Filomat 37:19 (2023), 6453–6461
https://doi.org/10.2298/FIL2319453A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remarks on the ring $B_1(X)$

Mohammad Reza Ahmadi Zand[a], Zahra Khosravi[a]

[a] *Department of Mathematical Sciences, Yazd University, Yazd, Iran*

Abstract. Let X be a nonempty topological space, $C(X)_F$ be the set of all real-valued functions on X which are discontinuous at most on a finite set and $B_1(X)$ be the ring of all real-valued Baire one functions on X. We show that any member of $B_1(X)$ is a zero divisor or a unit. We give an algebraic characterization of X when for every $p \in X$, there exists $f \in B_1(X)$ such that $\{p\} = f^{-1}(0)$ and we give some topological characterizations of minimal ideals, essential ideals and socle of $B_1(X)$. Some relations between $C(X)_F$, $B_1(X)$ and some interesting function rings on X are studied and investigated. We show that $B_1(X)$ is a regular ring if and only if every countable intersection of cozero sets of continuous functions can be represented as a countable union of zero sets of continuous functions.

1. Introduction

All our spaces are assumed to be T_1 unless otherwise stated and all rings will be assumed commutative with identity and semiprime. The set of natural numbers, rational numbers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. The set of all functions from a topological space X into $\mathbb{R}$ is denoted by $F(X)$ and the set of all $f \in F(X)$ such that $f^{-1}(U)$ is an F_σ-set for all open set U in $\mathbb{R}$ is denoted by $F_\sigma(X)$. $F(X)$ is a commutative ring with pointwise addition and multiplication and continuous members of $F(X)$ which is a subring of $F(X)$ is denoted by $C(X)$. Let $f \in F(X)$. Then $f^{-1}(0) = \{x \in X | f(x) = 0\}$ denoted by $Z(f)$ is called the *zero set* of f and a set is said to be a *cozero set* if it is the complement of a zero set of a function in $F(X)$. A subset M of X is called a *Zero*$_\sigma$*-set* or a *Coz*$_\delta$*-set* if $M = \bigcup_{n\in\mathbb{N}} F_n$ or $M = \bigcap_{n\in\mathbb{N}} G_n$, respectively, where each F_n is a zero set of a function in $C(X)$ and each G_n is a cozero set of a function in $C(X)$. The set of all $f \in F(X)$ such that $f^{-1}(U)$ is a *Zero*$_\sigma$-set for all open set U in $\mathbb{R}$ is denoted by $F^*_\sigma(X)$. Clearly, $F^*_\sigma(X) \subseteq F_\sigma(X)$. The set of points at which $f \in F(X)$ is continuous is denoted by $C(f)$. The ring of all $f \in F(X)$ such that $X \setminus C(f)$ is a finite set is denoted by $C(X)_F$ [18]. The ring of all real-valued functions which are continuous on some dense open subsets of X is denoted by $T'(X)$ [1]. The *characteristic function* of the subset S of X is denoted by χ_S.

Recall that a commutative ring R is called (von Neumann) *regular* if for each $r \in R$, there exists $s \in R$ such that $r = r^2 s$. Thus for any nonempty set X the commutative ring $F(X)$ is regular. By a minimal ideal of R we mean an ideal which is minimal in the poset of non-zero ideals of R. A non-zero ideal I in R is an *essential ideal* if I intersects every non-zero ideal of R non-trivially. The *socle* of R denoted by $Soc(R)$ is the sum of all minimal ideals of R, or the intersection of all essential ideals of R.
The set of all pointwise limit functions of sequences in $C(X)$ is denoted by $B_1(X)$ and the element of $B_1(X)$

2020 *Mathematics Subject Classification*. Primary 26A21; Secondary 54C40, 13C99.
Keywords. Baire one functions; P-space; Minimal ideal; Fixed ideal; $B_1(X)$;
Received: 30 September 2022; Revised: 09 February 2023; Accepted: 17 February 2023
Communicated by Ljubiša D.R. Kočinac
Email addresses: mahmadi@yazd.ac.ir (Mohammad Reza Ahmadi Zand), zahra.khosravi@stu.yazd.ac.ir (Zahra Khosravi)

are called *real-valued Baire one functions*. $B_1(X)$ was introduced and investigated by Baire [6] and studied extensively by many mathematiciants such as Hausdorff [20] and Lebesgue [27]. By generalizing definitions and ideas from the book " Rings of Continuous Functions" [19], Deb Ray and Mondal [10–12] studied $B_1(X)$ as a subring of $F(X)$. Our intention in the present paper is to pursue research on the ring $B_1(X)$, although $B_1(X)$ has been studied in several aspects, see for example [7, 17, 30].
In section 2, some topological characterizations of minimal ideals, essential ideals and socle of $B_1(X)$, where every point of X is a zero set of a function in $C(X)$ are given. For a topological space X, we show that any member of $B_1(X)$ is a zero divisor or a unit. In section 3, some relations between subrings $C(X)$, $B_1(X)$, $C(X)_F$ and $T'(X)$ of $F(X)$ are studied. We give an algebraic characterization of a normal space X when every subset of X is a G_δ-set. We give some examples of topological spaces in which the ring of Baire one functions is regular. We give a topological characterization of $B_1(X)$ when it is a regular ring.
As usual the Stone-Čech compactification of a $T_{3\frac{1}{2}}$-space X is denoted by βX. The reader is referred to [9], [14] and [19] for terms and notations not described here.

