Filomat 37:19 (2023), 6395–6415
https://doi.org/10.2298/FIL2319395A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On completeness of some pro-solvable Lie algebras

K.K. Abdurasulov[a], B.A.Omirov[b,d,a], I.S.Rakhimov[c], G.O.Solijanova[b]

[a] *V.I.Romanovski Institute of Mathematics Uzbekistan Academy of Sciences, University street 9, 100174 Tashkent, Uzbekistan*
[b] *National University of Uzbekistan, University street 4, 100174 Tashkent, Uzbekistan*
[c] *Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (UiTM), 40450 Shah Alam, Selangor Darul Ehsan, Malaysia*
[d] *New Uzbekistan University, Mustaqillik ave. 54, 100007 Tashkent, Uzbekistan*

Abstract. In the paper we describe the derivations of two $\mathbb{N}$-graded infinite-dimensional Lie algebras $\mathbf{n}_1$ and $\mathbf{n}_2$ which are the positive parts of the affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively. Then we construct all pro-solvable Lie algebras whose potential nilpotent ideals are $\mathbf{n}_1$ and $\mathbf{n}_2$ and compute low-dimensional (co)homology groups of the pro-solvable Lie algebras constructed.

1. Introduction

It is well-known that the classification problem of algebras can be split into two main tasks: classification of semisimple and solvable algebras. The most of the cases semisimple part is reduced to the classification of simple algebras. This theory is well-studied for Lie algebras. The problem of classification of simple finite-dimensional Lie algebras over the field of complex numbers was solved by the end of the 19th century by W. Killing and É. Cartan. There is a method originated by V.V. Morozov and C.M. Mubarakzjanov (see [19], [20]) on construction of finite-dimensional solvable Lie algebras by their nilradicals. A generalization of this method to infinite-dimensional case is of interest. It turned out what was noticed by V.V. Morozov and C.M. Mubarakzjanov that there is interrelations between a few invariants of Lie algebras: the dimension and the number of generators of the nilradical, the co-dimension of the nilradical, the number of nil-independent derivations, the existence of inner and outer derivations, the dimensions of the first and second (co)homology spaces. One of such kind relations states that the co-dimension of the nilradical of a Lie algebra is at most the number of its nill-independent derivations. The fact has been used to construct the solvable Lie algebras in [21–24]. By using these relationships the classification of solvable extensions has been given for the following classes of nilpotent Lie algebras in low-dimensions with Abelian [25, 26], Heisenberg [27], Borel [36], $\mathbb{N}$-graded [7], filiform and quasifiliform nilradicals [34, 35] (also see [37] and references therein). A few other nilradicals cases are given in [28–30]. In finite-dimensional case, the concept of filiform Lie algebra was introduced by Vergne in [38]. It turned out that the case of filiform and quasi-filiform nilradicals the relationships mentioned above much simplify the situation. Moreover, there

2020 *Mathematics Subject Classification*. 17B40, 17B56, 17B65
Keywords. Lie algebra; Potentially nilpotent Lie algebra; Pro-nilpotent (pro-solvable) Lie algebra; Cohomology group.
Received: 04 October 2022; Accepted: 15 February 2023
Communicated by Dijana Mosić
Research partially supported by Agencia Estatal de Investigación (Spain), grant PID2020-115155GB-I00 (European FEDER support included, UE)
Email addresses: abdurasulov0505@mail.ru (K.K. Abdurasulov), omirovb@mail.ru (B.A.Omirov), isamiddin@uitm.edu.my (I.S.Rakhimov), gulhayo.solijonova@mail.ru (G.O.Solijanova)

are cases when the co-dimension of the nilradical is equal to the number of its generators. In these cases the structure of solvable Lie algebras is quite rigid. It was observed that such Lie algebra is unique up to isomorphism (see [13]), its center is trivial and all derivations are inner. In addition, often such Lie algebras have trivial second Chevalley cohomology groups (see [4, 14]).

Unfortunately for infinite-dimensional cases non of the above either classification of semisimple or solvable is completed. There is É. Cartan's classification of simple infinite-dimensional Lie algebras of vector fields on a finite-dimensional space. Then B. Weisfeiler [39] gave an algebraic proof of Cartan's classification theorem reducing the problem to the classification of simple $\mathbb{Z}$-graded Lie algebras of finite "depth".

There are, however, four classes of infinite-dimensional Lie algebras that underwent a more or less intensive study due to their various applications, mostly in Physics. These are, first of all, the above-mentioned Lie algebras of vector fields, the second class consists of Lie algebras of smooth mappings of a given manifold into a finite-dimensional Lie algebra, the third class is the classical Lie algebras of operators in a Hilbert or Banach space and finally, the fourth class of infinite-dimensional Lie algebras is the class of the so-called Kac-Moody algebras.

In the classification theory of infinite-dimensional Lie algebras, several deep results were obtained with Galois cohomology methods exhibiting exciting connections between forms of multi-loop algebras and the Galois theory of forms of algebras over rings. This branch of structure theory is complemented by the connection between the classification of generalized Kac-Moody algebras and automorphic forms.

Note that there are some examples of the so-called pro-solvable Lie algebras whose maximal pro-nilpotent ideal is $\mathbb{N}$-graded Lie algebra of maximal class (infinite-dimensional filiform Lie algebra) the method described above for finite-dimensional solvable Lie algebras by means of its nilradical is applicable. It should be noted that in all of these mentioned examples, the codimension of the maximal pro-nilpotent ideal of a pro-solvable algebra coincides with the number of generators of the pro-nilpotent ideal.

The infinite-dimensional analogue of filiform Lie algebras has been introduced by A.Fialowski a long time ago in [9]. Nevertheless, the systematic study of infinite-dimensional cases has not been given. The attempts made were occasional depending mainly on some applications in Physics and Geometry. For instance, in [12], two classes of infinite-dimensional Lie algebras called potentially nilpotent and potentially solvable were introduced in connection with the study of their deformations.

All algebras considered in the paper are supposed to be over the field of complex numbers $\mathbb{C}$ unless otherwise specified.

Definition 1.1. *A Lie algebra $(L, [\cdot,\cdot])$ is called a positively graded ($\mathbb{N}$-graded) if it is represented as a direct sum*

$$L = \bigoplus_{i=1}^{\infty} L_i$$

of its homogonious subspaces L_i such that $[L_i, L_j] \subset L_{i+j}$ for all $i, j \in \mathbb{N}$.

Example 1.2. *Let $L = L(m)$ be the free Lie algebra with the generators $a_1, \dots, a_m$. Consider the following linear span of the k-words $L_k = \mathrm{Span}\{[a_{i_1}, [a_{i_2}, [\dots, [a_{i_{k-1}}, a_{i_k}] \dots]]]\}$. Then it is easy to see that L_k, where $k = 1, 2, \dots$ are desired homogeneous subspaces of $L = L(m)$ to be positively graded.*

Example 1.3. *Let $L = \mathrm{Span}\{e_1, e_2, e_3, e_4, \dots\}$ with the commutation rules*

$$[e_1, e_i] = e_{i+1}, \ i \geq 2, \ [e_i, e_k] = 0, \ \text{where } i, k \neq 1.$$

If as homogeneous subspaces we take $L_k = \mathrm{Span}\{e_k\}$, $k = 1, 2, \dots$ then L is positively graded.

Note that the quotient algebra L/I of an infinite-dimensional positively graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ by the ideal $I = \bigoplus_{i=k+1}^{\infty} L_i$ is nilpotent.

The concept of Lie algebra's width was introduced by Zel'manov and Shalev in [31, 32]. A positively graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ is said to be of width d if there is a positive number d such that $\dim L_i \leq d$, for all $i \in \mathbb{N}$. The problem of classifying graded Lie algebras of finite width was outlined by Zelmanov and Shalev as an important and difficult problem (In [32] it was called "a formidable challenge").

The classification of positive graded finite-dimensional Lie algebras has been given in [17]). The author also could manage to classify infinite-dimensional case with the width $\frac{3}{2}$.

Definition 1.4. *A grading* $L = \bigoplus_{i=1}^{\infty} L_i$ *is called natural if* $[L_1, L_i] \subset L_{i+1}$ *for all* $i \in \mathbb{N}$.

Example 1.5. *Let us consider the algebra* $L = \mathrm{Span}\{e_1, e_2, e_3, e_4, \dots\}$ *from Example 1.3.*

$$L_1 = \mathrm{Span}\{e_1, e_2\}, \;\; L_2 = \mathrm{Span}\{e_3\}, \;\; L_3 = \mathrm{Span}\{e_4\}, \;\; L_4 = \mathrm{Span}\{e_5\}, \dots$$

is the natural grading of L.

For a Lie algebra L we define the *lower central* and the *derived series* as follows

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \geq 1, \qquad L^{[1]} = L, \; L^{[s+1]} = [L^{[s]}, L^{[s]}], \; s \geq 1,$$

respectively.

Definition 1.6. *A Lie algebra* L *is said to be residually nilpotent (respectively, residually solvable) if* $\bigcap_{i=1}^{\infty} L^i = 0$ (*respectively,* $\bigcap_{i=1}^{\infty} L^{[i]} = 0$).

The following definition can be found in [12].

Definition 1.7. *An infinite dimensional Lie algebra* L *is said to be potentially nilpotent* (*respectively, solvable*), *if* $\bigcap_{i=1}^{\infty} L^i = 0$ (*respectively,* $\bigcap_{i=1}^{\infty} L^{[i]} = 0$) *and* $\dim(L^i/L^{i+1}) < \infty$ (*respectively,* $\dim(L^{[i]}/L^{[i+1]}) < \infty$) *for any* $i \geq 1$.

Here are two examples of potentially nilpotent Lie algebras (see [17]).

Example 1.8.

- *Lie algebra* $\mathbf{m}_2 = \mathrm{Span}\{e_1, e_2, e_3, \dots\}$ *with the commutation rules*

$$\mathbf{m}_2 := \begin{cases} [e_1, e_i] = e_{i+1}, & i \geq 2, \\ [e_2, e_j] = e_{j+2}, & i \geq 3. \end{cases}$$

is a potentially nilpotent Lie algebra.

