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Seperation, irreducibility, Urysohn's lemma and Tietze extension theorem for Cauchy spaces

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Abstract. In this paper, we introduce two notions of closure operators in the category of Cauchy spaces which satisfy (weak) hereditariness, productivity and idempotency, and we characterize each of $T_i, i = 0, 1, 2$ cauchy spaces by using these closure operators as well as show each of these subcategories are isomorphic. Furthermore, we characterize the irreducible Cauchy spaces and examine the relationship among each of irreducible, connected Cauchy spaces. Finally, we present Urysohn's lemma and Tietze extension theorem for Cauchy spaces.

1. Introduction

In general topology and analysis, a Cauchy space is a generalization of metric spaces and uniform spaces. The theory of Cauchy spaces was initiated by H. J. Kowalsky [19]. Cauchy spaces were introduced by H. Keller [17] in 1968, as an axiomatic tool derived from the idea of a Cauchy filter in order to study completeness in topological spaces.

In general topology, one of the most important usage of separation properties is theorems such as the Urysohn's Lemma and the Tietze Extension Theorem. In view of this, these results are presented in the category of **pqsMet**, extended pseudo-quasi-semi metric spaces [13] and **ConFCO**, constant filter convergence spaces [12].

The notions of closedness and strongly closedness in set based topological categories are introduced by Baran [2] and it is shown in [6] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [15] in some well-known topological categories. The aims of this paper are stated below:

(i) to introduce two notions of closure operators in the category of Cauchy spaces and to characterize each of $T_i, i = 0, 1, 2$ Cauchy spaces by using these closure operators as well as to show each of these subcategories are isomorphic,

(ii) to characterize the irreducible Cauchy spaces and to examine the relationship among each of irreducible, connected Cauchy spaces and the subcategories $\mathbf{CHY}_{ic}$ and $\mathbf{CHY}_{isc}$, $i = 0, 1, 2$,

(iii) to present Urysohn's lemma and Tietze extension theorem for Cauchy spaces.

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2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small (i.e., sets) fibers, and for which every $\mathcal{U}-$source has an initial lift or, equivalently, for which each $\mathcal{U}-$sink has a final lift [1].

Recall in [1] or [21], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}\,\mathcal{E}$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

A filter on a set X is a collection of subsets of X, containing X, which is closed under finite intersection and formation of supersets. Let $\mathbf{F}(X)$ denote the set of filters on X.

For filters α and β we denote by $\alpha \cup \beta$ the smallest filter containing both α and β.

Definition 2.1. (cf. [17]) Let A be a set and $K \subset \mathbf{F}(A)$ be subject to the following axioms:

1. $[x] = [\{x\}] \in K$ for each $x \in A$ (where $[x] = \{B \subset A \ : \ x \in B\}$);
2. if $\alpha,\ \beta \in K$ and $\alpha \vee \beta$ exists (i.e., $\alpha \cup \beta$ is proper), then $\alpha \cap \beta \in K$;
3. $\alpha \in K$ and $\beta \geq \alpha$ implies $\beta \in K$ (i.e., $\beta \supset \alpha \in K$ implies $\beta \in K$ for any filter β on A), then K is a precauchy (resp. Cauchy) structure if it obeys 1 and 3 (resp. 1-3) and the pair (A, K) is called a precauchy space (resp. Cauchy space), resp. members of K are called Cauchy filters. A map $f : (A, K) \to (B, L)$ between Cauchy spaces is said to be Cauchy continuous map or Cauchy map iff $\alpha \in K$ implies $f(\alpha) \in L$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) \ : D \in \alpha\}$).

The concrete category whose objects are the precauchy (resp. Cauchy) spaces and whose morphisms are the Cauchy continuous maps is denoted by *PCHY* (resp. *CHY*), respectively.

Definition 2.2. A source $\{f_i : (A, K) \to (A_i, K_i)\,,\ i \in I\}$ in *CHY* is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [20], [22] or [26].

Definition 2.3. An epimorphism $f : (A, K) \to (B, L)$ in *CHY* is a final lift iff $\alpha \in L$ implies that there exists a finite sequence $\alpha_1, ..., \ \alpha_n$ of Cauchy filters in K such that every member of α_i intersects every member of α_{i+1} for all $i < n$ and such that $\bigcap_{i=1}^{n} f(\alpha_i) \subset \alpha$ [20], [22] or [26].

Definition 2.4. We write Δ for the diagonal in B^2, where $B \in CHY$. For $B \in CHY$ we define the *wedge* $B^2 \vee_\Delta B^2$, as the final structure, with respect to the map $B^2 \coprod B^2 \to B^2 \vee_\Delta B^2$, that is the identification of the two copies of B^2 along the diagonal Δ. An epi sink $\left\{i_1,\ i_2 : \left(B^2, K\right) \to \left(B^2 \vee_\Delta B^2, L\right)\right\}$, where $i_1,\ i_2$ are the canonical injections, in *CHY* is a final lift if and only if the following statement holds. For any filter α on the wedge $B^2 \vee_\Delta B^2$, where either $\alpha \supset i_k(\alpha_1)$ for some $k = 1, 2$ and some $\alpha_1 \in K$, or $\alpha \in L$, we have that there exist Cauchy filters $\alpha_1, \alpha_2 \in K$ such that every member of α_1 intersects every member of α_2 (i.e., $\alpha_1 \cup \alpha_2$ is proper) and $\alpha \supset i_1\alpha_1 \cap i_2\alpha_2$. This is a special case of 2.3.

Definition 2.5. The discrete structure (A, K) on A in *CHY* is given by $K = \{[a] \mid a \in A\} \cup \{[\emptyset]\}$ [20] or [22].

Definition 2.6. The indiscrete structure (A, K) on A in *CHY* is given by $K = F(A)$ [20] or [22].

CHY is a topological category. The category of Cauchy spaces is cartesian closed, and contains the category of proximity spaces as a full subcategory [22].

