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Nonlinear Langevin time-delay differential equations with generalized Caputo fractional derivatives

Nguyen Minh Dien[a]

[a]*Faculty of Education, Thu Dau Mot University, Binh Duong Province, Vietnam*

Abstract. In a Banach space, we consider the nonlinear Langevin time-delay differential equations with ψ-Caputo fractional derivatives. Using weighted norms involving Mittag-Leffler functions, we obtain some existence and uniqueness of solutions of the problem. Besides, in some cases, the condition for the problem having a unique solution can be relaxed using Burton's method. We also obtain some Ulam-Hyers and Ulam-Hyers-Rassias Mittag-Leffler stability results for the main equation. Two examples are given to illustrate our theoretical findings.

1. Introduction

The Langevin equation plays an important role in describing the fluctuation phenomenon in Brownian motion. It brings many benefits in depicting the time evolution of the velocity of the Brownian motion. However, the classical Langevin equation has been restricted for some complex systems such as dynamical processes in media (see [15]). The fractional Langevin equations have been proposed as one of the ways to overcome the restriction (see e.g. [1, 11, 16] and reference therein). The fact shows that the fractional form of the Langevin equations has provided a useful tool for investigating anomalous diffusion.

In recent years, fractional Langevin delay equations have attracted the attention of many researchers. In fact that Kumar et al [13] investigated the problem of controllability of linear and nonlinear fractional Langevin delay dynamical systems with multiple delays and distributed delays. Mahmudov [14] introduced delayed Mittag-Leffler type functions and applied them to investigate non-homogeneous fractional delayed Langevin equations with Riemann-Liouville fractional derivatives. Recently, Ahmadova and Mahmudov [2] considered both homogeneous and inhomogeneous fractional Langevin differential equations, presented explicit analytical solutions, investigated Ulam-Hyers, and applied them to electric circuit theory

In 2017, Almeida [3] followed the idea in [12] to propose a concept of a Caputo fractional derivative of a function with respect to another (called ψ-Captuto fractional derivatives) and has attracted the attention of numerous researchers. In particular, Langevin equations involving ψ-Caputo fractional derivatives without delays have been studied in [6–9], but, Langevin time-delays differential equations regarding these derivatives are still not considered.

Inspired by the derivative concept and motivated by the above analyses, we study the nonlinear Langevin time-delay differential equation involving ψ-Caputo fractional derivatives. Explicitly, let $0 <$

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Email address: diennm@tdmu.edu.vn; nmdien81@gmail.com (Nguyen Minh Dien)

$\alpha, \beta \leq 1$ and $\mathbb{B}$ be a Banach space with the norm $\|\cdot\|$. Let μ, a, b be real numbers with $\mu < a < b$, and let $\varrho \in C([\mu, a], \mathbb{B})$, $\sigma \in C([a, b], [\mu, b])$ with $\sigma(t) \leq t$. We consider the following nonlinear Langevin time-delay differential equation

$$ {}^{C}D^{\psi,\beta}_{a+}\left({}^{C}D^{\psi,\alpha}_{a+} + \lambda\right)u(t) = f(t, u(t), u(\sigma(t))), \quad t \in [a, b] \tag{1} $$

subject to the conditions

$$ u(t) = \varrho(t), \ \mu \leq t \leq a \text{ and } {}^{C}D^{\psi,\alpha}_{a+}u(a) = \xi, \tag{2} $$

where ${}^{C}D^{\psi,\alpha}_{a+}$, ${}^{C}D^{\psi,\beta}_{a+}$ are ψ-Caputo fractional derivatives and $\lambda \in \mathbb{R}$ is a friction constant. The main aim of the current paper is to discuss the existence and uniqueness of solutions of the problem as well as investigate Ulam-Hyers and Ulam-Hyers-Rassias Mittag-Leffler stability for the equation.

Our main contributions are that: (i) we prove that the problem has a unique solution by using weighted norms involving Mittag-Leffler functions; (ii) if $\sigma(t) \geq r$ for some positive number r, then the problem has a unique solution under some weaker conditions than the general cases; (iii) we derive some Ulam-Hyers and Ulam-Hyers-Rassias Mittag-Leffler stability results for the main equation.

The paper is organized as follows. In section 2, we present the concept of ψ-Caputo fractional derivative and its properties. We also introduce some preparatory lemmas for the proof of the main results. In section 3, we present the main results of the paper. In section 4, we construct some examples to illustrate the theoretical findings.

