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The extremal t-uniform unicyclic hypergraph on Estrada index

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Abstract. For a t-uniform hypergraph $H = (V(H), E(H))$, the Estrada index $EE(H)$ of H is defined as $\sum_{i\in[n]} e^{\theta_i}$, where $\theta_1, \ldots, \theta_n$ are the eigevalues of the adjacency matrix of H. In this paper, the extremal t-uniform unicylic hypergraph which has maximum Estrada index are characterized.

1. Introduction

Let $H = (V(H), E(H))$ be a hypergraph with vertex set $V(H)(= [n] = \{1, 2, \ldots, n\})$ and edge set $E(H)$, where $E(H) \subseteq 2^{V(H)}$ and $2^{V(H)}$ stands for the power set of $V(H)$. A hypergraph H is t-uniform if $|e| = t$ for any $e \in E(H)$, and 2-uniform hypergraphs are well-known ordinary graphs. For any nonempty subset S of $V(H)$, the sub-hypergraph $H[S]$ induced by S is $(S, E(H[S]))$, where S is its vertex set and $E(H[S]) = \{V(e) \cap S : e \in E(H)\}$. Let $E(v) = \{e|v \in e \in E(H)\}$ and let $d(v)(= |E(v)|)$ be the degree of v in H. For any $e \in E(v) \subseteq E(H)$, if $|e| \geq 3$, let $E(v) \setminus \{v\} = \{e \setminus \{v\} : e \in E(v)\}$ be obtained from $E(v)$ by v-shrinking on $E(v)$. Especially, we say that $e \setminus \{v\}$ is obtained from e by v-shrinking on e. Let $N_H(v) = \{u|u, v \in e \in E(H)\}$. A vertex with degree 1 is called pendent vertex and an edge e is called a pendent edge if e contains exactly $|e| - 1$ pendent vertices.

A walk W in H is a sequence of alternating vertices and edges $v_0e_1v_1e_2 \cdots e_pv_p$, where $v_{i-1}, v_i \in e_i$ for $i \in [p]$. If v_i (resp. e_i) are all distinct for $i \in [p] \cup \{0\}$ (resp. $i \in [p]$), then W is called a path. If $|e_i \cap e_j| \leq 1$ for $i \neq j, i, j \in [p]$, W is also called a loose path. A cycle is a loose path satisfying $v_0 = v_p$. For any $u, v \in V(H)$, if there is at least one path connecting u with v, we say that H is connected. An edge $e \in E(H)$ is a cut edge if $H - e$ is disconnected, where $H - e = (V(H), E(H) \setminus \{e\})$. For more detailed notations and terminologies related to walks, please see Table 1.

For a connected t-uniform hypergraph H with order n and size ε, its cyclomatic number $c(H)$ is defined as $\epsilon(t - 1) - n + 1$. In particular, 0-cyclic hypergraph and 1-cyclic hypergraph are called supertree and t-uniform unicyclic hypergraph, respectively. Let $\mathcal{U}(n)$ be the class of t-uniform unicyclic hypergraphs of order n. For $m, t \geq 3$, let $C_{m,t} = v_1e_1v_2e_2v_3 \cdots v_me_mv_1$ be a t-uniform cycle of length m, for $i \in [m]$,

$$\begin{aligned} &e_i = \{v_i, v_{i+1}, u_{i1}, u_{i2}, \ldots, u_{i,t-2}\}, \;\; v_{m+1} = v_1, \\ &V_0 = \{v_i : i \in [m]\}. \end{aligned} \tag{1}$$