3. Closed subobjects

In this section, we introduce two notions of closure in the category of Cauchy spaces which satisfy (weak) hereditariness, productivity and idempotency, and we characterize each of $T_i, i = 0, 1, 2$ Cauchy spaces by using these closure operators as well as show each each of these subcategories are isomorphic.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p ([2] p.334), i.e., two disjoint copies of B identified at p, i.e., the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set** the category of sets and functions). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram [6]. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal $p-$axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed $p-$axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$.

The fold map at p, $\triangledown_p : B \vee_p B \to B$ is given by $\triangledown_p(x_i) = x$ for $i = 1, 2$ [2], [4].

Note that the maps S_p and $\triangledown_p$ are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\triangledown_p i_j = id$, $j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at p.

Remark 3.1. We define p_1, p_2 by $1 + p, p + 1 : B \vee_p B \to B$, respectively where $1 : B \to B$ is the identity map, $p : B \to B$ is constant map at p (i.e., having value p). Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \nabla_p$, where $\pi_i : B^2 \to B$ is the i-th projection, $i = 1$, 2. When showing A_p and S_p are initial it is sufficient to show that (p_1 and p_2) and (p_1 and ∇_p) are initial lifts, respectively [2], [4].

The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, ..., p, x, p, ...)$ (infinite principal p-axis map), and $\triangledown_p^\infty : \vee_p^\infty B \longrightarrow B$ by $\triangledown_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), [2], [4].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place [6].

Definition 3.2. (cf. [2], [4]) Let $\mathcal{U} : \mathcal{E} \longrightarrow$ ***Set***(the category whose objects are sets and morphisms are functions) be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}-$sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $*$ [2]. Let p be a point in B.

1. X is T_1 at p iff the initial lift of the $\mathcal{U}-$source $\{S_p : B \vee_p B \longrightarrow \mathcal{U}(X^2) = B^2$ and $\triangledown_p : B \vee_p B \longrightarrow \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
2. p is closed iff the initial lift of the $\mathcal{U}-$source $\{A_p^\infty : \vee_p^\infty B \longrightarrow \mathcal{U}(X^\infty) = B^\infty$ and $\triangledown_p^\infty : \vee_p^\infty B \longrightarrow \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.
3. $F \subset X$ is closed iff $\{*\}$, the image of F is closed in X/F or $F = \emptyset$.
4. $F \subset X$ is strongly closed iff X/F is T_1 at $\{*\}$ or $F = \emptyset$.

Remark 3.3. ([3], p. 106) Let α and β be filters on A. If $f : A \to B$ is a function, then $f(\alpha \cap \beta) = f\alpha \cap f\beta$.

Lemma 3.4. ([4], Lemma 3.3) *Let $\phi \neq F \subset B$, $q : B \to B/F$ be the identification map that identifies F to a point $*$, σ be a filter on B, and $a \in B$ with $a \notin F$. Then $[a] \cap [*] = q([a] \cap [F]) \supset q\sigma$ iff $\sigma \cup [F]$ is proper and $\sigma \subset [a]$.*

Lemma 3.5. (cf. [5], Lemma 3.2) *Let $f : A \to B$ be a map.*
(1) If α and β are filters on A, then $f(\alpha) \cup f(\beta) \subset f(\alpha \cup \beta)$.
(2) If δ is filter on B, then $\delta \subset ff^{-1}(\delta)$, where $f^{-1}(\delta)$ is the filter generated by $\{f^{-1}(D) : D \in \delta\}$.

Theorem 3.6. *Let (A, K) be a Cauchy space and $p \in A$. (A, K) is T_1 at p iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$ [18].*

Theorem 3.7. (cf. [18]) *Let (A, K) be a Cauchy space, $p \in A$, and $\emptyset \neq M \subset A$. Then,*

(1) $\{p\}$ in A is closed for (A, K) in CHY iff for each $\alpha \in K$ such that $\alpha \neq [p]$, there exists $U \in \alpha$ such that $p \notin U$.
(2) The followings are equivalent.
 (a) M is strongly closed.
 (b) M is closed.

(c) *for each $a \in A$ with $a \notin M$ and for all $\alpha \in K$, $\alpha \cup [M]$ is improper or $\alpha \not\subseteq [a]$.*

(3) *The followings are equivalent.*

(a) *M is strongly open.*

(b) *M is open.*

(c) *for each $a \in A$ with $a \notin M^c$ and for all $\alpha \in K$, $\alpha \cup [M^c]$ is improper or $\alpha \not\subseteq [a]$.*

Theorem 3.8. *Let (A, K) and (B, L) be a Cauchy spaces and $f : (A, K) \to (B, L)$ be a Cauchy map.*
(1) If $M \subset B$ is (strongly) closed, then $f^{-1}(M) \subset A$ is (strongly) closed.
(2) If $M \subset N$ and $N \subset B$ is (strongly) closed, then $M \subset B$ is (strongly) closed.

Proof. (1) Suppose $M \subset B$ is (strongly) closed, $a \in A$, $a \notin f^{-1}(M)$, and $\alpha \in K$. Note that $f(a) \notin M$, $f(\alpha) \in L$, and $f(\alpha) \not\subseteq [f(a)]$ or $f(\alpha) \cup [M]$ is improper since M is (strongly) closed. Note that, by Lemma 3.2, $f(\alpha) \cup [M] \subset f(\alpha) \cup [ff^{-1}(M)] \subset f(\alpha \cup [f^{-1}(M)])$.

If $\alpha \cup [f^{-1}(M)]$ is proper, then $f(\alpha \cup [f^{-1}(M)])]$ is proper (otherwise, $\emptyset \supset U \cup [f^{-1}(M)]$ for some $U \in \alpha$). It follows $U \cup [f^{-1}(M)]$ a contradiction and consequently, $f(\alpha) \cup [M]$ is proper.

