\kappa_2, \kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\kappa_1^+}X^{(n)}\Big)(\kappa_1 + \eta(\kappa_2, \kappa_1), .) \\ & +(-1)^n\Big({}^{c}D^{\alpha}_{(\kappa_1+\eta(\kappa_2,\kappa_1))^-}X^{(n)}\Big)(\kappa_1, .)\Big], \end{aligned}$$

which gives the first inequality in (6). In order to prove the second inequality of (6).
It is used the preinvex property of the stochastic process $X^{(n)}$:

$$\begin{aligned}& X^{(n)}(\kappa_1 + (1-\xi)\eta(\kappa_2,\kappa_1)),.) + X^{(n)}(\kappa_1 + \xi\eta(\kappa_2,\kappa_1)),.) \\ \leq\ & X^{(n)}(\kappa_1,.) + X^{(n)}(\kappa_2,.).\end{aligned}$$

Multiplying both sides of the above inequality by $\xi^{n-\alpha-1}$, and integrating over $[0,1]$ we have

$$\begin{aligned}& \int_0^1 X^{(n)}(\kappa_1 + (1-\xi)\eta(\kappa_2,\kappa_1)),.)d\xi + \int_0^1 X^{(n)}(\kappa_1 + \xi\eta(\kappa_2,\kappa_1)),.)d\xi \\ \leq\ & [X^{(n)}(\kappa_1,.) + X^{(n)}(\kappa_2,.)]\int_0^1 \xi^{n-\alpha-1}d\xi.\end{aligned}$$

From which one can have

$$\begin{aligned}& \frac{\Gamma(n-\alpha)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{\kappa_1^+}X^{(n)}\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1)),.\Big) \\ & +(-1)^n\Big({}^cD^{\alpha}_{(\kappa_1+\eta(\kappa_2,\kappa_1))^-}X^{(n)}\Big)(\kappa_1,.)\Big] \\ \leq\ & \frac{X^{(n)}(\kappa_1,.) + X^{(n)}(\kappa_2,.)}{2},\end{aligned}$$

The proof is completed. □

Remark 3.3. *If we take $\eta(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, then Theorem 3.2 reduces to [5].*

Lemma 3.4. *Let $X : [\kappa_1, \kappa_1 + \eta(\kappa_2,\kappa_1)] \times \Omega \to \mathbb{R}$ be a square mean differentiable stochastic process such that $X \in AC^n[\kappa_1, \kappa_1 + \eta(\kappa_2,\kappa_1)]$ with $\eta(\kappa_2,\kappa_1) > 0$. If $X^{(n+1)}$ is a square mean differentiable stochastic process, then the following inequality for Caputo fractional derivatives holds almost everywhere:*

$$\begin{aligned}& \frac{X^{(n)}(\kappa_1,.) + X^{(n)}(\kappa_1+\eta(\kappa_2,\kappa_1),.)}{2} \\ & -\frac{\Gamma(n-\alpha+1)}{2(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{\kappa_1^+}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big) \\ & +(-1)^n\Big({}^cD^{\alpha}_{\left(\kappa_1+\eta(\kappa_2,\kappa_1)\right)^-}X\Big)(\kappa_1,.)\Big] \\ =\ & \frac{\eta(\kappa_2,\kappa_1)}{2}\int_0^1 \Big[(1-t)^{n-\alpha} - t^{n-\alpha}\Big]X^{(n+1)}(\kappa_1 + (1-t)\eta(\kappa_2,\kappa_1)),.)dt.\end{aligned} \tag{9}$$

Proof. First, it must be noted that

$$\begin{aligned}& \int_0^1 \Big[(1-\xi)^{n-\alpha} - \xi^{n-\alpha}\Big]X^{(n+1)}(\kappa_1 + (1-\xi)\eta(\kappa_2,\kappa_1)),.)d\xi \\ =\ & \int_0^1 (1-\xi)^{n-\alpha}X^{(n+1)}(\kappa_1 + (1-\xi)\eta(\kappa_2,\kappa_1)),.)d\xi \\ & -\int_0^1 \xi^{n-\alpha}X^{(n+1)}(\kappa_1 + (1-\xi)\eta(\kappa_2,\kappa_1)),.)d\xi,\end{aligned}$$