- *The positive part* W^+ *of the Witt algebra is given by the commutation rules*

$$[e_i, e_j] = (i - j)e_{i+j}, \quad i, j \in \mathbb{N}.$$

The algebra W^+ *also is a potentially nilpotent Lie algebra.*

Another two algebras below give examples of potentially solvable but not necessarily potentially nilpotent algebras (see [5]).

Example 1.9.

- *Let* $L = \mathrm{Span}\{x, y, e_1, e_2, e_3, \dots\}$ *with the following rules of compositions*

$$L := \begin{cases} [e_i, e_1] = e_{i+1}, & i \geq 2, \\ [x, e_1] = e_1, \\ [x, e_i] = (i-1)e_i, & i \geq 2, \\ [y, e_i] = e_i, & i \geq 2. \end{cases}$$

- *Consider non-negative part* $\widetilde{W^+}$ *of the Witt algebra* W *given by the following rules*

$$[e_i, e_j] = (i - j)e_{i+j}, \quad i, j \geq 0.$$

The algebra $\widetilde{W^+}$ *is potentially solvable but is not potentially nilpotent.*

Note that finite-dimensional solvable (respectively, nilpotent) Lie algebras are also (potentially solvable) potentially nilpotent.

Definition 1.10. *[17] A Lie algebra* L *is said to be pro-nilpotent (respectively, pro-solvable) if* $\bigcap\limits_{i=1}^{\infty} L^i = 0$ *and* $\dim L/L^i < \infty$ *(respectively,* $\bigcap\limits_{i=1}^{\infty} L^{[i]} = 0$ *and* $\dim L/L^{[i]} < \infty$ *) for any* $i \geq 1$.

In [1], it was shown that the concepts of potentially nilpotency (respectively, potentially solvability) and pro-nilpotency (respectively, pro-solvability) are equivalent.

Let L be a pro-nilpotent Lie algebra. Then for the ideals L^k, where $k \geq 1$ one has

$$L = L^1 \supseteq L^2 \supseteq \dots \supseteq L^k \supseteq L^{k+1} \supseteq \dots, \text{ and } [L^i, L^j] \subset L^{i+j}, \ \ i, j \in \mathbb{N}.$$

Consider the associated graded Lie algebra $grL = \bigoplus\limits_{i=1}^{+\infty}(L^i/L^{i+1})$ with respect to the above filtration. The Lie bracket on grL is given by

$$[x + L^{i+1}, y + L^{j+1}] = [x, y] + L^{i+j+1}, \text{ for } x \in L^i, \ y \in L^j.$$

It is evident that grL is a $\mathbb{N}$-graded Lie algebra.

Definition 1.11. *A pro-nilpotent Lie algebra* L *is said to be a* naturally graded *if it is isomorphic to* grL*. The grading* $L = \bigoplus\limits_{i=1}^{\infty} L_i$ *of a naturally graded Lie algebra* L *is called natural if there exists a graded isomorphism*

$$\varphi : grL \longrightarrow L, \ \ \varphi((grL)_i) = L_i, \ \ i \in \mathbb{N}.$$

Let us define *natural grading* of the algebra L. Write

$$L_1 := L^1/L^2, \ L_i := L^i/L^{i+1}, \text{ where } i = 2, 3, \dots$$

In this case, $L \simeq L_1 \oplus L_2 \oplus L_3 \oplus \dots$ One can readily verify the validity of the inclusions $[L_i, L_j] \subset L_{i+j}$ and the grading is natural.

Let L be a pro-nilpotent Lie algebra. Suppose that there is a basis $\{e_1, e_2, e_3, \dots\}$ of L, $k_i = \dim L_i$ and $L^i = \mathrm{Span}\{e_{k_i}, e_{k_{i+1}}, e_{k_{i+2}}, \dots\}$, $i = 1, 2, 3, \dots$ Define

$$c(L) = (k_2 - k_1, k_3 - k_2, k_4 - k_3, \dots), \ where \ k_1 = 1.$$

Since L is residually nilpotent, it follows that $k_{i+1} - k_i \geq 1$ for any $i \geq 1$.

Here are two important examples of the naturally graded Lie algebras from [17].

The first of them is a Lie algebra $\mathbf{n}_1$ with the following non-zero brackets on a basis $\{e_1, e_2, e_3, \dots\}$

$$\mathbf{n}_1: \quad [e_i, e_j] = c_{i,j} e_{i+j}, \quad i, j \in \mathbb{N}, \quad c_{i,j} = \begin{cases} 1, & if \quad i - j \equiv 1 \pmod 3, \\ 0, & if \quad i - j \equiv 0 \pmod 3, \\ -1, & if \quad i - j \equiv -1 \pmod 3. \end{cases} \tag{1}$$

One can see that

$$(\mathbf{n}_1)_1 := \mathbf{n}_1^1/\mathbf{n}_1^2 = \mathrm{Span}\{e_1, e_2\}, \quad (\mathbf{n}_1)_i := \mathbf{n}_1^i/\mathbf{n}_1^{i+1} = \mathrm{Span}\{e_{i+1}\}, \ i = 2, 3, \dots .$$

Hence,

$$c(\mathbf{n}_1) = (2, 1, 1, \dots).$$

The second algebra is

$$\mathbf{n}_2: \qquad [f_q, f_l] = d_{q,l} f_{q+l}, \quad q, l \in \mathbb{N},$$

given by the following table of multiplications on a basis

$$\{f_{8i+1}, f_{8i+2}, f_{8i+3}, f_{8i+4}, f_{8i+5}, f_{8i+6}, f_{8i+7}, f_{8i+8}\}, \text{ where } i \in \mathbb{Z}^+$$

as follows

	f_{8j}	f_{8j+1}	f_{8j+2}	f_{8j+3}	f_{8j+4}	f_{8j+5}	f_{8j+6}	f_{8j+7}
f_{8i}	0	1	−2	−1	0	1	2	−1
f_{8i+1}	−1	0	1	1	−3	−2	0	1
f_{8i+2}	2	−1	0	0	0	1	−1	0
f_{8i+3}	1	−1	0	0	3	−1	1	−2
f_{8i+4}	0	3	0	−3	0	3	0	−3
f_{8i+5}	−1	2	−1	1	−3	0	0	1
f_{8i+6}	−2	0	1	−1	0	0	0	1
f_{8i+7}	1	−1	0	2	3	−1	−1	0

(2)

It is clear that

$$\mathbf{n}_2^2 = \mathrm{Span}\{f_3, f_4, f_5, \dots\}, \quad \mathbf{n}_2^3 = \mathrm{Span}\{f_4, f_5, f_6, \dots\}, \ \dots \ \mathbf{n}_2^i = \mathrm{Span}\{f_{i+1}, f_{i+2}, \dots\}, \dots$$

Hence,

$$(\mathbf{n}_2)_1 = \mathrm{Span}\{f_1, f_2\}, \ (\mathbf{n}_2)_i = \mathrm{Span}\{f_{i+1}\}, \ \ i = 2, 3, ...$$

$$c(\mathbf{n}_2) = (2, 1, 1, \dots)$$

The algebras $\mathbf{n}_1$ and $\mathbf{n}_2$ are naturally graded pro-nilpotent Lie algebras with homogeneous components $(\mathbf{n}_1)_j$ and $(\mathbf{n}_2)_j$, where $j = 1, 2, 3, \dots$, respectively. These two algebras are known as positive parts of the affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively. They also are famous with their special role in the theory of combinatoric identities (see [10, 15, 16]). The algebras $\mathbf{n}_1$ and $\mathbf{n}_2$ are two representatives (among a few others) of the isomorphism classes of natural graded algebras over fields of characteristic zero given by A. Fialowski and D.Millionshchikov (see [9, 17]). The pro-solvable extensions of n_1 and n_2 were considered in [18] with applications to the theory of characteristic Lie algebras. Note that similar problems to the ones of this paper but for another two representative $\mathbf{m}_0$ and $\mathbf{m}_2$ have been treated earlier in [2]. Thus the present paper along with the results mentioned above in some sense completes the classification problem of the naturally graded algebras with weight $\frac{3}{2}$.

Definition 1.12. *A linear map* $d\colon L \to L$ *of* $(L, [\cdot,\cdot])$ *is said to be a derivation if for all* $x, y \in L$*, the following differentiation rule*

$$d([x, y]) = [d(x), y] + [x, d(y)],$$

holds true.

The set of all derivations of L is denoted by $\mathrm{Der}(L)$. For $x \in L$, as usual, ad_x denotes the map $ad_x : L \to L$ defined by $ad_x(y) = [x, y],\ \forall y \in L$. Obviously, ad_x is a derivation called *inner derivations*. The set of all inner derivations of L is denoted by $\mathrm{Inner}(L)$. No inner derivations in $\mathrm{Der}(L)$ are called *outer derivations*. $\mathrm{Der}(L)$ is a Lie algebra with respect to the composition and $\mathrm{Inner}(L)$ is an ideal in $\mathrm{Der}(L)$.

Definition 1.13. *A Lie algebra L is called complete if* $\mathrm{Center}(L) = 0$ *and all the derivations of L are inner.*

Definition 1.14. *A linear map* $\rho : L \to L$ *is called residually nilpotent, if* $\cap_{i=1}^{\infty}(Im\ \rho^i) = 0$ *holds true.*

Below we introduce the analogue of the notion of nil-independency which had played a crucial role in the description of finite-dimensional solvable Lie algebras in [20].