2. Preliminaries

This section presents some definitions and preliminary lemmas that we will use in the subsequent section. We begin by recalling the Gamma function

$$ \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t, \quad \alpha > 0. $$

We also recall the definition of the Mittag-Leffler function.

Definition 2.1. *Let $\alpha, \beta > 0$ and $z \in \mathbb{C}$. The Mittag-Leffler function is defined by the power series as follows*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}. $$

Throughout the present paper, we write $E_\alpha(z)$ stands for $E_{\alpha,1}(z)$. Next, we give here some properties of the Mittag-Leffler function.

Lemma 2.2 (see [10, 18])**.** *Let α and β be two positive numbers.*
(i). Suppose that $\lambda \in \mathbb{C}$ is not eigenvalue of the Abel integral operator. Then, we have

$$ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}E_\alpha(\lambda\tau^\alpha) = E_\alpha(\lambda t^\alpha) - 1. $$

(ii). For any $t \geq 0$, we have

$$ 0 \leq E_{\alpha,\beta}(-t) \leq 1/\Gamma(\beta). $$

We continue by introducing the concepts of fractional integral and fractional derivative of a function depending on another function. To this aim, we firstly define the class of the function

$$ H^1_+[a, b] = \left\{\psi : \psi \in C[a, b] \cap C^1(a, b) \text{ and } \psi'(t) > 0 \text{ for all } t \in [a, b]\right\}. $$

Definition 2.3 (see [3, 12])**.** *Let $\alpha > 0$, $a < b$, $\psi \in H^1_+[a,b]$.*

(i). For $f \in L^1(a,b)$, the fractional integral of a function f with respect to the function ψ is defined by

$$I^{\alpha,\psi}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(\tau)\, \mathrm{d}\tau.$$

(ii). For $f \in C^n[a,b]$, the Caputo fractional derivative of a function f with respect to the function ψ is defined by

$${}^C D^{\psi,\alpha}_{a+} f(t) = I^{n-\alpha,\psi}_{a+} \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t),$$

where $n = [\alpha] + 1$ for $n \neq \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

To further investigate the properties of the fractional integrals and fractional derivatives of a function concerning another function, we refer to [3, 12].

To end this section, we define the concepts Ulam-Hyers and Ulam-Hyers-Rassias stability. These concepts were adapted from paper [17].

Definition 2.4. *Equation (1) is called Ulam-Hyers stable if there exists a positive number C_f such that for each $\epsilon > 0$ and for each solution $v \in C([\mu,b],\mathbb{B})$ of the following inequality*

$$\left\| {}^C D^{\psi,\beta}_{a+} \left({}^C D^{\psi,\alpha}_{a+} + \lambda\right) v(t) - f(t, v(t), v(\sigma(t))) \right\| \leq \epsilon, \quad t \in [a,b], \tag{3}$$

there exists a solution $u \in C([\mu,b],\mathbb{B})$ of Equation (1) such that

$$\|u(t) - v(t)\| \leq C_f \epsilon, \quad t \in [\mu,b].$$

Remark 2.5. *A function u is a solution of inequality (3) if there exists a function $h \in C([a,b],\mathbb{B})$ such that*

$$\|h(t)\| \leq \epsilon$$

for all $t \in [a,b]$ and satisfying the following equation

$${}^C D^{\psi,\beta}_{a+} \left({}^C D^{\psi,\alpha}_{a+} + \lambda\right) v(t) = f(t, v(t), v(\sigma(t))) + h(t), \quad t \in [a,b].$$

Definition 2.6. *Equation (1) is called Ulam-Hyers-Rassias stable with respect to $\varphi \in C([a,b],\mathbb{R})$ if there exists a positive number C_f such that for each $\epsilon > 0$ and for each $v \in C([\mu,b],\mathbb{B})$ satisfying the following inequality*

$$\left\| {}^C D^{\psi,\beta}_{a+} \left({}^C D^{\psi,\alpha}_{a+} + \lambda\right) v(t) - f(t, v(t), v(\sigma(t))) \right\| \leq \epsilon\varphi(t), \quad t \in [a,b], \tag{4}$$

there exists a solution $u \in C([\mu,b],\mathbb{B})$ of Equation (1) such that

$$\|u(t) - v(t)\| \leq C_f \epsilon\varphi(t), \quad t \in [\mu,b].$$

The Ulam-Hyers-Rassias stable with respect to $\varphi = E_p(\cdot)$ is called Ulam-Hyers-Rassias Mittag-Leffler stable.