1.1. Preliminaries

For our purpose we need the following results and definitions that will be used in this paper.

Theorem 1.1. *Let X be an arbitrary topological space. Then* $B_1(X) = F^*_\sigma(X)$.

Proof. See [28, Exercise 3.A.1]. □

Theorem 1.2. [31] *Let X be a normal topological space. Then* $B_1(X) = F_\sigma(X)$.

The collection of all zero sets of functions in $B_1(X)$ is denoted by $Z(B_1(X))$ [10]. In the following proposition we show that a subset A of X is a zero set of a function in $B_1(X)$ if and only if A can be represented in the form of a countable intersection of cozero sets of continuous functions.

Proposition 1.3. *For a topological space X we have* $Z(B_1(X)) = \{A \subseteq X : A \text{ is a } Coz_\delta\text{-set}\}$.

Proof. The inclusion $Z(B_1(X)) \subseteq \{A \subseteq X : A$ is a Coz_δ-set$\}$ immediately follows from Theorem 1.1. The inverse inclusion follows from [24, Proposition 2]. □

Definition 1.4. [11] *A nonempty subset* $\mathcal{F}$ *of* $Z(B_1(X))$ *is said to be a* Z_B*-filter if* $\mathcal{F}$ *satisfies the following conditions.*

(I) $\emptyset \notin \mathcal{F}$.

(II) If $Z_1, Z_2 \in \mathcal{F}$, *then* $Z_1 \cap Z_2 \in \mathcal{F}$.

(III) If $Z \in \mathcal{F}$ *and* $Z' \in Z(B_1(X))$ *which satisfies* $Z \subseteq Z'$, *then* $Z' \in \mathcal{F}$.

Theorem 1.5. [11] *Let X be a topological space.*

(I) Let I be a proper ideal in $B_1(X)$. *Then* $Z_B[I] = \{Z(f) : f \in I\}$ *is a* Z_B*-filter on X.*

(II) Let $\mathcal{F}$ *be a* Z_B*-filter on X. Then* $Z_B^{-1}[\mathcal{F}] = \{f \in B_1(X) : Z(f) \in \mathcal{F}\}$ *is a proper ideal in* $B_1(X)$.

Definition 1.6. [11] *Let I be a proper ideal of* $B_1(X)$. *Then I is called fixed if* $\cap Z[I] \neq \emptyset$, *otherwise, it is called free.*

2. Minimal ideals of $B_1(X)$

In this section, we show that every member of $B_1(X)$ is a zero-divisor or a unit and we give some topological and algebraic characterizations of minimal ideals, essential ideals and socle of $B_1(X)$.

Proposition 2.1. *Let M be a subset of a topological space X. Then* $\chi_M \in B_1(X)$ *if and only if M and* $X \setminus M$ *are* $Zero_\sigma$*-sets.*

Proof. By Theorem 1.1, $f = \chi_M \in B_1(X)$ if and only if $f \in F^*_\sigma(X)$. But $f^{-1}(U) = \{\emptyset, X, M, X \setminus M\}$ for any open subset U of $\mathbb{R}$ which completes the proof. □

Clearly, every zero set of a function in $C(X)$ is a $Zero_\sigma$-set and a Coz_δ-set.

Corollary 2.2. *Let K be a compact subset of a topological space X. Then for any $Zero_\sigma$-set A in X which is disjoint from K there exists $f \in B_1(X)$ such that $f[K] = \{1\}$ and $f[A] = \{0\}$.*

Proof. Let $f_n : X \to [0, 1]$ be continuous for any $n \in \mathbb{N}$, $A = \bigcup_{n=1}^{\infty} Z(f_n)$ and $K \cap A = \emptyset$. Then every f_n attains its minimum $b_n > 0$ on K and so $F = \bigcap_{n=1}^{\infty} f_n^{-1}([b_n, 1])$ is a zero set of a function in $C(X)$ such that $K \subseteq F \subseteq X \setminus A$ [19]. Let $f = \chi_F$. Then $f[K] = \{1\}$ and $f[A] = \{0\}$ and Proposition 2.1 implies that $f \in B_1(X)$. □

Proposition 2.3. *Let M be a nonempty proper subset of a topological space X which are a $Zero_\sigma$-set and a Coz_δ-set. Then $B_1(X)$ is isomorphic to the direct sum of two rings; moreover, there exists a nontrivial idempotent f in $B_1(X)$ such that $Z(f) = M$.*

Proof. Let $N = X \setminus M$. Then $\phi : B_1(X) \to B_1(M) \oplus B_1(N)$ defined by $\phi(g) = g|M + g|N$ is a one-one ring homomorphism. If $h \in B_1(M)$ and $k \in B_1(N)$, then $g = h \cup k \in F(X)$ and if U is open in $\mathbb{R}$, then $g^{-1}(U) = h^{-1}(U) \cup k^{-1}(U)$ is a $Zero_\sigma$-set in X. Thus by Theorem1.1, $g \in B_1(X)$ and $\phi(g) = h + k$, and so ϕ is a ring isomorphism. By Proposition 2.1, $\chi_M \in B_1(X)$ and so $f = 1 - \chi_M \in B_1(X)$ is a nontrivial idempotent such that $M = Z(f)$. □