Definition 1.15. *Derivations* $d_1, d_2, \ldots, d_n$ *of a Lie algebra L over a field* $\mathbb{F}$ *are said to be residually nil-independent, if a map* $f = \alpha_1 d_1 + \alpha_2 d_2 + \ldots + \alpha_n d_n$ *is not potentially nilpotent for any scalars* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$*. In other words,* $\bigcap_{i=1}^{\infty} Imf^i = 0$ *implies* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Recall that

$$\mathrm{H}^1(L, L) = \mathrm{Der}(L)/\mathrm{Inner}(L) \quad \text{and} \quad \mathrm{H}^2(L, L) = \mathrm{Z}^2(L, L)/\mathrm{B}^2(L, L)$$

where the set $\mathrm{Z}^2(L, L)$ consists of those elements $\varphi \in \mathrm{Hom}(\wedge^2 L, L)$ such that

$$Z(x, y, z) = [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y) = 0, \tag{3}$$

while $\mathrm{B}^2(L, L)$ consists of elements $\psi \in \mathrm{Hom}(\wedge^2 L, L)$ such that

$$\psi(x, y) = d([x, y]) - [d(x), y] - [x, d(y)] \text{ for some linear map } d \in \mathrm{Hom}(L, L). \tag{4}$$

In terms of cohomology groups the notion of completeness of a Lie algebra L means that it is centerless and $\mathrm{H}^1(L, L) = 0$.

The organization of the paper is as follows. The next section contains the description of derivations of two $\mathbb{N}$-graded infinite-dimensional Lie algebras $\mathbf{n}_1$ and $\mathbf{n}_2$ what are positive parts of affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively. This followed by the construction of all pro-solvable Lie algebras whose pro-nilpotent nilpotent ideals are $\mathbf{n}_1$ and $\mathbf{n}_2$. The final sections contains the results on the completeness of these classes of Lie algebras and the vanishing of their second cohomology groups.

Note that the similar problems as in this paper for Leibniz superalgebras have been treated recently in [6].

2. Results

2.1. *Derivations of pro-nilpotent Lie algebras* $\mathbf{n}_1$ *and* $\mathbf{n}_2$

In this section we describe the derivations of $\mathbf{n}_1$ and $\mathbf{n}_2$ that we need to use further in the constructions of all pro-solvable Lie algebras whose potential nilpotent ideals are $\mathbf{n}_1$ and $\mathbf{n}_2$.

Proposition 2.1. *The derivations of the algebra* $\mathbf{n}_1$ *are given as follows:*

$$\begin{aligned}
d(e_{3i-2}) &= \sum_{k=1}^{t} (((i-1)\beta_{3k-1} + i\alpha_{3k-2})e_{3k+3i-5} + \alpha_{3k}e_{3k+3i-3}),\\
d(e_{3i-1}) &= \sum_{k=1}^{t} (i\beta_{3k-1} + (i-1)\alpha_{3k-2})e_{3k+3i-4} + \beta_{3k}e_{3k+3i-3}),\\
d(e_{3i}) &= \sum_{k=1}^{t} (i(\beta_{3k-1} + \alpha_{3k-2})e_{3k+3i-3} - \beta_{3k}e_{3k+3i-2} - \alpha_{3k}e_{3k+3i-1}),\ i \in \mathbb{N}.
\end{aligned}$$

Proof. Since the algebra $\mathbf{n}_1$ has two generators $\{e_1, e_2\}$, any derivation d on $\mathbf{n}_1$ is completely determined by $d(e_1)$ and $d(e_2)$.

Let

$$d(e_1) = \sum_{i=1}^{p} \alpha_i e_i, \quad \text{and} \quad d(e_2) = \sum_{j=1}^{q} \beta_j e_j.$$

Without loss of generality one can assume that

$$d(e_1) = \sum_{i=1}^{3t} \alpha_i e_i, \quad d(e_2) = \sum_{j=1}^{3t} \beta_j e_j, \quad \max\{p, q\} \le 3t, \quad t \in \mathbb{N}.$$

Applying the derivation rule we have

$$\begin{array}{lllll} d(e_3) & = & d([e_2, e_1]) & = & \sum_{k=1}^{t} (\beta_{3k-1} + \alpha_{3k-2}) e_{3k} - \sum_{k=1}^{t} \beta_{3k} e_{3k+1} - \sum_{k=1}^{t} \alpha_{3k} e_{3k+2}, \\ d(e_4) & = & -d([e_3, e_1]) & = & \sum_{k=1}^{t} (\beta_{3k-1} + 2\alpha_{3k-2}) e_{3k+1} + \sum_{k=1}^{t} \alpha_{3k-1} e_{3k+2} - \sum_{k=1}^{t} \alpha_{3k} e_{3k+3} \end{array}$$

and $d([e_4, e_1]) = -2 \sum_{k=1}^{t} \alpha_{3k-1} e_{3k+3} = 0$ implies $\alpha_{3k-1} = 0, \ 1 \le k \le t$.

Then,

$$\begin{array}{lll} d(e_5) & = & d([e_3, e_2]) = [d(e_3), e_2] + [e_3, d(e_2)] \\ & = & \sum_{k=1}^{t} (\beta_{3k-1} + \alpha_{3k-2}) e_{3k+2} + \sum_{k=1}^{t} \beta_{3k} e_{3k+3} - \sum_{k=1}^{t} \beta_{3k-2} e_{3k+1} + \sum_{k=1}^{t} \beta_{3k-1} e_{3k+2} \\ & = & \sum_{k=1}^{t} (2\beta_{3k-1} + \alpha_{3k-2}) e_{3k+2} + \sum_{k=1}^{t} \beta_{3k} e_{3k+3} - \sum_{k=1}^{t} \beta_{3k-2} e_{3k+1}, \end{array}$$

and $d([e_5, e_2]) = \sum_{k=1}^{t} \beta_{3k-2} e_{3k+3} = 0$, gives $\beta_{3k-2} = 0, \ 1 \le k \le t$.

$$\begin{array}{lll} d(e_6) & = & d([e_5, e_1]) = [d(e_5), e_1] + [e_5, d(e_1)] \\ & = & [\sum_{k=1}^{t} (2\beta_{3k-1} + \alpha_{3k-2}) e_{3k+2} + \sum_{k=1}^{t} \beta_{3k} e_{3k+3}, e_1] + [e_5, \sum_{k=1}^{t} (\alpha_{3k-2} e_{3k-2} + \alpha_{3k} e_{3k})] \\ & = & \sum_{k=1}^{t} (2\beta_{3k-1} + \alpha_{3k-2}) e_{3k+3} - \sum_{k=1}^{t} \beta_{3k} e_{3k+4} + \sum_{k=1}^{t} (\alpha_{3k-2} e_{3k+3} - \alpha_{3k} e_{3k+5}) \\ & = & \sum_{k=1}^{t} (2(\beta_{3k-1} + \alpha_{3k-2}) e_{3k+3} - \beta_{3k} e_{3k+4} - \alpha_{3k} e_{3k+5}) \end{array}$$

Consequently,

$$\begin{array}{lll} d(e_{3i-2}) & = & \sum_{k=1}^{t} (((i-1)\beta_{3k-1} + i\alpha_{3k-2}) e_{3k+3i-5} + \alpha_{3k} e_{3k+3i-3}), \\ d(e_{3i-1}) & = & \sum_{k=1}^{t} (i\beta_{3k-1} + (i-1)\alpha_{3k-2}) e_{3k+3i-4} + \beta_{3k} e_{3k+3i-3}), \\ d(e_{3i}) & = & \sum_{k=1}^{t} (i(\beta_{3k-1} + \alpha_{3k-2}) e_{3k+3i-3} - \beta_{3k} e_{3k+3i-2} - \alpha_{3k} e_{3k+3i-1}) \end{array} \tag{5}$$

for $i = 3, 4, \ldots$

□

The proof of the following proposition is carried out similarly to that of Proposition 2.1.

Proposition 2.2. *The derivations of the algebra* $\mathbf{n}_2$ *are given as follows*

$$\begin{aligned}
d(f_{8i+1}) &= \sum_{k=0}^{t}(((4i+1)\alpha_{8k+1}+2i\beta_{8k+2})f_{8i+8k+1}+\alpha_{8k+3}f_{8i+8k+3}+\alpha_{8k+4}f_{8i+8k+4}\\
&\qquad +\alpha_{8k+5}f_{8i+8k+5}-2\beta_{8k+7}f_{8i+8k+6}+\alpha_{8k+8}f_{8k+8}),\\
d(f_{8i+2}) &= \sum_{k=0}^{t}((4i\alpha_{8k+1}+(2i+1)\beta_{8k+2})f_{8i+8k+2}+\beta_{8k+3}f_{8i+8k+3}+\beta_{8k+7}f_{8i+8k+7}+\beta_{8k+8}f_{8i+8k+8}),\\
d(f_{8i+3}) &= \sum_{k=0}^{t}(((4i+1)\alpha_{8k+1}+(2i+1)\beta_{8k+2})f_{8i+8k+3}+\beta_{8k+3}f_{8i+8k+4}-\alpha_{8k+5}f_{8i+8k+7}\\
&\qquad -\beta_{8k+7}f_{8i+8k+8}-\beta_{8k+8}f_{8i+8k+9}-2\alpha_{8k+8}f_{8i+8k+10}),\\
d(f_{8i+4}) &= \sum_{k=0}^{t}(((4i+2)\alpha_{8k+1}+(2i+1)\beta_{8k+2})f_{8i+8k+4}-3\beta_{8k+3}f_{8i+8k+5}-3\alpha_{8k+4}f_{8i+8k+7}\\
&\qquad +3\beta_{8k+7}f_{8i+8k+9}-3\alpha_{8k+8}f_{8i+8k+11}),\\
d(f_{8i+5}) &= \sum_{k=0}^{t}(((4i+3)\alpha_{8k+1}+(2i+1)\beta_{8k+2})f_{8i+8k+5}-2\beta_{8k+3}f_{8i+8k+6}-\alpha_{8k+3}f_{8i+8k+7}\\
&\qquad +\alpha_{8k+4}f_{8i+8k+8}+\alpha_{8k+5}f_{8i+8k+9}+\alpha_{8k+8}f_{8i+8k+12}),\\
d(f_{8i+6}) &= \sum_{k=0}^{t}(((4i+4)\alpha_{8k+1}+(2i+1)\beta_{8k+2})f_{8i+8k+6}+\alpha_{8k+3}f_{8i+8k+8}-\alpha_{8k+4}f_{8i+8k+9}\\
&\qquad +\alpha_{8k+8}f_{8i+8k+13}),\\
d(f_{8i+7}) &= \sum_{k=0}^{t}(((4i+3)\alpha_{8k+1}+(2i+2)\beta_{8k+2})f_{8i+8k+7}+\beta_{8k+3}f_{8i+8k+8}+2\alpha_{8k+4}f_{8i+8k+10}\\
&\qquad -\alpha_{8k+5}f_{8i+8k+11}-\beta_{8k+7}f_{8i+8k+12}+\beta_{8k+8}f_{8i+8k+13}),\\
d(f_{8i+8}) &= \sum_{k=0}^{t}(((4i+4)\alpha_{8k+1}+(2i+2)\beta_{8k+2})f_{8i+8k+8}-\beta_{8k+3}f_{8i+8k+9}-2\alpha_{8k+3}f_{8i+8k+10}\\
&\qquad -\alpha_{8k+4}f_{8i+8k+11}+\beta_{8k+7}f_{8i+8k+13}-2\beta_{8k+8}f_{8i+8k+14}-\alpha_{8k+8}f_{8i+8k+15}),
\end{aligned}$$

where $i \in \mathbb{N} \cup \{0\}$.