Remark 2.7. *A function u is a solution of inequality (4) if there exists a function $h \in C([a,b],\mathbb{B})$ such that*

$$\|h(t)\| \leq \epsilon\varphi(t)$$

for all $t \in [a,b]$ and satisfying the following equation

$${}^C D^{\psi,\beta}_{a+} \left({}^C D^{\psi,\alpha}_{a+} + \lambda\right) v(t) = f(t, v(t), v(\sigma(t))) + h(t), \quad t \in [a,b].$$

3. Uniqueness and Ulam-Hyers-Rassias stability

This section is devoted to state and prove the main results of the present paper. Thanks to Lemma 3.2 in [9], we find that if u is a solution of the problem (1) and (2) the u satisfies the following integral equation

$$u(t) = \varrho(0)E_\alpha\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) + (\lambda\varrho(0)+\xi)(\psi(t)-\psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t)-\psi(a))^\alpha\right)$$
$$+\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t)-\psi(\tau))^\alpha\right) f(\tau,u(\tau),u(\sigma(\tau)))\,\mathrm{d}\tau \tag{5}$$

for $t \in (a,b]$ and $u(t) = \varrho(t)$ for $t \in [\mu, a]$.

For convenience in stating the main results, we make the following assumptions.

- **Assumption** ($\mathcal{A}1$). $\psi \in H_+^1[a,b]$, $f \in C([a,b]\times\mathbb{B}\times\mathbb{B},\mathbb{B})$ and $\sigma \in C([a,b],[\mu,b])$ satisfying $\sigma(t) \le t$ on $[a,b]$.
- **Assumption** ($\mathcal{A}2$). There exists $L > 0$ such that
$$\left\| f(t,u_1,v_1) - f(t,u_2,v_2)\right\| \le L\left(\|u_1-u_2\| + \|v_1-v_2\|\right)$$
for any $u_1,u_2,v_1,v_2 \in \mathbb{B}$ and for every $t \ge a$.
- **Assumption** ($\mathcal{A}3$). There exists $L > 0$ such that
$$\left\| f(t,u_1,v) - f(t,u_2,v)\right\| \le L\|u_1-u_2\|$$
for any $u_1,u_2 \in \mathbb{B}$ and for every $t \ge a$.

To obtain the main results, we use the following weighted norm involving a Mittag-Leffler function as follows

$$|||u|||_{\omega,b} = \max\left\{\frac{\|u(t)\|}{G(\omega,t)} : \mu \le t \le b\right\}, \tag{6}$$

where ω is a positive number, and G is a positive and continuous function (with respect to t) defined by

$$G(\omega,t) = \begin{cases} E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) & \text{for } t \in [a,b] \\ 1 & \text{for } t \in [\mu,a). \end{cases}$$

We are now in position to state and prove the first result of the paper.

Theorem 3.1. *Suppose that Assumptions* ($\mathcal{A}1$) – ($\mathcal{A}2$) *are satisfied. Then the problem (1) and (2) have a unique solution belongs to* $C([\mu,b],\mathbb{B})$.

Proof. Let us define the following operator

$$Q : C([\mu,b],\mathbb{B}) \to C([\mu,b],\mathbb{B})$$

defined by

$$Qu(t) = \begin{cases} \text{the right hand side of equation (5)} & \text{for } t \in [a,b] \\ \varrho(t) & \text{for } t \in [\mu,a]. \end{cases}$$

We will show that Q is a contraction mapping on the Banach space $C([\mu,b],\mathbb{B})$ with the weighted norm given by (6). Note that $Qu(t) = Qv(t) = \varrho(t)$ on $[\mu,a]$ for all $u,v \in C([\mu,b],\mathbb{B})$, so we only consider $t \in (a,b]$. Using Lemma 2.2, we have

$$\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}G(\omega,\tau)\,\mathrm{d}\tau = \frac{\Gamma(\alpha+\beta)}{\omega}\left(E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right)-1\right)$$
$$\le \frac{\Gamma(\alpha+\beta)}{\omega}E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) \tag{7}$$

for any $t \in [a, b]$. Thanks to the inequality (7) and Lemma 2.2, we get that

$$\begin{aligned}
\|Qu(t) - Qv(t)\| &\le \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} \left\| f(\tau, u(\tau), u(\sigma(\tau))) - f(\tau, v(\tau), v(\sigma(\tau))) \right\| \mathrm{d}\tau \\
&\le \frac{L}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} G(\omega, \tau) \\
&\qquad \times (\|u(\tau)-v(\tau)\| + \|u(\sigma(\tau)) - v(\sigma(\tau))\|) / G(\omega,\tau) \, \mathrm{d}\tau \\
&\le \frac{2L}{\Gamma(\alpha+\beta)} \|u - v\|_{\omega,b} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} G(\omega,\tau) \, \mathrm{d}\tau \\
&\le \frac{2L}{\omega} E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) \|u - v\|_{\omega,b}.
\end{aligned}$$