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Table 1: **Nomenclature**

$W_k(H;u,v)$	the set of (u,v)-walks of length k in H
$M_k(H;u,v)$	$\lvert W_k(H;u,v)\rvert$, that is, the order of $W_k(H;u,v)$
$W_k(H;u)$	the set of (u,u)-walks of length k in H
$W_k(H)$	the set of $\bigcup_{u\in V(H)} W_k(H;u)$
$M_k(H)$	$\lvert W_k(H)\rvert$
$W_k(H;u,[v])$	the set of (u,u)-walks of length k which pass through v in H
$(H_1;u_1,v_1)\preceq(H_2;u_2,v_2)$	$M_k(H_1;u_1,v_1)\le M_k(H_2;u_2,v_2)$ for any positive integer k
$(H_1;u_1,v_1)\prec(H_2;u_2,v_2)$	$(H_1;u_1,v_1)\preceq(H_2;u_2,v_2)$ and $M_{k_0}(H_1;u_1,v_1)<M_{k_0}(H_2;u_2,v_2)$ for some positive integer k_0
$(H_1;u_1)\preceq(H_2;u_2)$	$M_k(H_1;u_1)\le M_k(H_2;u_2)$ for any positive integer k
$(H_1;u_1)\prec(H_2;u_2)$	$(H_1;u_1)\preceq(H_2;u_2)$ and $M_{k_0}(H_1;u_1)<M_{k_0}(H_2;u_2)$ for some positive integer k_0

Let $\mathcal{U}(m,n)$ be the class of t-uniform unicyclic hypergraph of order n obtained from $C_{m,t}$ (which is described as in (1.1)) by attaching $\frac{n}{t-1}-m$ pendent edges to some vertices in V_0. Note that $\epsilon(t-1)-n+1=1$, it has $\epsilon=\frac{n}{t-1}$ a nonnegative integer. For some other notations and terminologies which are not given in this paper, please refer to [1].

The adjacency matrix $A(H)=(a_{ij})$ of H is defined as

$$a_{ij}=\begin{cases}\lvert\{e\in E(H): i,j\in e\subseteq V(H)\rvert, & \text{if } i\neq j,\\ 0, & \text{if } i=j.\end{cases}$$

The characteristic polynomial of H is the determinant $\lvert xI-A(H)\rvert$, where I is the identity matrix of order n. The eigevalues $\theta_1,\ldots,\theta_n$ of H are the zeroes of $\lvert xI-A(H)\rvert$.

In [5], Estrada and Rodrgíuea-Veláquez proposed a measure $\frac{1}{n}\sum_{i\in[n]}e^{\theta_i}$ to describe the sub-hypergraph centrality of H, which has found extensively applications such as in chemistry, complex networks and other interdisciplinary fields [5, 8]. In [11], the authors named directly $\sum_{i\in[n]}e^{\theta_i}$ as Estrada index of H, denoted it by $EE(H)$. More applications on Estrada index for ordinary graphs have been proposed, for examples, see [6, 7, 9, 10]. At the same time, many mathematical properties about the Estrada index of ordinary graphs have been obtained, such as [2–4, 13, 14] and references therein. Note that graphs are limited to representing pairwise relationships between entities. However, in real-world there are more multi-way relationships, which can be modeled by hypergraphs. So the applications of theory of hypergraphs have attracted the interest of many researchers, but this research is still in its beginning stage. Since Li et al. [11] generalized the notation of Estrada index from graphs to hypergraphs, few results in this field have been obtained. Recently, some extremal properties of Estrada index on t-uniform hypergraphs are obtained[12, 15, 16]. In this paper, we will continue to study the properties on Estrada index for hypergraphs.

The following lemmas are useful in our main results.

Lemma 1.1. *[12] For a t-uniform hypergraph* $H=(V(H),E(H))$ *with a cut edge* $e=\{u_1,u_2,\ldots,u_t\}$, $t\geq 3$, *if* $d_H(u_1)=1$ *and* $d_H(u_2)\geq 2$, *then* $(H;u_1)\prec(H;u_2)$.

Lemma 1.2. *[12] For a hypergraph* $H=(V(H),E(H))$ *with* $u,v\in V(H)$ *and* $E_0=\{e_1,e_2,\ldots,e_s\}$, *where* $e_h\in 2^{V(H)}, e_h\cap\{u,v\}=\emptyset$ *for* $h\in[s]$. *Let*

$$\begin{aligned}E_u &= \{e'_h=e_h\cup\{u\}: e_h\in E_0, h\in[s]\},\\ E_v &= \{e'_h=e_h\cup\{v\}: e_h\in E_0, h\in[s]\};\end{aligned}$$

$H_u=H-E_0+E_u$ *and* $H_v=H-E_0+E_v$. *If for any* $w\in\bigcup_{h\in[s]}e_h$, $(H;u)\prec(H;v)$ *and* $(H;w,u)\preceq(H;w,v)$, *then* $EE(H_u)<EE(H_v)$.