From the propositions above we can immediately obtain the following corollary on the number of nil-independent derivations of $\mathbf{n}_1$ and $\mathbf{n}_2$.

Corollary 2.3. *The maximal number of residually nil-independent derivations of* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *is* 2.

Lemma 2.4. *The derivations* ad_x *and* ad_y *of pro-solvable Lie algebras with maximal pro-nilpotent ideals* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *with the subspaces* Q_1 *and* Q_2 *complementary to* $\mathbf{n}_1$ *and* $\mathbf{n}_2$*, respectively, are non-residually nilpotent for any* $x \in Q_1$ *and* $y \in Q_2$.

Proof. We prove the statement for ad_x, where $x \in Q_1$ and for ad_y, where $y \in Q_2$ being similar referring to Proposition 2.2.

Let us assume the contrary that ad_x, where $x \in Q_1$ is residually nilpotent, i.e., $\bigcap_{k=1}^{\infty} \operatorname{Im} ad_x^k = 0$. Consider $V = \mathbf{n}_1 \bigoplus \mathbb{C}x$ as vector spaces. From Proposition 2.1 we conclude that $ad_x = d$ for some $d \in \operatorname{Der}(\mathbf{n}_1)$. The condition $\bigcap_{k=1}^{\infty} \operatorname{Im} ad_x^k = 0$ implies that $\alpha_1 = \beta_2 = 0$. Hance, we have

$$\begin{aligned}
[x, e_{3i-2}] &= -\alpha_3 e_{3i} - \sum_{k=2}^{t}(((i-1)\beta_{3k-1} + i\alpha_{3k-2})e_{3k+3i-5} + \alpha_{3k}e_{3k+3i-3}),\\
[x, e_{3i-1}] &= -\beta_3 e_{3i} - \sum_{k=2}^{t}(i\beta_{3k-1} + (i-1)\alpha_{3k-2})e_{3k+3i-4} + \beta_{3k}e_{3k+3i-3}),\\
[x, e_{3i}] &= \beta_3 e_{3i+1} + \alpha_3 e_{3i+2} - \sum_{k=2}^{t}(i(\beta_{3k-1} + \alpha_{3k-2})e_{3k+3i-3} - \beta_{3k}e_{3k+3i-2} - \alpha_{3k}e_{3k+3i-1}),
\end{aligned}$$

where $i \in \mathbb{N}$.

One can easily verify that $\bigcap_{i=1}^{\infty}(V)^i = 0$. This implies that V is residually nilpotent which is a contradiction. □

As a consequence of the lemma above we obtain the following corollary on the dimensions of the subspaces Q_1 and Q_2.

Corollary 2.5. *The dimensions of Q_1 and Q_2 are not greater than the maximal number of residually nil-independent derivations of $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively.*

2.2. Maximal pro-solvable extensions of pro-nilpotent ideals $\mathbf{n}_1$ and $\mathbf{n}_2$

Here and onward we deal with maximal pro-solvable extensions $\mathbf{n}_1 \oplus Q_1$ and $\mathbf{n}_2 \oplus Q_2$ of pro-nilpotent ideals $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, where $\mathbf{n}_1 = \mathrm{Span}\{e_1, e_2, \dots\}$, $Q_1 = \mathrm{Span}\{x, y\}$ and $\mathbf{n}_2 = \mathrm{Span}\{f_1, f_2, \dots\}$, $Q_2 = \mathrm{Span}\{x, y\}$, i.e., $\mathbf{n}_i \oplus Q_i$ and $\dim Q_i = 2$, $i = 1, 2$.

Theorem 2.6.

- *An arbitrary maximal pro-solvable Lie algebra with maximal pro-nilpotent ideal $\mathbf{n}_1$ admits a basis $\{x, y, e_1, e_2, ...\}$ such that its table of multiplication is given as follows*

$$\mathfrak{M}_1(\alpha) = \begin{cases}
[e_i, e_j] = c_{i,j}e_{i+j} \text{ as in } (1.1),\\
[e_{3i-2}, x] = ie_{3i-2} + \sum_{k=2}^{t}(i-1)\alpha_k e_{3k+3i-5},\\
[e_{3i-1}, x] = (i-1)e_{3i-1} + \sum_{k=2}^{t} i\alpha_k e_{3k+3i-4},\\
[e_{3i}, x] = ie_{3i} + \sum_{k=2}^{t} i\alpha_k e_{3k+3i-3},\\
[e_{3i-2}, y] = -e_{3i-2} + \sum_{k=2}^{t}\alpha_k e_{3k+3i-5},\\
[e_{3i-1}, y] = e_{3i-1} - \sum_{k=2}^{t}\alpha_k e_{3k+3i-4},\\
[x, y] = \sum_{k=1}^{2t-1}(k\alpha_{k+1} + \sum_{i=2}^{k-j+2}\sum_{j=2}^{k}(i-1)\alpha_i\alpha_j)e_{3k},
\end{cases} \tag{6}$$

  where $i \in \mathbb{N}$, $\alpha = (\alpha_2, \alpha_3, \dots, \alpha_{2t}) \in \mathbb{C}^{2t-1}$ for some $t \in \mathbb{N}$;

- *The isomorphism classes of $\mathfrak{M}_1(\alpha)$ are represented by algebras with $\alpha = (\alpha_2, \alpha_3, ..., \alpha_{2t})$, where the first two non-zero components of α are equal to 1.*

Proof. By using Proposition 2.1 for $i \in \mathbb{N}$ we get

$$\begin{aligned}
[e_{3i-2}, x] &= ie_{3i-2} + \alpha_3 e_{3i} + \sum_{k=2}^{t}(((i-1)\beta_{3k-1} + i\alpha_{3k-2})e_{3k+3i-5} + \alpha_{3k}e_{3k+3i-3}),\\
[e_{3i-1}, x] &= (i-1)e_{3i-1} + \beta_3 e_{3i} + \sum_{k=2}^{t}((i\beta_{3k-1} + (i-1)\alpha_{3k-2})e_{3k+3i-4} + \beta_{3k}e_{3k+3i-3}),\\
[e_{3i}, x] &= ie_{3i} - \beta_3 e_{3i+1} - \alpha_3 e_{3i+2} + \sum_{k=2}^{t}(i(\beta_{3k-1} + \alpha_{3k-2})e_{3k+3i-3} - \beta_{3k}e_{3k+3i-2} - \alpha_{3k}e_{3k+3i-1}),\\
[e_{3i-2}, y] &= (i-1)e_{3i-2} + \alpha'_3 e_{3i} + \sum_{k=2}^{t}(((i-1)\beta'_{3k-1} + i\alpha'_{3k-2})e_{3k+3i-5} + \alpha'_{3k}e_{3k+3i-3}),\\
[e_{3i-1}, y] &= ie_{3i-1} + \beta'_3 e_{3i} + \sum_{k=2}^{t}((i\beta'_{3k-1} + (i-1)\alpha'_{3k-2})e_{3k+3i-4} + \beta'_{3k}e_{3k+3i-3}),\\
[e_{3i}, y] &= ie_{3i} - \beta'_3 e_{3i+1} - \alpha'_3 e_{3i+2} + \sum_{k=2}^{t}(i(\beta'_{3k-1} + \alpha'_{3k-2})e_{3k+3i-3} - \beta'_{3k}e_{3k+3i-2} - \alpha'_{3k}e_{3k+3i-1}),\\
[x, y] &= \sum_{k=1}^{s}\gamma_{3k-2}e_{3k-2} + \sum_{k=1}^{s}\gamma_{3k-1}e_{3k-1} + \sum_{k=1}^{s}\gamma_{3k}e_{3k} + \delta_1 x + \delta_2 y.
\end{aligned}$$

Let us consider the following base change

$$x' = x - \sum_{k=2}^{t}\alpha_{3k-2}e_{3k-3} + \sum_{k=1}^{t}(-\beta_{3k}e_{3k-2} + \alpha_{3k}e_{3k-1}), \quad y' = y + \sum_{k=2}^{t}\beta'_{3k-1}e_{3k-3} + \sum_{k=1}^{t}(-\beta'_{3k}e_{3k-2} + \alpha'_{3k}e_{3k-1}).$$

This yields

$$\begin{aligned}
[e_{3i-2}, x] &= ie_{3i-2} + (i-1)\sum_{k=2}^{t}\mu_{3k-2}e_{3k+3i-5},\\
[e_{3i-1}, x] &= (i-1)e_{3i-1} + i\sum_{k=2}^{t}\mu_{3k-2}e_{3k+3i-4},\\
[e_{3i}, x] &= ie_{3i} + i\sum_{k=2}^{t}\mu_{3k-2}e_{3k+3i-3},\\
[e_{3i-2}, y] &= (i-1)e_{3i-2} + i\sum_{k=2}^{t}\nu_{3k-1}e_{3k+3i-5},\\
[e_{3i-1}, y] &= ie_{3i-1} + (i-1)\sum_{k=2}^{t}\nu_{3k-1}e_{3k+3i-4},\\
[e_{3i}, y] &= ie_{3i} + i\sum_{k=2}^{t}\nu_{3k-1}e_{3k+3i-3},\\
[x, y] &= \sum_{k=1}^{s}\gamma_{3k-2}e_{3k-2} + \sum_{k=1}^{s}\gamma_{3k-1}e_{3k-1} + \sum_{k=1}^{s}\gamma_{3k}e_{3k} + \delta_1 x + \delta_2 y,
\end{aligned}$$

where $\mu_{3k-2} = \beta_{3k-1} + \alpha_{3k-2}$, $\nu_{3k-1} = \beta'_{3k-1} + \alpha'_{3k-2}$.