This implies that

$$\|Qu - Qv\|_{\omega,b} \le \frac{2L}{\omega} \|u - v\|_{\omega,b}.$$

Choosing $\omega > 2L$, we obtain from the latter inequality that Q is a contraction mapping. Hence, Q has a unique fixed point belongs to $C([\mu, b], \mathbb{B})$, which is a solution of the problem (1) and (2). The proof process is implemented completely. □

If there exist $r > 0$ such that $\sigma(t) \le t - r$, then the conditions in order to problem (1) and (2) having a unique solution can be relaxed. In fact, we have the following result.

Theorem 3.2. *Suppose that Assumptions ($\mathcal{A}$1) and ($\mathcal{A}$3) hold. Suppose further that there exists a positive number r such that $\sigma(t) \le t - r$ for all $t \in [a, b]$. Then the problem (1) and (2) has a unique solution in $C([\mu, b], \mathbb{B})$.*

Remark 3.3. *It is obvious that Assumption ($\mathcal{A}$3) is weaker than Assumption ($\mathcal{A}$2). This shows the conditions in Theorem 3.1 have been relaxed in Theorem 3.2 in the case $\sigma(t) \le t - r$.*

Proof. We follow the Burton's method (see [4, 5]) to prove the result of Theorem. Indeed, we divide $[a, b]$ into n parts by $a = T_0 < T_1 < .. < T_n = b$ with $T_k - T_{k-1} = l$ $(k = 1, 2, .., n)$ for some $l \le r$. Fixed $\omega > L$, we divide the proof into n steps as follow.

Step 1. Let $(\mathbb{B}_1, \|\cdot\|_{\omega,T_1})$ be the Banach space of all continuous functions $u : [\mu, T_1] \to \mathbb{B}$ such that $u(t) = \varrho(t)$ for $t \in [\mu, a]$, where the norm defined in (6). We consider the operator $Q_1 : \mathbb{B}_1 \to \mathbb{B}_1$ defined by

$$Q_1 u(t) = \begin{cases} \text{the right hand side of equation (5)} & \text{for } t \in [a, T_1] \\ \varrho(t) & \text{for } t \in [\mu, a]. \end{cases}$$

Since $Q_1 u(t) = Q_1 v(t) = \varrho(t)$ for all $u, v \in \mathbb{B}_1$ and $t \in [\mu, a]$, we only consider $t \in [a, T_1]$. We first have $\sigma(t) \le t - r \le t - l \le a$ for all $t \le T_1$. This implies $u(\sigma(t)) = v(\sigma(t)) = \varrho(t)$ for $t \le T_1$. Using (7), we obtain

$$\begin{aligned}
\|Q_1 u(t) - Q_1 v(t)\| &\le \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} \left\| f(\tau, u(\tau), u(\sigma(\tau))) - f(\tau, v(\tau), v(\sigma(\tau))) \right\| \mathrm{d}\tau \\
&\le \frac{L}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} G(\omega,\tau) \|u(\tau) - v(\tau)\| / G(\omega,\tau) \, \mathrm{d}\tau \\
&\le \frac{L}{\Gamma(\alpha+\beta)} \|u - v\|_{\omega,T_1} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1} G(\omega,\tau) \, \mathrm{d}\tau \\
&\le \frac{L}{\omega} E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) \|u - v\|_{\omega,T_1}.
\end{aligned}$$

This implies that

$$|||Q_1u - Q_1v|||_{\omega,T_1} \leq \frac{L}{\omega}|||u - v|||_{\omega,T_1}.$$

Since $\omega > L$, we conclude from the latter inequality that Q_1 is a contraction mapping on $\mathbb{B}_1$. Thus, there exists a unique $u_1 \in C([\mu, T_1], \mathbb{B})$ satisfying the problem (1) and (2) on $[\mu, T_1]$.