If $\alpha \subset [a]$, then $f(\alpha) \subset [f(a)]$ contradicting to $M \subset B$ is being (strongly) closed. Thus, $\alpha \not\subseteq [a]$ and by Theorem 3.2, $f^{-1}(M) \subset A$ is (strongly) closed.

(2) Suppose $M \subset N$ and $N \subset B$ is (strongly) closed, $a \notin M$ with $a \in B$ and $\alpha \in K$. If $a \notin N$, then by Theorem 3.2, $\alpha \not\subseteq [a]$ or $\alpha \cup [N]$ is improper since $N \subset B$ is (strongly) closed. Suppose $\alpha \cup [N]$ is improper. Since $M \subset [N]$, $\alpha \cup [M] \subset \alpha \cup [N]$ and consequently, $\alpha \cup [M]$ is improper.

Suppose $a \in N$. L_N be a subspace structure on N deduced by the inclusion map $i : (N, L_N) \to (B, L)$. Note that $i^{-1}(\alpha) = \alpha \cup [N]$ and by Lemma 3.2, $\alpha \subset i(i^{-1}(\alpha))$. Since $\alpha \in L$, it follows $i(i^{-1}(\alpha)) \in L$ and by 2.2, $i^{-1}(\alpha) \in L_N$. Note that $a \notin M$, $a \in N$, and that $i^{-1}(\alpha) \in L_N$, by Theorem 3.2, $i^{-1}(\alpha) \not\subseteq [a]$ or $i^{-1}(\alpha) \cup [M]$ is improper since $M \subset N$ is (strongly) closed. Notice that, $i^{-1}(\alpha) \cup [M] = \alpha \cup [N] \cup [M] = \alpha \cup [M]$ and $i^{-1}(\alpha) = \alpha \cup [N] \not\subseteq [a]$ implies $\alpha \not\subseteq [a]$ (otherwise, if $\alpha \subset [a]$, then $\alpha \cup [N] \subset [a]$ since $a \in N$).

Hence, $\alpha \not\subseteq [a]$ or $\alpha \cup [M]$ is improper and by Theorem 3.2, $M \subset B$ is (strongly) closed. □

Let $\mathcal{E}$ be a set-based topological category. We recall [14, 15] that, a closure operator $\mathbf{c}$ of $\mathcal{E}$ is an assignment to each subset K of (the underlying set of) any object X of a subset $\mathbf{c}(K)$ of X such that

(i) $K \subset \mathbf{c}(K)$,
(ii) $\mathbf{c}(L) \subset \mathbf{c}(K)$ whenever $L \subset K$,
(iii) $\mathbf{c}(f^{-1}(K)) \subset f^{-1}(\mathbf{c}(K))$, or equivalently, $f(\mathbf{c}(K)) \subset \mathbf{c}(f(K))$ for each $f : X \to Y$ in $\mathcal{E}$ and $K \subset Y$ (continuity condition).

Let $\mathbf{c}$ be a closure operator of $\mathcal{E}$. Hence,

1. $\mathcal{E}_{0c} = \{X \in \mathcal{E} \mid x \in \mathbf{c}(\{y\}) \text{ and } y \in \mathbf{c}(\{x\}) \implies x = y \text{ with } x, y \in X\}$ [14].
2. $\mathcal{E}_{1c} = \{X \in \mathcal{E} \mid \mathbf{c}(\{x\}) = \{x\}, \text{ for each } x \in X\}$ [14].
3. $\mathcal{E}_{2c} = \{X \in \mathcal{E} \mid \mathbf{c}(\Delta) = \Delta, \text{ the diagonal}\}$ [14].

Remark 3.9. Let $\mathcal{E} = \mathbf{Top}$ and $\mathbf{c}$ be the ordinary closure operator. Then, each $\mathbf{Top}_{ic}$ reduce to the class of T_i spaces for $i = 0, 1, 2$.

Definition 3.10. Let (A, K) be a Cauchy space and $M \subset A$.

(i) $\mathbf{c}_A(M) = \bigcap \{U \subset A \mid M \subset U \text{ and } U \text{ is closed}\}$ is called the closure of M.
(ii) $\mathbf{sc}_A(M) = \bigcap \{U \subset A \mid M \subset U \text{ and } U \text{ is strongly closed}\}$ is called the strong closure of M.

It is shown that the notion of closedness forms closure operator [14] in some topological categories [6, 9, 11, 16, 23, 24].

Theorem 3.11. $\mathbf{c}$ *and* $\mathbf{sc}$ *are (weakly) hereditary, productive and idempotent closure operators of* **CHY**.

Proof. Combine Theorem 3.3, Definition 3.2, and Theorems 2.3, 2.4, Proposition 2.5, Exercise 2.D of [14]. □

Theorem 3.12. *Let (A, K) be a Cauchy space. Then, the followings are equivalent.*
(1) $(A, K) \in \boldsymbol{CHY}_{0c}$,
(2) $(A, K) \in \boldsymbol{CHY}_{0sc}$,
(3) For each $x, y \in A$ with $x \neq y$, there exists a (strongly) closed $M \subset A$ such that $x \notin M$, $y \in M$ either $\alpha \not\subseteq [x]$ or $\alpha \cup [M]$ is improper for every $\alpha \in K$ or there exists a (strongly) closed $N \subset A$ such that $x \in N$, $y \notin N$ either $\alpha \not\subseteq [y]$ or $\alpha \cup [N]$ is improper for every $\alpha \in K$.

Proof. By Theorem 3.2 and Definition 3.2, $(A, K) \in \mathbf{CHY}_{0c}$ if and only if $(A, K) \in \mathbf{CHY}_{0sc}$ which shows (1) is equivalent to (2).