Observe that

$$\begin{aligned}
[e_1, [x, y]] &= [[e_1, x], y] - [[e_1, y], x] = [e_1, y] - [\sum_{i=2}^{t}\nu_{3i-1}e_{3i-2}, x]\\
&= \sum_{i=2}^{t}\nu_{3i-1}e_{3i-2} - \sum_{i=2}^{t}\nu_{3i-1}\Big(ie_{3i-2} + \sum_{k=2}^{t}(i-1)\mu_{3k-2}e_{3k+3i-5}\Big)\\
&= \sum_{i=2}^{t}(1-i)\nu_{3i-1}e_{3i-2} - \sum_{i=2}^{t}\sum_{k=2}^{t}(i-1)\nu_{3i-1}\mu_{3k-2}e_{3k+3i-5},
\end{aligned}$$

$$\begin{aligned}
[e_2, [x, y]] &= [[e_2, x], y] - [[e_2, y], x] = \sum_{i=2}^{t}\mu_{3i-2}[e_{3i-1}, y] - [e_2, x]\\
&= \sum_{i=2}^{t}\mu_{3i-2}(ie_{3i-1} + (i-1)\sum_{k=2}^{t}\nu_{3k-1}e_{3k+3i-4}) - \sum_{i=2}^{t}\mu_{3i-2}e_{3i-1}\\
&= \sum_{i=2}^{t}\mu_{3i-2}(i-1)(e_{3i-1} + \sum_{k=2}^{t}\nu_{3k-1}e_{3k+3i-4}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[e_1,[x,y]] &= [e_1, \sum_{k=1}^{s}\gamma_{3k-2}e_{3k-2} + \sum_{k=1}^{s}\gamma_{3k-1}e_{3k-1} + \sum_{k=1}^{s}\gamma_{3k}e_{3k} + \delta_1 x + \delta_2 y]\\
&= -\sum_{k=1}^{s}\gamma_{3k-1}e_{3k} + \sum_{k=1}^{s}\gamma_{3k}e_{3k+1} + \delta_1 e_1 + \delta_2\sum_{i=2}^{t}\nu_{3i-1}e_{3i-2},
\end{aligned}$$

$$\begin{aligned}
[e_2,[x,y]] &= [e_2, \sum_{k=1}^{s}\gamma_{3k-2}e_{3k-2} + \sum_{k=1}^{s}\gamma_{3k-1}e_{3k-1} + \sum_{k=1}^{s}\gamma_{3k}e_{3k} + \delta_1 x + \delta_2 y]\\
&= \sum_{k=1}^{s}\gamma_{3k-1}e_{3k} - \sum_{k=1}^{s}\gamma_{3k}e_{3k+2} + \delta_1\sum_{i=2}^{s}\mu_{3i-2}e_{3i-1} + \delta_2 e_2.
\end{aligned}$$

Comparing the coefficients at the appropriate basis elements we conclude that

$$\begin{aligned}
\gamma_{3k-2} &= \gamma_{3k-1} = \delta_1 = \delta_2 = 0,\\
\nu_{3i-1} &= \mu_{3i-2},\ 1 \le k \le s,\ \text{where } 2 \le i \le t,\\
\gamma_{3k} &= k\nu_{3k+2} + \sum_{j=2}^{k}\sum_{i=2}^{k-j+2}(i-1)\nu_{3i-1}\nu_{3j-2},\ \text{where } 1 \le k \le 2t-1,\\
\gamma_{3k} &= 0,\ \text{where } 2t \le k \le s.
\end{aligned}$$

Taking into account all the above and applying the basis change $y' = y - x$ we get the table of multiplications (6).

The second part of the theorem is proven as follows. Consider the base change $f : \{x, y, e_1, e_2, ...\} \to \{x', y', e'_1, e'_2, ...\}$ on the vector space $\mathfrak{M}_1(\alpha)$:

$$\begin{aligned}
e'_{3k} &= A_1^k A_2^k e_{3k}, && k \ge 1\\
e'_{3k-2} &= A_1^k A_2^{k-1} e_{3k-2}, && k \ge 1\\
e'_{3k-1} &= A_1^{k-1} A_2^k e_{3k-1}, && k \ge 1\\
x' &= \sum_{k=1}^{3s} B_k e_k + B_{1,1}x + B_{2,2}y &&\\
y' &= \sum_{k=1}^{3s} C_k e_k + C_{1,1}x + C_{2,2}y, && s > t.
\end{aligned}$$

Applying the table of multiplications we get

$$\begin{aligned}
B_{1,1} &= 1 + B_{2,2},\\
B_{3k-3} &= B_{2,2}\alpha_k,\ \text{where } 2 \le k \le t,\\
B_{3k-1} &= 0,\ \text{where } 1 \le k \le s,\\
B_{3k} &= 0,\ \text{where } t \le k \le s\\
B_{2,2} &= 0,\\
B_{3k-2} &= 0,\ \text{where } 1 \le k \le s,\\
B_{1,1}\alpha_k &= \alpha'_k A_1^{k-1} A_2^{k-1} + B_{3k-3},\ \text{where } 2 \le k \le t\\
C_k &= 0,\ \text{where } 1 \le k \le 3s,\\
C_{1,1} &= 0,\\
C_{1,2} &= 1.
\end{aligned}$$

These imply $B_{1,1} = 1,\ B_{3k-3} = 0$, where $2 \le k \le t$ and it yields

$$\alpha'_k = \frac{\alpha_k}{A_1^{k-1}A_2^{k-1}},\ 2 \le k \le t,\ \ f(x) = x \ \text{ and } f(y) = y.$$

Therefore, we can choose A_1 and A_2 such a way that $\alpha'_k = 1$ for the first two non-zero α_k.

□

Theorem 2.7.

- *Any maximal pro-solvable Lie algebra with the maximal pro-nilpotent ideal* $\mathbf{n}_2$ *admits a basis* $\{x, y, f_1, f_2, ...\}$ *with respect to that the table of multiplications is given as follows*

$$\mathfrak{M}_2(\beta) := \left\{ \begin{array}{ll} [f_i, f_j] = d_{i,j} f_{i+j} \text{ as in } (1.2), & [f_{8i+1}, y] = 2i f_{8i+1} + \sum_{k=1}^{t} 2i\beta_k f_{8i+8k+1}, \\ [f_{8i+1}, x] = f_{8i+1}, & [f_{8i+2}, y] = (2i+1) f_{8i+2} + \sum_{k=1}^{t} (2i+1)\beta_k f_{8i+8k+2}, \\ [f_{8i+2}, x] = -2f_{8i+2}, & [f_{8i+3}, y] = (2i+1) f_{8i+3} + \sum_{k=1}^{t} (2i+1)\beta_k f_{8i+8k+3}, \\ [f_{8i+3}, x] = -f_{8i+3}, & [f_{8i+4}, y] = (2i+1) f_{8i+4} + \sum_{k=1}^{t} (2i+1)\beta_k f_{8i+8k+4}, \\ [f_{8i+5}, x] = f_{8i+5}, & [f_{8i+5}, y] = (2i+1) f_{8i+5} + \sum_{k=1}^{t} (2i+1)\beta_k f_{8i+8k+5}, \\ [f_{8i+6}, x] = 2f_{8i+6}, & [f_{8i+6}, y] = (2i+1) f_{8i+6} + \sum_{k=1}^{t} (2i+1)\beta_k f_{8i+8k+6}, \\ [f_{8i+7}, x] = -f_{8i+7}. & [f_{8i+7}, y] = (2i+2) f_{8i+7} + \sum_{k=1}^{t} (2i+2)\beta_k f_{8i+8k+7}, \\ & [f_{8i+8}, y] = (2i+2) f_{8i+8} + \sum_{k=1}^{t} (2i+2)\beta_k f_{8i+8k+8}, \end{array} \right.$$

where $i \in \mathbb{N} \cup \{0\}$, $\beta = (\beta_1, \dots, \beta_t) \in \mathbb{C}^t$ *for some* $t \in \mathbb{N}$.

- *The isomorphism classes of* $\mathfrak{M}_2(\beta)$ *are represented by algebras with* $\beta = (\beta_1, \dots, \beta_t)$*, where the first two non-zero components of* β *are equal to* 1.