Step 2. We now extend the interval $[\mu, T_1]$ in Step 1 into $[\mu, T_1]$ by using u_1 as the new initial function. To this aim, we consider the the Banach space $(\mathbb{B}_2, |||\cdot|||_{\omega,T_2})$ of all continuous functions $u : [\mu, T_2] \to \mathbb{B}$ such that $u(t) = u_1(t)$ for $t \in [\mu, T_1]$, where the norm defined in (6). We consider the operator $Q_2 : \mathbb{B}_2 \to \mathbb{B}_2$ defined by

$$Q_2u(t) = \begin{cases} \text{the right hand side of equation (5)} & \text{for } t \in [T_1, T_2] \\ u_1(t) & \text{for } t \in [\mu, T_1]. \end{cases}$$

Since $Q_1u(t) = Q_1v(t) = u_1(t)$ for all $u, v \in \mathbb{B}_2$ and $t \in [\mu, T_1]$, we now consider $t \in [T_1, T_2]$. We first have $\sigma(t) \leq t - r \leq t - l \leq T_1$ for all $t \leq T_2$. So, $u(\sigma(t)) = v(\sigma(t)) = u_1(t)$ for $t \leq T_2$. Using the method as in Step 1, we obtain

$$|||Q_2u - Q_2v|||_{\omega,T_2} \leq \frac{L}{\omega}|||u - v|||_{\omega,T_2},$$

where $\omega > L$. Thus, Q_2 is also a contraction mapping on $\mathbb{B}_2$ and there is a unique $u_2 \in C([\mu, T_2], \mathbb{B})$ satisfying the problem (1) and (2) on $[\mu, T_2]$.

Step 3. Repeating this process n-times, we can find a unique $u = u_n \in C([\mu, T_n], \mathbb{B})$ satisfying the problem (1) and (2) on $[\mu, T_n] = [\mu, b]$. This finishes the proof of Theorem. □

We continue by presenting results on Ulam-Hyers and Ulam-Hyers-Rassias stability.

Theorem 3.4. *Equation (1) is Ulam-Hyers stable when Assumptions* $(\mathcal{A}1) - (\mathcal{A}2)$ *are true.*

Proof. Since Assumptions $(\mathcal{A}1) - (\mathcal{A}2)$ hold, by virtue of Theorem 3.1, we conclude that the problem (1) and (2) has a unique solution $u \in C([\mu, T], \mathbb{B})$, which satisfies the following integral equation

$$\begin{aligned} u(t) &= \varrho(0)E_\alpha\left(-\lambda(\psi(t) - \psi(a))^\alpha\right) + (\lambda\varrho(0) + \xi)(\psi(t) - \psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t) - \psi(a))^\alpha\right) \\ &\quad + \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t) - \psi(\tau))^\alpha\right) f(\tau, u(\tau), u(\sigma(\tau)))\, \mathrm{d}\tau \end{aligned}$$

on interval $[a, b]$.

Let v is a solution of the inequality (3). Then, for each $\epsilon > 0$, it follows from Remark 2.5 that v satisfies the following integral equation on the interval $[a, b]$

$$\begin{aligned} v(t) &= \varrho(0)E_\alpha\left(-\lambda(\psi(t) - \psi(a))^\alpha\right) + (\lambda\varrho(0) + \xi)(\psi(t) - \psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t) - \psi(a))^\alpha\right) \\ &\quad + \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t) - \psi(\tau))^\alpha\right)\left(f(\tau, v(\tau), v(\sigma(\tau))) + h(\tau)\right) \mathrm{d}\tau, \end{aligned}$$

where $h \in C([a, b], \mathbb{B})$ with $\|h(t)\| \leq \epsilon$ for all $t \in [a, b]$. Thus, using Lemma 2.2, we get

$$\begin{aligned} &||v(t) - \varrho(0)E_\alpha\left(-\lambda(\psi(t) - \psi(a))^\alpha\right) - (\lambda\varrho(0) + \xi)(\psi(t) - \psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t) - \psi(a))^\alpha\right) \\ &- \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t) - \psi(\tau))^\alpha\right) f(\tau, v(\tau), v(\sigma(\tau)))\, \mathrm{d}\tau|| \\ &\leq \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t) - \psi(\tau))^\alpha\right)\|h(\tau)\|\, \mathrm{d}\tau \\ &\leq \frac{(\psi(t) - \psi(a))^{\alpha+\beta}}{(\alpha + \beta)\Gamma(\alpha + \beta)}\epsilon \leq \frac{(\psi(b) - \psi(a))^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)}\epsilon. \end{aligned}$$