2. Extremal t-uniform unicyclic hypergraph with maximum Estrada index

In this section, we consider the extremal structure of t-uniform unicyclic hypergraph with maximum Estrada index.

Lemma 2.1. *For a t-uniform unicyclic hypergraph H with unique cycle $C_{m,t}$ (which is described as in (1.1)), let $u \in V_0$ and u_1 be a pendent neighbor of u in H, then $(H;u_1) \prec (H;u)$.*

Proof. Let $u_1 \in f = \{u, u_1, u_2, \ldots, u_{t-1}\}$.

Case 1. $f \notin E(C_{m,t})$, then f must be a cut edge of H, by Lemma 1.1, we have $(H;u_1) \prec (H;u)$.

Case 2. $f \in E(C_{m,t})$.

Let A_1 be the hypergraph obtained from H by u-shrinking on $E(u)$, and A_2 be the hypergraph obtained from H by u_1-shrinking on $E(u_1)$. Then

$$\begin{aligned} W_k(H;u_1) &= W_k(A_1;u_1) \cup W_k(H;u_1,[u]), \\ W_k(H;u) &= W_k(A_2;u) \cup W_k(H;u,[u_1]). \end{aligned}$$

Further we have

$$\begin{aligned} M_k(H;u_1) &= M_k(A_1;u_1) + M_k(H;u_1,[u]), \\ M_k(H;u) &= M_k(A_2;u) + M_k(H;u,[u_1]). \end{aligned}$$

It is easy to see that

$$M_k(A_1;u_1) \le M_k(A_2;u) \ \ (resp., M_k(A_1;u_1,u_i)) \le M_k(A_2;u,u_i))$$

for any nonegative integer k and $i \in [t-2] \setminus \{1\}$, and for $k = 2$, by direct calculation, we have

$$M_2(A_1;u_1) = t-2 < t-2 + (d_H(u)-1)(t-1) = M_2(A_2;u).$$

For any $W \in W_k(H;u_1,[u])$, let $W = W_1W_2$, where W_1 is either a walk which consists of (u_1,u_1)-section of length $l_1 - 1$ in A_1 and a (u_1,f,u)-section of length 1 or (u_1,u_i)-section of length $l_1 - 1$ in A_1 and a (u_i,f,u)-section of length 1 for $i \in \{2,\ldots,t-1\}$; and W_2 is a (u,u_1)-section with length l_2 in H. Then we have

$$\begin{aligned} M_k(H;u_1) &= M_k(A_1;u_1) + \sum_{l_1+l_2=k, l_1,l_2 \ge 1} M_{l_1-1}(A_1;u_1)M_{l_2}(H;u,u_1) + \\ &\quad \sum_{i=2}^{t-1} \sum_{l_1+l_2=k,l_1,l_2\ge 1} M_{l_1-1}(A_1;u_1,u_i)M_{l_2}(H;u,u_1). \end{aligned}$$

Similarly, we have

$$\begin{aligned} M_k(H;u) &= M_k(A_2;u) + \sum_{l_1+l_2=k, l_1,l_2 \ge 1} M_{l_1-1}(A_2;u)M_{l_2}(H;u_1,u) + \\ &\quad \sum_{i=2}^{t-1} \sum_{l_1+l_2=k,l_1,l_2\ge 1} M_{l_1-1}(A_2;u,u_i)M_{l_2}(H;u_1,u). \end{aligned}$$

Hence $M_k(H;u_1) \le M_k(H;u)$ for any nonegative integer k and $M_2(H;u_1) < M_2(H;u)$. So $(H;u_1) \prec (H;u)$. □

Lemma 2.2. *For $m \ge 2, t \ge 3$, let H_0 be a t-uniform unicyclic hypergraph with maximum Estrada index among t-uniform unicyclic hypergraphs of order n with unique cycle $C_{m,t}$ (which is described as in (1)), then $H_0 \in \mathcal{U}(m,n)$.*