and using integration by parts it is obtained

$$\begin{aligned}
&\int_0^1 (1-\xi)^{n-\alpha} X^{(n+1)}(\kappa_1+(1-\xi)\eta(\kappa_2,\kappa_1)),.)d\xi \\
=&\ \frac{X^{(n)}(\kappa_1+\eta(\kappa_2,\kappa_1),.)}{\eta(\kappa_2,\kappa_1)} - (n-\alpha)\int_{\kappa_1}^{\kappa_1+\eta(\kappa_2,\kappa_1)} \Big(\frac{\kappa_1+\eta(\kappa_2,\kappa_1)-u}{\eta(\kappa_2,\kappa_1)}\Big)^{n-\alpha-1}\frac{X^{(n)}(u,.)}{\eta(\kappa_2,\kappa_1)}du \\
=&\ \frac{X^{(n)}(\kappa_1+\eta(\kappa_2,\kappa_1),.)}{\eta(\kappa_2,\kappa_1)} - \frac{\Gamma(n-\alpha+1)}{2(\eta(\kappa_2,\kappa_1))^{n-\alpha+1}}(-1)^n\Big({}^cD^{\alpha}_{(\kappa_1+\eta(\kappa_2,\kappa_1))^-}X\Big)(\kappa_1,.)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int_0^1 t^{n-\alpha} X^{(n+1)}(\kappa_1+(1-t)\eta(\kappa_2,\kappa_1)),.)dt \\
=&\ -\frac{X^{(n)}(\kappa_1,.)}{\eta(\kappa_2,\kappa_1)} + \frac{\Gamma(n-\alpha+1)}{2(\eta(\kappa_2,\kappa_1))^{n-\alpha+1}}\Big({}^cD^{\alpha}_{\kappa_1^+}X\Big)(\kappa_1+\eta(\kappa_2,\kappa_1),.),
\end{aligned}$$

Adding these get results we get the desired result.
The proof is completed. □

Theorem 3.5. *Let $X : [\kappa_1, \kappa_1+\eta(\kappa_2,\kappa_1)] \times \Omega \to \mathbb{R}$ be a square mean differentiable stochastic process such that $X \in AC^n[\kappa_1, \kappa_1+\eta(\kappa_2,\kappa_1)]$ with $\eta(\kappa_2,\kappa_1) > 0$. If $|X^{(n+1)}|$ is a preinvex stochastic process, then the following inequality for Caputo fractional derivatives holds almost everywhere:*

$$\begin{aligned}
&\Big|\frac{X^{(n)}(\kappa_1,.)+X^{(n)}(\kappa_1+\eta(\kappa_2,\kappa_1),.)}{2} \\
&-\frac{\Gamma(n-\alpha+1)}{2(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{\kappa_1^+}X\Big)(\kappa_1+\eta(\kappa_2,\kappa_1),.) \\
&+(-1)^n\Big({}^cD^{\alpha}_{(\kappa_1+\eta(\kappa_2,\kappa_1))^-}X\Big)(\kappa_1,.)\Big]\Big| \\
\leq&\ \frac{\eta(\kappa_2,\kappa_1)}{2(n-\alpha+1)}\Big(1-\frac{1}{2^{n-\alpha}}\Big)\Big[|X^{(n+1)}(\kappa_1,.)|+|X^{(n+1)}(\kappa_2,.)|\Big].
\end{aligned} \tag{10}$$

Proof. Using Lemma 3.4 and the convexity of $|X^{n+1}|$, it is obtained that

$$\begin{aligned}
&\Big|\frac{X^{(n)}(\kappa_1,.)+X^{(n)}(\kappa_1+\eta(\kappa_2,\kappa_1),.)}{2} \\
&-\frac{\Gamma(n-\alpha+1)}{2(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{\kappa_1^+}X\Big)(\kappa_1+\eta(\kappa_2,\kappa_1),.) \\
&+(-1)^n\Big({}^cD^{\alpha}_{(\kappa_1+\eta(\kappa_2,\kappa_1))^-}X\Big)(\kappa_1,.)\Big]\Big| \\
\leq&\ \frac{\eta(\kappa_2,\kappa_1)}{2}\int_0^1 |(1-\xi)^{n-\alpha}-\xi^{n-\alpha}||X^{(n+1)}(\kappa_1+(1-\xi)\eta(\kappa_2,\kappa_1),.)|d\xi \\
\leq&\ \frac{\eta(\kappa_2,\kappa_1)}{2}\int_0^{\frac{1}{2}} \Big((1-\xi)^{n-\alpha}-\xi^{n-\alpha}\Big)\Big(\xi|X^{(n+1)}(\kappa_1,.)|+(1-\xi)|X^{(n+1)}(\kappa_2,.)|\Big)dv \\
&+\frac{\eta(\kappa_2,\kappa_1)}{2}\int_{\frac{1}{2}}^1 \Big(\xi^{n-\alpha}-(1-\xi)^{n-\alpha}\Big)\Big(\xi|X^{(n+1)}(\kappa_1,.)|+(1-v)|X^{(n+1)}(\kappa_2,.)|\Big)dv,
\end{aligned} \tag{11}$$

