Filomat 37:19 (2023), 6537–6567
https://doi.org/10.2298/FIL2319537B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some properties of degenerate Hermite Appell polynomials in three variables

Gizem Baran[a], Zeynep Özat[b], Bayram Çekim[b], Mehmet Ali Özarslan[c,*]

[a]*Cyprus Science University, Faculty of Engineering, Department of Software Engineering, Kyrenia, via Mersin 10, Turkey*
[b]*Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey*
[c]*Eastern Mediterranean University, Faculty of Arts and Sciences, Department of Mathematics, Famagusta, via Mersin 10, Turkey*

Abstract. This study is about degenerate Hermite Appell polynomials in three variables or Δ_h-Hermite Appell polynomials which include both discrete and degenerate cases. After we recall the definition of these polynomials and special cases, we investigate some properties of them such as recurrence relation, lowering operators (LO), raising operators (RO), difference equation (DE), integro-difference equation (IDE) and partial difference equation (PDE). We also obtain the explicit expression in terms of the Stirling numbers of the first kind. Moreover, we introduce $3D$- Δ_h-Hermite λ-Charlier polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Euler polynomials and $3D$-Δ_h-Hermite λ-Boole polynomials as special cases of Δ_h-Hermite Appell polynomials. Furthermore, we derive the explicit representation, determinantal form, recurrence relation, LO, RO and DE for these special cases. Finally, we introduce new approximating operators based on h-Hermite polynomials in three variables and examine the weighted Korovkin theorem. The error of approximation is also calculated in terms of the modulus of continuity and Peetre's K-functional.

1. Introduction

Appell polynomials have attracted intensive interest in recent years because of their diverse application areas. Among the families of polynomials, Bernoulli, Euler, Genocchi and Hermite polynomials [1] are the best known and they have many applications in numerical analysis, asymptotic approximation and special function theory, and thanks to these applications, they have a wide range of uses in engineering and applied sciences. It is because of these application areas that many extensions of Appell polynomials such as Δ_h-Appell polynomials in [2], twice iterated Appell polynomials [3], Hermite-based Appell polynomials in [4], Laguerre-based Appell polynomials [5, 6], truncated exponential-based Appell polynomials in [7], twice iterated Δ_h-Appell polynomials [8] and Δ_h-Gould-Hopper Appell polynomials (Δ_h-GHAP) [9] have been the subject of intensive research especially in the last decade. In the literature, extensions of several structures are very important, if this extension is also a unification of the existing structures, since this

2020 *Mathematics Subject Classification*. Primary 33C65; Secondary 11B68, 11B83
Keywords. Δ_h-Appell polynomials; Δ_h-Gould-Hopper Appell polynomials; Partial difference equations; Integro difference equations; h-Hermite operators; Modulus of continuity.
Received: 11 September 2022; Accepted: 28 Februaary 2023
Communicated by Miodrag Spalević
* Corresponding author: Mehmet Ali Özarslan
Email addresses: gizembaran@csu.edu.tr (Gizem Baran), zeynepozat95@gmail.com (Zeynep Özat), bayramcekim@gazi.edu.tr (Bayram Çekim), mehmetali.ozarslan@emu.edu.tr (Mehmet Ali Özarslan)

unification focuses researchers on the investigation of advanced properties rather than focusing on the investigation of modified families having similar properties to the existing area.

Hermite-based special numbers and polynomials are based on the work done by Milne-Thomson, in 1933 [10]. Taking into account Milne-Thomson polynomials, Dere et al. have done research in from 2011 to 2013 [11–13]. Later, in 2019, Milne-Thomson type polynomials and numbers were defined by Simsek, ending the generalization of almost all Hermite-based numbers [14]. Moreover, in 2020, Kilar and Simsek built the special numbers and polynomials based on multivariate Hermite polynomials [15]. For some of the work and done in this field, see papers [16–21].

It is the main purpose of this paper to examine $3D$-Δ_h-Hermite Appell polynomials, which unify (therefore include) existing Appell polynomials such as usual Appell polynomials, Δ_h-Appell polynomials, bivariate Appell polynomials, Hermite-based Appell polynomials and Δ_h-GHAP, etc. ([2, 4, 9, 22–30]). Degenerate Hermite Appell polynomials in three variables first introduced and studied in Baran's Ph.D thesis in September 2021 [31] and later, by Riyaset et al. in November 2022 [32]. The degenerate Hermite Appell polynomials in three variables $\mathcal{A}_j(x,y,z;h) := \mathcal{A}_j^h$ introduced in [31, 32] via the generating relation

$$\vartheta(t)\left(1+ht^3\right)^{\frac{z}{h}}\left(1+ht^2\right)^{\frac{y}{h}}(1+ht)^{\frac{x}{h}} = \sum_{j=0}^{\infty}\mathcal{A}_j(x,y,z;h)\frac{t^j}{j!}, \tag{1}$$

where $\mathcal{A}_j(0,0,0;h) = \lambda_{j,h}$ $(j = 0,1,2,\cdots)$ are the degenerate numbers given by the series

$$\vartheta(t) = \sum_{i=0}^{\infty}\lambda_{i,h}\frac{t^i}{i!},\ \ \lambda_{0,h}\neq 0. \tag{2}$$

Throughout the paper, we assume that the conditions $|ht|<1$, $\left|ht^2\right|<1$, $\left|ht^3\right|<1$ are satisfied and if there is a need for additional condition on the variable t which might come from $\vartheta(t)$ will be mentioned in addition. Δ_h is the finite difference operator (see for instance [33]) given by

$$\Delta_h[k](x) = k(x+h) - k(x),\ h\in\mathbb{R}^+.$$

We have from [33] that, the rate of m $(m\in\mathbb{N})$ the operator finite difference can be derived from the above definition to give

$$\Delta^m[f](x) = \Delta\left(\Delta^{m-1}[f](x)\right) = \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}f(x+rh) \tag{3}$$

and $\Delta^0 : I$, $\Delta^1 : \Delta$, I is the identity operator. $3D$-Δ_h-Hermite Appell polynomials provide the following difference operator properties [31]:

$$_x\Delta_h\left(\mathcal{A}_j^h\right) = jh\mathcal{A}_{j-1}^h, \tag{4}$$

$$_y\Delta_h\left(A_j^h\right) = j(j-1)h\mathcal{A}_{j-2}^h \tag{5}$$

and

$$_z\Delta_h\left(\mathcal{A}_j^h\right) = j(j-1)(j-2)h\mathcal{A}_{j-3}^h. \tag{6}$$

It can be seen from equalities (4)-(6) that $3D$-Δ_h-Hermite Appell polynomials satisfy the following difference equations [31]:

$$\left(h\,{}_y\Delta_h - {}_x\,\Delta_h^2\right)\mathcal{A}_j^h = 0 \tag{7}$$

and

$$\left(h^2\,{}_z\Delta_h - {}_x\,\Delta_h^3\right)\mathcal{A}_j^h = 0. \tag{8}$$

It should be noted that $3D$-Δ_h-Hermite Appell type polynomials include the following polynomial families:

- Taking $z = 0$ in (1), we have the Δ_h-GHAP [9]

$$\vartheta(t)\left(1+ht^2\right)^{\frac{y}{h}}(1+ht)^{\frac{x}{h}} = \sum_{j=0}^{\infty} A_j(x,y;h)\frac{t^j}{j!}, \tag{9}$$

the numbers $A_j(0,0;h) := \vartheta_{j,h}$ $(j = 0,1,2,...)$ which are degenerate numbers in (2).

Özarslan and Yaşar [9] obtained explicit representation, determinantal form, recurrence relation, shift operators, DE and PDE for Δ_h-GHAP.

- Taking the limit $h \to 0$ in (1), we get the Hermite-based Appell polynomials [4, 22]

$$\vartheta(t)\exp\left(xt+yt^2+zt^3\right) = \sum_{j=0}^{\infty} {}_A\mathcal{H}_j(x,y,z)\frac{t^j}{j!}.$$

In [22], for Hermite-based Appell polynomials, they found DE, IDE and PDE by using the factorization method. Also they derived the Hermite-based Bernoulli polynomials, the Hermite-based Euler polynomials and the Hermite-based Genocchi polynomials.

In the case $\vartheta(t) = 1$, we recover the $3D$-Hermite polynomials [34],

$$\exp\left(xt+yt^2+zt^3\right) = \sum_{j=0}^{\infty} \mathcal{H}_j(x,y,z)\frac{t^j}{j!}.$$

- Taking $y = z = 0$, we get Δ_h-Appell polynomials [23]. In [2], the power series expansion of $\vartheta(t)(1+ht)^{\frac{x}{h}}$ is used for defining Δ_h-Appell polynomial were defined

$$\sum_{j=0}^{\infty} \mathcal{A}_j(x;h)\frac{t^j}{j!} = \vartheta(t)(1+ht)^{\frac{x}{h}} = \mathcal{A}_0(x;h) + \frac{t}{1!}\mathcal{A}_1(x;h) + ... + \frac{t^j}{j!}\mathcal{A}_j(x;h) + ..., \tag{10}$$

where $\vartheta(t)$ is the power series of t given as

$$\vartheta(t) = \vartheta_{0,h} + \frac{t}{1!}\vartheta_{1,h} + \frac{t^2}{2!}\vartheta_{2,h} + ... + \frac{t^j}{j!}\vartheta_{j,h} + ..., \ \vartheta_{0,h} \neq 0.$$

In [2], Costabile and Longo defined Δ_h-Appell sequences by

$$\Delta_h\left(\mathcal{A}_j(x)\right) = jh\mathcal{A}_{j-1}(x).$$

- Taking $h \to 1$ in (1), we get Δ-Appell polynomials in three variables as

$$\vartheta(t)\left(1+t^3\right)^z\left(1+t^2\right)^y(1+t)^x = \sum_{j=0}^{\infty} A_j(x,y,z)\frac{t^j}{j!},$$

where in the case $\vartheta(t) = \exp(-at)$ and then taking $y = z = 0$, we obtain Charlier polynomials $C_j^{(a)}(x)$ in [37]

$$\exp(-at)(1+t)^x = \sum_{n=0}^{\infty} C_j^{(a)}(x)\frac{t^j}{j!}, \quad a \neq 1.$$

- Setting $z = 0$ and then letting $h \to 0$, we obtain bivariate Appell polynomials [24]

$$\vartheta(t)e^{xt+yt^2} = \sum_{j=0}^{\infty} R_j^{(2)}(x,y)\frac{t^j}{j!}$$

- Taking $y = z = 0$ and then letting $h \to 0$, we get the usual Appell polynomials which were investigated extensively from different aspects. Appell polynomials [35, 36] are defined via the generating relation as

$$\vartheta(t) e^{xt} = \sum_{j=0}^{\infty} A_j(x) \frac{t^j}{j!}, \tag{11}$$

where $\vartheta(t)$ is a formal power series as follows

$$\vartheta(t) = \sum_{i=0}^{\infty} \lambda_i \frac{t^i}{i!}, \quad \lambda_0 \neq 0. \tag{12}$$

An equivalent definition of the Appell polynomials is the following

$$D_x \left(A_j(x) \right) = j A_{j-1}(x), \; j = 1, 2, 3, ...,$$

where $D_x := \frac{d}{dx}$ is the derivative operator.

- Taking $\vartheta(t) = 1$, we have h-Hermite polynomials in three variables in [31] defined by

$$(1 + ht)^{\frac{x}{h}} \left(1 + ht^2\right)^{\frac{y}{h}} \left(1 + ht^3\right)^{\frac{z}{h}} = \sum_{j=0}^{\infty} G_j^h(x, y, z) \frac{t^j}{j!}. \tag{13}$$

Note that these polynomials include three variable Hermite polynomials as $h \to 0$, Δ_h-GHAP for $z = 0$, bivariate Hermite polynomials for $z = 0$ and $h \to 0$.

By making special choices of $\vartheta(t)$, we construct some new families of polynomials in three variables. The special cases that we consider in the present investigation are $3D$- Δ_h-Hermite λ-Charlier polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Euler polynomials and $3D$-Δ_h-Hermite λ-Boole polynomials

We organize the paper as follows:
In section 2, after we recall $3D$-h-Hermite polynomials and their explicit representation, we investigate some properties of $3D$-Δ_h-Hermite Appell polynomials recurrence relation, LO, RO, DE, IDE and PDE. In section 3, we obtain an explicit representation in terms of the Stirling numbers of the first kind and some addition formulas. In section 4, we consider some special examples of $3D$-Δ_h-Hermite Appell polynomials. More precisely, we introduce $3D$- Δ_h-Hermite λ-Charlier polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Euler polynomials and $3D$-Δ_h-Hermite λ-Boole polynomials and obtain their explicit representation, determinantal form, recurrence relation, lowering and raising operators, difference equations. In section 5, we construct new operators based on h-Hermite polynomials in three variables and examine the weighted Korovkin theorem for them. We also obtain the error of approximation with the help of Peetre's K-functional and modulus of continuity.

2. 3*D*-*h*-Hermite Appell polynomials and their properties

The properties given in this section were studied in a part of Baran's doctoral thesis [31]. For the completion of the paper, we recall the results with their proofs. We should also mention that Theorem 2.6 is new result.