Proof. Let u be a vertex in V_0, if there exists a non-pendent neighbor u_1 of u outside $C_{m,t}$ or in $V(C_{m,t}) \setminus V_0$, for $i \in [d_{H_0}(u_1) - 1]$, let $\tilde{e}_i$ be the edge which is incident to u_1, $e_i = \tilde{e}_i \setminus \{u_1\}$ but $u \notin e_i$. For convenience, let

$$\begin{aligned} E_0 &= \{e_1, e_2, \ldots, e_{d_{H_0}(u_1)-1}\}, \\ E_u &= \{e_i \cup \{u\}, e_i \in E_0, i \in [d_{H_0}(u_1) - 1]\}, \\ E_{u_1} &= \{e_i \cup \{u_1\}, e_i \in E_0, i \in [d_{H_0}(u_1) - 1]\}. \end{aligned}$$

Further let $H = H_0 - E_0$. Then we have the following results:

(i) $d_H(u_1) = 1$ and $d_H(u) \geq 2$. By Lemma 2.1, $(H; u_1) \prec (H; u)$;

(ii) For any $w \in V(E_0)$, $M_k(H; w, u_1) = M_k(H; w, u) = 0$, then $(H; w, u_1) \preceq (H; w, u)$;

(iii) $H + E_u$ is also a t-uniform unicyclic hypergraphs of order n with unique cycle $C_{m,t}$.
By Lemma 1.2, we have $EE(H_0) < EE(H + E_u)$, a contradiction. Thus $H_0 \in \mathcal{U}(m, n)$. □

From Lemma 2.2, in order to obtain t-uniform unicyclic hypergraph with maximum Estrada index in $\mathcal{U}(n)$, we only need to consider the hypergraphs in $\mathcal{U}(m, n)$.

Lemma 2.3. *For $U \in \mathcal{U}(m, n)$ with unique cycle $C_{m,t}$ (which is described as in (1)), there exist $n_i (n_i \geq 0)$ pendent edges attaching at v_i for $i \in [m]$. Let H be the hypergraph obtained from U by v_1-shrinking on e_m. If $m \geq 4$ and $n_1 \leq n_3$, then*

(i) $(H; v_1) \prec (H; v_3)$;

(ii) $(H; w, v_1) \preceq (H; w, v_3)$ for $w \in \{v_m, u_{m1}, u_{m2}, \ldots, u_{m,t-2}\}$.

Proof. (i) Let $\hat{H}$ be the hypergraph obtained from H by v_3-shrinking on e_2 and H_1 be the component of $\hat{H}$ containing v_2, and $\tilde{H}$ be the hypergraph obtained from H by v_1-shrinking on e_1 and H_2 be the component of $\tilde{H}$ containing v_2. Obviously, H_1 is a proper sub-hypergraph of H_2. Then

(1) $(H_1; v_1) \prec (H_2; v_3)$, $(H_1; v_1, v_2) \prec (H_2; v_3, v_2)$ and $(H_1; v_1, u_{1i}) \preceq (H_2; v_3, u_{2i})$ for $i \in [t - 2]$.

(2) For any positive integer k,

$$\begin{aligned} M_k(H; v_1) &= M_k(H_1; v_1) + M_k(H; v_1, [v_3]), \\ M_k(H; v_3) &= M_k(H_2; v_3) + M_k(H; v_3, [v_1]). \end{aligned}$$

For any $W \in W_k(H; v_1, [v_3])$, let $W = W_1 W_2$, where W_1 is the shortest (v_1, v_3)-section, which consists of (v_1, v_2)-section (or (v_1, u_{2i}) -section, $i \in [t - 2]$) of length $l_1 - 1$ in H_1 and (v_2, e_2, v_3)(or (u_{2i}, e_2, v_3))-section of length 1, and W_2 is the remaining (v_3, v_1) -section of length l_2 in H. Then we have

$$\begin{aligned} M_k(H; v_1, [v_3]) = & \sum_{l_1+l_2=k, l_1, l_2 \geq 2} M_{l_1-1}(H_1; v_1, v_2) M_{l_2}(H; v_3, v_1) + \\ & \sum_{i=1}^{t-2} \sum_{l_1+l_2=k, l_1, l_2 \geq 2} M_{l_1-1}(H_1; v_1, u_{2i}) M_{l_2}(H; v_3, v_1). \end{aligned}$$