Suppose $(A, K) \in \mathbf{CHY}_{0sc}$ and $x, y \in A$ with $x \neq y$. Since $(A, K) \in \mathbf{CHY}_{0sc}$, $x \notin \mathbf{c}_A(\{y\})$ or $y \notin \mathbf{c}_A(\{x\})$. Suppose $x \notin \mathbf{c}_A(\{y\})$. By Definition 3.2, there exists a (strongly) closed $M \subset A$ such that $x \notin M$ and $y \in M$. By Theorem 3.2, either $\alpha \not\subseteq [x]$ or $\alpha \cup [M]$ is improper for every $\alpha \in K$. Suppose $y \notin \mathbf{c}_A(\{x\})$. By Definition 3.2, there exists a (strongly) closed $N \subset A$ such that $x \notin N$ and $y \in N$. By Theorem 3.2, either $\alpha \not\subseteq [x]$ or $\alpha \cup [N]$ is improper for every $\alpha \in K$. This shows (2) implies (3).

Suppose (3) holds and $x, y \in A$ with $x \neq y$. If the first condition in (3) holds, then by Theorem 3.2, $M \subset A$ is a strongly closed and by Definition 3.2, $y \notin \mathbf{c}_A(\{x\})$.

If the second condition in (3) holds, then by Theorem 3.2, $N \subset A$ is a strongly closed and by Definition 3.2, $x \notin \mathbf{c}_A(\{y\})$. Hence, $(A, K) \in \mathbf{CHY}_{0sc}$ which shows (3) implies (2). □

Theorem 3.13. *Let (A, K) be a Cauchy space. Then, the followings are equivalent.*
(1) $(A, K) \in \boldsymbol{CHY}_{1c}$,
(2) $(A, K) \in \boldsymbol{CHY}_{1sc}$,
(3) $[x] \cap [y] \notin K$ for all $x, y \in A$ with $x \neq y$.

Proof. By Theorem 3.2 and Definition 3.2, $(A, K) \in \mathbf{CHY}_{1c}$ if and only if $(A, K) \in \mathbf{CHY}_{1sc}$ which shows (1) is equivalent to (2).

Suppose $(A, K) \in \mathbf{CHY}_{1sc}$ and $x \in A$. Note that $\mathbf{sc}_A(\{x\}) = x$, i.e., $\{x\}$ is strongly closed (**sc**-closed). By Theorem 3.2, $[x] \cap [y] \notin K$ for all $x, y \in A$ with $x \neq y$ which shows (2) implies (3).

Suppose $[x] \cap [y] \notin K$ for all $x, y \in A$ with $x \neq y$. By Theorem 3.2, in particular, $\{x\}$ is strongly closed, i.e., $\mathbf{sc}_A(\{x\}) = x$ and consequently, $(A, K) \in \mathbf{CHY}_{1sc}$ which shows (3) implies (2). □

Theorem 3.14. *Let (A, K) be a Cauchy space. Then, the followings are equivalent.*
(1) $(A, K) \in \boldsymbol{CHY}_{2c}$,
(2) $(A, K) \in \boldsymbol{CHY}_{2sc}$,
(3) For all $x, y \in A$ with $x \neq y$ and $\alpha, \beta \in K$, if $\alpha \subset [x]$ and $\beta \subset [y]$, then $\alpha \cup \beta$ is improper.

Proof. By Theorem 3.2 and Definition 3.2, $(A, K) \in \mathbf{CHY}_{2c}$ if and only if $(A, K) \in \mathbf{CHY}_{2sc}$ which shows (1) is equivalent to (2).

Suppose $(A, K) \in \mathbf{CHY}_{2sc}$ and for all $x, y \in A$ with $x \neq y$ and for any $\alpha, \beta \in K$, $\alpha \subset [x]$ and $\beta \subset [y]$. Let $\sigma = \pi_1^{-1}\alpha \cup \pi_2^{-1}\beta$, where π_1 and π_2 are projection maps. Note that $\pi_1\sigma = \alpha \in K$ and $\pi_2\sigma = \beta \in K$ and by 2.2, $\sigma \in K^2$, the product structure on B^2. If $V \in \sigma$, then there exists $V_1 \in \sigma$ and $V_2 \in \beta$, $V \supset V_1 \times V_2$. Since $\alpha \subset [x]$ and $\beta \subset [y]$, $x \in V_1$ and $y \in V_2$ and consequently, $\alpha \subset [(x, y)]$. Since Δ is closed in B^2, by Theorem 3.2, $\alpha \cup [\Delta]$ is improper. Therefore, there exists $V \in \sigma$ such that $V \cap \Delta = \emptyset$. Thus,

$(V_1 \times V_2) \cap \Delta = \emptyset$ if and only if $V_1 \cap V_2 = \emptyset$, i.e., $\alpha \cup \beta$ is improper.

Conversely, suppose that for all $x, y \in A$ with $x \neq y$ and $\alpha, \beta \in K$, if $\alpha \subset [x]$ and $\beta \subset [y]$, then $\alpha \cup \beta$ is improper. We show that $(A, K) \in \mathbf{CHY}_{2sc}$, i.e., Δ is **sc**-closed, i.e., by Theorem 3.2, for any $(x, y) \in B^2$, $(x, y) \notin \Delta$ and every $\sigma \in K^2$, i.e., $\alpha \cup [\Delta]$ is improper or $\sigma \not\subseteq [(x, y)]$. Since $\sigma \in K^2$, the product structure on B^2, by 2.2, $\pi_1\sigma, \pi_2\sigma \in K$ and $x \neq y$. By assumption, $\pi_1\sigma \cup \pi_2\sigma$ is improper if $\pi_1\sigma \subset [x]$ and $\pi_2\sigma \subset [y]$.