We consider the $3D$-h-Hermite polynomials $G_n^h(x, y, z) := G_n^h$ defined in (13) since they are the usual members of the $3D$-h-Hermite Appell polynomials (in the case $\vartheta(t) = 1$), satisfy the following difference operator properties:

$$ {}_x\Delta_h \left(G_j^h \right) = hjG_{j-1}^h, \tag{14}$$

$$ {}_y\Delta_h \left(G_j^h \right) = hj(j-1) G_{j-2}^h \tag{15}$$

and

$$ {}_{z}\Delta_{h}\left(G_{j}^{h}\right)=hj\left(j-1\right)\left(j-2\right)G_{j-3}^{h}. \tag{16}$$

Therefore $3D$-h-Hermite polynomials satisfy the following difference equations

$$\left(h\,{}_{y}\Delta_{h}-{}_{x}\Delta_{h}^{2}\right)G_{j}^{h}=0 \tag{17}$$

and

$$\left(h^{2}\,{}_{z}\Delta_{h}-{}_{x}\Delta_{h}^{3}\right)G_{j}^{h}=0. \tag{18}$$

The polynomial G_{j}^{h} have the following explicit representation

$$G_{j}^{h}=\sum_{l=0}^{\left[\frac{j}{3}\right]}\sum_{s=0}^{\left[\frac{j-3l}{2}\right]}\binom{j-3l}{2s}\binom{j}{3l}(x)_{j-2s-3l}^{h}(y)_{s}^{h}(z)_{l}^{h}\frac{(2s)!}{s!}\frac{(3l)!}{l!} \tag{19}$$

where

$$(\eta)_{j}^{h}=\left(-\frac{\eta}{h}\right)_{j}(-h)^{j},$$

with

$$(\eta)_{j}=\eta\left(\eta+1\right)\left(\eta+2\right)\cdots\left(\eta+j-1\right)=\frac{\Gamma\left(\eta+j\right)}{\Gamma\left(\eta\right)},\quad j\geq 1,\quad (\eta)_{0}:=1.$$

The polynomial sequence $\left\{\mathcal{A}_{j}\left(x,y,z;h\right)\right\}_{j\in\mathbb{N}}$ has the following explicit representation

$$\mathcal{A}_{j}^{h}=\sum_{s=0}^{j}\binom{j}{s}\sum_{l=0}^{\left[\frac{s}{3}\right]}\sum_{m=0}^{\left[\frac{s-3l}{2}\right]}\lambda_{j-s,h}\binom{s-3l}{2m}\binom{s}{3l}(x)_{s-2m-3l}^{h}(y)_{m}^{h}(z)_{l}^{h}\frac{(2m)!}{m!}\frac{(3l)!}{l!} \tag{20}$$

where

$$\vartheta\left(t\right)=\sum_{j=0}^{\infty}\lambda_{j,h}\frac{t^{j}}{j!}.$$

For $j=0,1,2,\cdots$, the determinantal form of the $3D$-Δ_h-Hermite Appell polynomials is given by

$$\mathcal{A}_{j}^{h}=\frac{(-1)^{j}}{\left(\gamma_{0,h}\right)^{j+1}}\begin{vmatrix} G_{0}^{h} & G_{1}^{h} & G_{2}^{h} & \cdots & G_{j-1}^{h} & G_{j}^{h}\\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{j-1,h} & \gamma_{j,h}\\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{j-1}{1}\gamma_{j-2,h} & \binom{j}{1}\gamma_{j-1,h}\\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{j-1}{2}\gamma_{j-3,h} & \binom{j}{2}\gamma_{j-2,h}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{j}{j-1}\gamma_{1,h} \end{vmatrix} \tag{21}$$

where the coefficients of the Maclaurin series of $\frac{1}{\vartheta(t)}$ are the numbers $\gamma_{s,h}$ $(s=0,1,2,\cdots)$ and

$$\frac{1}{\vartheta(t)}=\sum_{s=0}^{\infty}\gamma_{s,h}\frac{t^s}{s!}, \tag{22}$$

$$G_j^h=\sum_{s=0}^{j}\binom{j}{s}\gamma_{s,h}\mathcal{A}_{j-s}^h. \tag{23}$$

In the following theorem, we obtain the recurrence equation of the $3D$-Δ_h-Hermite Appell polynomials.

Theorem 2.1. *3D-h-Hermite Appell polynomials have the following recurrence relation*

$$\begin{aligned}\mathcal{A}_{j+1}^h &= (x+\beta_{0,h})\mathcal{A}_j^h+\sum_{s=0}^{j-1}\binom{j}{s}\beta_{j-s,h}\mathcal{A}_s^h+xj!\sum_{s=1}^{j}(-h)^s\frac{\mathcal{A}_{j-s}^h}{(j-s)!}\\ &+2jy\mathcal{A}_{j-1}^h+2yj!\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{\mathcal{A}_{j-2s-1}^h}{(j-2s-1)!}+3zj(j-1)\mathcal{A}_{j-2}^h\\ &+3zj!\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{\mathcal{A}_{j-3s-2}^h}{(j-3s-2)!},\quad j\geq 2\end{aligned} \tag{24}$$

with

$$\mathcal{A}_{-s}^h=0,\ \ (s=1,2,\cdots),$$

where

$$\frac{\vartheta'(t)}{\vartheta(t)}=\sum_{s=0}^{\infty}\beta_{s,h}\frac{t^s}{s!}. \tag{25}$$

Proof. Differentiating (1) wrt t, we get

$$\begin{aligned}\sum_{j=0}^{\infty}\mathcal{A}_{j+1}^h\frac{t^j}{j!} &= \frac{\vartheta'(t)}{\vartheta(t)}\vartheta(t)(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}+\frac{1}{1+ht}x\vartheta(t)(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}\\ &+2y\frac{t}{1+ht^2}\vartheta(t)(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}+3z\frac{t^2}{1+ht^3}\vartheta(t)(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}.\end{aligned} \tag{26}$$

Considering the expansions

$$\frac{1}{1+ht}=\sum_{s=0}^{\infty}(-ht)^s,\quad \frac{1}{1+ht^2}=\sum_{s=0}^{\infty}(-1)^s h^s t^{2s},\quad \frac{1}{1+ht^3}=\sum_{s=0}^{\infty}(-1)^s h^s t^{3s},$$

in (26), we obtain

$$\begin{aligned}\sum_{j=0}^{\infty}\mathcal{A}_{j+1}^h\frac{t^j}{j!} &= \sum_{s=0}^{\infty}\beta_{s,h}\frac{t^s}{s!}\sum_{j=0}^{\infty}\mathcal{A}_j^h\frac{t^j}{j!}+x\sum_{s=0}^{\infty}(-ht)^s\sum_{j=0}^{\infty}\mathcal{A}_j^h\frac{t^j}{j!}\\ &+2y\sum_{s=0}^{\infty}(-1)^s h^s t^{2k}\sum_{j=0}^{\infty}\mathcal{A}_j^h\frac{t^{j+1}}{j!}+3z\sum_{s=0}^{\infty}(-1)^s h^s t^{3s}\sum_{j=0}^{\infty}\mathcal{A}_j^h\frac{t^{j+2}}{j!}.\end{aligned}$$

Then using Cauchy product rule in the above equation, we get

$$\begin{aligned}\mathcal{A}_1^h+\mathcal{A}_2^h t+\sum_{j=2}^{\infty}A_{j+1}^h\frac{t^j}{j!} &= (x+\beta_{0,h})A_0^h+\sum_{j=2}^{\infty}\sum_{s=0}^{j}\binom{j}{s}\beta_{s,h}\mathcal{A}_{j-s}^h\frac{t^j}{j!}(x+\beta_{0,h})\mathcal{A}_1^h t+(\beta_{1,h}-xh+2y)\mathcal{A}_1^h t\\ &+x\sum_{j=2}^{\infty}\sum_{s=0}^{j}(-h)^s\frac{j!}{(j-s)!}\mathcal{A}_{j-s}^h\frac{t^j}{j!}+2y\sum_{j=2}^{\infty}\sum_{s=0}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{j!}{(j-2s-1)!}\mathcal{A}_{j-2s-1}^h\frac{t^j}{j!}\\ &+3z\sum_{j=2}^{\infty}\sum_{s=0}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{j!}{(j-3s-2)!}\mathcal{A}_{j-3s-2}^h\frac{t^j}{j!},\ j\geq 2.\end{aligned} \tag{27}$$

Besides, by (20), we have the equalities $A_0^h=\lambda_{0,h}$, $\mathcal{A}_1^h=\lambda_{1,h}+\lambda_0 x$ and $\mathcal{A}_1^h=\lambda_{2,h}+2\lambda_{1,h}+\lambda_{0,h}x(x-h)+2y\lambda_{0,h}$. Then, using the equations (2) and (27), we have

$$\mathcal{A}_1^h+\mathcal{A}_2^h t=(x+\beta_{0,h})A_0^h+(x+\beta_{0,h})\mathcal{A}_1^h t+(\beta_{1,h}-xh+2y)\mathcal{A}_1^h t. \tag{28}$$

Using (28) in (27), if polynomial equality is used in terms of $\frac{t^j}{j!}$, we get the recurrence relation for the $3D$-Δ_h-Hermite Appell polynomials. □

Remark 2.2. *Since the last four terms of the right hand side of* (24) *can be written as*

$$\begin{aligned}&2jy\mathcal{A}_{j-1}^h+2yj!\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{\mathcal{A}_{j-2s-1}^h}{(j-2s-1)!}+3zj(j-1)\mathcal{A}_{j-2}^h+3zj!\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{\mathcal{A}_{j-3s-2}^h}{(j-3s-2)!}\\ &=2yj!\sum_{s=0}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{\mathcal{A}_{j-2s-1}^h}{(j-2s-1)!}+3zj!\sum_{s=0}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{\mathcal{A}_{j-3s-2}^h}{(j-3s-2)!},\end{aligned}$$

it should be remarked that (24) *is valid for* $j\geq 2$. *Similar consideration will be taken into account in the rest of the paper.*

Theorem 2.3. *For the* $3D$-Δ_h*-Hermite Appell polynomials, we have the LO as*

$${}_xL_j^-=\frac{1}{jh}\,{}_x\Delta_h,$$

and the RO as

$$\begin{aligned}{}_xL_j^+ &= x+\beta_{0,h}+\sum_{s=0}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!h^{j-s}}\,{}_x\Delta_h^{j-s}+x\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s\\ &+2y\frac{{}_x\Delta_h}{h}+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+1}}\,{}_x\Delta_h^{2s+1}+3z\frac{{}_x\Delta_h^2}{h^2}+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+2}}\,{}_x\Delta_h^{3s+2},\ j\geq 2.\end{aligned} \tag{29}$$

The difference equation satisfied by the $3D$-Δ_h*-Hermite Appell polynomials is given by*

$$\begin{aligned}&\left[\left(\frac{x}{h}+1+\frac{\beta_{0,h}}{h}\right){}_x\Delta_h+\sum_{s=1}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!h^{j-s+1}}\,{}_x\Delta_h^{j-s+1}+\left(\frac{x}{h}+1\right)\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^{s+1}\right.\\ &\left.+\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s+2y\frac{{}_x\Delta_h^2}{h^2}+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+2}}\,{}_x\Delta_h^{2s+2}+3z\frac{{}_x\Delta_h^3}{h^3}+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+3}}\,{}_x\Delta_h^{3s+3}-j\right]\mathcal{A}_j^h=0,\quad j\geq 2.\end{aligned} \tag{30}$$

Proof. The LO can be seen easily from (4). To give the RO, we write $\mathcal{A}_s^h$, $\mathcal{A}_{j-s}^h$, $\mathcal{A}_{j-1}^h$, $\mathcal{A}_{j-2s-1}^h$, $\mathcal{A}_{j-2}^h$ and $\mathcal{A}_{j-3s-2}^h$ in terms of the lowering operator as follows

$$\begin{aligned}
\mathcal{A}_s^h &= \left[{}_xL_{s+1}^- {}_xL_{s+2}^- \cdots {}_xL_j^-\right]\mathcal{A}_j^h \\
&= \left[\frac{1}{(s+1)h}\,{}_x\Delta_h \frac{1}{(s+2)h}\,{}_x\Delta_h \cdots \frac{1}{jh}\,{}_x\Delta_h\right]\mathcal{A}_j^h \\
&= \left[\frac{s!}{j!h^{j-s}}\,{}_x\Delta_h^{j-s}\right]\mathcal{A}_j^h, \\
\mathcal{A}_{j-s}^h &= \frac{(j-s)!}{j!h^s}\,{}_x\Delta_h^s\left(\mathcal{A}_j^h\right), \\
\mathcal{A}_{j-1}^h &= \frac{1}{jh}\,{}_x\Delta_h\left(\mathcal{A}_j^h\right), \\
\mathcal{A}_{j-2s-1}^h &= \frac{(j-2s-1)!}{j!h^{2s+1}}\,{}_x\Delta_h^{2s+1}\left(\mathcal{A}_j^h\right), \\
\mathcal{A}_{j-2}(x,y,z;h) &= \frac{1}{j(j-1)h^2}\,{}_x\Delta_h^2\left(\mathcal{A}_j^h\right), \\
\mathcal{A}_{j-3s-2}^h &= \frac{(j-3k-2)!}{j!h^{3s+2}}\,{}_x\Delta_h^{3s+2}\left(\mathcal{A}_j^h\right).
\end{aligned}$$

Using these relations in (24), we get

$$\begin{aligned}
\mathcal{A}_{j+1}^h &= (x+\beta_{0,h})\mathcal{A}_j^h + \sum_{s=0}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!h^{j-s}}\,{}_x\Delta_h^{j-s}\left(\mathcal{A}_j^h\right) + x\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s\left(\mathcal{A}_j^h\right) + 2y\frac{{}_x\Delta_h\left(\mathcal{A}_j^h\right)}{h} \\
&\quad +2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^k}{h^{s+1}}\,{}_x\Delta_h^{2s+1}\left(\mathcal{A}_j^h\right) + 3z\frac{{}_x\Delta_h^2\left(\mathcal{A}_j^h\right)}{h^2} + 3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+2}}\,{}_x\Delta_h^{3s+2}\left(\mathcal{A}_j^h\right), \quad n\geq 2. \qquad (31)
\end{aligned}$$

Hence, we have the equation (29).
To obtain the DE, we apply the factorization method, which is given simply by

$$ {}_xL_{j+1}^-\left({}_xL_j^+\mathcal{A}_j^h\right) = \mathcal{A}_j^h. $$

Then with the help of the product rule

$$ {}_x\Delta_h\left(u(x)\,v(x,y,z)\right) = u(x+h)\ {}_x\Delta_h v(x,y,z) + v(x,y,z)\ {}_x\Delta_h u(x), $$

we can write that difference equation (30). So the resultant equation gives the claim. □

Theorem 2.4. *3D-Δ_h-Hermite Appell polynomials satisfy the following integro-LO, integro-RO, and integro-DE*

$$\begin{aligned}
{}_xL_j^- &= \frac{1}{j}\,{}_x\Delta_h^{-1}\,{}_y\Delta_h, \\
{}_xL_j^+ &= x+\beta_{0,h}+\sum_{s=0}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!}\,{}_x\Delta_h^{-(j-s)}\,{}_y\Delta_h^{j-s} + x\sum_{s=1}^{n}(-h)^s\,{}_x\Delta_h^{-s}\,{}_y\Delta_h^s \\
&\quad +2y\,{}_x\Delta_h^{-1}\,{}_y\Delta_h + 2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\,{}_x\Delta_h^{-(2s+1)}\,{}_y\Delta_h^{2s+1} + 3z\,{}_x\Delta_h^{-2}\,{}_y\Delta_h^2 \\
&\quad +3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\,{}_x\Delta_h^{-(3s+2)}\,{}_y\Delta_h^{3s+2}, \quad j\geq 2,
\end{aligned}$$

$$\left[(x+\beta_{0,h}) \frac{{}_y\Delta_h}{h} + \frac{1}{h}\sum_{s=1}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!}\, {}_x\Delta_h^{-(j-s)}\, {}_y\Delta_h^{j-s+1} + x\sum_{s=1}^{j} (-1)^s h^{s-1}\, {}_x\Delta_h^{-s}\, {}_y\Delta_h^{s+1} + 2\left(\frac{y}{h}+1\right) {}_x\Delta_h^{-1}\, {}_y\Delta_h^{2} \right.$$
$$+2\, {}_x\Delta_h^{-1}\, {}_y\Delta_h + 2\left(\frac{y}{h}+1\right)\sum_{s=1}^{\left[\frac{j-1}{2}\right]} (-h)^s\, {}_x\Delta_h^{-(2s+1)}\, {}_y\Delta_h^{2s+2} + 2\sum_{s=1}^{\left[\frac{j-1}{2}\right]} (-h)^s\, {}_x\Delta_h^{-(2s+1)}\, {}_y\Delta_h^{2s+1}$$
$$\left. +\frac{3z}{h}\, {}_x\Delta_h^{-2}\, {}_y\Delta_h^{3} + 3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]} (-1)^s h^{s-1}\, {}_x\Delta_h^{-(3s+2)}\, {}_y\Delta_h^{3s+3} - (j+1)\frac{{}_x\Delta_h}{h} \right] \mathcal{A}_j^h = 0, \quad j \geq 2,$$

respectively.