Similarly,

$$\begin{aligned} M_k(H; v_3, [v_1]) = & \sum_{l_1+l_2=k, l_1, l_2 \geq 2} M_{l_1-1}(H_2; v_3, v_2) M_{l_2}(H; v_1, v_3) + \\ & \sum_{i=1}^{t-2} \sum_{l_1+l_2=k, l_1, l_2 \geq 2} M_{l_1-1}(H_2; v_3, u_{1i}) M_{l_2}(H; v_1, v_3). \end{aligned}$$

Then $M_k(H; v_1, [v_3]) < M_k(H; v_3, [v_1])$. Hence $(H; v_1) \prec (H; v_3)$.

(ii) For $h \in [m]$ and $j \in [n_h]$, let $e_{hj} = \{v_h, v_{hj}^1, \ldots, v_{hj}^{t-1}\}$ be the pendent edges attaching at v_h, and

$$V_{v_h} = \bigcup_{j \in [n_h]} (e_{hj} \setminus \{v_h\}).$$

For a $W \in W_k(H; w, v_1)$ and $w \in \{v_m, u_{m1}, u_{m2}, \ldots, u_{m,t-2}\}$, let $W = W_1W_2$, where W_1 is the longest (w, v_2)-section of W, and W_2 is the the remaining (v_2, v_1)-section of W which the internal vertices (if exist) are only possible to belong to $V_{v_1} \cup \{u_{11}, u_{12}, \ldots, u_{1,t-2}\}$. Let W_2' be obtained from W_2 by replacing v_1 by v_3, u_{1k} by u_{2k} for $k \in [t-2]$, and v_{1j}^s by v_{3j}^s for $j \in [n_1]$ and $s \in [t-1]$. Then $W' = W_1W_2' \in W_k(H; w, v_3)$. Thus $(H; w, v_1) \preceq (H; w, v_3)$. □

By Lemma 2.3 and Lemma 1.2, we have the following result.

Lemma 2.4. *For a $U \in \mathcal{U}(m, n)$ with unique cycle $C_{m,t}$ (which is described as in (1)), there exist $n_i (n_i \geq 0)$ pendent edges attaching at v_i for $i \in [m]$. Let H be the hypergraph obtained from U by v_1-shrinking on e_m and $e_0 = e_m \setminus \{v_1\}$. If $m \geq 4$ and $n_3 = \max_{i \in [m]} n_i$, then $EE(U) < EE(H - e_0 + (e_0 \cup \{v_3\}))$.*

Remark 1. $H - e_0 + (e_0 \cup \{v_3\})$ is a t-uniform unicyclic hypergraphs of order n and the length of its unique cycle is $m-2$.

Let $C_3(n_1, n_2, n_3)$ be a t-uniform unicyclic hypergraphs obtained from $C_{3,t}$ by attaching n_i pendent edges at v_i for $i \in [3]$. Without loss of generality, let $n_1 \geq n_2 \geq n_3$ in the following.

Lemma 2.5. *If $n_1 \geq 1$ and $n_2 \geq 0$, then $(C_3(n_1, n_2, 0); v_3) \prec (C_3(n_1, n_2, 0); v_1)$.*

Proof. Let $\tilde{H}$ (resp. $\check{H}$) be the hypergraph obtained from $C_3(n_1, n_2, 0)$ by v_1-shrinking on e_3 and v_2-shrinking on e_1 (resp. v_3-shrinking on e_3 and v_2-shrinking on e_2) at the same time, and H_1 (resp. H_2) be the component of $\tilde{H}$ (resp. $\check{H}$) containing v_2. Obviously, H_1 is a proper sub-hypergraph of H_2. Then