Proposition 2.4. *If p is a point in a topological space X, then $\chi_{\{p\}} \in B_1(X)$ if and only if $\{p\}$ is a zero set of a function in $C(X)$.*

Proof. Let $\chi_{\{p\}} \in B_1(X)$. Then by Proposition 2.1, $\{p\}$ is a $Zero_\sigma$-set and so $\{p\}$ is a zero set of a function in $C(X)$. The converse follows from Proposition 2.1. □

Corollary 2.5. *Let X be a completely regular space and $\{p\}$ be a G_δ-set. Then $\chi_{\{p\}} \in B_1(X)$*

Proof. Since $\{p\}$ is a G_δ-set in X, $\{p\}$ is a zero set of a function in $C(X)$. Thus by Proposition 2.4 we are done. □

Proposition 2.6. *Let X be a topological space. Then any member of $B_1(X)$ is a zero-divisor or a unit.*

Proof. Suppose that $0 \neq f \in B_1(X)$ is not a unit element. Then there exists $p \in Z(f)$ by [2, Theorem 1.1] and $Z(f)$ is a G_δ-set by [10, Theorem 4.1]. Thus by [19, 3.11(b)], there is $g \in C(X)$ such that $p \in Z(g) \subseteq Z(f)$. If $M = Z(g)$, then by Proposition 2.1, $\chi_M \in B_1(X)$ since $\emptyset \neq M$ is a zero set of a function in $C(X)$. Therefore $f\chi_M = 0$ which completes the proof. □

Let every singleton set of X be a zero set of a function in $C(X)$. Some topological characterizations of minimal ideals in $B_1(X)$, essential ideals in $B_1(X)$ and the socle of $B_1(X)$ are given in the following result.

Proposition 2.7. *Suppose that X is a topological space in which every singleton set is a zero set of a function in $C(X)$. Then the following hold.*

(i) *An ideal I of $B_1(X)$ is minimal if and only if for some $p \in X$, I is generating by $\chi_{\{p\}}$.*

(ii) *An ideal I of $B_1(X)$ is minimal if and only if $|Z[I]| = 2$.*

(iii) *The socle of $B_1(X)$ consists of all functions which vanish everywhere except on a finite subset of X .*

(iv) *The socle of $B_1(X)$ is an essential ideal.*

(v) *The socle of $B_1(X)$ is a free ideal.*

(vi) *The socle of $B_1(X)$ is the intersection of all free ideals in $B_1(X)$, and of all free ideals in $B^*_1(X)$.*

Proof. If I is an ideal in $B_1(X)$ and $r = f(p)$ for some $p \in X$, then $\chi_{\{p\}} \in B_1(X)$ by Proposition 2.4 and

$$r\chi_{\{p\}} = f\chi_{\{p\}} \in I. \tag{1}$$

Thus if $r \neq 0$, then the ideal generated by $\chi_{\{p\}}$ is contained in I.
(i) If J is the ideal generated by $\chi_{\{p\}}$ in $B_1(X)$, then by (1), $J = \{r\chi_{\{p\}} : r \in \mathbb{R}\}$ which is a minimal ideal in $B_1(X)$. Conversely, if I is a minimal ideal in $B_1(X)$, then there exists $f \in I$ and $p \in X$ such that $0 \neq r = f(p)$. From (1) we infer that I is the ideal generated by $\chi_{\{p\}}$.
(ii) Let $|Z[I]| = 2$. Then there exists $f \in I$ and $p \in X$ such that $0 \neq r = f(p)$, and so $\chi_{\{p\}} = \frac{1}{r}\chi_{\{p\}}f \in I$ by (1). Thus from $0 \neq g \in I$ it follows that $Z(g) = Z(\chi_{\{p\}}) = X \setminus \{p\}$ and so I is generated by $\chi_{\{p\}}$. Now, (i) implies that I is minimal. The converse is obvious by (i).
(iii) It follows from (i) since the socle of a commutative ring is the sum of its minimal ideals.
(iv) If $0 \neq f \in B_1(X)$, then $r = f(p) \neq 0$ for some $p \in X$. By (1), $f\chi_{\{p\}} \neq 0$ and so by (iii) $f\chi_{\{p\}}$ is in the socle of $B_1(X)$, i.e., the socle of $B_1(X)$ is essential.
(v) For any $p \in X$, $\chi_{\{p\}}(p) \neq 0$ and the ideal generated by $\chi_{\{p\}}$ is minimal by (i). Thus by (iii), the socle of $B_1(X)$ is free.
(vi) If I is a free ideal in $B_1(X)$ or $B_1^*(X)$, then for every $p \in X$ there exists $f \in I$ such that $r = f(p) \neq 0$. By (1), $\chi_{\{p\}} = \frac{1}{r}\chi_{\{p\}}f \in I$. Thus, by (iii), the socle of $B_1(X)$ is contained in the intersection of all free ideals in $B_1(X)$ and of all free ideals in $B_1^*(X)$. By (v) the socle of $B_1(X)$ is free which completes the proof. □