Proof. Applying the inverse difference operator on both sides of (4), we get

$$\begin{aligned} {}_x\Delta_h^{-1}\left(\mathcal{A}_{j-1}^h\right) &= \frac{1}{jh}\mathcal{A}_j^h, \\ {}_x\Delta_h^{-s}\left(\mathcal{A}_{j-1}^h\right) &= \frac{1}{j(j+1)\cdots(j+s-1)h^s}\mathcal{A}_{j+s-1}^h, \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{A}_s^h &= \frac{s!}{j!}\, {}_x\Delta_h^{-(j-s)}\, {}_y\Delta_h^{j-s}\left(\mathcal{A}_j^h\right), \\ \mathcal{A}_{j-s}^h &= \frac{(j-s)!}{j!}\, {}_x\Delta_h^{-s}\, {}_y\Delta_h^{s}\left(\mathcal{A}_j^h\right), \\ \mathcal{A}_{j-1}^h &= \frac{1}{j}\, {}_x\Delta_h^{-1}\, {}_y\Delta_h\left(A_j^h\right), \\ \mathcal{A}_{j-2s-1}^h &= \frac{(j-2s-1)!}{j!}\, {}_x\Delta_h^{-(2s+1)}\, {}_y\Delta_h^{2s+1}\left(\mathcal{A}_j^h\right), \\ \mathcal{A}_{j-2}^h &= \frac{1}{j(j-1)}\, {}_x\Delta_h^{-2}\, {}_y\Delta_h^{2}\left(\mathcal{A}_j^h\right), \\ \mathcal{A}_{j-3s-2}^h &= \frac{(j-3s-2)!}{j!}\, {}_x\Delta_h^{-(3s+2)}\, {}_y\Delta_h^{3s+2}\left(\mathcal{A}_j^h\right), \end{aligned}$$

by writing the above equation in the recurrence relation (24), we get the integro-raising operator.

To obtain the integro-DE, we use the factorization method

$${}_xL_{j+1}^{-}\left({}_xL_j^{+}\left(\mathcal{A}_j^h\right)\right) = \mathcal{A}_j^h,$$

and the product rule given in the proof of Theorem 2.3. □

Theorem 2.5. *3D-Δ_h-Hermite Appell polynomials satisfy the PDE*

$$\left[\left(\frac{x}{h^j}+\frac{\beta_{0,h}}{h^j}\right) {}_x\Delta_h^{j-1}\, {}_y\Delta_h + \frac{1}{h^j}\sum_{s=1}^{j}\frac{\beta_{j-s,h}}{(j-s)!}\, {}_x\Delta_h^{s-1}\, {}_y\Delta_h^{j-s+1}\right]\mathcal{A}_j^h$$
$$+\frac{1}{h^{j-1}}\sum_{i=0}^{j-1}(-1)^{j-1-i}\binom{j-1}{i}(x+ih)\sum_{s=1}^{j}(-1)^s h^{s-1}\, {}_x\Delta_h^{-s}\, {}_y\Delta_h^{s+1} A_j(x+ih,y,z;h)$$

$$+\frac{1}{h^{j-1}}\left[2\left(\frac{y}{h}+1\right){}_x\Delta_h^{j-2}\,{}_y\Delta_h^2+2\,{}_x\Delta_h^{n-2}\,{}_y\Delta_h\right.$$
$$\left.+2\left(\frac{y}{h}+1\right)\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\,{}_x\Delta_h^{-2s+j-2}\,{}_y\Delta_h^{2s+2}\right]\mathcal{A}_j(x,y,z;h)$$
$$+\frac{1}{h^{j-1}}\left[2\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\,{}_x\Delta_h^{-2s+j-2}\,{}_y\Delta_h^{2s+1}+\frac{3z}{h}\,{}_x\Delta_h^{j-3}\,{}_y\Delta_h^3\right.$$
$$\left.+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-1)^k h^{s-1}\,{}_x\Delta_h^{-3s+j-3}\,{}_y\Delta_h^{3s+3}-(j+1)\frac{{}_x\Delta_h^j}{h}\right]\mathcal{A}_j^h=0,\quad j\geq 2.$$

Proof. We use the IDE and taking $(j-1)$ multiples of the operator wrt x in (3) and use the h-summation by parts formula [9] given by

$${}_x\Delta_h^{j-1}[uv](x)=\sum_{i=0}^{j-1}(-1)^{j-1-i}\binom{j-1}{i}u(x+ih)\,v(x+ih),$$

and lastly, dividing both sides by h^{j-1}, we get the PDE for $3D$-Δ_h-Hermite Appell polynomials. □

Theorem 2.6. *$3D$-Δ_h-Hermite Appell polynomials satisfy the following integro-LO, İntegro-RO and IDE in terms of ${}_y\Delta_h^{-1}\,{}_z\Delta_h$:*

$${}_yL_j^-=\frac{1}{j}\,{}_y\Delta_h^{-1}\,{}_z\Delta_h,\tag{32}$$

$$\begin{aligned}{}_yL_j^+ &= x+\beta_{0,h}+\sum_{s=0}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!}\,{}_y\Delta_h^{-(j-s)}\,{}_z\Delta_h^{j-s}+x\sum_{s=1}^{j}(-h)^s\,{}_y\Delta_h^{-s}\,{}_z\Delta_h^s\\ &\quad+2y\,{}_y\Delta_h^{-1}\,{}_z\Delta_h+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\,{}_y\Delta_h^{-(2s+1)}\,{}_z\Delta_h^{2s+1}+3z\,{}_y\Delta_h^{-2}\,{}_z\Delta_h^2\\ &\quad+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\,{}_y\Delta_h^{-(3s+2)}\,{}_z\Delta_h^{3s+2},\quad j\geq 2,\end{aligned}\tag{33}$$

$$\begin{aligned}&\left[(x+\beta_{0,h})\frac{{}_z\Delta_h}{h}+\frac{1}{h}\sum_{s=1}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!}\,{}_y\Delta_h^{-(j-s)}\,{}_z\Delta_h^{j-s+1}+x\sum_{s=1}^{j}(-1)^s h^{s-1}\,{}_y\Delta_h^{-s}\,{}_z\Delta_h^{s+1}+\frac{2y}{h}\,{}_y\Delta_h^{-1}\,{}_z\Delta_h^{-1}\right.\\ &+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-1)^s h^{s-1}\,{}_y\Delta_h^{-(2s+1)}\,{}_z\Delta_h^{2s+2}+3\left(\frac{z}{h}+1\right){}_y\Delta_h^{-1}\,{}_z\Delta_h^2+3\,{}_y\Delta_h^{-1}\,{}_z\Delta_h^2\\ &\left.+3\left(\frac{z}{h}+1\right)\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-1)^s(h)^{s-1}\,{}_y\Delta_h^{-(3s+2)}\,{}_z\Delta_h^{3+3}+3\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-1)^s h^s\,{}_y\Delta_h^{-(3s+2)}\,{}_z\Delta_h^{3s+2}-(j+1)\frac{{}_y\Delta_h}{h}\right]\mathcal{A}_j^h=0,\quad j\geq 2,\end{aligned}\tag{34}$$

respectively.

Proof. Replacing $(j-1)$ instead of j for the integro-lowering operator

$$ {}_{y}\Delta_{h}\left(\mathcal{A}_{j}(x,y,z;h)\right)=j(j-1)h\mathcal{A}_{j-2}(x,y,z;h), $$

we get

$$ \begin{aligned} {}_{y}\Delta_{h}\left(\mathcal{A}_{j-1}(x,y,z;h)\right) &= (j-1)(j-2)h\mathcal{A}_{j-3}(x,y,z;h), \\ \mathcal{A}_{j-3}(x,y,z;h) &= \frac{1}{(j-1)(j-2)h}\,{}_{y}\Delta_{h}\left(\mathcal{A}_{j-1}(x,y,z;h)\right) \end{aligned} $$

and

$$ \begin{aligned} {}_{z}\Delta_{h}\left(\mathcal{A}_{j}(x,y,z;h)\right) &= j(j-1)(j-2)h\mathcal{A}_{j-3}(x,y,z;h) \\ &= j(j-1)(j-2)h\frac{1}{(j-1)(j-2)h}\,{}_{y}\Delta_{h}\left(\mathcal{A}_{j-1}(x,y,z;h)\right), \\ {}_{z}\Delta_{h}\left(\mathcal{A}_{j}(x,y,z;h)\right) &= j\,{}_{y}\Delta_{h}\left(\mathcal{A}_{j-1}(x,y,z;h)\right), \\ \mathcal{A}_{j-1}(x,y,z;h) &= \frac{1}{j}\,{}_{y}\Delta_{h}^{-1}\,{}_{z}\Delta_{h}\left(\mathcal{A}_{j}(x,y,z;h)\right). \end{aligned} $$

Then we can write the integro-lowering operator (32). Also we give the integro-raising operator

$$ \begin{aligned} \mathcal{A}_{s}(x,y,z;h) &= \left[{}_{y}L_{s+1}^{-}\,{}_{y}L_{s+2}^{-}\cdots{}_{y}L_{j}^{-}\right]\mathcal{A}_{j}(x,y,z;h) \\ &= \left[\frac{1}{s+1}\,{}_{y}\Delta_{h}^{-1}\,{}_{z}\Delta_{h}\frac{1}{s+2}\,{}_{y}\Delta_{h}^{-1}\,{}_{z}\Delta_{h}\cdots\frac{1}{j}\,{}_{y}\Delta_{h}^{-1}\,{}_{z}\Delta_{h}\right]\mathcal{A}_{j}(x,y,z;h) \\ &= \frac{s!}{j!}\,{}_{y}\Delta_{h}^{-(j-s)}\,{}_{z}\Delta_{h}^{j-s}\mathcal{A}_{j}(x,y,z;h) \\ \mathcal{A}_{s}(x,y,z;h) &= \frac{s!}{j!}\,{}_{y}\Delta_{h}^{-(j-s)}\,{}_{z}\Delta_{h}^{j-s}\mathcal{A}_{j}(x,y,z;h). \end{aligned} $$

We can write the following equations instead of $\mathcal{A}_{j-s}(x,y,z;h)$, $\mathcal{A}_{j-1}(x,y,z;h)$, $\mathcal{A}_{j-2s-1}(x,y,z;h)$, $\mathcal{A}_{j-2}(x,y,z;h)$ and $\mathcal{A}_{j-3s-2}(x,y,z;h)$

$$ \begin{aligned} \mathcal{A}_{j-s}(x,y,z;h) &= \frac{(j-s)!}{j!}\,{}_{y}\Delta_{h}^{-(j-j+s)}\,{}_{z}\Delta_{h}^{j-j+s}\mathcal{A}_{j}(x,y,z;h) \\ &= \frac{(j-s)!}{j!}\,{}_{y}\Delta_{h}^{-s}\,{}_{z}\Delta_{h}^{s}\mathcal{A}_{j}(x,y,z;h), \\ \mathcal{A}_{j-1}(x,y,z;h) &= \frac{(j-1)!}{j!}\,{}_{y}\Delta_{h}^{-(j-j+1)}\,{}_{z}\Delta_{h}^{j-j+1}\mathcal{A}_{j}(x,y,z;h) \\ &= \frac{(j-1)!}{j!}\,{}_{y}\Delta_{h}^{-1}\,{}_{z}\Delta_{h}\mathcal{A}_{j}(x,y,z;h), \\ \mathcal{A}_{j-2s-1}(x,y,z;h) &= \frac{(j-2s-1)!}{j!}\,{}_{y}\Delta_{h}^{-(2s+1)}\,{}_{z}\Delta_{h}^{2s+1}\mathcal{A}_{j}(x,y,z;h), \\ \mathcal{A}_{j-2}(x,y,z;h) &= \frac{(j-2)!}{j!}\,{}_{y}\Delta_{h}^{-2}\,{}_{z}\Delta_{h}^{2}\mathcal{A}_{j}(x,y,z;h), \\ \mathcal{A}_{j-3s-2}(x,y,z;h) &= \frac{(j-3s-2)!}{j!}\,{}_{y}\Delta_{h}^{-(3s+2)}\,{}_{z}\Delta_{h}^{3s+2}\mathcal{A}_{j}(x,y,z;h). \end{aligned} $$

Using the recurrence relation

$$\begin{aligned}\mathcal{A}_{j+1}(x,y,z;h) &= (x+\beta_{0,h})\mathcal{A}_j(x,y,z;h)+\sum_{s=0}^{j-1}\frac{\beta_{j-s,h}}{(j-s)!}\,{}_y\Delta_h^{-(j-s)}\,{}_z\Delta_h^{j-s}\mathcal{A}_j(x,y,z;h)\\ &+x\sum_{s=1}^{j}(-h)^s\,{}_y\Delta_h^{-s}\,{}_z\Delta_h^{s}\mathcal{A}_j(x,y,z;h)+2y\,{}_y\Delta_h^{-1}\,{}_z\Delta_h\mathcal{A}_j(x,y,z;h)\\ &+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\,{}_y\Delta_h^{-(2s+1)}\,{}_z\Delta_h^{2s+1}\mathcal{A}_j(x,y,z;h)+3z\,{}_y\Delta_h^{-2}\,{}_z\Delta_h^{2}\mathcal{A}_j(x,y,z;h)\\ &+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\,{}_y\Delta_h^{-(3s+2)}\,{}_z\Delta_h^{3s+2}\mathcal{A}_j(x,y,z;h),\quad j\geq 2,\end{aligned}\tag{35}$$

we can write that the integro-raising operator (33). For the integro-difference equation, in (35) applying to both sides the following equation in [9]

$${}_z\Delta_h\left(f(z)\,g(y,z)\right)=f(z+h)\,{}_z\Delta_h g(y,z)+g(y,z)\,{}_z\Delta_h f(z).$$