which allow us to write

$$\begin{aligned}&\int_0^{\frac{1}{2}}\Big((1-\xi)^{n-\alpha}-\xi^{n-\alpha}\Big)\Big(\xi|X^{(n+1)}(\kappa_1,.)|+(1-\xi)|X^{(n+1)}(\kappa_2,.)|\Big)d\xi\\ =\ & |X^{(n+1)}(\kappa_1,.)|\Big[\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\Big]\\ &+|X^{(n+1)}(\kappa_2,.)|\Big[\frac{1}{(n-\alpha+2)}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\Big].\end{aligned}$$

Analogously

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\Big(\xi^{n-\alpha}-(1-\xi)^{n-\alpha}\Big)\Big(\xi|X^{(n+1)}(\kappa_1,.)|+(1-\xi)|X^{(n+1)}(\kappa_2,.)|\Big)d\xi\\ =\ & |X^{(n+1)}(\kappa_1,.)|\Big[\frac{1}{(n-\alpha+2)}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\Big]\\ &+|X^{(n+1)}(\kappa_2,.)|\Big[\frac{1}{(n-\alpha+2)(n-\alpha+1)}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\Big].\end{aligned}$$

Suitable rearrangements completes the proof. □

Remark 3.6. *If we take $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, then Theorem 3.5 reduces to [5].*

Theorem 3.7. *Let $\alpha>0$ and $X:[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]\times\Omega\to\mathbb{R}$ be a positive stochastic process with $\eta(\kappa_2,\kappa_1)>0$ and $X(t,.)\in AC^n[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]$. If $X^{(n)}(\xi,.)$ is preinvex and Condition C holds, then the following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned}&X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2},.\Big)\\ \le\ & \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^+}X\Big)(\kappa_2,.)\\ &+\Big({}^{c}D^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^-}X\Big)(\kappa_1,.)\Big]\\ \le\ & \frac{X^{(n)}(\kappa_1,.)+X^{(n)}(\kappa_2,.)}{2}.\end{aligned}\tag{12}$$

Proof. Since $X^{(n)}$ is preinvex stochastic process, therefore for $x,y\in[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]$ we have

$$X^{(n)}\Big(\frac{2x+\eta(y,x)}{2},.\Big)\le\frac{X^{(n)}(x,.)+X^{(n)}(y,.)}{2},\tag{13}$$

Using Condition C and putting $x=\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1)$ and $y=\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1)$ for $\xi\in[0,1]$, we have $x,y\in[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]$. The above inequality becomes

$$\begin{aligned}&2X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2},.\Big)\\ \le\ & X^{(n)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)+X^{(n)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big).\end{aligned}$$

Multiplying both sides of above inequality with $\xi^{n-\alpha-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&2X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2},.\Big)\int_0^1 \xi^{n-\alpha-1}d\xi\\
\leq\ & \int_0^1 \xi^{n-\alpha-1}X^{(n)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi\\
&+\int_0^1 \xi^{n-\alpha-1}X^{(n)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi,
\end{aligned}$$

By using the change of variables we have

$$\begin{aligned}
&\frac{2}{n-\alpha}X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2},.\Big)\\
\leq\ & \frac{2^{n-\alpha}}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\int_{\kappa_1}^{\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}-u\Big)^{n-\alpha-1}X(u,.)du\\
&+\int_{\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}}^{\kappa_1+\eta(\kappa_2,\kappa_1)}\Big(v-\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)^{n-\alpha-1}X(v,.)dv\Big]\\
=\ & \frac{2^{n-\alpha}\Gamma(n-\alpha)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^+}X\Big)(v,.)\\
&+\Big({}^cD^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^-}X\Big)(\kappa_1,.)\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2},.\Big)\\
\leq\ & \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^+}X\Big)(\kappa_2,.)\\
&+\Big({}^cD^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^-}X\Big)(\kappa_1,.)\Big].
\end{aligned} \tag{14}$$