Theorem 2.8. *Let X be an infinite topological space. If every singleton set of X is a cozero set of a Baire one function, then the following statements hold.*

(i) *For any $f \in B_1(X)$ which is not a unit in $B_1(X)$ there exists $g \in B_1(X)$ such that $g \neq 1$ and $f = g^r f$, where r is a positive real number.*

(ii) *Let I be a fixed ideal in $B_1(X)$ such that $\bigcap Z[I]$ is a finite set. Then $Ann(I)$ is a proper ideal generated by $g \in B_1(X)$ such that $g^r = g$, where r is an arbitrary positive real number.*

(iii) *For any subset A of X there exists a subset S of $B_1(X)$ such that $A = \bigcup_{f\in S}(X \setminus Z(f))$, in particular if A is countable, then A is a cozero set of a function in $B_1(X)$.*

Proof. (i) If $0 \neq f \in B_1(X)$ is not a unit element, then by the proof of Proposition 2.6, there exists a nonempty proper subset M of $Z(f)$ such that $\chi_M \in B_1(X)$. Hence, $g = 1 - \chi_M \neq 1$ is a function with corresponding properties.
(ii) Let $S = \bigcap Z[I] = \{x_1, \cdots, x_n\}$ be an n-element set, where $n \in \mathbb{N}$. Then by our hypothesis, there exists $g_i \in B_1(X)$ such that $X \setminus Z(g_i) = \{x_i\}$ for $i = 1, \cdots n$. Thus, $g = \sum_{i=1}^n |\frac{1}{g_i(x_i)}g_i| \in B_1(X)$ and $g^r = g$ for any positive real number r since $g = \chi_S$. Thus for any $f \in I$, we have $gf = 0$, i.e., $g \in Ann(I)$ and so if the ideal generated by g is denoted by J, then $J \subseteq Ann(I)$. If $s \notin S$, then there exists $f \in I$ such that $f(s) \neq 0$ and so $h \in Ann(I)$ implies that $h(s)f(s) = 0$, i.e., $h(s) = 0$. Thus from $X \setminus S \subseteq Z(h)$ it follows that $h = gh \in J$ and so $Ann(I) \subseteq J$ which implies that $Ann(I) = J$.
(iii) For any subset A of X, $S = \{\chi_{\{x\}} | x \in A\} \subseteq B_1(X)$ as we have shown above. Clearly, $A = \bigcup_{f\in S}(X \setminus Z(f))$. By [10, Theorem 4.5] $Z(B_1(X))$ is closed under countable intersection and so if A is countable, then $X \setminus A \in Z(B_1(X))$. □

3. Some relations between $B_1(X)$ and $C(X)_F$

Let X be a T_1-space. Then we have the following.

$$C(X) \subseteq C(X)_F \subseteq T'(X) \subseteq F(X),$$

$$C(X) \subseteq B_1(X) \subseteq F(X).$$

Recently, some nice properties of some overrings of $C(X)$ that are subrings of $F(X)$ are studied and investigated, see for example [1–4, 10, 11, 18]. In this section, we will investigate some new relations between some nice subrings of $F(X)$. We now define an interesting subring of $B_1(X)$.

Definition 3.1. *$TB_1(X)$ denotes the set of all $f \in B_1(X)$ such that the restriction of f to a dense open subset of X is continuous, i.e., $TB_1(X) = B_1(X) \bigcap T'(X)$.*

Let X be a $T_{\frac{1}{2}}$-space, i.e., each singleton set of X is either closed or open [13]. Then by [4, Remark 3.3], $C(X)_F$ is a subring of $T'(X)$. Thus we have the following result.

Proposition 3.2. *Let X be a $T_{\frac{1}{2}}$-space. Then $C(X)_F$ is a subring of $B_1(X)$ if and only if $C(X)_F$ is a subring of $TB_1(X)$.*

Recall that a space X has *countable pseudocharacter* [23] if each singleton set of X is a G_δ-set. Let R be a semiprime commutative ring. Recall that a ring S as an overring of R is called a ring of quotients of R if and only if for every $0 \neq s \in S$ there is an element $r \in R$ such that $0 \neq sr \in R$ [16]. Now we are ready to give a generalization of [32, Corollary 1] see also [12, Theorem 4.18].

Proposition 3.3. *If X has countable pseudocharacter and X is a $T_{3\frac{1}{2}}$-space, then the following statements are equivalent.*

(i) $C(X) = B_1(X)$.

(ii) $C(X) = TB_1(X)$.

(iii) *X is a discrete space.*

(iv) *$B_1(X)$ is a ring of quotients of $C(X)$.*

(vi) $C(X) = T'(X)$.

(v) $C(X) = C(X)_F$.