We can write the integro-difference equation (34). □

3. Some summation formulas for $3D$-Δ_h-Hermite Appell polynomials

Now, we investigate the connections between the Stirling numbers of the first kind and $3D$-Δ_h-Hermite Appell polynomials. We recall that the Stirling numbers $S(i,k)$ of the first kind are defined the following generating function [38]

$$\{\log(1+z)\}^k=k!\sum_{i=k}^{\infty}S(i,k)\frac{z^i}{i!},\quad |z|<1.$$

From the definition of $S(i,k)$ [38], the Pochhammer symbol can be written as

$$(z)_i=\sum_{k=0}^{i}(-1)^{i-k}S(i,k)\,z^k.\tag{36}$$

Theorem 3.1. *Δ_h-GHAP can be represented by*

$$\mathcal{A}_j(x,y;h)=\sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{s=0}^{\left[\frac{j-2l}{2}\right]}\frac{j!}{(j-2s-2l)!\,(s+l)!}\mathcal{A}_{j-2s-2l}(x;h)\,h^sS(s+l,l)\,y^l.\tag{37}$$

Proof. Using the definition of Δ_h-Appell polynomials (10) and Δ_h-GHAP (9), we have

$$\begin{aligned}\sum_{j=0}^{\infty}\mathcal{A}_j(x,y;h)\frac{t^j}{j!} &= \vartheta(t)\left(1+ht^2\right)^{\frac{y}{h}}(1+ht)^{\frac{x}{h}}\\ &= \left(\sum_{j=0}^{\infty}\mathcal{A}_j(x;h)\frac{t^j}{j!}\right)\left(\sum_{s=0}^{\infty}\left(-\frac{y}{h}\right)_s(-h)^s\frac{t^{2s}}{s!}\right).\end{aligned}$$

Using Cauchy product rule and then if polynomial equality is used in terms of $\frac{t^j}{j!}$ in the resultant equation, we get

$$\mathcal{A}_j(x,y;h) = \sum_{s=0}^{\left[\frac{j}{2}\right]} \frac{j!}{(j-2s)!s!}(-h)^s \mathcal{A}_{j-2s}(x;h)\left(-\frac{y}{h}\right)_s.$$

Using in the above equality (36), we get

$$\begin{aligned}\mathcal{A}_j(x,y;h) &= \left(\sum_{s=0}^{\left[\frac{j}{2}\right]} \frac{j!}{(j-2s)!s!}(-h)^s \mathcal{A}_{j-2s}(x;h)\right)\left(\sum_{l=0}^{s}(-1)^{s-l}S(s,l)(-h)^{-l}y^l\right)\\ &= \sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{s=l}^{\left[\frac{j}{2}\right]} \frac{j!}{(j-2s)!s!}(-h)^{s-l}\mathcal{A}_{j-2s}(x;h)S(s,l)(-1)^{s-l}y^l\\ &= \sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{s=0}^{\left[\frac{j-2l}{2}\right]} \frac{j!}{(j-2s-2l)!(s+l)!}(-h)^s\\ &\quad\times\mathcal{A}_{j-2s-2l}(x;h)(-1)^s S(s+l,l)y^l.\end{aligned}$$

Whence the result. □

Theorem 3.2. *The 3D-Δ_h-Hermite Appell polynomials have the following representation*

$$\begin{aligned}\mathcal{A}_j^h = &\sum_{v=0}^{\left[\frac{j}{3}\right]}\sum_{m=0}^{\left[\frac{j-3v}{3}\right]} \frac{j!}{(j-3m-3v)!(m+v)!}\sum_{l=0}^{\left[\frac{j-3m-3v}{2}\right]}\sum_{s=0}^{\left[\frac{j-3m-3v-2l}{2}\right]}\frac{(j-3m-3v)!}{(j-3m-3v-2s-2l)!}\\ &\times\frac{1}{(s+l)!}(h)^{s+m}\mathcal{A}_{j-3m-3v-2s-2l}(x;h)s(m+v,v)S(s+l,l)y^l z^v.\end{aligned} \tag{38}$$

Proof. Using the equation (9) in the equation (1), we have

$$\begin{aligned}\sum_{j=0}^{\infty}\mathcal{A}_j^h\frac{t^j}{j!} &= \vartheta(t)(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}\\ &= \left(\sum_{j=0}^{\infty}\mathcal{A}_j(x,y;h)\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\left(-\frac{z}{h}\right)_m(-h)^m\frac{t^{3m}}{m!}\right)\\ &= \sum_{j=0}^{\infty}\sum_{m=0}^{\left[\frac{j}{3}\right]}\frac{j!}{(j-3m)!m!}\mathcal{A}_{j-3m}(x,y;h)(-h)^m\left(-\frac{z}{h}\right)_m\frac{t^n}{n!}.\end{aligned}$$

If polynomial equality is used in terms of $\frac{t^j}{j!}$ in this equation, we obtain

$$\mathcal{A}_j^h = \sum_{m=0}^{\left[\frac{n}{3}\right]}\frac{j!}{(j-3m)!m!}\mathcal{A}_{j-3m}(x,y;h)(-h)^m\left(-\frac{z}{h}\right)_m.$$

Using (36) for the term $\left(-\frac{z}{h}\right)_m$ in the above equality, we have

$$\mathcal{A}_j^h = \sum_{v=0}^{\left[\frac{j}{3}\right]}\sum_{m=0}^{\left[\frac{n-3v}{3}\right]}\frac{j!}{(j-3m-3v)!(m+v)!}\mathcal{A}_{j-3m-3v}(x,y;h)(h)^m S(m+v,v)z^v.$$

Using the equation corresponding to $\mathcal{A}_{j-3m-3v}(x,y;h)$ from (37), we have

$$\mathcal{A}_j^h = \sum_{v=0}^{\left[\frac{j}{3}\right]}\sum_{m=0}^{\left[\frac{j-3v}{3}\right]}\frac{j!}{(j-3m-3v)!\,(m+v)!}\sum_{l=0}^{\left[\frac{j-3m-3v}{2}\right]}\sum_{s=0}^{\left[\frac{j-3m-3v-2l}{2}\right]}\frac{(j-3m-3v)!}{(j-3m-3v-2s-2l)!}$$
$$\times\frac{1}{(s+l)!}(h)^{s+m}\mathcal{A}_{j-3m-3v-2s-2l}(x;h)\,S(m+v,v)\,S(s+l,l)\,y^l z^v,$$

which completes the proof. □

Theorem 3.3. *The* 3D-Δ_h*-Hermite Appell polynomials satisfy the following identity*

$$\mathcal{A}_j^h = \sum_{l=0}^{\left[\frac{j}{3}\right]}\sum_{s=0}^{\left[\frac{j-3l}{3}\right]}\frac{j!}{(j-3s-3l)!\,(s+l)!}\mathcal{A}_{j-3s-3l}(x,y;h)\,h^s S(s+l,l)\,z^l. \tag{39}$$

Proof. *Using the definition of* 3D-Δ_h*-Hermite-Appell polynomials in* (1)*, we get*

$$\sum_{j=0}^{\infty}\mathcal{A}_j^h\frac{t^j}{j!} = \left(\sum_{j=0}^{\infty}\mathcal{A}_j(x,y;h)\frac{t^j}{j!}\right)\left(\sum_{s=0}^{\infty}\left(\frac{-z}{h}\right)_s(-h)^s\frac{t^{3s}}{s!}\right).$$

Then using the Cauchy rule, we have

$$\mathcal{A}_j^h = \sum_{s=0}^{\left[\frac{j}{3}\right]}\binom{j}{3s}\frac{(3s)!}{s!}\mathcal{A}_{j-3s}(x,y;h)\left(\frac{-z}{h}\right)_s(-h)^s.$$

Using Stirling numbers of the first kind in (36) *instead of* $\left(\frac{-z}{h}\right)_s$ *and then using Cauchy product rule, we arrive the desired result.* □

Corollary 3.4. *Taking* $z=1$ *in* (39)*, we have*

$$\mathcal{A}_j(x,y,1;h) = \sum_{l=0}^{\left[\frac{j}{3}\right]}\sum_{s=0}^{\left[\frac{j-3l}{3}\right]}\frac{j!}{(j-3s-3l)!\,(s+l)!}\mathcal{A}_{j-3s-3l}(x,y;h)\,h^s S(s+l,l).$$

Theorem 3.5. *The* 3D-Δ_h*-Hermite Appell polynomials have the following addition formulas*

$$\mathcal{A}_j(x+\omega,y,z;h) = \sum_{l=0}^{j}\sum_{s=0}^{j-l}\frac{j!}{(j-s-l)!\,(s+l)!}\mathcal{A}_{j-s-l}(x,y,z;h)\,h^s S(s,l)\,\omega^l, \tag{40}$$

$$\mathcal{A}_j(x,y+\omega,z;h) = \sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{s=0}^{\left[\frac{j-2l}{2}\right]}\frac{j!}{(j-2s-2l)!\,(s+l)!}\mathcal{A}_{j-2s-2l}(x,y,z;h)\,h^s S(s,l)\,\omega^l, \tag{41}$$

$$\mathcal{A}_j(x,y,z+\omega;h) = \sum_{l=0}^{\left[\frac{j}{3}\right]}\sum_{s=0}^{\left[\frac{j-3l}{3}\right]}\frac{j!}{(j-3s-3l)!\,(s+l)!}\mathcal{A}_{j-3s-3l}(x,y,z;h)\,h^s S(s,l)\,\omega^l. \tag{42}$$

Proof. Taking $x+\omega$ instead of x in the equation (40), we have

$$\sum_{j=0}^{\infty} A_j(x+\omega,y,z;h) = \left(\sum_{j=0}^{\infty} A_j^h \frac{t^j}{j!}\right)\left(\sum_{s=0}^{\infty} (\omega)_s^h \frac{t^s}{s!}\right).$$

Using the Cauchy rule and after comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the resulting equation, we have

$$\mathcal{A}_j(x+\omega,y,z;h) = \sum_{s=0}^{j} \binom{j}{s} \mathcal{A}_{j-s}(x,y,z;h)(\omega)_s^h.$$

Then, using (36) for $(\omega)_s^h$, we have the claim (40). In a similar way, we can obtain the equations (41) and (42). □

4. Special examples of 3D-Δ_h-Hermite Appell polynomials

We introduce some special cases of 3D-Δ_h-Hermite Appell polynomials and obtain their properties such as explicit representation, determinantal form, recurrence relation, LO, RO and DE for them. The special cases, we consider are the 3D- Δ_h-Hermite λ-Charlier polynomials, 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials, 3D-Δ_h-Hermite degenerate Apostol-Euler polynomials and 3D-Δ_h-Hermite λ-Boole polynomials

4.1. 3D-Δ_h-Hermite λ-Charlier polynomials

Here, we examine explicit representation, determinantal form, recurrence relation, LO, RO and DE provied by 3D-Δ_h-Hermite Charlier polynomials $C_{j,\lambda}^a(x,y,z;h)$.

We introduce 3D-Δ_h-Hermite λ-Charlier polynomials via the following generating function

$$\lambda^{-a^h t}(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}} = \sum_{j=0}^{\infty} C_{j,\lambda}^a(x,y,z;h)\frac{t^j}{j!}, \quad \lambda>0.$$

If $\lambda=e$, $h\to 1$, $y=0$ and $z=0$ in generating function, we obtain the Charlier polynomials.

Corollary 4.1. *3D-Δ_h-Hermite λ-Charlier polynomials satisfy the explicit representation as follows*

$$C_{j,\lambda}^a(x,y,z;h) = \sum_{s=0}^{j}\binom{j}{s}\sum_{l=0}^{\left[\frac{s}{3}\right]}\sum_{m=0}^{\left[\frac{s-3l}{2}\right]} C_{j-s,\lambda}^{a,h}\binom{s-3l}{2m}\binom{s}{3l}(x)_{s-2m-3l}^h(y)_m^h(z)_l^h\frac{(2m)!}{m!}\frac{(3l)!}{l!}$$

in which the λ-Charlier numbers $C_{j,\lambda}^a(0,0,0;h) = C_{j,\lambda}^{a,h}$ for $x=y=z=0$, $\lambda=e$, [37, 39], are given by the following series

$$\lambda^{-a^h t} = \sum_{j=0}^{\infty} C_{j,\lambda}^{a,h}\frac{t^j}{j!}.$$

The first four 3D-Δ_h-Hermite λ-Charlier polynomials are as follows.