(1) $(H_1; v_3) \prec (H_2; v_1)$;

(2) $(H_1; w, v_3) \prec (H_2; w, v_1)$ for $w \in \{u_{31}, u_{32}, \ldots, u_{3,t-2}\}$;

(3) For any positive integer k,

$$\begin{aligned} M_k(C_3(n_1, n_2, 0); v_3) &= M_k(H_1; v_3) + M_k(C_3(n_1, n_2, 0); v_3, [v_1]), \\ M_k(C_3(n_1, n_2, 0); v_1) &= M_k(H_2; v_1) + M_k(C_3(n_1, n_2, 0); v_1, [v_3]). \end{aligned}$$

By (1), we know that $M_k(H_1; v_3) \leq M_k(H_2; v_1)$, and there exists at least one k_0 satisfying

$$M_{k_0}(H_1; v_3) \leq M_{k_0}(H_2; v_1).$$

Now we only need to prove $M_k(C_3(n_1, n_2, 0); v_3, [v_1]) \leq M_k(C_3(n_1, n_2, 0); v_1, [v_3])$.

In fact, for any $W \in W_k(C_3(n_1, n_2, 0); v_3, [v_1])$, let $W = W_1W_2$, where W_1 is the shortest (v_3, v_1)-section, which consists of (v_3, v_3)-section (or (v_3, u_{3i})-section, $i \in [t-2]$) of length $l_1 - 1$ in H_1 and (v_3, e_3, v_1)(or (u_{3i}, e_3, v_1))-section of length 1, and W_2 is the remaining (v_1, v_3)-section of length l_2 in $C_3(n_1, n_2, 0)$. Then we have

$$\begin{aligned} M_k(C_3(n_1, n_2, 0); v_3, [v_1]) &= \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_1; v_3) M_{l_2}(C_3(n_1, n_2, 0); v_1, v_3) + \\ &\quad \sum_{i=1}^{t-2} \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_1; v_3, u_{3i}) M_{l_2}(C_3(n_1, n_2, 0); v_1, v_3). \end{aligned}$$

Similarly,

$$\begin{aligned} M_k(C_3(n_1, n_2, 0); v_1, [v_3]) &= \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_2; v_1) M_{l_2}(C_3(n_1, n_2, 0); v_3, v_1) + \\ &\quad \sum_{i=1}^{t-2} \sum_{l_1 + l_2 = k, l_1, l_2 \geq 1} M_{l_1 - 1}(H_2; v_1, u_{3i}) M_{l_2}(C_3(n_1, n_2, 0); v_3, v_1). \end{aligned}$$

Then by (1) and (2), we have $M_k(C_3(n_1, n_2, 0); v_3, [v_1]) \leq M_k(C_3(n_1, n_2, 0); v_1, [v_3])$. This completes the proof. □

Lemma 2.6. *If $n_1 \geq n_2 \geq n_3 \geq 1$, then $EE(C_3(n_1, n_2, n_3)) < EE(C_3(n_1 + n_3, n_2, 0))$.*

Proof. Let $E_1 = \{e_{31}, \ldots, e_{3,n_3}\}$ be set of the pendent edges attaching at v_3 and

$$E_0 = \{e_{31} \setminus \{v_3\}, \ldots, e_{3,n_3} \setminus \{v_3\}\}.$$

Denote $H = C_3(n_1, n_2, n_3) - E_1 + E_0$, now we have the following results.

(1) The component of H containing v_3 is $C_3(n_1, n_2, 0)$. By Lemma 2.5, we know that

$$(C_3(n_1, n_2, 0); v_3) \prec (C_3(n_1, n_2, 0); v_1);$$

(2) For any vertex $w \in V(E_0)$, $M_k(C_3(n_1, n_2, 0); w, v_3) = M_k(C_3(n_1, n_2, 0); w, v_1) = 0$, then

$$(C_3(n_1, n_2, 0); w, v_3) \preceq (C_3(n_1, n_2, 0); w, v_1).$$

Further by Lemma 1.2, we obtain $EE(C_3(n_1, n_2, n_3)) < EE(C_3(n_1 + n_3, n_2, 0))$. □

Remark 2. For $H \in \mathcal{U}(3, n)$, repeated by lemmas 2.5 and 2.6, we have $EE(H) \leq EE(C_3(n_1 + n_2 + n_3, 0))$ with equality if and only if $H \cong C_3(n_1 + n_2 + n_3, 0)$, where $n = (t-1)(n_1 + n_2 + n_3 + 3)$.