Moreover, in the process of the proof of Theorem 3.1, we have proved that

$$\begin{aligned}&\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}\left\| f(\tau,v(\tau),v(\sigma(\tau))) - f(\tau,u(\tau),u(\sigma(\tau)))\right\| \mathrm{d}\tau \\ &\leq \frac{2L\Gamma(\alpha+\beta)}{\omega} E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) \|\!|u-v|\!\|_{\omega,b}. \end{aligned} \tag{8}$$

Thanks to the two latter inequalities, we get

$$\begin{aligned}\|v(t)-u(t)\| &\leq ||v(t) - \varrho(0)E_\alpha\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) - (\lambda\varrho(0)+\xi)(\psi(t)-\psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) \\ &- \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t)-\psi(\tau))^\alpha\right) f(\tau,u(\tau),u(\sigma(\tau)))\,\mathrm{d}\tau|| \\ &\leq ||v(t) - \varrho(0)E_\alpha\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) - (\lambda\varrho(0)+\xi)(\psi(t)-\psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) \\ &- \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t)-\psi(\tau))^\alpha\right) f(\tau,v(\tau),v(\sigma(\tau)))\,\mathrm{d}\tau|| \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}\left\| f(\tau,v(\tau),v(\sigma(\tau))) - f(\tau,u(\tau),u(\sigma(\tau)))\right\| \mathrm{d}\tau \\ &\leq \frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\epsilon + \frac{2L}{\omega}E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right)\|\!|u-v|\!\|_{\omega,b}.\end{aligned}$$

Using the fact that $E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) \geq 1/\Gamma(\alpha+\beta)$ for all $t \in [a,b]$ and $u(t) = v(t)$ on $[\mu, a]$, we get

$$\|\!|u-v|\!\|_{\omega,b} \leq \frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{\alpha+\beta}\epsilon + \frac{2L}{\omega}\|\!|u-v|\!\|_{\omega,b}.$$

Choosing $\omega > 2L$, then $\kappa = 2L/\omega < 1$ and we obtain

$$\|v(t)-u(t)\|/E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right) \leq \|\!|u-v|\!\|_{\omega,b} \leq \frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{(\alpha+\beta)(1-\kappa)}\epsilon$$

or

$$\begin{aligned}\|v(t)-u(t)\| &\leq \frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{(\alpha+\beta)(1-\kappa)}E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right)\epsilon \\ &\leq \frac{(\psi(b)-\psi(a))^{\alpha+\beta}}{(\alpha+\beta)(1-\kappa)}E_{\alpha+\beta}\left(\omega(\psi(b)-\psi(a))^{\alpha+\beta}\right)\epsilon\end{aligned}$$

due to the function $E_{\alpha+\beta}(\cdot)$ is increasing on $(0,+\infty)$. The latter inequality shows that Equation (1) is Ulam-Hyers stable. The proof of Theorem is completed. □

Theorem 3.5. *Assume that Assumptions* $(\mathcal{A}1)-(\mathcal{A}2)$ *hold and* $\omega > 2L$. *Then, Equation (1) is Ulam-Hyers-Rassias Mittag-Leffler stable with respect to* $\varphi(t) = E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right)$.

Proof. We denote by v a solution of the inequality (3). For each $\epsilon > 0$, it follows from Remark 2.7 that v satisfies the following integral equation

$$\begin{aligned}v(t) &= \varrho(0)E_\alpha\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) + (\lambda\varrho(0)+\xi)(\psi(t)-\psi(a))^\alpha E_{\alpha,\alpha+1}\left(-\lambda(\psi(t)-\psi(a))^\alpha\right) \\ &\quad + \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t)-\psi(\tau))^\alpha\right)\left(f(\tau,v(\tau),v(\sigma(\tau))) + h(\tau)\right)\mathrm{d}\tau,\end{aligned}$$

where $h \in C([a,b],\mathbb{B})$ with $\|h(t)\| \leq \epsilon\varphi(t) = \epsilon E_{\alpha+\beta}\left(\omega(\psi(t)-\psi(a))^{\alpha+\beta}\right)$ for all $t \in [a,b]$. By using Lemma 2.2 and direct computation, we have

$$\begin{aligned}
&\|v(t) - \varrho(0)E_{\alpha}\left(-\lambda(\psi(t) - \psi(a))^{\alpha}\right) - (\lambda\varrho(0) + \xi)(\psi(t) - \psi(a))^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda(\psi(t) - \psi(a))^{\alpha}\right) \\
&\quad - \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t) - \psi(\tau))^{\alpha}\right) f(\tau, v(\tau), v(\sigma(\tau)))\, \mathrm{d}\tau\| \\
&\quad \leq \frac{\epsilon}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1} E_{\alpha+\beta}\left(\omega(\psi(\tau) - \psi(a))^{\alpha+\beta}\right) \mathrm{d}\tau \\
&\quad \leq \frac{\epsilon}{\omega} E_{\alpha+\beta}\left(\omega(\psi(t) - \psi(a))^{\alpha+\beta}\right). \qquad (9)
\end{aligned}$$