Table 1: The first four $3D$-Δ_h-Hermite λ-Charlier polynomials.

j	$C^a_{j,\lambda}(x,y,z;h)$
0	1
1	$x - a^h \ln\lambda$
2	$x^2 - a^{2h}\ln^2\lambda - 2a^h x\ln\lambda - hx + +2y$
3	$x^3 - 3hx^2 + 2h^2x + 3xa^{2h}\ln^2\lambda + 3xa^h\ln\lambda h - 3a^h\ln\lambda x^2 - a^{3h}\ln^3\lambda - 6ya^h\ln\lambda + 6xy + 6z$

Corollary 4.2. *$3D$-Δ_h-Hermite λ-Charlier polynomials satisfy the following determinantal form*

$$C^a_{j,\lambda}(x,y,z;h) = (-1)^j \begin{vmatrix} 1 & G^h_1 & G^h_2 & \cdots & G^h_{j-1} & G^h_j \\ 1 & a^h\ln\lambda & a^{2h}\ln^2\lambda & \cdots & a^{h(j-1)}\ln^{j-1}\lambda & a^{hj}\ln^j\lambda \\ 0 & 1 & \binom{2}{1}a^h\ln\lambda & \cdots & \binom{j-1}{1}a^{h(j-2)}\ln^{j-2}\lambda & \binom{j}{1}a^{h(j-1)}\ln^{j-1}\lambda \\ 0 & 0 & 1 & \cdots & \binom{j-1}{2}a^{h(j-3)}\ln^{j-3}\lambda & \binom{j}{2}a^{h(j-2)}\ln^{j-2}\lambda \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{j}{j-1}a^h\ln\lambda \end{vmatrix},$$

where

$$\frac{1}{\lambda^{-a^h t}} = \sum_{s=0}^{\infty} a_{s,h}(\lambda)\frac{t^s}{s!} \quad \text{and} \quad G^h_j = \sum_{s=0}^{j}\binom{j}{s} a_{s,h}(\lambda)\, C^a_{j-s,\lambda}(x,y,z;h).$$

Proof. Taking $\vartheta(t) = \lambda^{-a^h t}$, it gives

$$\frac{1}{\lambda^{-a^h t}} = \sum_{s=0}^{\infty} a_{s,h}(\lambda)\frac{t^s}{s!}$$

from equation (22) *and*

$$G^h_j = \sum_{s=0}^{j}\binom{j}{s} a_{s,h}(\lambda)\, C^a_{j-s,\lambda}(x,y,z;h)$$

from equation (23). *Therefore, these equations are provided by substituting them in equation* (21) . □

Corollary 4.3. *The recurrence relation for $3D$-Δ_h-Hermite λ-Charlier polynomials is given by*

$$\begin{aligned} C^a_{j+1,\lambda}(x,y,z;h) &= \left(x - a^h\ln\lambda\right) C^a_{j,\lambda}(x,y,z;h) \\ &+ xj!\sum_{s=1}^{j}(-h)^s\frac{C^a_{j-s,\lambda}(x,y,z;h)}{(j-s)!} + 2jyC^a_{j-1,\lambda}(x,y,z;h) \\ &+ 2yj!\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{C^a_{j-2s-1,\lambda}(x,y,z;h)}{(j-2s-1)!} + 3j(j-1)z\,C^a_{j-2,\lambda}(x,y,z;h) \\ &+ 3zj!\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{C^a_{j-3s-2,\lambda}(x,y,z;h)}{(j-3s-2)!},\ j \geq 2. \end{aligned}$$

Proof. It is proved when $\lambda^{-a^h t}$ is written in case of $\vartheta(t)$ in Theorem 2.1. □

Corollary 4.4. *The 3D-Δ_h-Hermite λ-Charlier polynomials have the following LO, RO and DE*

$$\begin{aligned}
{}_xL_j^- &= \frac{1}{jh}\,{}_x\Delta_h,\\
{}_xL_j^+ &= x - a^h \ln\lambda + x\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s + 2y\frac{{}_x\Delta_h}{h} + 2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+1}}\,{}_x\Delta_h^{2s+1}\\
&\quad +3z\frac{{}_x\Delta_h^2}{h^2} + 3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+2}}\,{}_x\Delta_h^{3s+2},\ \ j\geq 2,
\end{aligned}$$

$$\begin{aligned}
&\left[\left(\frac{x}{h}+1-\frac{a^h\ln\lambda}{h}\right){}_x\Delta_h + \left(\frac{x}{h}+1\right)\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^{s+1} + \sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s + 2y\frac{{}_x\Delta_h^2}{h^2}\right.\\
&\left.+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+2}}\,{}_x\Delta_h^{2s+2} + 3z\frac{{}_x\Delta_h^3}{h^3} + 3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+3}}\,{}_x\Delta_h^{3s+3} - j\right]C_{j,\lambda}^a(x,y,z;h)=0,\quad j\geq 2,
\end{aligned}$$

respectively.

Proof. It is proved when $\lambda^{-a^h t}$ is written in case of $\vartheta(t)$ in Theorem 2.3. □

4.2. 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials

This part includes determinantal form, recurrence relation, LO, RO and DE for degenerate 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials $\mathcal{B}_j(x,y,z;\lambda;h)$.

We introduce the 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials via the following generating function

$$\frac{t}{\lambda(1+ht)^{\frac{1}{h}}-1}(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}} = \sum_{j=0}^{\infty}\mathcal{B}_j(x,y,z;\lambda;h)\frac{t^j}{j!},\quad |t|<\left|\ln\left(\frac{1}{\lambda}\right)\right|.$$

If $h\to 0$ and $y=0$, $z=0$ in generating function, we get the degenerate Apostol-Bernoulli polynomials [40].

Corollary 4.5. *The polynomials $\mathcal{B}_j(x,y,z;\lambda;h)$ have the explicit representation*

$$\mathcal{B}_j(x,y,z;\lambda;h) = \sum_{s=0}^{j}\binom{j}{s}\sum_{l=0}^{\left[\frac{s}{3}\right]}\sum_{m=0}^{\left[\frac{s-3l}{2}\right]}\mathcal{B}_{j-s,h}(\lambda)\binom{s-3l}{2m}\binom{s}{3l}(x)_{s-2m-3l}^h(y)_m^h(z)_l^h\frac{(2m)!}{m!}\frac{(3l)!}{l!}$$

in which $\mathcal{B}_j(0,0,0;\lambda;h)=\mathcal{B}_{j,h}(\lambda)$ (degenerate Apostol-Bernoulli numbers) [40] as follows

$$\frac{t}{\lambda(1+ht)^{\frac{1}{h}}-1} = \sum_{j=0}^{\infty}\mathcal{B}_{j,h}(\lambda)\frac{t^j}{j!}.$$

The first four 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials are as follows.

Table 2: The first four 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials.

j	$\mathcal{B}_j(x,y,z;\lambda;h)$
0	0
1	$\frac{1}{\lambda-1}$
2	$-\frac{2}{(\lambda-1)^2}(x+\lambda-x\lambda)$
3	$\frac{3}{(\lambda-1)^3}\left[(\lambda-1)^2x^2+(\lambda-1)(1-\lambda h-2\lambda)x+\lambda^2(1+h)+\lambda(1-h)+2y(\lambda-1)^2\right]$

Corollary 4.6. *The* 3D-Δ_h*-Hermite degenerate Apostol-Bernoulli polynomials satisfy the following determinantal form*

$$\mathcal{B}_j(x,y,z;\lambda;h)=\frac{(-1)^j}{\lambda^{j+1}}\begin{vmatrix} 0 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h \\ \lambda & \lambda\frac{(1)_2^h}{2} & \lambda\frac{(1)_3^h}{3} & \cdots & \frac{(1)_j^h}{j} & \lambda\frac{(1)_{j+1}^h}{j+1} \\ 0 & \lambda & \lambda\binom{2}{1}\frac{(1)_2^h}{2} & \cdots & \lambda\binom{j-1}{1}\frac{(1)_{j-1}^h}{j-1} & \lambda\binom{j}{1}\frac{(1)_j^h}{j} \\ 0 & 0 & \lambda & \cdots & \lambda\binom{j-1}{2}\frac{(1)_{j-2}^h}{j-2} & \lambda\binom{j}{2}\frac{(1)_{j-1}^h}{j-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & \lambda\binom{j}{j-1}\frac{(1)_2^h}{2} \end{vmatrix},$$

where

$$\frac{\lambda(1+ht)^{\frac{1}{h}}-1}{t}=\sum_{s=0}^{\infty}a_{s,h}(\lambda)\frac{t^s}{s!}\quad and\quad G_j^h=\sum_{s=0}^{j}\binom{j}{s}a_{s,h}(\lambda)\mathcal{B}_{j-s}(x,y,z;\lambda;h).$$

Proof. *Taking* $\vartheta(t)=\frac{t}{\lambda(1+ht)^{\frac{1}{h}}-1}$, *it gives*

$$\frac{\lambda(1+ht)^{\frac{1}{h}}-1}{t}=\sum_{s=0}^{\infty}a_{s,h}(\lambda)\frac{t^s}{s!}$$

from equation (22) *and*

$$\frac{\lambda(1+ht)^{\frac{1}{h}}-1}{t}=\sum_{s=0}^{\infty}a_{s,h}(\lambda)\frac{t^s}{s!}\quad and\quad G_j^h=\sum_{s=0}^{j}\binom{j}{s}a_{s,h}(\lambda)\mathcal{B}_{j-s}(x,y,z;\lambda;h)$$

from equation (23). *Therefore, these equations are proved by substituting them in equation* (21). □

Corollary 4.7. *3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials satisfy the following recurrence relation*

$$\begin{aligned}
&\left(1+\frac{1}{j}\right)(x-1)\mathcal{B}_j(x,y,z;\lambda;h)-\left(1+\frac{1}{j}\right)\sum_{s=0}^{j-1}\binom{j}{s}(-h)^{j-s}(j-s)!\mathcal{B}_s(x,y,z;\lambda;h)\\
&-\left(1+\frac{1}{j}\right)\sum_{s=0}^{j}\sum_{k=0}^{j-s+1}\binom{j}{s}\frac{(j-s)!\,(-h)^k}{(j-s+1-k)!}\mathcal{B}_{j-s+1-k,h}(\lambda)\ \mathcal{B}_s(x,y,z;\lambda;h)+\left(1+\frac{1}{j}\right)xj!\sum_{s=1}^{j}(-h)^s\frac{\mathcal{B}_{j-s}(x,y,z;\lambda;h)}{(j-s)!}\\
&+\left(1+\frac{1}{j}\right)2jy\mathcal{B}_{j-1}(x,y,z;\lambda;h)+\left(1+\frac{1}{j}\right)2yj!\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{\mathcal{B}_{j-2s-1}(x,y,z;\lambda;h)}{(j-2s-1)!}\\
&+\left(1+\frac{1}{j}\right)3j(j-1)z\mathcal{B}_{j-2}(x,y,z;\lambda;h)+\left(1+\frac{1}{j}\right)3zj!\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{\mathcal{B}_{j-3s-2}(x,y,z;\lambda;h)}{(j-3s-2)!}\\
&=\mathcal{B}_{j+1}(x,y,z;\lambda;h),\quad j\geq 2,
\end{aligned}$$

where

$$\frac{t}{\lambda(1+ht)^{\frac{1}{h}}-1}=\sum_{j=0}^{\infty}\mathcal{B}_{j,h}(\lambda)\frac{t^j}{j!}.$$

Proof. It is proved when $\frac{t}{\lambda(1+ht)^{\frac{1}{h}}-1}$ is written in case of $\vartheta(t)$ in Theorem 2.1. □

Corollary 4.8. *The LO, RO and DE of the 3D-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials are*

$$\begin{aligned}
{}_xL_j^- &= \frac{1}{jh}\,{}_x\Delta_h,\\
{}_xL_j^+ &= \left(1+\frac{1}{j}\right)(x-1)-\left(1+\frac{1}{j}\right)\sum_{s=0}^{j-1}(-1)^{j-s}\,{}_x\Delta_h^{j-s}-\left(1+\frac{1}{j}\right)\sum_{s=0}^{j}\sum_{k=0}^{j-s+1}(-h)^k\frac{\mathcal{B}_{j-s+1-k,h}(\lambda)}{(j-s+1-k)!\,h^{j-s}}\,{}_x\Delta_h^{j-s}\\
&\quad+\left(1+\frac{1}{j}\right)x\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s+\left(1+\frac{1}{j}\right)2y\frac{{}_x\Delta_h}{h}+2\left(1+\frac{1}{j}\right)y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+1}}\,{}_x\Delta_h^{2s+1}+\left(1+\frac{1}{j}\right)3z\frac{{}_x\Delta_h^2}{h^2}\\
&\quad+\left(1+\frac{1}{j}\right)3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+2}}\,{}_x\Delta_h^{3s+2},\ j\geq 2,
\end{aligned}$$

$$\begin{aligned}
&\Bigg[\left(1+\frac{1}{j}\right)\left(\frac{x}{h}+1-\frac{1}{h}\right){}_x\Delta_h-\left(1+\frac{1}{j}\right)\sum_{s=0}^{j}\sum_{k=0}^{j-s+1}(-h)^k\frac{\mathcal{B}_{j-s+1-k,h}(\lambda)}{(j-s+1-k)!\,h^{j-s+1}}\,{}_x\Delta_h^{j-s+1}\\
&+\left(1+\frac{1}{j}\right)\left(\frac{x}{h}+1\right)\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^{s+1}+\left(1+\frac{1}{j}\right)\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s-\left(1+\frac{1}{j}\right)\sum_{s=0}^{j-1}(-1)^{j-s}\frac{{}_x\Delta_h^{j-s+1}}{h}+2\left(1+\frac{1}{j}\right)y\frac{{}_x\Delta_h^2}{h^2}\\
&+2\left(1+\frac{1}{j}\right)y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+2}}\,{}_x\Delta_h^{2s+2}+\left(1+\frac{1}{j}\right)3z\frac{{}_x\Delta_h^3}{h^3}\\
&+\left(1+\frac{1}{j}\right)3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+3}}\,{}_x\Delta_h^{3s+3}-\left(j-\frac{1}{j}\right)\Bigg]\mathcal{B}_j(x,y,z;\lambda;h)=0,\ j\geq 2,
\end{aligned}$$

respectively.

Proof. It is proved when $\frac{t}{\lambda(1+ht)^{\frac{1}{h}}-1}$ is written in case of $\vartheta(t)$ in Theorem 2.3. □

4.3. 3D-Δ_h-Hermite degenerate Apostol-Euler polynomials

Now, we introduce the determinantal form, recurrence relation, LO, RO and DE for 3D-Δ_h-Hermite degenerate Apostol-Euler polynomials $E_j(x, y, z; \lambda; h)$.

We introduce the 3D-Δ_h-Hermite degenerate Apostol-Euler polynomials via the following generating function

$$\frac{2}{\lambda(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}=\sum_{j=0}^{\infty}E_j(x,y,z;\lambda;h)\frac{t^j}{j!},\quad |t|<\left|\ln\left(\frac{-1}{\lambda}\right)\right|.$$

Corollary 4.9. *3D-Δ_h-Hermite degenerate Apostol-Euler polynomials have the following explicit representation*

$$E_j(x,y,z;\lambda;h)=\sum_{s=0}^{j}\binom{j}{s}\sum_{l=0}^{\left[\frac{s}{3}\right]}\sum_{m=0}^{\left[\frac{s-3l}{2}\right]}\mathcal{E}_{j-s,h}(\lambda)\binom{s-3l}{2m}\binom{s}{3l}(x)^h_{s-2m-3l}(y)^h_m(z)^h_l\frac{(2m)!}{m!}\frac{(3l)!}{l!},$$

in which $E_j(0,0,0;\lambda;h)=\mathcal{E}_{j,h}(\lambda)$ (degenerate Apostol-Euler numbers) [40] are given by

$$\frac{2}{\lambda(1+ht)^{\frac{1}{h}}+1}=\sum_{j=0}^{\infty}\mathcal{E}_{j,h}(\lambda)\frac{t^j}{j!}.$$

The first four 3D-Δ_h-Hermite degenerate Apostol-Euler polynomials are as follows.