Let $\sigma_0 = \pi_1^{-1}\pi_1\sigma \cup \pi_2^{-1}\pi_2\sigma$. By Lemma 3.2, we have $\sigma_0 \subset \sigma$, $\pi_1\sigma_0 = \pi_1\sigma \in K$ and $\pi_2\sigma_0 = \pi_2\sigma \in K$ and by 2.2, $\sigma_0 \in K^2$ and $\sigma_0 \subset [(x, y)]$. Since $\pi_1\sigma_0 \cup \pi_2\sigma_0 = \pi_1\sigma \cup \pi_2\sigma$ is improper, there exists $V_1 \in \pi_1\sigma_0$ and $V_2 \in \pi_2\sigma_0$ such that $V_1 \cap V_2 = \emptyset$. It follows that $(V_1 \times V_2) \cap \Delta = \emptyset$, which means that, $\alpha_0 \cup \Delta$ is improper. By Theorem 3.2, Δ is **sc**-closed, i.e., $(A, K) \in \mathbf{CHY}_{2sc}$. □

Theorem 3.15. *A Cauchy space $(A, K) \in \mathbf{CHY}_{0c}$ if and only if $(A, K) \in \mathbf{CHY}_{ic}$, $i = 1, 2$.*

Proof. Suppose $(A, K) \in \mathbf{CHY}_{0c}$ and $x, y \in A$ with $x \neq y$. Then there exists a closed subset $M \subset A$ such that $x \notin M$, $y \in M$ either $\alpha \nsubseteq [x]$ or $\alpha \cup [M]$ is improper for every $\alpha \in K$ or there exists a closed subset $N \subset A$ such that $x \in N$ and $y \notin N$ either $\alpha \nsubseteq [y]$ or $\alpha \cup [N]$ is improper for every $\alpha \in K$. Let the first case holds and $M = \{y\}$. Then we have for each $a \in A$ with $a \notin M$ and for all $\alpha \in K$, $\alpha \cup [M]$ is improper or $\alpha \nsubseteq [a]$ by Theorem 3.2 (2) since $M = \{y\}$ is closed. It follows that $(A, K) \in \mathbf{CHY}_{ic}, i = 1, 2$ by Theorem 3.6 and 3.7. If the second case holds, then similarly we have (A, K) is in $\mathbf{CHY}_{ic}, i = 1, 2$.

Conversely, suppose $(A, K) \in \mathbf{CHY}_{ic}, i = 1, 2$, i.e., $[x] \cap [y] \notin K$ for all $x, y \in A$ with $x \neq y$. By Theorem 3.2 (1), $\{x\}$ and $\{y\}$ is closed. Let $M = \{y\}$ or $N = \{x\}$. It follows that $x \notin M$ and $y \in M$ or $x \in N$ and $y \notin N$, and consequently $(A, K) \in \mathbf{CHY}_{0c}$ by Theorem 3.5. □

Theorem 3.16. *Let (A, K) be a Cauchy space. Then, $(A, K) \in \mathbf{CHY}_{ic}$ if and only if $(A, K) \in \mathbf{CHY}_{isc}$ for $i = 0, 1, 2$.*

Proof. It follows from Theorem 3.2 and Definition 3.10. □

Remark 3.17. $\mathbb{T}$-**CHY** is the full subcategory of **CHY** consisting of all $\mathbb{T}$ objects, where $\mathbb{T}$ is $\overline{T}_0$ (resp. T_0 , T_1, $\overline{T}_2$ which were defined in [2]).

Theorem 3.18. *Let (A, K) be a Cauchy space. Then, (A, K) is $\overline{T}_0$ or T_0 (T_1) iff for each distinct pair x and y in A, $[x] \cap [y] \notin K$ [18].*

Theorem 3.19. *Let (A, K) be a Cauchy space. (A, K) is $\overline{T}_2$ iff for each distinct points x and y in A, we have $[x] \cap [y] \notin K$ [18].*

Theorem 3.20. *The following categories are isomorphic.*

1. $\mathbf{CHY}_{ik}$ *for $i = 0, 1, 2$ and $k = c$ or sc.*
2. $\mathbb{T}$-**CHY** *for $\mathbb{T} = \overline{T}_0, T_0, T_1, \overline{T}_2$.*

Proof. It follows from Theorems 3.15, 3.16, 3.10, 3.11 and Remark 3.4. □

Remark 3.21.

1. By Remark 3.17 and Theorems 3.15, 3.16, 3.10, 3.11, we have

$$\mathbf{CHY}_{2c} = \mathbf{CHY}_{2sc} = \mathbf{CHY}_{1c} = \mathbf{CHY}_{1sc} = \mathbf{CHY}_{0c} = \mathbf{CHY}_{0sc}.$$

2. For the category **Top**, by Theorem 2.2.11 of [2] and Remarks 3.4 and 3.5 of [6],

$$\mathbf{Top}_{2c} = \mathbf{Top}_{2sc} \subset \mathbf{Top}_{1c} = \mathbf{Top}_{1sc} \subset \mathbf{Top}_{0c} = \mathbf{Top}_{0sc}.$$

3. For the category of preordered spaces, **Prord**, by Theorem 4.5 of [8],

$$\mathbf{Prord}_{2c} = \mathbf{Prord}_{2sc} \subset \mathbf{Prord}_{0c} = \mathbf{Prord}_{0sc},$$

$$\mathbf{Prord}_{1c} = \mathbf{Prord}_{1sc} \subset \mathbf{Prord}_{0c} = \mathbf{Prord}_{0sc}.$$

4. For the category of bornological spaces, **Born**, by Lemma 2.11 of [5],

$$\mathbf{Born}_{0c} = \mathbf{Born}_{1c} = \mathbf{Born}_{2c} \subset \mathbf{Born}_{0sc} = \mathbf{Born}_{1sc} = \mathbf{Born}_{2sc}.$$

5. For the category of filter convergence spaces, **FCO**, by Theorem 2.9 of [5],

$$\mathbf{FCO}_{2sc} \subset \mathbf{FCO}_{2c} = \mathbf{FCO}_{1sc} = \mathbf{FCO}_{1c} \subset \mathbf{FCO}_{0sc} = \mathbf{FCO}_{0c}.$$

6. For the category of constant filter convergence spaces, **ConFCO**, by Remark 4.8 of [16],

$$\mathbf{ConFCO}_{2c} = \mathbf{ConFCO}_{2sc} \subset \mathbf{ConFCO}_{1c} = \mathbf{ConFCO}_{1sc} \subset \mathbf{ConFCO}_{0c} = \mathbf{ConFCO}_{0sc}.$$