Proof. We use Proposition 2.2 to get the following products

$$\begin{aligned}
[f_{8i+1}, x] &= (4i+1) f_{8i+1} + \sum_{k=1}^{t} ((4i+1)\alpha_{1,8k+1} + 2i\beta_{1,8k+2}) f_{8i+8k+1} + \sum_{k=0}^{t} (\alpha_{1,8k+3} f_{8i+8k+3} \\
&\quad + \alpha_{1,8k+4} f_{8i+8k+4} + \alpha_{1,8k+5} f_{8i+8k+5} - 2\beta_{1,8k+7} f_{8i+8k+6} + \alpha_{1,8k+8} f_{8i+8k+8}), \\
[f_{8i+2}, x] &= 4i f_{8i+2} + \sum_{k=1}^{t} (4i\alpha_{1,8k+1} + (2i+1)\beta_{1,8k+2}) f_{8i+8k+2} \\
&\quad + \sum_{k=0}^{t} (\beta_{1,8k+3} f_{8i+8k+3} + \beta_{1,8k+7} f_{8i+8k+7} + \beta_{1,8k+8} f_{8i+8k+8}). \\
[f_{8i+3}, x] &= (4i+1) f_{8i+3} + \sum_{k=1}^{t} ((4i+1)\alpha_{1,8k+1} + (2i+1)\beta_{1,8k+2}) f_{8i+8k+3} \\
&\quad + \sum_{k=0}^{t} (\beta_{1,8k+3} f_{8i+8k+4} - \alpha_{1,8k+5} f_{8i+8k+7} - \beta_{1,8k+7} f_{8i+8k+8} - \beta_{1,8k+8} f_{8i+8k+9} - 2\alpha_{1,8k+8} f_{8i+8k+10}). \\
[f_{8i+4}, x] &= (4i+2) f_{8i+4} + \sum_{k=1}^{t} ((4i+2)\alpha_{1,8k+1} + (2i+1)\beta_{1,8k+2}) f_{8i+8k+4} \\
&\quad + \sum_{k=0}^{t} (-3\beta_{1,8k+3} f_{8i+8k+5} - 3\alpha_{1,8k+4} f_{8i+8k+7} + 3\beta_{1,8k+7} f_{8i+8k+9} - 3\alpha_{1,8k+8} f_{8i+8k+11}). \\
[f_{8i+5}, x] &= (4i+3) f_{8i+5} + \sum_{k=1}^{t} ((4i+3)\alpha_{1,8k+1} + (2i+1)\beta_{1,8k+2}) f_{8i+8k+5} + \sum_{k=0}^{t} (-2\beta_{1,8k+3} f_{8i+8k+6} \\
&\quad - \alpha_{1,8k+3} f_{8i+8k+7} + \alpha_{1,8k+4} f_{8i+8k+8} + \alpha_{1,8k+5} f_{8i+8k+9} + \alpha_{1,8k+8} f_{8i+8k+12}). \\
[f_{8i+6}, x] &= (4i+4) f_{8i+6} + \sum_{k=1}^{t} ((4i+4)\alpha_{1,8k+1} + (2i+1)\beta_{1,8k+2}) f_{8i+8k+6} \\
&\quad + \sum_{k=0}^{t} (\alpha_{1,8k+3} f_{8i+8k+8} - \alpha_{1,8k+4} f_{8i+8k+9} + \alpha_{1,8k+8} f_{8i+8k+13}). \\
[f_{8i+7}, x] &= (4i+3) f_{8i+7} + \sum_{k=1}^{t} ((4i+3)\alpha_{1,8k+1} + (2i+2)\beta_{1,8k+2}) f_{8i+8k+7} + \sum_{k=0}^{t} (\beta_{1,8k+3} f_{8i+8k+8} \\
&\quad + 2\alpha_{1,8k+4} f_{8i+8k+10} - \alpha_{1,8k+5} f_{8i+8k+11} - \beta_{1,8k+7} f_{8i+8k+12} + \beta_{1,8k+8} f_{8i+8k+13}).
\end{aligned}$$

$$
\begin{aligned}
[f_{8i+8}, x] &= (4i+4)f_{8i+8} + \sum_{k=1}^{t}((4i+4)\alpha_{1,8k+1} + (2i+2)\beta_{1,8k+2})f_{8i+8k+8} + \sum_{k=0}^{t}(-\beta_{1,8k+3}f_{8i+8k+9} \\
&\quad -2\alpha_{1,8k+3}f_{8i+8k+10} - \alpha_{1,8k+4}f_{8i+8k+11} + \beta_{1,8k+7}f_{8i+8k+13} - 2\beta_{1,8k+8}f_{8i+8k+14} - \alpha_{1,8k+8}f_{8i+8k+15}).\\
[f_{8i+1}, y] &= 2if_{8i+1} + \sum_{k=1}^{t}((4i+1)\alpha_{2,8k+1} + 2i\beta_{2,8k+2})f_{8i+8k+1} + \sum_{k=0}^{t}(\alpha_{2,8k+3}f_{8i+8k+3} \\
&\quad +\alpha_{2,8k+4}f_{8i+8k+4} + \alpha_{2,8k+5}f_{8i+8k+5} - 2\beta_{2,8k+7}f_{8i+8k+6} + \alpha_{2,8k+8}f_{8i+8k+8}),\\
[f_{8i+2}, y] &= (2i+1)f_{8i+2} + \sum_{k=1}^{t}(4i\alpha_{2,8k+1} + (2i+1)\beta_{2,8k+2})f_{8i+8k+2} \\
&\quad + \sum_{k=0}^{t}(\beta_{2,8k+3}f_{8i+8k+3} + \beta_{2,8k+7}f_{8i+8k+7} + \beta_{2,8k+8}f_{8i+8k+8}).\\
[f_{8i+3}, y] &= (2i+1)f_{8i+3} + \sum_{k=1}^{t}((4i+1)\alpha_{2,8k+1} + (2i+1)\beta_{2,8k+2})f_{8i+8k+3} \\
&\quad + \sum_{k=0}^{t}(\beta_{2,8k+3}f_{8i+8k+4} - \alpha_{2,8k+5}f_{8i+8k+7} - \beta_{2,8k+7}f_{8i+8k+8} - \beta_{2,8k+8}f_{8i+8k+9} - 2\alpha_{2,8k+8}f_{8i+8k+10}).\\
[f_{8i+4}, y] &= (2i+1)f_{8i+4} + \sum_{k=1}^{t}((4i+2)\alpha_{2,8k+1} + (2i+1)\beta_{2,8k+2})f_{8i+8k+4} \\
&\quad + \sum_{k=0}^{t}(-3\beta_{2,8k+3}f_{8i+8k+5} - 3\alpha_{2,8k+4}f_{8i+8k+7} + 3\beta_{2,8k+7}f_{8i+8k+9} - 3\alpha_{2,8k+8}f_{8i+8k+11}).\\
[f_{8i+5}, y] &= (2i+1)f_{8i+5} + \sum_{k=1}^{t}((4i+3)\alpha_{2,8k+1} + (2i+1)\beta_{2,8k+2})f_{8i+8k+5} + \sum_{k=0}^{t}(-2\beta_{2,8k+3}f_{8i+8k+6} \\
&\quad -\alpha_{2,8k+3}f_{8i+8k+7} + \alpha_{2,8k+4}f_{8i+8k+8} + \alpha_{2,8k+5}f_{8i+8k+9} + \alpha_{2,8k+8}f_{8i+8k+12}).\\
[f_{8i+6}, y] &= (2i+1)f_{8i+6} + \sum_{k=1}^{t}((4i+4)\alpha_{2,8k+1} + (2i+1)\beta_{2,8k+2})f_{8i+8k+6} \\
&\quad + \sum_{k=0}^{t}(\alpha_{2,8k+3}f_{8i+8k+8} - \alpha_{2,8k+4}f_{8i+8k+9} + \alpha_{2,8k+8}f_{8i+8k+13}).\\
[f_{8i+7}, y] &= (2i+2)f_{8i+7} + \sum_{k=1}^{t}((4i+3)\alpha_{2,8k+1} + (2i+2)\beta_{2,8k+2})f_{8i+8k+7} + \sum_{k=0}^{t}(\beta_{2,8k+3}f_{8i+8k+8} \\
&\quad +2\alpha_{2,8k+4}f_{8i+8k+10} - \alpha_{2,8k+5}f_{8i+8k+11} - \beta_{2,8k+7}f_{8i+8k+12} + \beta_{2,8k+8}f_{8i+8k+13}).\\
[f_{8i+8}, y] &= (2i+2)f_{8i+8} + \sum_{k=1}^{t}((4i+4)\alpha_{2,8k+1} + (2i+2)\beta_{2,8k+2})f_{8i+8k+8} + \sum_{k=0}^{t}(-\beta_{2,8k+3}f_{8i+8k+9} \\
&\quad -2\alpha_{2,8k+3}f_{8i+8k+10} - \alpha_{2,8k+4}f_{8i+8k+11} + \beta_{2,8k+7}f_{8i+8k+13} - 2\beta_{2,8k+8}f_{8i+8k+14} - \alpha_{2,8k+8}f_{8i+8k+15}).\\
[x, y] &= \sum_{k=0}^{t}(\alpha_{8k+1}f_{8k+1} + \alpha_{8k+2}f_{8k+2} + \alpha_{8k+3}f_{8k+3} + \alpha_{8k+4}f_{8k+4} + \alpha_{8k+5}f_{8k+5} \\
&\quad +\alpha_{8k+6}f_{8k+6} + \alpha_{8k+7}f_{8k+7} + \alpha_{8k+8}f_{8k+8}) + \delta_1 x + \delta_2 y,
\end{aligned}
$$

where $i \in \mathbb{N} \cup \{0\}, \quad t \in \mathbb{N}$.