Table 3: The first four 3D-Δ_h-Hermite degenerate Apostol-Euler polynomials.

j	$E_j(x,y,z;\lambda;h)$
0	$\frac{2}{\lambda+1}$
1	$\frac{2}{(\lambda+1)^2}(x+x\lambda-\lambda)$
2	$\frac{2}{(\lambda+1)^3}\left[(\lambda+1)^2x^2-(\lambda+1)(h+2\lambda+h\lambda)x+2y(\lambda+1)^2+\lambda^2(1+h)-\lambda(1-h)\right]$
3	$\frac{2}{(\lambda+1)^4}\left[(\lambda+1)^3x^3-3(\lambda+1)^2(\lambda+h\lambda+h)x^2+(\lambda+1)\left(2(\lambda+1)^2h^2\right)+6\lambda(\lambda+1)h+6y(\lambda+1)^2+3\lambda(\lambda-1)x\right.$ $\left.+\left(-2h^2-3h-1-6y+6z\right)\lambda^3+\left(-4h^2+4-12y+18z\right)\lambda^2+\left(-2h^2+3h-1-6y+18z\right)\lambda+6z\right]$

Corollary 4.10. *3D-Δ_h-Hermite degenerate Apostol-Euler polynomials satisfy the following determinantal form*

$$E_j(x,y,z;\lambda;h)=(-1)^j\left(\frac{2}{\lambda+1}\right)^{j+1}\begin{vmatrix}1 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h\\ \frac{(\lambda+1)}{2} & \frac{\lambda}{2} & \frac{\lambda}{2}(1)_2^h & \cdots & \frac{\lambda}{2}(1)_{j-1}^h & \frac{\lambda}{2}(1)_n^h\\ 0 & \frac{(\lambda+1)}{2} & \binom{2}{1}\frac{1}{2} & \cdots & \binom{j-1}{1}\frac{\lambda}{2}(1)_{j-2}^h & \binom{j}{1}\frac{\lambda}{2}(1)_{j-1}^h\\ 0 & 0 & \frac{(\lambda+1)}{2} & \cdots & \binom{j-1}{2}\frac{\lambda}{2}(1)_{j-3}^h & \binom{j}{2}\frac{\lambda}{2}(1)_{j-2}^h\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{(\lambda+1)}{2} & \binom{j}{j-1}\frac{\lambda}{2}\end{vmatrix},$$

where

$$\frac{(1+ht)^{\frac{1}{h}}+1}{2}=\sum_{s=0}^{\infty}a_{s,h}(\lambda)\frac{t^s}{s!}\quad and\quad G_j^h=\sum_{s=0}^{j}\binom{j}{s}a_{s,h}(\lambda)E_{j-s}(x,y,z;\lambda;h).$$

Proof. *Taking* $\vartheta(t) = \frac{2}{\lambda(1+ht)^{\frac{1}{h}}+1}$, *it gives*

$$\frac{\lambda(1+ht)^{\frac{1}{h}}+1}{2} = \sum_{s=0}^{\infty} a_{s,h}(\lambda)\frac{t^s}{s!}$$

from equation (22) *and*

$$G_j^h = \sum_{s=0}^{j}\binom{j}{s} a_{s,h}(\lambda) E_{j-s}(x,y,z;\lambda;h)$$

from equation (23)*. Therefore, these equations are proved by substituting them in equation* (21). □

Corollary 4.11. *The following recurrence relation is satisfied for* 3*D*-Δ_h*-Hermite degenerate Apostol-Euler polynomials*

$$\begin{aligned}
&\left(x-\frac{\lambda}{\lambda+1}\right)E_j(x,y,z;\lambda;h)-\sum_{s=0}^{j-1}\binom{j}{s}(-h)^{j-s}(j-s)!E_s(x,y,z;\lambda;h)\\
&+\frac{1}{2}\sum_{s=0}^{j-1}\sum_{k=0}^{j-s}\binom{j}{s}\binom{j-s}{k}(-h)^k k!\,\mathcal{E}_{j-k-s,h}(\lambda)E_s(x,y,z;\lambda;h)\\
&+xj!\sum_{s=1}^{j}(-h)^s\frac{E_{j-s}(x,y,z;\lambda;h)}{(j-s)!}+2jyE_{j-1}(x,y,z;\lambda;h)\\
&+2yj!\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{E_{j-2s-1}(x,y,z;\lambda;h)}{(j-2s-1)!}+3j(j-1)zE_{j-2}(x,y,z;\lambda;h)\\
&+3zj!\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{E_{j-3s-2}(x,y,z;\lambda;h)}{(j-3s-2)!}=E_{j+1}(x,y,z;\lambda;h),\quad j\geq 2
\end{aligned}$$

where

$$\frac{2}{\lambda(1+ht)^{\frac{1}{h}}+1} = \sum_{j=0}^{\infty}\mathcal{E}_{j,h}(\lambda)\frac{t^j}{j!}$$

Proof. It is proved when $\frac{2}{\lambda(1+ht)^{\frac{1}{h}}+1}$ is written in case of ϑ in Theorem 2.1. □

Corollary 4.12. *The LO, RO and DE satisfied by* $E_j(x,y,z;\lambda;h)$ *are given as*

$$\begin{aligned}
{}_xL_j^- &= \frac{1}{jh}\,{}_x\Delta_h,\\
{}_xL_j^+ &= x-\frac{\lambda}{\lambda+1}+\frac{1}{2}\sum_{s=0}^{j-1}\sum_{k=0}^{j-s}(-1)^k\frac{\mathcal{E}_{j-k-s,h}(\lambda)}{(j-k-s)!\,h^{j-k-s}}\,{}_x\Delta_h^{j-s}+x\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s+2y\frac{{}_x\Delta_h}{h}+\sum_{s=0}^{j-1}(-1)^{j-s+1}\,{}_x\Delta_h^{j-s}\\
&\quad+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+1}}\,{}_x\Delta_h^{2s+1}+3z\frac{{}_x\Delta_h^2}{h^2}+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+2}}\,{}_x\Delta_h^{3s+2},\quad j\geq 2,
\end{aligned}$$

$$\left[\left(\frac{x}{h}+1-\frac{\lambda}{h(\lambda+1)}\right){}_x\Delta_h+\frac{1}{2h}\sum_{s=0}^{j-1}\sum_{k=0}^{j-s}(-1)^k\frac{\mathcal{E}_{j-k-s,h}(\lambda)}{(j-k-s)!\,h^{j-k-s}}\,{}_x\Delta_h^{j-s+1}+\left(\frac{x}{h}+1\right)\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^{s+1}\right.$$
$$+\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s+2y\frac{{}_x\Delta_h^2}{h^2}+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+2}}\,{}_x\Delta_h^{2s+2}+3z\frac{{}_x\Delta_h^3}{h^3}+\sum_{s=0}^{j-1}(-1)^{j-s+1}\frac{{}_x\Delta_h^{j-s+1}}{h}$$
$$\left.+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+3}}\,{}_x\Delta_h^{3s+3}-j\right]E_j(x,y,z;\lambda;h)=0,\quad j\geq 2.$$

Proof. It is proved when $\frac{2}{\lambda(1+ht)^{\frac{1}{h}}+1}$ is written in case of ϑ in Theorem 2.3. □

4.4. 3D-Δ_h-Hermite λ-Boole polynomials

In this part we derive explicit representation, determinantal form, recurrence relation, LO, RO and DE for $3D$-Δ_h-Hermite λ-Boole polynomials $\mathcal{B}l_j(x,y,z;\mu;\lambda;h)$.

We introduce $3D$-Δ_h-Hermite λ-Boole polynomials via the following generating function

$$\frac{1}{1+(\lambda+ht)^{\frac{\mu}{h}}}(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}=\sum_{j=0}^{\infty}\mathcal{B}l_j(x,y,z;\mu;\lambda;h)\frac{t^j}{j!}.$$

If $h=1$, $y=0$ and $z=0$ in generating function, we give the λ-Boole polynomials with the generating function as follows

$$\frac{1}{1+(\lambda+t)^{\mu}}(1+t)^x=\sum_{j=0}^{\infty}\mathcal{B}l_j(x;\mu;\lambda)\frac{t^j}{j!}.$$

Corollary 4.13. *3D-Δ_h-Hermite λ-Boole polynomials have the following explicit representation*

$$\mathcal{B}l_j(x,y,z;\mu;\lambda;h)\sum_{s=0}^{j}\binom{j}{s}\sum_{l=0}^{\left[\frac{s}{3}\right]}\sum_{m=0}^{\left[\frac{s-3l}{2}\right]}\mathcal{B}l_{j-s,h}(\mu;\lambda)\binom{s-3l}{2m}\binom{m}{3l}(x)^h_{s-2m-3l}(y)^h_m(z)^h_l\frac{(2m)!}{m!}\frac{(3l)!}{l!},$$

where the λ-Boole numbers $\mathcal{B}l_j(0,0,0;\mu;\lambda;h)=\mathcal{B}l_{j,h}(\mu;\lambda)$ (for $\lambda=1$ in [9]) , are given by the

$$\frac{1}{1+(\lambda+ht)^{\frac{\mu}{h}}}=\sum_{j=0}^{\infty}\mathcal{B}l_{j,h}(\mu;\lambda)\frac{t^j}{j!}.$$

Corollary 4.14. *3D-Δ_h-Hermite λ-Boole polynomials satisfy the determinantal form as follows*

$$\mathcal{B}l_j(x,y,z;\mu;\lambda;h)=\frac{(-1)^j}{\left(\lambda^{\frac{\mu}{h}}+1\right)^{j+1}}\begin{vmatrix}1 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h\\ \lambda^{\frac{\mu}{h}}+1 & \lambda^{\frac{1}{h}(\mu-h)}(\mu)_1^h & \lambda^{\frac{1}{h}(\mu-2h)}(\mu)_2^h & \cdots & \lambda^{\frac{1}{h}(\mu-(j-1)h)}(\mu)_{j-1}^h & \lambda^{\frac{1}{h}(\mu-jh)}(\mu)_j^h\\ 0 & \lambda^{\frac{\mu}{h}}+1 & \binom{2}{1}\lambda^{\frac{1}{h}(\mu-h)}(\mu)_1^h & \cdots & \binom{j-1}{1}\lambda^{\frac{1}{h}(\mu-(j-2)h)}(\mu)_{j-2}^h & \binom{j}{1}\lambda^{\frac{1}{h}(\mu-(j-1)h)}(\mu)_{j-1}^h\\ 0 & 0 & \lambda^{\frac{\mu}{h}}+1 & \cdots & \binom{j-1}{2}\lambda^{\frac{1}{h}(\mu-(j-3)h)}(\mu)_{j-3}^h & \binom{j}{2}\lambda^{\frac{1}{h}(\mu-(j-2)h)}(\mu)_{j-2}^h\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda^{\frac{\mu}{h}}+1 & \binom{j}{j-1}\lambda^{\frac{1}{h}(\mu-h)}(\mu)_1^h\end{vmatrix},$$

where

$$1+(\lambda+ht)^{\frac{\mu}{h}}=\sum_{s=0}^{\infty}a_{s,h}(\mu;\lambda)\frac{t^s}{s!}\quad and\quad G_j^h=\sum_{s=0}^{j}\binom{j}{s}a_{s,h}(\mu;\lambda)\mathcal{B}l_{j-s}(x,y,z;\mu;\lambda;h).$$

Proof. *Taking* $\vartheta(t) = \frac{1}{\lambda+(1+ht)^{\frac{\mu}{h}}}$, *it gives*

$$\lambda + (1+ht)^{\frac{\mu}{h}} = \sum_{s=0}^{\infty} a_{s,h}(\mu;\lambda)\frac{t^s}{s!}$$

from equation (22) *and*

$$G_j^h = \sum_{s=0}^{j}\binom{j}{s} a_{s,h}(\mu;\lambda)\mathcal{B}l_{j-s}(x,y,z;\mu;\lambda;h)$$

from equation (23). *Therefore, these equations are proved by substituting them in main determinantal form.* □

Corollary 4.15. *3D-Δ_h-Hermite λ-Boole polynomials satisfied the following recurrence relation as follows*

$$\begin{aligned}
&\left(x-\frac{\mu}{\lambda}\right)\mathcal{B}l_j(x,y,z;\mu;\lambda;h) - \frac{\mu}{\lambda}\sum_{s=0}^{j-1}\binom{j}{s}\frac{(-h)^{j-s}}{\lambda^{j-s}}(j-s)!\mathcal{B}l_s(x,y,z;\mu;\lambda;h) \\
&+\frac{\mu}{\lambda}\sum_{s=0}^{j}\sum_{k=0}^{j-s}\binom{j}{s}\binom{j-s}{k}\frac{(-h)^k}{\lambda^k}k!\,\mathcal{B}l_{j-k-s,h}(\mu;\lambda)\mathcal{B}l_s(x,y,z;\mu;\lambda;h) \\
&+xj!\sum_{s=1}^{j}(-h)^s\frac{\mathcal{B}l_{j-s}(x,y,z;\mu;\lambda;h)}{(j-s)!} + 2jy\mathcal{B}l_{j-1}(x,y,z;\mu;\lambda;h) \\
&+2yj!\sum_{s=1}^{\left[\frac{j-1}{2}\right]}(-h)^s\frac{\mathcal{B}l_{j-2s-1}(x,y,z;\mu;\lambda;h)}{(j-2s-1)!} + 3j(j-1)z\mathcal{B}l_{j-2}(x,y,z;\mu;\lambda;h) \\
&+3zj!\sum_{s=1}^{\left[\frac{j-2}{3}\right]}(-h)^s\frac{\mathcal{B}l_{j-3s-2}(x,y,z;\mu;h)}{(j-3s-2)!} = \mathcal{B}l_{j+1}(x,y,z;\mu;\lambda;h), \quad j\geq 2
\end{aligned}$$

where

$$\frac{1}{1+(\lambda+ht)^{\frac{\mu}{h}}} = \sum_{j=0}^{\infty}\mathcal{B}l_{j,h}(\mu;\lambda)\frac{t^j}{j!}, \quad \left|\frac{ht}{\lambda}\right| < 1.$$

Proof. It is proved when $\frac{1}{\lambda+(1+ht)^{\frac{\mu}{h}}}$ is written in case of ϑ in Theorem 2.1. □