Let $C_2(n_1, n_2)$ be a t-uniform unicyclic hypergraphs obtained from $C_{2,t}$ by attaching n_i pendent edges at v_i for $i \in [2]$.

Lemma 2.7. *If $n_1 \geq n_2 \geq 1$, then $EE(C_2(n_1, n_2)) < EE(C_2(n_1 + n_2, 0))$.*

Proof. Let $E_1 = \{e_{21}, \ldots, e_{2,n_2}\}$ be set of the pendent edges attaching at v_2 and

$$E_0 = \{e_{21} \setminus \{v_2\}, \ldots, e_{2,n_2} \setminus \{v_2\}\}.$$

Denote $H = C_2(n_1, n_2) - E_1 + E_0$, now we have the following results.

(1) The component of H containing v_2 is $C_2(n_1, 0)$ and $(C_2(n_1, 0); v_2) \prec (C_2(n_1, 0); v_1)$;

(2) For any vertex $w \in V(E_0)$, $M_k(C_2(n_1, 0); w, v_2) = M_k(C_2(n_1, 0); w, v_1) = 0$, then

$$(C_2(n_1, 0); w, v_2) \preceq (C_2(n_1, 0); w, v_1).$$

Further by Lemma 1.2, we obtain $EE(C_2(n_1, n_2)) < EE(C_2(n_1 + n_2, 0))$. □

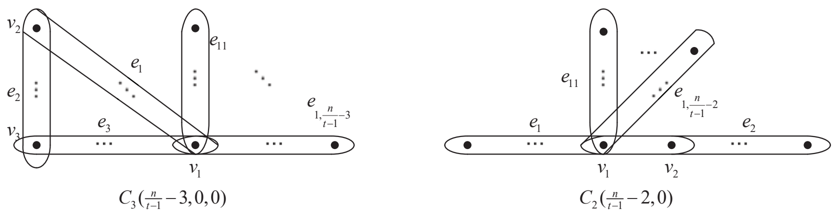


Figure 1: The hypergraphs $C_3(\frac{n}{t-1} - 3, 0, 0)$ and $C_2(\frac{n}{t-1} - 2, 0)$.

Let $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ (resp. $G[C_2(\frac{n}{t-1} - 2, 0)]$) be the graph obtained from $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$) by replacing each edge of $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$) by a complete graph K_t, as shown in Figure 2. Then $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ (resp. $G[C_2(\frac{n}{t-1} - 2, 0)]$) has the same adjacent matrix as $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$). Hence $G[C_3(\frac{n}{t-1} - 3, 0, 0)]$ (resp. $G[C_2(\frac{n}{t-1} - 2, 0)]$) has the same characteristic polynomial as $C_3(\frac{n}{t-1} - 3, 0, 0)$ (resp. $C_2(\frac{n}{t-1} - 2, 0)$). Then

$$EE(G[C_3(\frac{n}{t-1} - 3, 0, 0)]) = EE(C_3(\frac{n}{t-1} - 3, 0, 0)), \tag{2}$$

$$EE(G[C_2(\frac{n}{t-1} - 2, 0)]) = EE(C_2(\frac{n}{t-1} - 2, 0)). \tag{3}$$

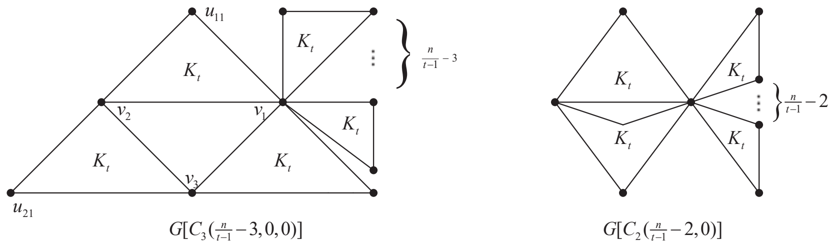
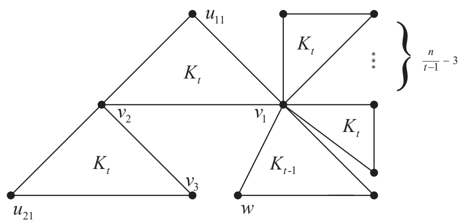