7. For the category of extended pseudo-quasi-semi metric spaces, **pqsMet**, by Remark 3.12 of [11],

$$\mathbf{pqsMet}_{1sc} = \mathbf{pqsMet}_{2sc} \subset \mathbf{pqsMet}_{1c} = \mathbf{pqsMet}_{2c},$$

$$\mathbf{pqsMet}_{0sc} \subset \mathbf{pqsMet}_{0c}.$$

8. For the category of semiuniform convergence spaces, **SUConv**, by Theorem 4.5 of [9],

$$\mathbf{SUConv}_{1c} = \mathbf{SUConv}_{1sc} = \mathbf{SUConv}_{0c}.$$

9. For the category of convergence approach spaces, **CApp**, by Remark 4.15 of [23],

$$\mathbf{CApp}_{2sc} \subset \mathbf{CApp}_{1sc} \subset \mathbf{CApp}_{0sc},$$

$$\mathbf{CApp}_{2c} \subset \mathbf{CApp}_{1c} \subset \mathbf{CApp}_{0c}.$$

10. For the category of reflexive relation spaces, **RRel** and for the category of pre-bornological spaces, **PBorn**, by Theorems 3.6 and 3.8 of [10],

$$\mathbf{RRel}_{2c} = \mathbf{RRel}_{2sc} = \mathbf{RRel}_{1sc} \subset \mathbf{RRel}_{1c}.$$

$$\mathbf{PBorn}_{0c} = \mathbf{PBorn}_{1c} = \mathbf{PBorn}_{2c} \subset \mathbf{PBorn}_{0sc} = \mathbf{PBorn}_{1sc} = \mathbf{PBorn}_{2sc}.$$

4. Irreducible Cauchy spaces

Definition 4.1. ([11]) Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be a topological functor, X be an object in $\mathcal{E}$.

1. X is said to be irreducible if A, B are closed subobjects of X and $X = A \cup B$, then $A = X$ or $B = X$.
2. X is said to be strongly irreducible if A, B are strongly closed subobjects of X and $X = A \cup B$, then $A = X$ or $B = X$.

Irreducibility play an important role in algebraic geometry. For example, according to a fundamental theorem of classical algebraic geometry, every algebraic set can be expressed in an unique way as a finite union of irreducible components. Also, the Zariski topologies are irreducible.

In **Top**, the notion of irreducibility coincides with the usual irreducibility [11].

Note that if a topological space (X, τ) is irreducible, then (X, τ) is connected, and if (X, τ) is T_1, then the notions of irreducible spaces and strongly irreducible spaces coincide [11]. Additionally, if (X, τ) is nonempty irreducible and T_2, then (X, τ) must be a one-point space [11].

Theorem 4.2. *A Cauchy space (A, K) is (strongly) irreducible if and only if for any nonempty proper subset F of A, either the condition (1) or (2) holds:*

1. *There exists a proper filter α in K such that $\alpha \cup [F]$ is proper and $\alpha \subset [a]$ for some $a \in F^c$.*
2. *There exists a proper filter α in K such that $\alpha \cup [F^c]$ is proper and $\alpha \subset [b]$ for some $b \in F$.*

Proof. Suppose that (A, K) is (strongly) irreducible but the conditions (1) and (2) do not hold for some nonempty proper subset F of X. Since the condition (1) does not hold, we get for each $a \in A$ with $a \notin F$ and for all α in K such that $\alpha \cup [F]$ is improper or $\alpha \not\subset [a]$, which means that subset a F is (strongly) closed by Theorem 3.2 (2). Similarly, since the condition (2) does not hold, we get F^c is (strongly) closed. Hence, $X = F \cup F^c$, but $X \neq F$ and $X \neq F^c$. This is a contradiction since (A, K) is (strongly) irreducible.

Conversely, suppose that the condition (1) holds. Then, F is not (strongly) closed by Theorem 3.2 (2). Similarly, suppose that the condition (2) holds. Then, F^c is not (strongly) closed. Hence, the only subsets of X both (strongly) open and (strongly) closed are $\emptyset$ and X. It follows that if A, B are closed subset of X and $X = A \cup B$, then $A = X$ or $B = X$. Thus, (A, K) is (strongly) irreducible. □

Theorem 4.3. *A Cauchy space (A, K) is irreducible if and only if (A, K) is strongly irreducible.*

Proof. It follows from Theorem 3.2 (2) and Definition 4.1. □

Example 4.4. Let $A = IR$ be the set of real numbers and $K_1 = F(A)$ and $K_2 = \{[a] \mid a \in A\} \cup \{[\emptyset]\}$. Then (A, K_1) is (strongly) irreducible, but (A, K_2) is not (strongly) irreducible. Because, both conditions in Theorem 4.2 do not hold for $F = \{5\}$.

Theorem 4.5. *Let (A, K) be in **CHY**. (A, K) is $PreT_2'$ iff for each pair of distinct points x and y in A, $[x] \cap [y] \in K$(equivalently, for each finite subset F of A, we have $[F] \in K$) [18].*

Theorem 4.6. *Let (A, K) be a Cauchy space. (A, K) is T_2' iff for each distinct points x and y in A, we have $[x] \cap [y] \in K$(equivalently, for each finite subset F of A, we have $[F] \in K$) [18].*

Theorem 4.7. *A Cauchy space (A, K) is (strongly) irreducible if and only if (A, K) is (strongly) connected.*

Proof. It follows from Theorems 4.2, 4.3 and Theorem 3.6 of [18]. □

Theorem 4.8. *Let (A, K) be a nonempty (strongly) irreducible Cauchy space.*

1. *If $(A, K) \in \mathbf{CHY}_{ik}$, $i = 0, 1, 2$ and $k = c$ or sc, then (A, K) must be a one-point space.*
2. *If (A, K) is $\overline{T}_0$ (resp. T_0 or T_1 or $\overline{T}_2$), then (A, K) must be a one-point space.*
3. *If (A, K) is $PreT_2'$ or T_2', then (A, K) may not be a one-point space.*

Proof. 1. Suppose that (A, K) is nonempty (strongly) irreducible, $(A, K) \in \mathbf{CHY}_{ik}$, $i = 0, 1, 2$, $k = c$ or sc, and A has least two points, x and y. By Theorems 3.15, 3.16 and Remarks 3.4 and 3.5, all subsets of A are (strongly) closed. It follows that $\{x\}$ and $\{y\}$ are proper (strongly) closed subsets and $A = \{x\} \cup \{x\}^c$. This is a contradiction since (A, K) is (strongly) irreducible.