Corollary 4.16. *3D-Δ_h-Hermite λ-Boole polynomials satisfy the following LO, RO and DE as follows*

$$\begin{aligned}
{}_xL_j^- &= \frac{1}{jh}\,{}_x\Delta_h, \\
{}_xL_j^+ &= x-\frac{\mu}{\lambda}+\frac{\mu}{\lambda}\sum_{s=0}^{j}\sum_{k=0}^{j-s}\frac{(-1)^k}{\lambda^k}\frac{\mathcal{B}l_{j-k-s,h}(\mu;\lambda)}{(j-s-k)!h^{j-s-k}}\,{}_x\Delta_h^{j-s} + x\sum_{s=1}^{j}(-1)^s\,{}_x\Delta_h^s - \frac{\mu}{\lambda}\sum_{s=0}^{j-1}\frac{(-1)^{j-s}}{\lambda^{j-s}}\,{}_x\Delta_h^{j-s} \\
&\quad +2y\frac{{}_x\Delta_h}{h}+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+1}}\,{}_x\Delta_h^{2s+1}+3z\frac{{}_x\Delta_h^2}{h^2}+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+2}}\,{}_x\Delta_h^{3s+2}, \quad j\geq 2,
\end{aligned}$$

$$\left[\left(\frac{x}{h}+1-\frac{\mu}{\lambda h}\right){}_x\Delta_h+\frac{\mu}{h\lambda}\sum_{s=0}^{j}\sum_{k=0}^{j-s}\frac{(-1)^k}{\lambda^k}\frac{\mathcal{B}l_{j-k-s,h}(\mu;\lambda)}{(j-s-k)!h^{j-s-k}}{}_x\Delta_h^{j-s+1}+\left(\frac{x}{h}+1\right)\sum_{s=1}^{j}(-1)^s{}_x\Delta_h^{s+1}\right.$$
$$-\frac{\mu}{h\lambda}\sum_{s=0}^{j-1}\frac{(-1)^{j-s}}{\lambda^{j-s}}{}_x\Delta_h^{j-s+1}+\sum_{s=1}^{j}(-1)^s{}_x\Delta_h^s+2y\frac{{}_x\Delta_h^2}{h^2}+2y\sum_{s=1}^{\left[\frac{j-1}{2}\right]}\frac{(-1)^s}{h^{s+2}}{}_x\Delta_h^{2s+2}$$
$$\left.+3z\frac{{}_x\Delta_h^3}{h^3}+3z\sum_{s=1}^{\left[\frac{j-2}{3}\right]}\frac{(-1)^s}{h^{2s+3}}{}_x\Delta_h^{3s+3}-j\right]\mathcal{B}l_{j,\lambda}(x,y,z;\mu;\lambda;h)=0,\quad j\geq 2.$$

Proof. It is proved when $\frac{1}{\lambda+(\lambda+ht)^{\frac{\mu}{h}}}$ is written in case of ϑ in Theorem 2.3. □

5. Approximating operators on *h*-Hermite polynomials in three variables

In this section, we examine new operators including h-Hermite polynomials in three variables and give the weighted Korovkin theorem, modulus of continuity and Peetre's K-functional for these operators.

Throughout this section let $n\in\mathbb{N}:=\{1,2,\cdots\}$, $\forall x\in[0,\infty)$ and fixed $y,z\geq 0$, $h\in(-1,0)$. Consider the following linear positive operators:

$$\mathfrak{L}_n(u;x)=\frac{1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}\sum_{k=0}^{\infty}\frac{G_k(nx,y,z;h)}{k!}u\left(\frac{k(h+1)}{n}\right),\tag{43}$$

where f is sufficiently nice function which guaranties the convergence of the above series.

Lemma 5.1. *We have the following properties for these operators:*

$$\mathfrak{L}_n(1;x) = 1,\tag{44}$$
$$\mathfrak{L}_n(t;x) = x+\frac{2y+3z}{n},\tag{45}$$
$$\begin{aligned}\mathfrak{L}_n\left(t^2;x\right) &= x^2+\frac{1}{n^2}\left[4yh\left(\frac{y}{h}-1\right)+12yz+9zh\left(\frac{z}{h}-1\right)\right.\\ &\quad\left.+(1+h)(2y+6z)+(1+h)(2y+3z)\right]\\ &\quad+\frac{1}{n}\left[x+4xy+6xz\right],\end{aligned}\tag{46}$$

where for each $x\in[0,\infty)$.

Proof. For $\mathfrak{L}_n(1;x)$, if take $u=1$ in (43), we get

$$\mathfrak{L}_n(1;x)=\frac{1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}\sum_{k=0}^{\infty}\frac{G_k(nx,y,z;h)}{k!}.$$

On the other hand, since $-1<h<0$, $|-ht|<1$,$|-ht^2|<1$, $|-ht^3|<1$, it is seen that it converges when replacing x by nx and taking $t=1$ in (13)

$$\sum_{k=0}^{\infty}\frac{G_k(nx,y,z,h)}{k!}=(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}},$$

and thus we get

$$\begin{aligned}\mathfrak{L}_n(1;x) &= \frac{1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}\\ &= 1.\end{aligned}$$

For $\mathfrak{L}_n(t;x)$, if take $u=t$ in (43), we get

$$\begin{aligned}\mathfrak{L}_n(t;x) &= \frac{1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}\sum_{k=0}^{\infty}\frac{G_k(nx,y,z;h)}{k!}\frac{k(h+1)}{n}\\ &= \frac{h+1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}n}\sum_{k=0}^{\infty}\frac{G_{k+1}(nx,y,z;h)}{k!}.\end{aligned}$$

Taking derivative with respect to t on both sides of the generating function, we have

$$\begin{aligned}\sum_{k=0}^{\infty}G_{k+1}(x,y,z;h)\frac{t^k}{k!} &= x(1+ht)^{\frac{x}{h}-1}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}}\\ &\quad+2yt(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}-1}\left(1+ht^3\right)^{\frac{z}{h}}\\ &\quad+3zt^2(1+ht)^{\frac{x}{h}}\left(1+ht^2\right)^{\frac{y}{h}}\left(1+ht^3\right)^{\frac{z}{h}-1}.\end{aligned}\tag{47}$$

In (47), we replacing x by nx and take $t=1$, we get

$$\sum_{k=0}^{\infty}\frac{G_{k+1}(x,y,z;h)}{k!}=(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}(1+h)^{-1}[nx+2y+3z],$$

and thus we have

$$\mathfrak{L}_n(t;x)=x+\frac{2y+3z}{n}.$$

For $\mathfrak{L}_n\left(t^2;x\right)$, if take $u=t^2$ in (43), we get

$$\begin{aligned}\mathfrak{L}_n\left(t^2;x\right) &= \frac{(h+1)^2}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}n^2}\sum_{k=0}^{\infty}\frac{G_{k+2}(nx,y,z;h)}{k!}\\ &\quad+\frac{(h+1)^2}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}n^2}\sum_{k=0}^{\infty}\frac{G_{k+1}(nx,y,z;h)}{k!}.\end{aligned}$$

On the other hand, in (47), taking derivative with respect to t then replace x by nx and $t=1$ to obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{G_{k+2}(nx,y,z;h)}{k!} &= \frac{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}{(1+h)^2}\Big[n^2x^2-nxh+4nxy+6nxz\\ &\quad+4yh\left(\frac{y}{h}-1\right)+12yz+9zh\left(\frac{z}{h}-1\right)\Big]\\ &\quad+\frac{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}{(1+h)}(2y+6z),\end{aligned}$$

and thus we get

$$\begin{aligned}\mathfrak{L}_n\left(t^2;x\right) &= x^2+\frac{1}{n^2}\left[4yh\left(\frac{y}{h}-1\right)+12yz+9zh\left(\frac{z}{h}-1\right)+(1+h)(2y+6z)\right.\\ &\quad\left.+(1+h)(2y+3z)\right]+\frac{1}{n}[x+4xy+6xz].\end{aligned}$$

Hence the proof is completed. □

Lemma 5.2. *We have the following results*

$$\mathfrak{L}_n(t-x;x)=\frac{2y+3z}{n} \tag{48}$$

and

$$\begin{aligned}\mathfrak{L}_n\left((t-x)^2;x\right) &= \frac{1}{n^2}\left[4yh\left(\frac{y}{h}-1\right)+12yz+9zh\left(\frac{z}{h}-1\right)+(1+h)(2y+6z)\right.\\ &\quad \left.+(1+h)(2y+3z)\right]+\frac{x}{n},\end{aligned} \tag{49}$$

where for each $x\in[0,\infty)$.

Let $\varphi(x)=1+x^2$. Then the function spaces $B_\varphi[0,\infty)$, $C_\varphi[0,\infty)$ and $C_\varphi^k[0,\infty)$ are defined as follows:

$$\begin{aligned}B_\varphi[0,\infty) &= \{u:[0,\infty)\to\mathbb{R}, |u(x)|\le M_u\varphi(x)\},\\ C_\varphi[0,\infty) &= \left\{u\in B_\varphi[0,\infty): u \text{ is continuous}\right\},\\ C_\varphi^k[0,\infty) &= \left\{u\in C_\varphi[0,\infty): \lim_{x\to\infty}\frac{u(x)}{\varphi(x)}=k\right\},\end{aligned}$$

where $\varphi(x)=1+x^2$ is a weight function and k and M_u are constants and the norm on $B_\varphi[0,\infty)$ is

$$\|u\|_\varphi=\sup_{x\geqslant 0}\frac{|u(x)|}{\varphi(x)},$$

see [42].

Lemma 5.3. $\mathfrak{L}_n: C_\varphi\to B_\varphi$ *is a sequnce of linear positive operators.*

Proof. We need to prove that linear positive operators $\mathfrak{L}_n$, from C_φ to B_φ.

$$\begin{aligned}\|\mathfrak{L}_n(u;x)\|_{\varphi=1+x^2} &= \sup_{x\in[0,\infty)}\frac{\left|\mathfrak{L}_n\left(\left(\frac{u}{\varphi}\right)(t)\varphi(t);x\right)\right|}{\varphi(x)}\\ &\le \|u\|_\varphi\sup_{x\in[0,\infty)}\frac{\mathfrak{L}_n(1;x)+\mathfrak{L}_n\left(t^2;x\right)}{1+x^2}\\ &\le \|u\|_\varphi\sup_{x\in[0,\infty)}\frac{1+x^2+C}{1+x^2}\\ &\le M\|u\|_\varphi\end{aligned}$$

where

$$\begin{aligned}C &= \frac{1}{n^2}\left[4yh\left(\frac{y}{h}-1\right)+12yz+9zh\left(\frac{z}{h}-1\right)+(1+h)(2y+6z)\right.\\ &\quad\left.+(1+h)(2y+3z)\right]+\frac{1}{n}\left[x+4xy+6xz\right].\end{aligned}$$

Therefore the given operators are uniformly bounded. □

Theorem 5.4. *Let* $\mathfrak{L}_n: C_\varphi\to B_\varphi$ *linear positive operators and* $\varphi(x)=1+x^2$. *If*

$$\lim_{n\to\infty}\left\|\mathfrak{L}_n\left(t^i;x\right)-x^i\right\|_\varphi=0,\quad i=0,1,2 \tag{50}$$

then for all $u\in C_\varphi^k$, *we have*

$$\lim_{n\to\infty}\|\mathfrak{L}_n(u;x)-u\|_\varphi=0. \tag{51}$$

Proof. From (44)-(46), we obtain

$$\|\mathfrak{L}_n(1;x)-1\|_\varphi=0,$$

$$\|\mathfrak{L}_n(t;x)-x\|_\varphi\leq\frac{1}{n}\to 0,\quad\text{as}\quad n\to\infty,$$

and

$$\left\|\mathfrak{L}_n\left(t^2;x\right)-x^2\right\|_\varphi\leq\frac{A+Bn}{n^2}\to 0,\quad\text{as}\quad n\to\infty,$$

where

$$A=4yh\left(\frac{y}{h}-1\right)+12yz+9zh\left(\frac{z}{h}-1\right)+(1+h)(2y+6z)+(1+h)(2y+3z)$$

and

$$B=x+4xy+6xz.$$

The proof completed by applying Korovkin's theorem [41] (see also [42],[43]). □

Let $u\in\widetilde{C}[0,\infty)$. Then for $\zeta>0$ the modulus of continuity of u defined by $\omega(u;\zeta)$ in [44],

$$\omega(u;\zeta)=\sup_{\substack{x,y\in[0,\infty)\\|x-y|\leq\zeta}}\left|u(x)-u(y)\right|$$

where $\widetilde{C}[0,\infty)$ denotes the space of uniformly continuous functions on $[0,\infty)$. Thereafter it is best known that one can write

$$\left|u(x)-u(y)\right|\leq\left(1+\frac{\left|x-y\right|}{\zeta}\right)\omega(u;\zeta). \tag{52}$$

for any $\zeta>0$ and each $x\in[0,\infty)$.

Theorem 5.5. *For $u\in\widetilde{C}[0,\infty)$, we have*

$$|\mathfrak{L}_n(u;x)-u(x)|\leq 2\omega(u;\zeta), \tag{53}$$

with $\zeta=\zeta_n(x)=\sqrt{\mathfrak{L}_n\left((t-x)^2;x\right)}$ where $\mathfrak{L}_n\left((t-x)^2;x\right)$ is obtained in Lemma 5.1.

Proof. Using linearity of the operators $\mathfrak{L}_n$, (44) and (52), we get

$$\begin{aligned}|\mathfrak{L}_n(u;x)-u(x)| &\leq \mathfrak{L}_n(|u(t)-u(x)|;x)\\ &\leq \left(1+\frac{1}{\zeta}\mathfrak{L}_n(|t-x|;x)\right)\omega(u;\zeta).\end{aligned}$$

According to the Cauchy-Schwarz inequality for $\mathfrak{L}_n(|t-x|;x)$, we obtain that

$$|\mathfrak{L}_n(u;x)-u(x)|\leq\left(1+\frac{1}{\zeta}\sqrt{\mathfrak{L}_n\left((t-x)^2;x\right)}\right)\omega(u;\zeta).$$

If we choose $\zeta=\zeta_n(x)=\sqrt{\mathfrak{L}_n\left((t-x)^2;x\right)}$, we get the result. □

Let

$$C_B^2[0,\infty) = \{v \in C_B[0,\infty) : v', v'' \in C_B[0,\infty)\}, \tag{54}$$

with the norm

$$\|v\|_{C_B^2[0,\infty)} = \|v\|_{C_B[0,\infty)} + \|v'\|_{C_B[0,\infty)} + \|v''\|_{C_B[0,\infty)} \tag{55}$$

also

$$\|v\|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} |v(x)|, \tag{56}$$

(see [44]) .