Figure 2: The graphs $G[C_3(\frac{n}{t-1}-3,0,0)]$ and $G[C_2(\frac{n}{t-1}-2,0)]$.

Lemma 2.8. $EE(C_3(\frac{n}{t-1}-3,0,0)) < EE(C_2(\frac{n}{t-1}-2,0))$.



A

Figure 3: The graph A.

Proof. Let A_1 be the hypergraph obtained from $C_3(\frac{n}{t-1}-3,0,0)$ (as shown in Figure 1) by v_3-shrinking on e_3, and A (as shown in Figure 3) be the graph obtained from A_1 by replacing each edge e of A_1 by a complete graph $K_{|e|}$.

Let $V_1 = \{v_3, u_{21}, u_{22}, \ldots, u_{2,t-2}\}$. Since v_2 is a cut-vertex of A, then for any $W \in W_k(A; v_3)$, it can be decomposed into two closed sections: (v_3, v_3)-walk W_1 whose internal vertices (if exist) are only possible to belong to $V_1 \cup \{v_2\}$; (v_2, v_2)-walk W_2 whose internal vertices (if exist) are only possible to belong to $V(A) - V_1$. Note that if the length of W_2 is at least 1, then v_2 must be an internal vertex of W_1.

Let W_1' be a walk which is obtained from W_1 by replacing v_3 by u_{11}, u_{21} by v_1 and u_{2i} by u_{1i} for $i \in \{2, 3, \ldots, t-2\}$. Obviously, $W' = W_1'W_2 \in W_k(A; u_{11})$. It is easy to see that $M_4(A; v_3) < M_4(A; u_{11})$. Thus $(A; v_3) \prec (A; u_{11})$.

Let $V_2 = \{v_1, u_{31}, u_{32}, \ldots, u_{3,t-2}\}$. For any $W \in W_k(A; w, v_3)$ and $w \in V_2$, let $W = W_1W_2$, where W_1 is the longest (w, v_2)-section of W which the internal vertices (if exist) are only possible to belong to $V(A) - V_1$, and W_2 is the the remaining (v_2, v_3)-section of W which the internal vertices (if exist) are only possible to belong to $V_1 \cup \{v_2\}$. W_2' is obtained from W_2 by replacing v_3 by u_{11}, u_{21} by v_1 and u_{2i} by u_{1i} for $i \in \{2, 3, \ldots, t-2\}$. Obviously, $W' = W_1W_2' \in W_k(A; w, u_{11})$. Thus $(A; w, v_3) \prec (A; w, u_{11})$.

Let $E_{v_3} = \{wv_3 : w \in \{v_1, u_{31}, \ldots, u_{3,t-2}\}\}$ and $E_{u_{11}} = \{wu_{11} : w \in \{v_1, u_{31}, \ldots, u_{3,t-2}\}\}$. By Lemma 1.2, we have $EE(A + E_{v_3}) < EE(A + E_{u_{11}})$.

Further by lemmas 2.2, 2.7 and (2), (3), we have $EE(C_3(\frac{n}{t-1}-3,0,0)) < EE(C_2(\frac{n}{t-1}-2,0))$. □

From lemmas 2.2, 2.7 and 2.8, Remark 2, we have our main result.

Theorem 2.9. *Let H be a t-uniform unicyclic hypergraph, $EE(H) \le EE(C_2(\frac{n}{t-1}-2,0))$ and the equality if and only if $H \cong C_2(\frac{n}{t-1}-2,0)$.*

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3. Statements and Declarations

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Conflicts of Interest: The authors declare no conflict of interest.

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