2. The proof is similar to the proof of (1) by using Theorem 3.10, 3.11, 3.12 and Remark 3.4, 3.5.

3. By Theorems 4.3 and 4.4, a cauchy space is $PreT_2'$ or T_2'. Let (A, K_1) be the cauchy space defined in Example 4.1. Then (A, K_1) is (strongly) irreducible and $PreT_2'$ (resp. T_2'), but (A, K_1) is not a one-point space. □

Let $\mathbb{IR}\mathcal{E}$ be the full subcategory of $\mathcal{E}$ consisting of all irreducible objects, and $\mathbf{T}\mathcal{E}$ be the full subcategory of $\mathcal{E}$ consisting of all $\mathbf{T}$ objects, where $\mathbf{T} = T_0, \overline{T}_0, T_0', T_1, \overline{T}_2, T_2'$.

Remark 4.9. 1. By Theorem 4.6, for $i = 1, 2$ and $k = \mathbf{c}$ or $\mathbf{sc}$, we have

$$\mathbb{IR}\mathbf{CHY}_{ik} = \overline{T}_0\mathbb{IR}\mathbf{CHY} = T_1\mathbb{IR}\mathbf{CHY} = \overline{T}_2\mathbb{IR}\mathbf{CHY} = T_2'\mathbb{IR}\mathbf{CHY} \subset T_0'\mathbb{IR}\mathbf{CHY}.$$

2. For the category **Top**, by Remark 3.5 of [6] and Theorem 3.12 of [10],

$$T_2\mathbb{IR}\mathbf{Top} = \mathbb{IR}\mathbf{Top}_{2cl} = \mathbb{IR}\mathbf{Top}_{2scl} \subset T_2\mathbf{Top} \cap \mathbb{IR}\mathbf{Top} \subset T_0\mathbf{Top}.$$

3. For the category of prebornological spaces, **PBorn**, by Theorem 3.6 and 3.9 of [10] and Theorem 3.7 of [7], for $i = 1, 2$,

$$\mathbf{PBorn}_{icl} = T_0\mathbf{PBorn} = T_2'\mathbf{PBorn} \subset \overline{T}_2\mathbf{PBorn} \subset \mathbb{IR}\mathbf{PBorn} =$$

$$\overline{T}_0\mathbf{PBorn} = T_0'\mathbf{PBorn} = T_1\mathbf{PBorn} = \mathbf{PBorn}_{iscl}.$$

4. For the category of pair spaces, **CP**, by [7] and Theorem 3.7 and 3.10 of [10], for $i = 0, 1, 2$,

$$T_0\mathbf{CP} = \mathbb{IR}\mathbf{CP} \subset \overline{T}_0\mathbf{CP} = T_0'\mathbf{CP} = T_1\mathbf{CP} = \overline{T}_2\mathbf{CP} = T_2'\mathbf{CP} = \mathbf{CP}_{icl}.$$

5. For the category of reflexive spaces, **RRel**, by Theorem 3.8 and 3.11 of [10] and Theorem 3.7 of [7], for $i = 1, 2$,

$$\mathbb{IR}\mathbf{RRel}_{iscl} = T_1\mathbb{IR}\mathbf{RRel} = \overline{T}_2\mathbb{IR}\mathbf{RRel} = T_2'\mathbb{IR}\mathbf{RRel} \subset \overline{T}_0\mathbf{RRel} \subset T_0\mathbf{RRel} = \mathbf{RRel}_{icl} \subset T_0'\mathbf{RRel}.$$

6. For the category of extended pseudo quasi semi metric spaces, **pqsMet**, by [7] and Theorem 3.10 of [11], for $i = 1, 2$,

$$\mathbb{IR}\mathbf{pqsMet}_{icl} = T_1\mathbb{IR}\mathbf{pqsMet} = \overline{T}_2\mathbb{IR}\mathbf{pqsMet} = T_2'\mathbb{IR}\mathbf{pqsMet} \subset T_0'\mathbb{IR}\mathbf{pqsMet}.$$

7. For the category of proximity spaces **Prox**, by Theorem 4.7 and Remark 4.8 of [25], for $k = \mathbf{c}$ or $\mathbf{sc}$,

$$\mathbb{IR}\mathbf{Prox}_{ik} = \overline{T}_0\mathbb{IR}\mathbf{Prox} = T_1\mathbb{IR}\mathbf{Prox} = \overline{T}_2\mathbb{IR}\mathbf{Prox} = T_2'\mathbb{IR}\mathbf{Prox} \subset T_0'\mathbb{IR}\mathbf{Prox}.$$

5. Urysohn's lemma and Tietze extention theorem

In this section, we present Urysohn's Lemma and Tietze Extention Theorem for the Cauchy spaces.

Theorem 5.1. *(Urysohn's Lemma) Let (A, K) be a Cauchy space and $M, N \subset A$ be nonempty disjoint closed subset of A. Then, there exists a Cauchy map $f : (A, K) \to ([0, 1], L)$, where L is any Cauchy structure on [0,1], such that $f(M) = 0$ and $f(N) = 1$.*

Proof. Define $f : (A, K) \to ([0, 1], L)$, by $f(x) = \begin{cases} 0 & , x \in M \\ 1 & , x \notin M \end{cases}$ for $x \in A$.