Similarly, preinvexity of $X^{(n)}$ gives

$$\begin{aligned}
&X^{(n)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)+X^{(n)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\\
\leq\ & X^{(n)}(\kappa_1,.)+X^{(n)}(\kappa_2,.).
\end{aligned}$$

Multiplying both sides of the above inequality with $\xi^{n-\alpha-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \xi^{n-\alpha-1}X^{(n)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi\\
&+\int_0^1 \xi^{n-\alpha-1}X^{(n)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi
\end{aligned}$$

$$\leq \Big[X^{(n)}(\kappa_1,.)+X^{(n)}(\kappa_2,.)\Big]\int_0^1 \xi^{n-\alpha}d\xi,$$

from which one can have

$$\begin{aligned}&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^{+}}X\Big)(\kappa_2,.)\\&+\Big({}^{c}D^{\alpha}_{(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2})^{-}}X\Big)(\kappa_1,.)\Big]\\ \leq\ &\frac{X^{(n)}(\kappa_1,.)+X^{(n)}(\kappa_2,.)}{2}.\end{aligned} \tag{15}$$

Combining inequality (14) and (15), we get inequality (12). □

Remark 3.8. *If we take $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, then Theorem 3.7 reduces to [5].*

Following lemma is useful for our next results.

Lemma 3.9. *Let $X:[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]\times\Omega\to\mathbb{R}$ be a square mean differentiable stochastic process such that $X\in AC^n[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]$ with $\eta(\kappa_2,\kappa_1)>0$. If $X^{(n+1)}$ is a square mean differentiable stochastic process, then the following inequality for Caputo fractional derivatives holds almost everywhere:*

$$\begin{aligned}&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\&+(-1)^n\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\&\quad-X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\\=\ &\frac{\eta(\kappa_2,\kappa_1)}{4}\Big[\int_0^1\xi^{n-\alpha}X^{(n+1)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi\\&-\int_0^1\xi^{n-\alpha}X^{(n+1)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi\Big].\end{aligned} \tag{16}$$

Proof. Since

$$\begin{aligned}&\frac{\eta(\kappa_2,\kappa_1)}{4}\int_0^1\xi^{n-\alpha}X^{(n+1)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi\\=\ &\frac{\eta(\kappa_2,\kappa_1)}{4}\Big[\frac{-2\xi^{n-\alpha}X^{(n)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)}{\eta(\kappa_2,\kappa_1)}\Big|_0^1\\&+\frac{2(n-\alpha)}{\eta(\kappa_2,\kappa_1)}\int_0^1\xi^{n-\alpha-1}X^{(n)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,v),.\Big)d\xi\Big]\\=\ &\frac{\eta(\kappa_2,\kappa_1)}{4}\Big[\frac{-2}{\eta(\kappa_2,\kappa_1)}X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\end{aligned}$$

$$+\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha+1}}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)(\kappa_2,.)\Big].$$

Analogously, we have

$$\begin{aligned}
&\frac{-\eta(\kappa_2,\kappa_1)}{4}\int_0^1 \xi^{n-\alpha}X^{(n+1)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)d\xi\\
=\;&\frac{-\eta(\kappa_2,\kappa_1)}{4}\Big[\frac{2}{\eta(\kappa_2,\kappa_1)}X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\\
&-(-1)^n\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha+1}}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\Big].
\end{aligned}$$

Adding above equalities we get the required equality. □

Theorem 3.10. *Let $\alpha>0$ and $X:[\kappa_1,\kappa_1+\eta(\kappa_2,a)]\times\Omega\to\mathbb{R}$ be a positive stochastic process with $\eta(\kappa_2,\kappa_1)>0$ and $X(\xi,.)\in AC^n[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]$. If $X^{(n)}(\xi,.)$ is preinvex on $[\kappa_1,\kappa_1+\eta(\kappa_2,\kappa_1)]$ for $k\geq 1$, then the following inequalities for Caputo fractional derivatives holds*

$$\begin{aligned}
&\Big|X^{(n)}\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\
&+(-1)^n\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\
&\quad -X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\
\leq\;&\frac{\eta(\kappa_2,\kappa_1)}{4(n-\alpha+1)}\Big(\frac{1}{2(n-\alpha+2)}\Big)^{\frac{1}{k}}\Bigg[\Big(|X^{(n+1)}(\kappa_1,.)|^k\\
&+(2(n-\alpha)+3)|X^{(n+1)}(\kappa_2,.)|^k\Big)^{\frac{1}{k}}\\
&\quad+\Big(2(n-\alpha)+3)|X^{(n+1)}(\kappa_1,.)|^k+|X^{(n+1)}(\kappa_2,.)|^k\Big)^{\frac{1}{k}}\Bigg].
\end{aligned}\tag{17}$$