By straightforward, we obtain from (8) and (9) that

$$\begin{aligned}
&\|v(t) - u(t)\| \leq \|v(t) - \varrho(0)E_{\alpha}\left(-\lambda(\psi(t) - \psi(a))^{\alpha}\right) - (\lambda\varrho(0) + \xi)(\psi(t) - \psi(a))^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda(\psi(t) - \psi(a))^{\alpha}\right) \\
&\quad - \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}\left(-\lambda(\psi(t) - \psi(\tau))^{\alpha}\right) f(\tau, v(\tau), v(\sigma(\tau)))\, \mathrm{d}\tau\| \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1} \left\| f(\tau, v(\tau), v(\sigma(\tau))) - f(\tau, u(\tau), u(\sigma(\tau))) \right\| \mathrm{d}\tau \\
&\quad \leq \frac{\epsilon}{\omega} E_{\alpha+\beta}\left(\omega(\psi(t) - \psi(a))^{\alpha+\beta}\right) + \frac{2L}{\omega} E_{\alpha+\beta}\left(\omega(\psi(t) - \psi(a))^{\alpha+\beta}\right) \|u - v\|_{\omega,b}.
\end{aligned}$$

This deduces that

$$\|u - v\|_{\omega,b} \leq \frac{\epsilon}{\omega} + \frac{2L}{\omega} \|u - v\|_{\omega,b}.$$

It implies that

$$\|u - v\|_{\omega,b} \leq \frac{\epsilon}{(1-\kappa)\omega},$$

where $\kappa = 1 - 2L/\omega$. As a consequent

$$\|v(t) - u(t)\| / E_{\alpha+\beta}\left(\omega(\psi(t) - \psi(a))^{\alpha+\beta}\right) \leq \|u - v\|_{\omega,b} \leq \frac{\epsilon}{(1-\kappa)\omega}$$

or

$$\|v(t) - u(t)\| \leq \frac{\epsilon}{(1-\kappa)\omega} E_{\alpha+\beta}\left(\omega(\psi(t) - \psi(a))^{\alpha+\beta}\right).$$

The proof of Theorem is done. □

4. Examples

In this section, we present two examples to show the applicability of the obtained results of the paper.

Example 4.1. *Let $\mathbb{B} = \mathbb{R}$ and $\psi(t) = \ln t$. We consider the following problem*

$$\begin{cases} {}^{C}D_{1+}^{\ln,3/4}\left({}^{C}D_{1+}^{\ln,4/5} + 1\right)u(t) = e^t/(|u(t)| + 1) + u(t^2/e), & t \in [1, e] \\ u(t) = \sin^2 t + t, \ t \in [0, 1], \ {}^{C}D_{1+}^{\ln,4/5} u(1) = 2, \end{cases} \qquad (10)$$

where $a = 1, b = e$, $\mu = 0$, $\alpha = 4/5$, $\beta = 2/3$, $\lambda = 1$, and $f(t, u, v) = e^t/(|u| + 1) + v$ with $\sigma(t) = t^2/e$. We can easily verify that $f \in C([1, e] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and

$$\left| f(t, u_1, v_1) - f(t, u_2, v_2) \right| \leq e(|u_1 - u_2| + |v_1 - v_2|)$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $t \in [1, e]$. Therefore, we conclude from Theorem 3.1 that the problem (10) has a unique solution. Moreover, based on Theorem 3.4 and Theorem 3.5, we also conclude that the first equation of the problem is Ulam-Hyers and Ulam-Hyers-Rassias Mittag-Leffler stable.