Note that $f(M) = 0$ and $f(N) = 1$. We show that f is a Cauchy map. Let $\alpha \in K$. If α is improper, then $f(\alpha)$ is improper. Suppose α is proper. Since $M \subset A$ closed, by Theorem 3.2, for $x \notin M$ with $x \in A$ and $\alpha \in K$, $\alpha \cup [M]$ is improper or $\alpha \not\subseteq [x]$.

Suppose $\alpha \cup [M]$ is improper. Then, there exists $V \in \alpha$ such that $V \bigcap M = \emptyset$. Hence, $f(\alpha) = [1] \in L$, since $\{1\} = f(V) \in f(\alpha)$ and $V \subset M^c$.

Suppose $\alpha \not\subseteq [x]$ ($x \notin M$ with $x \in A$). Then, $f(\alpha) \not\subseteq [f(x)] = [1]$. Hence, $f(\alpha) \subseteq [0]$ and $f(\alpha) \bigcup [[0, 1]] \subseteq [0] \bigcup [[0, 1]] = [0]$. It follows $f(\alpha) \bigcup [[0, 1]] \in L$ is proper. Since L is a Cauchy structure on [0,1], $f(\alpha) \bigcap [[0, 1]] \in L$ and $f(\alpha) \in L$.

Consequently, f is a Cauchy mapping. □

Theorem 5.2. *Let (A, K) be a $\overline{T}_0$ (resp. T_0 or T_1 or $\overline{T}_2$), Cauchy space and $M, N \subset A$ be nonempty disjoint subset of A. Then, there exists a Cauchy mapping $f : (A, K) \to ([0, 1], L)$, where L is any Cauchy structure on $[0, 1]$, such that $f(M) = \{0\}$ and $f(N) = \{1\}$.*

Proof. The proof is similar to the proof of Theorem 5.1 by using Theorem 3.2, 3.10 and 3.11. □

Definition 5.3. Let (A, K), (B, L) be Cauchy spaces, $M \subset A$ and $g : (M, K_M) \to (B, L)$ be a Cauchy map where K_M is the initial Cauchy structure on M induced by the inclusion, $i : M \to (A, K)$. If there exists a Cauchy map $f : (A, K) \to (B, L)$ such that for all $x \in M$ $f(x) = g(x)$, then f is called a Cauchy extension of g.

Theorem 5.4. *(Tietze Extension Theorem) Let (A, K) be a Cauchy space and $M \subset A$ be nonempty closed subset of A. Then, every Cauchy map $f : (M, K_M) \to ([0, 1], L)$, where L is any Cauchy structure on [0,1] and K_M is the initial Cauchy structure on M induced by the inclusion, has a Cauchy extention mapping $g : (A, K) \to ([0, 1], L)$.*

Proof. Suppose (A, K) is a Cauchy space, $M \subset A$ is a nonempty closed subspace of A and $f : (M, K_M) \to ([0, 1], L)$ is a Cauchy mapping and K_M is the initial Cauchy structure on M induced by the inclusion, $i : M \to (A, K)$.

Define $g : (A, K) \to ([0, 1], L)$ by

$g(x) = \begin{cases} f(x) & , x \in M \\ 0 & , x \notin M \end{cases}$ for $x \in A$.

Note that $g(x) = f(x)$ for all $x \in M$. We show that g is a Cauchy mapping. Let $\alpha \in K$. If α is improper, then $f(\alpha)$ is improper. Suppose α is proper. Let $x \in A$. If $x \in M$, then $g(x) = f(x)$ is a Cauchy map.

Suppose $x \notin M$. Since $M \subset A$ closed, $x \notin M$ with $x \in A$ and $\alpha \in K$, By Theorem 3.2, $\alpha \cup [M]$ is improper or $\alpha \not\subseteq [x]$.

Suppose $\alpha \cup [M]$ is improper. Then, there exists $V \in \alpha$ such that $V \bigcap M = \emptyset$. Hence, $g(\alpha) = [0] \in L$, since $\{0\} = g(V) \in g(\alpha)$ and $V \subset M^c$.

Suppose $\alpha \not\subseteq [x]$. Then $\exists U \in \alpha$ with $x \notin U$. Note that $x \notin U \cup M$ and $U \cup M \in \alpha$ since α is filter. $g(x) = \{0, f(x)\} \subset g(U \cup M) \in g(\alpha)$. Hence $g(\alpha) \subset [f(x)] \cap [0]$ and $g(\alpha) \bigcup [[0, 1]]$ is proper. Since L is a Cauchy structure on [0,1], then $g(\alpha) \bigcap [0, 1] \in L$ and $g(\alpha) \in L$.

Hence, g is a Cauchy mapping. □

Theorem 5.5. *Let (A, K) be a $\overline{T}_0$ Cauchy space and $M \subset A$ be nonempty subspace of A. Then, every Cauchy mapping $f : (M, K_M) \to ([0, 1], L)$, where L is any Cauchy structure on $[0, 1]$ and K_M is the initial Cauchy structure on M induced by the inclusion, has a Cauchy extention mapping $g : (A, K) \to ([0, 1], L)$.*

Proof. The proof is similar to the proof of Theorem 5.4 by using Theorem 3.2, 3.10 and 3.11. □

6. Conclusion

In this paper, we defined two notions of closure operators in the category of Cauchy spaces which satisfy (weak) hereditariness, productivity and idempotency, and we characterized each of $T_i, i = 0, 1, 2$ Cauchy spaces by using these closure operators as well as showed how these subcategories are related. Moreover, we characterized the irreducible Cauchy spaces and investigated the relationship among each of irreducible and connected Cauchy spaces. Furthermore, we compared our results with results in some topological categories. Finaly, we presented Urysohn's lemma and Tietze extension theorem for Cauchy spaces.

In the future work, it will be interesting to characterize each of hereditarily disconnected [7] , totally disconnected [7] and sober [10] Cauchy spaces.

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