Proof. Using Lemma 3.9 we have

$$\begin{aligned}
&\Big|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\
&+(-1)^n\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)-X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\
\leq\;&\frac{\eta(\kappa_2,\kappa_1)}{4}\int_0^1\xi^{n-\alpha}\Big|X^{(n+1)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|d\xi\\
&\quad+\int_0^1\xi^{n-\alpha}\Big|X^{(n+1)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|d\xi.
\end{aligned}\tag{18}$$

Applying power mean's inequality we have

$$\begin{aligned}
&\Big|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\
&+(-1)^{n}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\
&\quad -X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\
\leq\;& \frac{\eta(\kappa_2,\kappa_1)}{4}\Big(\frac{1}{n-\alpha+1}\Big)^{1-\frac{1}{k}}\Big[\Big[\int_0^1 \xi^{n-\alpha}\Big|X^{(n+1)}\Big(\kappa_1\\
&+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|^{k}d\xi\Big]^{\frac{1}{k}}\\
&\quad+\Big[\int_0^1 \xi^{n-\alpha}\Big|X^{(n+1)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|^{k}d\xi\Big]^{\frac{1}{k}}\Big]
\end{aligned}$$

Also from preinvexity of $|X^{(n+1)}|^{q}$, we have

$$\begin{aligned}
&\Big|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\
&+(-1)^{n}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\
&\quad -X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\
\leq\;& \frac{\eta(\kappa_2,\kappa_1)}{4}\Big(\frac{1}{n-\alpha+1}\Big)^{1-\frac{1}{k}}\Big[\Big[\int_0^1 \xi^{n-\alpha}\Big[\frac{1-\xi}{2}|X^{(n+1)}(\kappa_1,.)|^{k}\\
&+\frac{1+\xi}{2}|X^{(n+1)}(\kappa_2,.)|^{k}\Big]d\xi\Big]^{\frac{1}{k}}\\
&+\Big[\int_0^1 \xi^{n-\alpha}\Big[\frac{1+\xi}{2}|X^{(n+1)}(\kappa_1,.)|^{k}+\frac{1-\xi}{2}|X^{(n+1)}(\kappa_2,.)|^{k}\Big]d\xi\Big]^{\frac{1}{k}}\Big]\\
=\;& \frac{\eta(\kappa_2,\kappa_1)}{4(n-\alpha+1)}\Big(\frac{1}{2(n-\alpha+2)}\Big)^{\frac{1}{k}}\Big[\Big(|X^{(n+1)}(\kappa_1,.)|^{k}\\
&+(2(n-\alpha)+3)|X^{(n+1)}(\kappa_2,.)|^{k}\Big)^{\frac{1}{k}}\\
&\quad+\Big(2(n-\alpha)+3)|X^{(n+1)}(\kappa_1,.)|^{k}+|X^{(n+1)}(\kappa_2,.)|^{k}\Big)^{\frac{1}{k}}\Big].
\end{aligned}$$

The proof is completed. □

Next we use the Hӧlder inequality along with Lemma 3.9 to obtain the following result.

Theorem 3.11. *Let $\alpha > 0$ and $X : [\kappa_1, \kappa_1 + \eta(\kappa_2, \kappa_1)] \times \Omega \to \mathbb{R}$ be a positive stochastic process with $\eta(\kappa_2, \kappa_1) > 0$ and $X(\xi, .) \in AC^{n}[\kappa_1, \kappa_1 + \eta(\kappa_2, a)]$. If $|X^{(n)}(\xi, .)|^{k}$ is preinvex on $[\kappa_1, \kappa_1 + \eta(\kappa_2, \kappa_1)]$ for $k \geq 1$ with $k^{-1} + s^{-1} = 1$ then the following inequalities for Caputo fractional derivatives holds*

$$\begin{aligned}&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\&+(-1)^{n}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\&\quad-X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\\\leq\;&\frac{\eta(\kappa_2,\kappa_1)}{4}\Big(\frac{1}{ns-\alpha s+1}\Big)^{\frac{1}{s}}\Bigg[\Big[\frac{\big|X^{(n+1)}(\kappa_1,.)\big|^{k}+3\big|X^{(n+1)}(\kappa_2,.)\big|^{k}}{4}\Big]^{\frac{1}{k}}\\&\quad+\Big[\frac{3\big|X^{(n+1)}(\kappa_1,.)\big|^{k}+\big|X^{(n+1)}(\kappa_2,.)\big|^{k}}{4}\Big]^{\frac{1}{k}}\Bigg].\end{aligned}\tag{19}$$