Taking the base change

$$
\begin{aligned}
x' &= x + \sum_{k=1}^{t}\alpha_{1,8k+1}f_{8k} + \sum_{k=0}^{t}(\beta_{1,8k+3}f_{8k+1} - \alpha_{1,8k+3}f_{8k+2} - \alpha_{1,8k+4}f_{8k+3} + \tfrac{1}{3}\alpha_{1,8k+5}f_{8k+4} \\
&\qquad -\beta_{1,8k+7}f_{8k+5} + \beta_{1,8k+8}f_{8k+6} - \alpha_{1,8k+8}f_{8k+7}),\\
y' &= y + \sum_{k=1}^{t}\alpha_{2,8k+1}f_{8k} + \sum_{k=0}^{t}(\beta_{2,8k+3}f_{8k+1} - \alpha_{2,8k+3}f_{8k+2} - \alpha_{2,8k+4}f_{8k+3} + \tfrac{1}{3}\alpha_{2,8k+5}f_{8k+4} \\
&\qquad -\beta_{2,8k+7}f_{8k+5} + \beta_{2,8k+8}f_{8k+6} - \alpha_{2,8k+8}f_{8k+7}).
\end{aligned}
$$

we get $\beta_{j,8k+3} = \alpha_{j,8k+3} = \alpha_{j,8k+4} = \alpha_{j,8k+5} = \beta_{j,8k+7} = \alpha_{j,8k+8} = \beta_{j,8k+8} = 0,$ where $1 \leq j \leq 2$.
Under this base change some basis vectors appear with the coefficients $2\alpha_{1,8k+1} + \beta_{1,8k+2}$ and $2\alpha_{2,8k+1} + \beta_{2,8k+2}$. We denote them as follows

$$\beta_{1,8k} := 2\alpha_{1,8k+1} + \beta_{1,8k+2}, \qquad \beta_{2,8k} := 2\alpha_{2,8k+1} + \beta_{2,8k+2}.$$

Consider the identity

$$[f_1, [x, y]] = [[f_1, x], y] - [[f_1, y], x] = [f_1, y] = 0.$$

The left hand side of the identity can also be computed as follows

$$\begin{aligned}
[f_1,[x,y]] &= [f_1, \sum_{k=0}^{t}(\alpha_{8k+1}f_{8k+1}+\alpha_{8k+2}f_{8k+2}+\alpha_{8k+3}f_{8k+3}+\alpha_{8k+4}f_{8k+4}+\alpha_{8k+5}f_{8k+5}\\
&\quad +\alpha_{8k+6}f_{8k+6}+\alpha_{8k+7}f_{8k+7}+\alpha_{8k+8}f_{8k+8})+\delta_1 x+\delta_2 y]\\
&= \sum_{k=0}^{t}(\alpha_{8k+2}f_{8k+3}+\alpha_{8k+3}f_{8k+4}-3\alpha_{8k+4}f_{8k+5}-2\alpha_{8k+5}f_{8k+6}+\alpha_{8k+7}f_{8k+8}\\
&\quad -\alpha_{8k+8}f_{8k+9})+\delta_1 f_1.
\end{aligned}$$

Therefore, we get

$$\alpha_{8k+2}=\alpha_{8k+3}=\alpha_{8k+4}=\alpha_{8k+5}=\alpha_{8k+7}=\alpha_{8k+8}=\delta_1=0.$$

Let us now compute $[f_2,[x,y]]$ applying the identity $[f_2,[x,y]]=[[f_2,x],y]-[[f_2,y],x]$ as follows

$$\begin{aligned}
[f_2,[x,y]] &= [[f_2,x],y]-[[f_2,y],x]\\
&= \left[\sum_{i=1}^{t}\beta_{1,8i}f_{8i+2},y\right]-\left[f_2+\sum_{i=1}^{t}\beta_{2,8i}f_{8i+2},x\right]\\
&= \sum_{i=1}^{t}\beta_{1,8i}\Big((2i+1)f_{8i+2}+\sum_{k=1}^{t}(2i+1)\beta_{2,8k}f_{8i+8k+2}\Big)\\
&\qquad -\sum_{i=1}^{t}\beta_{1,8i}f_{8i+2}-\sum_{i=1}^{t}\beta_{2,8i}\Big(4if_{8i+2}+\sum_{k=1}^{t}(2i+1)\beta_{1,8k}f_{8i+8k+2}\Big)\\
&= \sum_{i=1}^{t}2i(\beta_{1,8i}-2\beta_{2,8i})f_{8i+2}+\sum_{i=1}^{t}\Big(\sum_{k=1}^{t}(2i+1)(\beta_{1,8i}\beta_{2,8k}-\beta_{1,8k}\beta_{2,8i})f_{8i+8k+2}\Big).
\end{aligned}$$

The same time for $[f_2,[x,y]]$ we have

$$\begin{aligned}
[f_2,[x,y]] &= [f_2,\sum_{k=0}^{t}(\alpha_{8k+1}f_{8k+1}+\alpha_{8k+6}f_{8k+6})+\delta_2 y]\\
&= \sum_{k=0}^{t}(-\alpha_{8k+1}f_{8k+3}-\alpha_{8k+6}f_{8k+8})+\delta_2(f_2+\sum_{i=1}^{t}\beta_{2,8i+2}f_{8i+2}).
\end{aligned}$$

Comparing the coefficients we obtain

$$\beta_{1,8k}=2\beta_{2,8k},\quad \alpha_{8k+1}=\alpha_{8k+6}=\delta_2=0.$$

Now applying the basis change $x'=x-2y$ to get the required table of multiplication for $\mathfrak{M}_2(\beta)$.

As for the second part we consider a base change in $\mathfrak{M}_2(\beta)$ as follows:

$$\begin{aligned}
f_1' &= A_1f_1\\
f_2' &= A_2f_2\\
f_{8k-7}' &= A_1^{4k-3}A_2^{2k-2}f_{8k-7}, & f_{8k-6}' &= A_1^{4k-4}A_2^{2k-1}f_{8k-6},\\
f_{8k-5}' &= A_1^{4k-3}A_2^{2k-1}f_{8k-5}, & f_{8k-4}' &= A_1^{4k-2}A_2^{2k-1}f_{8k-4},\\
f_{8k-3}' &= A_1^{4k-1}A_2^{2k-1}f_{8k-3}, & f_{8k-2}' &= A_1^{4k}A_2^{2k-1}f_{8k-2},\\
f_{8k-1}' &= A_1^{4k-1}A_2^{2k}f_{8k-1}, & f_{8k}' &= A_1^{4k}A_2^{2k}f_{8k},\\
x' &= \sum_{k=1}^{8s}B_kf_k+B_{1,1}x+B_{2,2}y\\
y' &= \sum_{k=1}^{8s}C_kf_k+C_{1,1}x+C_{2,2}y, & s&>t.
\end{aligned}$$

Apply the table of multiplications to get the following constrains for the entries of the base change's matrix

$$C_{1,1}=0,\quad C_{8k-6}=C_{8k-5}=C_{8k-4}=C_{8k-3}=C_{8k-1}=C_{8k}=0,\ 1\le k\le s,$$

$$C_{2,2} = 1, \quad C_{8k-7} = C_{8k-2} = 0,\ 1 \le k \le s,$$

$$B_i = 0,\ 1 \le i \le 8s,\ B_{1,1} = 1,\ B_{1,2} = 0,$$

$$\beta'_k = \frac{\beta_k}{A_1^{4k} A_2^{2k+1}},\ 1 \le k \le t.$$

Now we choose appropriate values of A_1 and A_2 and scale the first two non-zero β'_k to 1. □

2.3. On low-dimensional (co)homology groups of $\mathfrak{M}_1(\alpha)$ and $\mathfrak{M}_2(\beta)$

In this section we compute low-dimensional cohomology groups of $\mathfrak{M}_1(\alpha)$ and $\mathfrak{M}_2(\beta)$. We claim that the cohomology groups $\mathrm{H}^1(L, L)$ and $\mathrm{H}^2(L, L)$, where $L = \mathfrak{M}_1(\alpha)$ or $\mathfrak{M}_2(\beta)$, are trivial and the algebras are complete.

Consider the finite dimensional Lie algebras $(\mathfrak{M}_1(\alpha))_k = \mathfrak{M}_1(\alpha)/\mathbf{n}_1^k$ and $(\mathfrak{M}_2(\beta))_k = \mathfrak{M}_2(\beta)/\mathbf{n}_2^k$. Since, $(\mathfrak{M}_1(\alpha))_{3t-1}$ and $(\mathfrak{M}_1(\alpha))_{3t-2}$ are given as $(\mathfrak{M}_1(\alpha))_{3t}$ with some zero coefficients, we will be dealing with $(\mathfrak{M}_1(\alpha))_{3t}$.

Let

$$(\mathfrak{M}_1(\alpha))_{3t} = \begin{cases} [e_{3i-2}, x] &= ie_{3i-2} + \sum\limits_{k=2}^{t-i+1} (i-1)\alpha_k e_{3k+3i-5}, \\ [e_{3i-1}, x] &= (i-1)e_{3i-1} + \sum\limits_{k=2}^{t-i+1} i\alpha_k e_{3k+3i-4}, \\ [e_{3i}, x] &= ie_{3i} + \sum\limits_{k=2}^{t-i+1} i\alpha_k e_{3k+3i-3}, \\ [e_{3i-2}, y] &= -e_{3i-2} + \sum\limits_{k=2}^{t-i+1} \alpha_k e_{3k+3i-5} \\ [e_{3i-1}, y] &= e_{3i-1} - \sum\limits_{k=2}^{t-i+1} \alpha_k e_{3k+3i-4}, \\ [x, y] &= \sum\limits_{k=1}^{t} (k\alpha_{k+1} + \sum\limits_{i=2}^{k-j+2} \sum\limits_{j=2}^{k} (i-1)\alpha_i\alpha_j)e_{3k}, \quad 1 \le i \le t. \end{cases}$$

Accordingly, in the case of $\mathfrak{M}_2(\beta)$ we refer to $(\mathfrak{M}_2(\beta))_{8t}$.

Proposition 2.8.