Proof. (iii) ⇔ (vi) ⇔ (v) see [18, Proposition 3.1].
(i)⇒ (ii) It is obvious since $C(X) \subseteq TB_1(X) \subseteq B_1(X)$.
(ii) ⇒ *(iii)* If $x \in X$, then by Proposition 2.4, the characteristic function $f = \chi_{\{x\}}$ belongs to $B_1(X)$ and so $f \in TB_1(X)$. Thus f must be a continuous function since $C(X) = TB_1(X)$ by hypothesis in (ii). Hence it shows that x is an isolated point.
(iii) ⇒ (iv) It is straightforward.
(iv) ⇒ (i) Let $t \in X$. So by Proposition 2.4, $\chi_{\{t\}} \in B_1(X)$. But by hypothesis, $B_1(X)$ is a ring of quotients of $C(X)$ and $C(X)$ is a commutative semiprime ring, so there is $f \in C(X)$ such that $0 \neq f\chi_{\{t\}} \in C(X)$. But $f(t)\chi_{\{t\}} = f\chi_{\{t\}}$ is continuous and so t is an isolated point of X. Therefore X is a discrete space which completes the proof. □

Example 3.10 below shows that the sentence " X has countable pseudocharacter" cannot be dropped from the hypotheses of Proposition 3.3. If X is a $T_{\frac{1}{2}}$-space, then the conditions (iii), (vi) and (v) of Proposition 3.3 are equivalent and they are not equivalent in general [4]. The one-point compactification of $\mathbb{N}$ is not discrete and so any conditions of Proposition 3.3 don't hold for this space.

In the following example we show that the conditions (i) and (ii) of Proposition 3.3 are not equivalent with other it's conditions in the class of regular spaces.

Example 3.4. *There is an example of a regular space X on which every continuous real-valued function is constant* [22]. *Thus, $R = C(X) = TB_1(X) = B_1(X) \subsetneq C(X)_F \subseteq T'(X)$, where R denotes the set of all constant functions on X.*

An algebraic characterization of T_1-spaces that singleton sets of them are zero sets of continuous functions are given in the following theorem.

Theorem 3.5. *Let X be a T_1-space. Then $C(X)_F$ is a subring of $TB_1(X)$ if and only if every singleton set of X is a zero set of a function in $C(X)$.*

Proof. $\Rightarrow$ Let $C(X)_F \subseteq TB_1(X)$ and $a \in X$. Then $f = \chi_{\{a\}} \in TB_1(X) \subseteq B_1(X)$ since it is in $C(X)_F$. Thus by Proposition 2.4, $\{a\}$ is a zero set of a function in $C(X)$.
$\Leftarrow$ Let every singleton set of X be a zero set of a function in $C(X)$. Then every finite subset of X is a zero set of a function in $C(X)$. Let U be a nonempty open set of $\mathbb{R}$ and $f \in C_F(X)$. Then there exists a subset D of X such that $f|D \in C(D)$ and $X \setminus D$ is finite. Thus $X \setminus D$ is a zero set of a continuous function and so it is a $Zero_\sigma$-set and a Coz_δ-set. But $f^{-1}(U) = (f|D)^{-1}(U) \bigcup K$, where K is a subset of $X \setminus D$ and so K is a zero set of a function in $C(X)$. Therefore $f^{-1}(U)$ is a $Zero_\sigma$-set in X since it is a union of two $Zero_\sigma$-sets $(f|D)^{-1}(U) = f^{-1}(U) \cap D$ and K and so $f \in B_1(X)$ by Theorem 1.1. Thus $f \in TB_1(X)$ which completes the proof. □

Now we give an example of a normal space X, where $C(X)_F$ and $B_1(X)$ cannot be compared by inclusion.

Example 3.6. *Let $X = \beta\mathbb{N}$ and $y \in \beta\mathbb{N} \setminus \mathbb{N}$. Then, it is well known that $\{y\}$ is not a G_δ-set* [14, 19] *and so $\{y\}$ is not a zero-set of a function in $B_1(X)$ by* [10, Theorem 4.1]. *Thus by* Theorem 3.5, *$C(X)_F \nsubseteq B_1(X)$. On the other hand for every $n \in \mathbb{N}$, a mapping $f_n : X \to \mathbb{R}$ defined by $f_n(x) = \sum_{i=1}^{i=n} i\chi_{\{i\}}(x)$ is continuous. If $f : X \to \mathbb{R}$ is a mapping defined by*

$$f(x) = \begin{cases} x & x \in \mathbb{N}, \\ 0 & x \in \beta\mathbb{N} \setminus \mathbb{N}. \end{cases}$$

then for any $x \in X$, $f(x) = \lim_{n\to\infty} f_n(x)$ and so $f \in B_1(X)$. We note that the cardinality of $X \setminus C(f) = \beta\mathbb{N} \setminus \mathbb{N}$ is $2^{\mathfrak{c}}$, where $\mathfrak{c}$ denotes the cardinality of the continuum and so $f \notin C(X)_F$. Thus, $B_1(X) \nsubseteq C(X)_F$ and we have the following relations:

$$C(X) \subsetneq TB_1(X) = B_1(X) \subsetneq F(X) = T'(X), \quad C(X) \subsetneq C(X)_F \subsetneq F(X) = T'(X).$$

Remark 3.7. *Let X be a topological space and $C(X)_{\mathfrak{c}}$ denotes the set of all $f \in F(X)$ such that the cardinality of $X \setminus C(f)$ is not greater than $\mathfrak{c}$. Then $C(X)_{\mathfrak{c}}$ is a subring of $F(X)$ that contains $C(X)_F$. Similar to the above example, we note that $B_1(\beta\mathbb{N}) \nsubseteq C(\beta\mathbb{N})_{\mathfrak{c}}$ and $C(\beta\mathbb{N})_{\mathfrak{c}} \nsubseteq B_1(\beta\mathbb{N})$.*
Let X be a $T_{3\frac{1}{2}}$-space. If there is a point p in X such that $\{p\}$ is not G_δ, then $C(X)_F$ is not a subring of $B_1(X)$ by Theorem 3.5.