Proof. Using Lemma 3.9 we have

$$\begin{aligned}&\Big|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\&+(-1)^{n}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\&\quad-X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\\leq\;&\frac{\eta(\kappa_2,\kappa_1)}{4}\int_0^1\xi^{n-\alpha}\Big|X^{(n+1)}\Big(\kappa_1+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|d\xi\\&-\int_0^1\xi^{n-\alpha}\Big|X^{(n+1)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|d\xi.\end{aligned}\tag{20}$$

Applying Hölder's inequality we have

$$\begin{aligned}&\Big|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{+}}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\&+(-1)^{n}\Big({}^{c}D^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^{-}}X\Big)(\kappa_1,.)\\&\quad-X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\\leq\;&\frac{\eta(\kappa_2,\kappa_1)}{4}\Big(\int_0^1\xi^{ns-\alpha s}d\xi\Big)^{\frac{1}{s}}\Bigg[\Big[\int_0^1\Big|X^{(n+1)}\Big(\kappa_1\\&+\frac{1-\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|^{k}d\xi\Big]^{\frac{1}{k}}\end{aligned}$$

$$+\Big[\int_0^1 \Big|X^{(n+1)}\Big(\kappa_1+\frac{1+\xi}{2}\eta(\kappa_2,\kappa_1),.\Big)\Big|^k d\xi\Big]^{\frac{1}{k}}\Big].$$

Since $|X^{(n+1)}|^k$ is preinvex, so we have

$$\begin{aligned}
&\Big|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\eta(\kappa_2,\kappa_1))^{n-\alpha}}\Big[\Big({}^cD^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^+}X\Big)\Big((\kappa_1+\eta(\kappa_2,\kappa_1),.\Big)\\
&+(-1)^n\Big({}^cD^{\alpha}_{\left(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\right)^-}X\Big)(\kappa_1,.)\\
&\quad -X^{(n)}\Big(\frac{2\kappa_1+\eta(\kappa_2,\kappa_1)}{2}\Big)\Big]\Big|\\
\leq\ & \frac{\eta(\kappa_2,\kappa_1)}{4}\Big(\frac{1}{ns-\alpha s+1}\Big)^{\frac{1}{s}}\Big[\Big[\int_0^1\Big(\frac{1-\xi}{2}|X^{(n+1)}(\kappa_1,.)\Big|^k\\
&+\frac{1+\xi}{2}|X^{(n+1)}(\kappa_2,.)\Big|^k\Big)d\xi\Big]^{\frac{1}{k}}\\
&\quad +\Big[\int_0^1\Big(\frac{1+\xi}{2}|X^{(n+1)}(\kappa_1,.)\Big|^k+\frac{1-\xi}{2}|X^{(n+1)}(\kappa_2,.)\Big|^k\Big)d\xi\Big]^{\frac{1}{k}}\Big]\\
=\ & \frac{\eta(\kappa_2,\kappa_1)}{4}\Big(\frac{1}{ns-\alpha s+1}\Big)^{\frac{1}{s}}\Big[\Big[\frac{|X^{(n+1)}(\kappa_1,.)\Big|^k+3|X^{(n+1)}(\kappa_2,.)\Big|^k}{4}\Big]^{\frac{1}{k}}\\
&\quad +\Big[\frac{3|X^{(n+1)}(\kappa_1,.)\Big|^k+|X^{(n+1)}(\kappa_2,.)\Big|^k}{4}\Big]^{\frac{1}{k}}\Big].
\end{aligned}$$

The proof is completed. □

Remark 3.12. *If we take $\eta(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, then Theorem 3.11 reduces to [5].*

Conclusion:

In this article, we have derived a few inequalities of Hermite-Hadamard type for preinvex stochastic processes that possess a first derivatives on the interior of an interval of real numbers, by utilizing the Caputo fractional derivatives and assumptions that the mappings $|f'|^k, k\geq 1$ are prinvex. The resulting inequalities exhibited here surely give new bounds.

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