Theorem 5.6. *For any $v \in C_B^2[0,\infty)$, we have*

$$|\mathfrak{L}_n(v;x) - v(x)| \le \frac{1}{2}\zeta_n(2+\zeta_n)\|v\|_{C_B^2[0,\infty)},$$

where

$$\zeta_n = \left[\frac{1}{n^2}\left(4yh\left(\frac{y}{h}-1\right) + 12yz + 9zh\left(\frac{z}{h}-1\right) + (1+h)(2y+6z) + (1+h)(2y+3z)\right) + \frac{x}{n}\right]^{\frac{1}{2}}.$$

Proof. From the Taylor's series expansion of the function $v \in C_B^2[0,\infty)$, we have

$$v(t) = v(x) + (t-x)v'(x) + \frac{(t-x)^2}{2!}v''(u),$$

where u between x and t, from which it follows

$$|v(t) - v(x)| \le M_1|t-x| + \frac{1}{2}M_2(t-x)^2,$$

where

$$\begin{aligned} M_1 &= \sup_{x\in[0,\infty)}|v'(x)| = \|v'\|_{C_B[0,\infty)} \le \|v\|_{C_B^2[0,\infty)}, \\ M_2 &= \sup_{x\in[0,\infty)}|v''(x)| = \|v''\|_{C_B[0,\infty)} \le \|v\|_{C_B^2[0,\infty)} \end{aligned}$$

because of (55). Thus,

$$|v(t) - v(x)| \le \left(|t-x| + \frac{1}{2}(t-x)^2\right)\|v\|_{C_B^2[0,\infty)}.$$

Since

$$|\mathfrak{L}_n(v;x) - v(x)| = |\mathfrak{L}_n(v(t) - v(x);x)| \le \mathfrak{L}_n(|v(t) - v(x)|;x),$$

and $\mathfrak{L}_n(|t-x|;x) \le \left(\mathfrak{L}_n\left((t-x)^2;x\right)\right)^{\frac{1}{2}} = \zeta_n$ we get

$$\begin{aligned} |\mathfrak{L}_n(v;x) - v(x)| &\le \left(\mathfrak{L}_n(|t-x|;x) + \frac{1}{2}\mathfrak{L}_n\left((t-x)^2;x\right)\right)\|v\|_{C_B^2[0,\infty)} \\ &\le \frac{1}{2}\zeta_n(2+\zeta_n)\|v\|_{C_B^2[0,\infty)}. \end{aligned}$$

This complete the proof. □

In the proof of the following theorem in the use relation between Peetre's K-functional and the second of the modulus continuity, is defined by

$$\omega_2(u;\tau) := \sup_{0<t\leqslant\tau} \|u(.+2t) - 2u(.+t) + u(.)\|_{C_B[0,\infty)}$$

for $u \in C_B[0,\infty)$, which is

$$K(u;\tau) \leq C\left\{\omega_2\left(u;\sqrt{\tau}\right) + \min(1,\tau)\|u\|_{C_B}\right\}, \tag{57}$$

see [44].

Theorem 5.7. *If $u \in C_B[0,\infty)$, then we obtain*

$$|\mathfrak{L}_n(u;x) - u(x)| \leq 2M\left\{\omega_2\left(u;\sqrt{\tau}\right) + \min(1,\tau)\|u\|_{C_B}\right\}, \tag{58}$$

where

$$\tau = \tau_n = \frac{1}{2}\zeta_n \tag{59}$$

with ζ is same as Theorem 5.6 and $M > 0$ is a constant which is independent of the function u and τ.

Proof. We prove this by using Theorem 5.6. Let $v \in C_B^2[0,\infty)$. Since

$$|\mathfrak{L}_n(u;x) - u(x)| \leq |\mathfrak{L}_n(u-v;x) - (u-v)(x)| + |\mathfrak{L}_n(v;x) - v(x)|. \tag{60}$$

For $|\mathfrak{L}_n(u-v;x) - (u-v)(x)|$, we have

$$\begin{aligned} |\mathfrak{L}_n(u-v;x) - (u-v)(x)| &\leq |\mathfrak{L}_n(u-v;x)| + |(u-v)(x)| \\ &\leq 2\|u-v\|_{C_B}. \end{aligned} \tag{61}$$

Using (61) and write into (60), we get

$$\begin{aligned} |\mathfrak{L}_n(u;x) - u(x)| &\leq 2\|u-v\|_{C_B} + \frac{1}{2}\zeta_n(2+\zeta_n)\|v\|_{C_B^2[0,\infty)} \\ &\leq 2\left(\|u-v\|_{C_B} + \frac{1}{4}\zeta_n(2+\zeta_n)\|v\|_{C_B^2[0,\infty)}\right) \\ &\leq 2\inf_{v\in C_B^2[0,\infty)}\left\{\|u-v\|_{C_B} + \tau\|v\|_{C_B^2}\right\} \\ &\leq 2K(u;\tau), \end{aligned}$$

where $K(u;\delta)$ is Peetre's K-functional defined by (57). Then we have

$$|\mathfrak{L}_n(u;x) - u(x)| \leq 2M\left\{\omega_2\left(f;\sqrt{\tau}\right) + \min(1,\delta)\left\|f\right\|_{C_B}\right\}.$$

□

6. Conclusion

Δ_h-GHAP were defined in [38] and interesting properties of these polynomials containing special polynomials were obtained. In this paper, we studied $3D$-Δ_h-Hermite Appell polynomials. We give some of their properties such as recurrence relation, determinantal form, shift operators, etc. As special cases, we introduce $3D$- Δ_h-Hermite λ-Charlier polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Bernoulli polynomials, $3D$-Δ_h-Hermite degenerate Apostol-Euler polynomials and $3D$-Δ_h-Hermite λ-Boole polynomials. We exhibit certain of their properties such as explicit and determinantal forms, recurrence relation, raising and lowering operators and difference equation.

It can be noted that the r-variable case can be defined via generating relation

$$\vartheta(t)\prod_{i=1}^{r}\left(1+ht^{i}\right)^{\frac{x_i}{h}}=\sum_{j=0}^{\infty}\mathcal{A}_j\left(x_1,x_2,...,x_r\right)\frac{t^j}{j!},\ \left|ht^i\right|<1,\quad (i=0,1,...,r). \tag{62}$$

Starting from this definition and their properties, it is possible to derive families with r variables. Its properties such as explicit representation, determinantal form, recurrence relation, summation formulas and shift operators can be investigated. In addition to these, more r-variable polynomials can be defined as special cases of $\mathcal{A}_j\left(x_1,x_2,\cdots x_r\right)$ which may have some potential applications in number theory, special function theory and approximation theory.

Acknowledgements

The second section of this work is a part of the doctorate thesis of the first author. We would like to thank the reviewer for his/her valuable suggestions and comments. We devote is paper to late Professor Recep ŞAHİN who passed away this year.

References

[1] G. B. Djordjević, G.V Milovanović, Special Classes of Polynomials, University of Nis, Faculty of Technology, Leskovac, 2014.
[2] F. A. Costabile, E. Longo, Δ_h-Appell sequences and related interpolation problem, Numerical Algorithms 63(1) (2013) 165–186.
[3] S. Khan, N. Raza, 2-Iterated Appell polynomials and related numbers, Appl. Math. Comput. 219(17) (2013) 9469–9483.
[4] S. Khan, G. Yasmin, R. Khan, N. A. M. Hassan, Hermite-based Appell polynomials: properties and applications, J. Math. Anal. Appl. 351 (2009) 756–764.
[5] S. Khan, M. W. M. Al-Saad, R. Khan, Laguerre-based Appell polynomials: properties and applications, Math. Comput. Modelling 52(1–2) (2010) 247–259.
[6] T. Nahid, S. Khan, Construction of some hybrid relatives of Laguerre-Appell polynomials associated with Gould-Hopper matrix polynomials, The Journal of Analysis 29(3) (2021) 927–946.
[7] S. Khan, G. Yasmin, N. Ahmad, A note on truncated exponential based Appell polynomials, Bull. Malays. Math. Sci. Soc. 40 (2017) 373–388.
[8] H. M. Srivastava, M. A. Özarslan, B. Y. Yaşar, Difference equations for a class of twice iterated Δ_h-Appell sequences of polynomials, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113 (2019) 1851–1871.
[9] M. A. Özarslan, B. Y. Yaşar, Δ_h-Gould-Hopper Appell polynomials, Acta Mathematica Scientia 41B(4) (2021) 1196–1222.
[10] L.M. Milne-Thomson, Two classes of generalized polynomials, Proceedings of the London Mathematical Society s2-35 (1)(1933) 514–522.
[11] R. Dere, Y. Simsek, Genocchi polynomials associated with the umbral algebra, Applied Mathematics and Computation 218(3) (2011)756–761.
[12] R. Dere, Y. Simsek, Hermite base Bernoulli type polynomials on the umbral algebra, Russian Journal of Mathematical Physics 22(1) (2015) 1–5.
[13] R. Dere, Y. Simsek, H.M. Srivastava, A unied presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, Journal of Number Theory 133(10) (2013) 3245–3263.
[14] Y. Simsek, Formulas for Poisson-Charlier, Hermite, Milne-Thomson and other type polynomials by their generating functions and p-adic integral approach, RACSAM 113(2) (2019) 931–948.
[15] N. Kilar, Y. Simsek, Computational formulas and identities for new classes of Hermite-based Milne-Thomson type polynomials: Analysis of generating functions with Euler's formula. Mathematical Methods in the Applied Sciences 44(8) (2021) 6731–6762.
[16] N. Khan, T. Usman, J. Choi, A new class of generalized polynomials, Turkish Journal of Mathematics 42(3) (2018) 1366–1379.
[17] B. Kurt, Y. Simsek, On the Hermite based Genocchi polynomials, Advanced Studies in Contemporary Mathematics 23(1) (2013) 13–17.
[18] Y. Simsek, N. Cakic, Identities associated with Milne-Thomson type polynomials and special numbers, Journal of Inequalities and Applications 84 (2018) 1–13.
[19] Y. Simsek, Peters type polynomials and numbers and their generating functions: Approach with p-adic integral method, Mathematical Methods in the Applied Sciences 42(18) (2019) 7030–7046.
[20] S. Khan, T. Nahid, M. Riyasat, Properties and graphical representations of the 2-variable form of the Simsek polynomials, Vietnam Journal of Mathematics 50 (2022) 95–109.
[21] Y. Simsek, Applications of constructed new families of generating-type functions interpolating new and known classes of polynomials and numbers, Mathematical Methods in the Applied Sciences 44(14) (2021) 11245–11268.
[22] H. M. Srivastava, M. A. Özarslan, B. Yılmaz, Some families of differential equations associated with the Hermite based Appell polynomials and other classes of Hermite-based polynomials, Filomat 28(4) (2014) 695–708.
[23] F. A. Costabile, Modern Umbral Calculus: An Elementary Introduction With Applications to Linear Interpolation and Operator Approximation Theory, Gruyter-Studies In Mathematics. De Gruyter, Berlin, 2019.

[24] G. Bretti, P.E. Ricci, Multidimensional extension of the Bernoulli and Appell polynomials, Taiwan J. Math. 8(3) (2004) 415–428.
[25] P. Jr. Boas, R. C. Buck, Polynomial Expansions of Analytic Functions, Heidelberg: Springer, Berlin, Göttingen, 1964.
[26] N. Kilar, Y. Şimşek, New computational formulas for special numbers and polynomials derived from applying trigonometric functions to generating functions, Milan J. Math. 89 (2021) 217–239.
[27] S. Khan, M. Ali, A linear algebra approach to the hybrid Sheffer-Appell polynomials, Mathematical Sciences 13(2) (2019) 153–164.
[28] W. A. Khan, A Note on degenerate Hermite Poly-Bernoulli numbers and polynomials, Journal of Classical Analysis, 8(1) (2016) 65–76.
[29] M. A. Özarslan, B. Yılmaz, A set of finite order differential equations for the Appell polynomials, Journal of Computational and Applied Mathematics, 259 (2014) 108–116.
[30] M. A. Özarslan, Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials, Advances in Difference Equations, 116 (2013) 1–13.
[31] G. Baran, On the ω-Multiple Charlier Polynomials, Doctor of Philosophy Thesis in Mathematics, Eastern Mediterranean University, Gazimağusa, North Cyprus, September 2021.
[32] M. Riyasat, T. Nahid, S. Khan, An algebraic approach to degenerate Appell polynomials and their hybrid forms via determinants, Acta Mathematica Scientia, 43B (2022) 1–17.
[33] C. Jordan, Calculus of Finite Differences, Chelsea Publishing Company, New York, 1965.
[34] G. Dattoli, Generalized polynomials operational identities and their applications, J. Comput. Appl. Math. 118 (2000) 111–123.
[35] I. M. Sheffer, Note on Appell polynomials, Bulletin of the American Mathematical Society, 51(10) (1945) 739–744.
[36] S. Roman, The umbral calculus, Academic Press Inc. New York, 1984.
[37] Y.B. Cheikh, A. Zaghouani, Some discrete d-orthogonal polynomial sets, Journal of Computational and Applied Mathematics, 156 (2003) 253–263.
[38] H. M. Srivastava, J., Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Inc, Amsterdam, 2012.
[39] M. A. Özarslan, G. Baran, On the w-multiple Charlier polynomials, Advences in Difference Equations, 119 (2021) 1-16.
[40] S. Khan, T. Nahid, M. Riyasat, On degenerate Apostol-type polynomials and applications, Boletín de la Sociedad Matemática Mexicana, 25 (2019) 509-–528.
[41] A. D. Gadzhiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogues to that PP Korovkin, Soviet Mathematics Doklady 15(5) (1974) 1453-1436.
[42] A. D. Gadzhiev, Weighted approximation of continuous functions by positive linear operators on the whole real axis, Izv. Akad. Nauk Azerbaizhan, SSR Ser. Fiz.-Tehn. Mat. Nauk 5 (1975) 41-45 (Russian).
[43] A. D. Gadzhiev, Theorems of the type of P. P. Korovkin's theorems, Mat, Zametki 20(5) (1976) 781-786 (Russian). Presented at the International Conferance on the Theory of Approximation of Functions (Kaluga, 1975).
[44] A. Ciupa, A class of integral Favard-Szasz type operators, Studia Univ. Babeş-Bolyai Math. 40(1) (1995) 39-47.