3.1. *When $B_1(X)$ is a regular subring of $F(X)$*

Definition 3.8. *A space X is called a B_1P-space if $B_1(X)$ is a regular ring.*

We recall that X is a *P-space* if and only if $C(X)$ is a regular ring, see [19, 4J].

Remark 3.9. *Let X be a $T_{3\frac{1}{2}}$-space. Then, X is a P-space if and only if $B_1(X) = C(X)$* [32]. *Thus every P-space is a B_1P-space.*

Now we give an example of a B_1P-space which is normal, $T'(X)$ is not a subring of $B_1(X)$ and $C(X)_F$ is not a subring of $B_1(X)$.

Example 3.10. *Recall that a totally ordered set T is called an η_1-set if for any countable subsets A and B such that $a < b$ for each $a \in A$ and $b \in B$, there exists $c \in T$ such that $a < c < b$ for each $a \in A$, $b \in B$. Let X be an η_1-set in the order topology such that $|X| = 2^{\aleph_0}$ as it is constructed in* [19, page 187]. *Then X is a P-space without isolated points by* [19, Problem 13.P]. *Thus $B_1(X) = C(X)$* [32], *X is normal and each singleton set of X is not G_δ and so*

$$C(X) = B_1(X) = TB_1(X) \subsetneq C(X)_F \subseteq T'(X) \subsetneq F(X).$$

Recall that a topological space X is called a *Q-space* [5] if every subset of X is G_δ. Clearly, every σ-discrete space is a Q-space. An algebraic characterization of a Q-space in the class of normal spaces is given in the following result.

Theorem 3.11. *Let X be a normal topological space. Then X is a Q-space if and only if*
$TB_1(X) = T'(X) \subseteq B_1(X) = F(X)$.

Proof. $\Rightarrow$ Let X be a Q-space, U be a nonempty open subset of $\mathbb{R}$ and $f \in F(X)$. Then $f^{-1}(U)$ is an F_σ-set in X and so by Theorem 1.2, $f \in B_1(X)$. Thus $B_1(X) = F(X)$ and so $T^{'}(X) = TB_1(X) \subseteq F(X) = B_1(X)$.
$\Leftarrow$ Let $TB_1(X) = T^{'}(X) \subseteq B_1(X) = F(X)$ and A be a nonempty subset of X. Then from $f = \chi_A \in F(X)$ it follows that $f \in B_1(X)$ and so $A = f^{-1}(0, 2)$ is an F_σ-set by Theorem 1.2. Thus X is a Q-space. □

If QX is the non-σ-discrete Q-space, which is perfectly normal as it is constructed in [8], then by Theorem 3.11 we have the following relations:

$$TB_1(QX) = T^{'}(QX) \subsetneq B_1(QX) = F(QX).$$

Clearly, if $B_1(X) = F(X)$, then X is a B_1P-space. Thus by Theorem 3.11, we have the following result.

Corollary 3.12. *If a normal space X is a Q-space, then X is a B_1P-space.*

Remark 3.13. *If X is a Q-space, then it is well known and easy to prove that X is a P-space if and only if X is a discrete space. Thus by* Corollary 3.12, *any non-discrete normal Q-space is an example of a B_1P-space which is not a P-space. Thus a countable normal non-discrete space is a B_1P-space which is not a P-space, and so we have the following relations:*

$$C(\mathbb{Q}) \subsetneq C(\mathbb{Q})_F \subsetneq TB_1(\mathbb{Q}) = T^{'}(\mathbb{Q}) \subsetneq F(\mathbb{Q}) = B_1(\mathbb{Q}),$$

and

$$C(\mathbb{N}^*) \subsetneq C(\mathbb{N}^*)_F = T^{'}(\mathbb{N}^*) = F(\mathbb{N}^*) = TB_1(\mathbb{N}^*) = B_1(\mathbb{N}^*),$$

where $\mathbb{N}^*$ is the one-point compactification of $\mathbb{N}$.

Theorem 3.14. *X is a B_1P-space if and only if every Coz_δ-set is a $Zero_\sigma$-set.*

Proof. $\Rightarrow$ Let X be a B_1P-space and A be a Coz_δ-set. Then there exists $f \in B_1(X)$ such that $A = Z(f)$ by Proposition 1.3, and so there is $g \in B_1(X)$ such that $f^2 g = f$. Thus, $fg - 1 \in B_1(X)$ since $B_1(X)$ is a subring of $F(X)$. It is straightforward that $Z(fg-1) = X \setminus Z(f)$. By Proposition 1.3, $Z(fg-1)$ is a Coz_δ-set which implies that $A = Z(f)$ is a $Zero_\sigma$-set.