Example 4.2. *Let $\mathbb{B} = \mathbb{R}$ and $\psi(t) = t$. Consider the following problem*

$$\begin{cases} {}^{C}D_{0+}^{t,1/3}\left({}^{C}D_{0+}^{t,3/5} + 1/3\right)u(t) = \sin u(t) + u^3(t - 1/(t+2)), \ \ t \in [0, 2] \\ u(t) = t + t^2, \ t \in [-1, 0], \ {}^{C}D_{0+}^{t,3/5}u(0) = 0, \end{cases} \tag{11}$$

where $a = 0, b = 2, \mu = -1, \alpha = 3/5, \beta = 1/3, \lambda = 1/3$, and $f(t, u, v) = \sin u + v^3$ with $\sigma(t) = t - 1/(t+2) \leq t - 1/4$. We can easily verify that $f \in C([0, 2] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and

$$\left| f(t, u_1, v) - f(t, u_2, v) \right| \leq |u_1 - u_2|$$

for all $u_1, u_2, v \in \mathbb{R}$ and $t \in [0, 2]$. Thus, we conclude from Theorem 3.2 that the problem (11) has a unique solution.

5. Conclusions

We have investigated the nonlinear Langevin time-delay differential equations connected to the generalized Caputo fractional derivatives. By using weighted norm, we have obtained the existence and uniqueness of solutions of the problem. In some cases, we have proved that the condition for the problem to have a unique solution can be relaxed. We have also obtained the results on Ulam-Hyers and Ulam-Hyers-Rassias Mittag-Leffler stability. In the future works, we would like to consider Langevin time-delay differential equations where source functions may have a singularity.

Acknowledgements

References

[1] M. I. Abbas, M. A. Ragusa, *Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions*, Appl. Anal., 101(9), 3231-3245 (2020).
[2] A. Ahmadova, N. I. Mahmudov, *Langevin differential equations with general fractional orders and their applications to electric circuit theory*, J. Comput. Appl. Math., **388**, 113299 (2021).
[3] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simulat., **44**, 460-481 (2017).
[4] T. A. Burton, Existence and uniqueness results by progressive contractions for integro-differential equations, Nonlinear Dyn. Syst. Theory., **16** (4), 366-371, (2016).
[5] T. A. Burton, *A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions*, Fixed Point Theory, **20** (1), 107-113, (2019).
[6] N. M. Dien, D. D. Trong, *On the nonlinear generalized Langevin equation involving ψ-Caputo fractional derivatives*, Fractals, **29** (6), 2150128 (2021). DOI: 10.1142/S0218348X21501280
[7] N. M. Dien, T. Q. Viet, *On mild solutions of the p-Laplacian fractional Langevin equations with anti-periodic type boundary conditions*, Int. J. Appl. Comput. Math., **99** (9), 1823-1848 (2022). DOI: 10.1080/00207160.2021.2012167
[8] N. M. Dien, *Generalized weakly singular Gronwall-type inequalities and their applications to fractional differential equations*, Rocky Mountain J. Math., **51** (2), 689-707 (2021). DOI: 10.1216/rmj.2021.51.689
[9] N. M. Dien *Existence and continuity results for a nonlinear fractional Langevin equation with a weakly singular source*, J. Integral Equ. Appl., **33**(3): 349-369 (2021). DOI: 10.1216/jie.2021.33.349
[10] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Heidelberg New York Dordrecht London (2014).
[11] E. Ilhan, *Analysis of the spread of Hookworm infection with Caputo-Fabrizio fractional derivative*, Turkish Journal of Science, 7(1), 43-52, (2022)
[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland mathematics studies, **207**, Amsterdam: Elsevier (2006).
[13] P. S. Kumar, K. Balachandran, N. Annapoorani, *Controllability of nonlinear fractional Langevin delay systems*, Nonlinear Anal.: Model. Control., **23** (3), 321-340 (2018).
[14] N. I. Mahmudov, *Fractional Langevin type delay equations with two fractional derivatives*, Appl. Math. Lett., **103**, 106215 (2020).
[15] E. Lutz, *Fractional Langevin equation*, Phys. Rev. E., **64** (5), 051106 (2001). https://doi.org/10.1103/PhysRevE.64.051106
[16] R. Ullah, M. Waseem, N. B. Rosli, J. Kafle, *Analysis of COVID-19 fractional model pertaining to the Atangana-Baleanu-Caputo fractional derivatives*, J. Funct. Spaces, Vol. 2021, art.n.2643572 (2021).
[17] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator*, J. Fixed Point Theory Appl., **96**, (2018).
[18] J. Wang, M. Feckan, Y. Zhou, *Presentation of solutions of impulsive fractional Langevin equations and existence results*, Eur. Phys. J.: Spec. Top., **222** (8): 1857-1874 (2013).