$\Leftarrow$ Let every Coz_δ-set be a $Zero_\sigma$-set and $f \in B_1(X)$. Then by Theorem 1.1, $f^{-1}(V)$ is an $Zero_\sigma$-set for any open set V in $\mathbb{R}$ and so $Z(f)$ is a $Zero_\sigma$-set since $Z(f)$ is a Coz_δ-set. Now consider a mapping $g \in F(X)$ defined by

$$g(x) = \begin{cases} \frac{1}{f(x)} & x \notin Z(f), \\ 0 & x \in Z(f). \end{cases}$$

It is sufficient to prove that $g \in B_1(X)$ since $gf^2 = f$. Let $U = (a, b)$ be any open interval in $\mathbb{R}$. Then it is straightforward that

$$g^{-1}(U) = \begin{cases} f^{-1}(\frac{1}{b}, \frac{1}{a}) & 0 \notin (a, b), \\ f^{-1}(-\infty, \frac{1}{a}) \cup f^{-1}(\frac{1}{b}, \infty) \cup Z(f) & 0 \in (a, b). \end{cases}$$

Thus by our hypothesis, in any case $g^{-1}(U)$ is an $Zero_\sigma$-set in X and so by Theorem 1.1, $g \in B_1(X)$ which completes the proof. □

Due to Kuratowski [26, chapter 40, VI] a topological space is called σ-space, if every G_δ-set is F_σ. Hence by Theorem 3.14, in the class of perfectly normal spaces, B_1P-spaces are exactly σ-spaces. We now give an example of a B_1P-space X which is not a P-space and $C(X)_F = T^{'}(X)$ is not a subring of $B_1(X)$.

Example 3.15. *Let Y be an uncountable discrete space and let $X = Y \cup \{y\}$ be it's one-point compactification, where $y \notin Y$. Let A be a Coz_δ-set in X. Then by* Proposition 1.3, *there is $f \in B_1(X)$ such that $A = Z(f)$. Thus, every continuous, and consequently every Baire-one function on X satisfies $f(x) = f(y)$ for all but countably many points $x \in X$ and so $Z(f)$ is countable or $X \setminus Z(f)$ is countable. In any case, $A = Z(f)$ is a $Zero_\sigma$-set in X and so by* Theorem 3.14, *X is a B_1P-space and we have the following relations:*

$$C(X) \subsetneq TB_1(X) = B_1(X) \subsetneq F(X) = C(X)_F = T'(X).$$

Thus X is not a Q-space by Theorem 3.11.

To give some conditions on a topological space X under which $T'(X) \nsubseteq B_1(X)$ we recall the following definitions. A space is called *Baire* if the intersection of countably many open dense subsets of the space is dense. A space is called *irresolvable* if it does not admit disjoint dense sets, otherwise it is called *resolvable* [21]. It is well known that every locally compact space without isolated points is resolvable [21]. A *perfect* subset of the space is a closed subset which in its relative topology has no isolated points.

Proposition 3.16. *Let X be a topological space and $\emptyset \neq A$ be a nowhere dense and perfect subset of X which is locally compact. Then $T'(X)$ is not a subring of $B_1(X)$.*

Proof. $D = X \setminus A$ is an open dense subset of X since A is a nowhere dense and perfect subset of X. A is resolvable [21] and so there exists a subset B of A such that B and $A \setminus B$ are two disjoint dense subsets of A. If $f : X \to \mathbb{R}$ is a mapping defined by

$$f(x) = \begin{cases} 1 & x \in D \bigcup B, \\ 0 & x \in A \setminus B, \end{cases}$$

then $f \in T'(X)$. We claim that $f \notin B_1(X)$ which completes the proof. On the contrary let $f \in B_1(X)$, so $f|A \in B_1(A)$. Since A is locally compact, A is a Baire space and so $C(f|A)$ must be dense in A by [29, Theorem 48.5]. But $C(f|A) = \emptyset$ which is a contradiction. □

The following example shows that $T'(\mathbb{R})$ is not a subring of $B_1(\mathbb{R})$ and $B_1(\mathbb{R})$ is not a subring of $T'(\mathbb{R})$.

Example 3.17. *Let $f_0 : \mathbb{R} \to \mathbb{R}$ be defined as,*

$$f_0(x) = \begin{cases} \frac{1}{q} & if\ x = \frac{p}{q}, where\ p \in \mathbb{Z},\ q \in \mathbb{N}\ \ and\ g.c.d.\ (p, q) = 1, \\ 1 & if\, x = 0, \\ 0 & otherwise. \end{cases}$$

Clearly, $f_0 \notin T'(\mathbb{R})$. By [15], *$f_0 \in B_1(\mathbb{R})$, and so $B_1(\mathbb{R}) \nsubseteq T'(\mathbb{R})$. With slight changes in* [11, Example 2.7] *we observe that there is no $g_0 \in B_1(X)$ such that $f_0^2 g_0 = f_0$, i.e., $\mathbb{R}$ is not a B_1P-space. It is well known that the Cantor set is a perfect subset of $\mathbb{R}$ and it is compact and nowhere dense so by* Proposition 3.16, *$T'(\mathbb{R}) \nsubseteq B_1(\mathbb{R})$. Thus by* Theorem 3.5, *we have the following relations between some subrings of $F(\mathbb{R})$.*

$$C(\mathbb{R}) \subsetneq C(\mathbb{R})_F \subsetneq T'(\mathbb{R}) \subsetneq F(\mathbb{R})\ and\ C(\mathbb{R}) \subsetneq C(\mathbb{R})_F \subsetneq TB_1(\mathbb{R}) \subsetneq B_1(\mathbb{R}) \subsetneq F(\mathbb{R}).$$

Acknowledgements. We record our pleasure to the anonymous referee for his or her constructive report and many helpful suggestions on the main results of the earlier version of the manuscript which improved the presentation of the paper.

References

[1] M.R. Ahmadi Zand, *An algebraic characterization of Blumberg spaces*, Quaest. Math. **33** (2010), 223–230.
[2] M.R. Ahmadi Zand, *Common properties of some subrings of* $\mathbb{R}^X$, Math. and Soc. **6** (2022), 29–47 (in Persian).
[3] M.R. Ahmadi Zand, R. Khayyeri, *Spaces in which every dense subset is a* G_δ, Quaest. Math. **39** (2016), 251–260.

[4] M.R. Ahmadi Zand, Z. Khosravi, *Remarks on the rings of functions which have a finite number of discontinuities*, Appl. Gen. Topol. **22** (2021), 139–147.
[5] A.V. Arhangel'skiĭ, *Divisibility and cleavability of spaces*, Recent Developments of General Topology and Applications, Math. Res., vol. **67**, Akademie-Verlag, Berlin, (1992), 13–26.
[6] R. Baire, *Sur les fonctions de variable reelles*, Ann. di Mat. Pura ed Appl. **3** (1899).
[7] T. Banakh, S. Gabriyelyan, *Baire category of some Baire type function spaces* , Topol. Appl. **272** (2020), 107078.
[8] Z. Balogh, *There is a paracompact Q-set space in ZFC*, Proc. Amer. Math. Soc. **126** (1998), 1827–1833.
[9] A. J. Berrick, M. E. Keating, *An Introduction to Rings and Modules*. Cambridge University Press, 2000.
[10] A. Deb Ray, A. Mondal, *On rings of Baire one functions*, Appl. Gen. Topol. **20** (2019), 237–249.
[11] A. Deb Ray, A. Mondal, *Ideals in $B_1(X)$ and residue class rings of $B_1(X)$ modulo an ideal*, Appl. Gen. Topol. **20** (2019), 379–393.
[12] A. Deb Ray, A. Mondal, *On Structure Space of the ring $B_1(X)$*, Filomat **35** (2021), 2911–2918.
[13] W. Dunham, *$T_{\frac{1}{2}}$-spaces*, Kyung. Math. J. **17** (1977), 161–169.
[14] R. Engelking, *General Topology*, Heldermann-Verlag, Berlin, 1989.
[15] J. P. Fenecios, E. A. Cabral, *On some properties of Baire-1 functions*, Int. J. Math. Anal. **7** (2013), 393–402.
[16] N.J. Fine, L. Gillman and J. Lambek, Rings of Quotients of rings of functions, McGill University Press, Montreal/Quebec, 1966.
[17] S. Gabriyelyan, A.V. Osipov, *Topological properties of some function spaces*, Topol. Appl. **279** (2020), 107248.
N.J. Fine, L. Gillman and J. Lambek, *Rings of Quotients of rings of functions*, McGill University Press, Montreal/Quebec, 1966.
[18] Z. Gharabaghi, M. Ghirati. and A. Taherifar, *On the rings of functions which are discontinuous on a finite set*, Houston J. Math. **44** (2018), 721–739.
[19] L. Gillman and M. Jerison, *Rings of continuous functions*, New York: Van Nostrand Rein- hold Co., 1960.
[20] F. Hausdorff, *Set Theory*, 2nd ed., Chelsea, New York, 1962.
[21] E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309–333.
[22] E. Hewitt, *On two problems of Urysohn*, Ann. of Math. **47** (1946), 503–509.
[23] R. Hodel, *Cardinal functions I*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. van Mill (eds.), North-Holland, Amsterdam, 1984, 1–61.
[24] O. F. K. Kalenda, J. Spurny, *Extending Baire-one functions on topological spaces*, Topol. Appl. **149** (2005), 195–216.
[25] O. Karlova, V. Mykhaylyuk, *Functions of the first Baire class with values in metrizable spaces*, Ukr. J. Math. **58** (2006), 567–571 (in Ukrainian).
[26] K. Kuratowski, *Topology, Vol. I*, Academic Press, 1960.
[27] H. Lebesgue, *Sur les fonctions reprèsentables analytiquement*, J. Math. Pure Appl. **1** (1905), 139–216.
[28] J. Lukeš, J. Mulý and L. Zajíček, *Fine Topological Methods in Real Analysis and Ptential Theory*, Springer-Verlag, 1986.
[29] J. R. Munkres, *Topology a first course*, Second edition, Pearson Education, Delhi, 2003.
[30] A.V. Osipov, *Baire property of space of Baire-one functions*, arXiv:2110.15496.
[31] L. Vesely, *Characterization of Baire-One Functions Between Topological Spaces*, Acta Univ. Carol. Math. Phys. **33** (1992), 143–156.
[32] M. Wojtowicz and W. Sieg, *P*-spaces and an unconditional closed graph theorem, Rev. de la Real Acad. de Cien. Exactas, Fisc. Y Nat. **104** (2010), 13–18.