- *The algebra $(\mathfrak{M}_1(\alpha))_{3t}$ is isomorphic to $(\mathfrak{M}_1(0))_{3t} = (\mathfrak{M}_1(0))/\mathbf{n}_1^{3t}$.*
- *The algebra $(\mathfrak{M}_2(\beta))_{8t+8}$ is isomorphic to $(\mathfrak{M}_2(0))_{8t} = \mathfrak{M}_2(0))/\mathbf{n}_2^{8t}$.*

Proof. First of all by setting $x' = x + y$ we get

$$(\mathfrak{M}_1(\alpha))_{3t} = \begin{cases} [e_{3i-2}, x] &= (i-1)e_{3i-2} + \sum\limits_{k=2}^{t-i+1} i\alpha_k e_{3k+3i-5}, \\ [e_{3i-1}, x] &= ie_{3i-1}, \\ [e_{3i}, x] &= ie_{3i} + \sum\limits_{k=2}^{t-i+1} i\alpha_k e_{3k+3i-3}, \\ [e_{3i-2}, y] &= -e_{3i-2} + \sum\limits_{k=2}^{t-i+1} \alpha_k e_{3k+3i-5} \\ [e_{3i-1}, y] &= e_{3i-1} - \sum\limits_{k=2}^{t-i+1} \alpha_k e_{3k+3i-4}, \\ [x, y] &= \sum\limits_{k=1}^{t} (k\alpha_{k+1} + \sum\limits_{i=2}^{k-j+2} \sum\limits_{j=2}^{k} (i-1)\alpha_i\alpha_j)e_{3k}, \quad 1 \le i \le t. \end{cases}$$

Consider the base change

$$e'_2 = e_2, \quad e'_i = \nu_i(e_i + \sum_{k=1}^{t-\gamma_i} A_k e_{3k+i}), \quad i \in \{1, 3, 4, \dots, 3t\},$$

where

$$\nu_i = \begin{cases} -1, & \text{if} \quad i \equiv 1 \pmod 3), \\ 1, & \text{if otherwise}, \end{cases}$$

$$A_1 = -\alpha_2, \quad A_i = -(\alpha_{i+1} + \sum_{k=2}^{i} k A_{k-1}\alpha_{i-k+2}).$$

We obtain

$$(\mathfrak{M}_1(\alpha))_{3t} = \begin{cases} [e_{3i-2}, x] &= (i-1)e_{3i-2}, \\ [e_{3i-1}, x] &= ie_{3i-1}, \\ [e_{3i}, x] &= ie_{3i}, \\ [e_{3i-2}, y] &= -e_{3i-2}, \\ [e_{3i-1}, y] &= e_{3i-1}, \quad 1 \le i \le t. \end{cases}$$

with the multiplication table $\mathbf{n}_1$. The second part of the proposition is proven similarly. □

Lemma 2.9. *All the derivations of* $(\mathfrak{M}_1(\alpha))_{3t}$ *and* $(\mathfrak{M}_2(\beta))_{8t}$ *are inner.*

The proof of the lemma can be found in [13]. We will be using the lemma to prove the following two theorems.

Theorem 2.10. *All the derivations of* $\mathfrak{M}_1(\alpha)$ *are inner.*

Proof. First, we note that $\mathfrak{M}_1(\alpha) = \mathbf{n}_1 \oplus Q$, where $\{x, y\}$ is the basis of Q and $\{e_1, e_2, x, y\}$ are generators of $\mathfrak{M}_1(\alpha)$. Since a derivation is completely defined by its value on generators it is sufficient to prove the existence $c \in \mathfrak{M}_1(\alpha)$ such that $d(z) = ad_c(z)$ for $z \in \{e_1, e_2, x, y\}$.

For any $k \in \mathbb{N}$ the quotient algebra

$$(\mathfrak{M}_1(\alpha))_{3k} = (\mathfrak{M}_1(\alpha))/\mathbf{n}_1^{3k} = \mathfrak{n}_1/\mathbf{n}_1^{3k} \oplus Q = \overline{\mathbf{n}}_1 \oplus Q \text{ with } \mathbf{n}_1^{3k} =< e_{3k+1}, \cdots >$$

is finite-dimensional solvable Lie algebra, which is maximal solvable extension of the nilpotent Lie algebra $\mathbf{n}_1$. By a base change it is easy to see that any algebra of the family $(\mathfrak{M}_1(\alpha))_{3k}$ is isomorphic to $(\mathfrak{M}_1(0))_{3k}$ and according to Lemma 2.9, the derivations of this algebra are inner.

Any $d \in \mathrm{Der}(\mathfrak{M}_1(\alpha))$ induces a derivation $\bar{d} \in \mathrm{Der}((\mathfrak{M}_1)_k(\alpha))$ such that $\bar{d}(\bar{v}) = \overline{d(v)}, \ \ \bar{v} = v + \mathfrak{n}_1^{3k}$. We claim that $\bar{d}$ is well-defined. Indeed, let set

$$d(\mathbf{n}_1) = d_{\mathbf{n}_1}(\mathfrak{n}_1) + d_Q(\mathbf{n}_1), \text{ where } d_{\mathbf{n}_1} : \mathbf{n}_1 \to \mathbf{n}_1, \quad d_Q : \mathbf{n}_1 \to Q.$$

Taking into account that $[Q, Q] \subseteq \mathbf{n}_1$ and $\mathbf{n}_1$ to be an ideal of $M(\alpha)$ such that $[\mathbf{n}_1, Q] = \mathbf{n}_1$ we derive

$$d(\mathbf{n}_1) = d([\mathbf{n}_1, Q]) = [d(\mathbf{n}_1), Q] + [\mathbf{n}_1, d(Q)] = [d_{\mathbf{n}_1}(\mathbf{n}_1), Q] + [\mathbf{n}_1, d(Q)] \subseteq \mathbf{n}_1.$$

Therefore,

$$d_Q(\mathbf{n}_1) = 0, \quad d(\mathbf{n}_1) \subseteq \mathbf{n}_1.$$

Now applying the derivation rule we get $d(\mathbf{n}_1^{3k}) \subseteq \mathbf{n}_1^{3k}$ for any $k \in \mathbb{N}$, that gives the well-definedness of $\bar{d}$.

Let

$$d(e_1) = \sum_{i=1}^{s} a_i e_i, \quad d(e_2) = \sum_{i=1}^{s} b_i e_i, \quad d(x) = \sum_{i=1}^{s} \gamma_i e_i + \gamma_{1,1}x + \gamma_{2,2}y, \quad d(y) = \sum_{i=1}^{s} \tau_i e_i + \tau_{1,1}x + \tau_{2,2}y.$$

Note that if $3k \geq \max\{s,t\}$ then $\overline{d(v)} = \bar{d}(\bar{v}) = ad_{\bar{c}_k}$ for some $\bar{c}_k = c_k + \mathfrak{n}_1^{3k}$ and any $\bar{v} = v + \mathfrak{n}_1^{3k}$. Let

$$\bar{c}_k = \sum_{i=1}^{3k} \rho_{k,i} e_i + \lambda_k x + \mu_k y + \mathfrak{n}_1^{3k}.$$

Then the equalities

$$\overline{d(e_1)} = [\bar{e}_1, \bar{c}_k], \quad \overline{d(e_2)} = [\bar{e}_2, \bar{c}_k], \quad \overline{d(y)} = [\bar{y}, \bar{c}_k].$$

imply

$$\begin{array}{l}\sum\limits_{i=1}^{s} a_i e_i - \lambda_k e_1 + \sum\limits_{i=2}^{k}(-\rho_{k,3i-1}e_{3i} + \rho_{k,3i}e_{3i+1}) + \mu_k(e_1 - \sum\limits_{i=2}^{t}\alpha_i e_{3i-2}) \in \mathfrak{n}_1^{3k},\\ \sum\limits_{i=1}^{s} b_i e_i - \sum\limits_{i=1}^{k}(\rho_{k,3i-2}e_{3i} + \rho_{k,3i}e_{3i+2}) - \lambda_k \sum\limits_{i=2}^{t}\alpha_i e_{3i-1} - \mu_k(e_2 - \sum\limits_{i=2}^{t}\alpha_i e_{3i-1}) \in \mathfrak{n}_1^{3k},\\ \sum\limits_{i=1}^{s} \tau_i e_i + \tau_{1,1}x + \tau_{2,2}y + \sum\limits_{i=1}^{t}(\rho_{k,3i-2}(-e_{3i-2} + \sum\limits_{j=2}^{t}\alpha_j e_{3j+3i-5}) + \rho_{3i-1}(e_{3i-1} - \sum\limits_{j=2}^{t}\alpha_j e_{3j+3i-4}))\\ +\lambda_k \sum\limits_{j=2}^{t}(1-j)\alpha_j e_{3j} \in \mathfrak{n}_1^{3k}.\end{array}$$

Comparing the coefficient at the basis elements we derive

$$\begin{cases} \lambda_k = a_1 + b_2, \quad \mu_k = b_2, \quad \rho_{k,3i-2} = b_{3i},\ 1 \leq i \leq k, \\ \rho_{k,2} = \tau_2, \quad \rho_{k,3i-1} = a_{3i},\ 2 \leq i \leq k, \quad \rho_{k,3i} = -a_1\alpha_{3i+2} + b_{3i+2},\ 1 \leq i \leq k. \end{cases} \tag{7}$$

Therefore, we conclude that $c_k = c_{k+1}$ for any $3k \geq \max\{s,t\}$. Now setting $c := c_k$ and $W_k = \mathrm{Span}\{x, y, e_1, \ldots, e_k\}$ we get $d(z)_{|W_k} = ad_c(z)_{|W_k}$ for any $z \in \{e_1, e_2, x, y\}$ and $k \geq \max\{s,t\}$. Now taking into account $\bigcup\limits_{k=1}^{\infty} W_k = \mathfrak{M}_1(\alpha)$ we obtain $d = ad_c$. □

Here is counterpart of the theorem above for $\mathfrak{M}_2(\beta)$. To prove that we make use the same arguments as in Theorem 2.10.

Theorem 2.11. *All the derivations of $\mathfrak{M}_2(\beta)$ are inner.*

Proof. Note $\mathfrak{M}_2(\beta) = \mathbf{n}_2 \oplus Q_2$, where $\{x, y\}$ is the basis of Q_2 and $\{e_1, e_2, x, y\}$ are generators of $\mathfrak{M}_2(\beta)$. Consider the quotient algebra $(\mathfrak{M}_2(\beta))_{8k} = \overline{\mathbf{n}}_2 \oplus Q_2$, where $\overline{\mathbf{n}}_2 = \mathbf{n}_2/\mathbf{n}_2{}^{8k}$ for $k \in \mathbb{N}$. It is isomorphic to $(\mathfrak{M}_2(0))_{8k}$ and all the derivations are inner. Therefore, all the derivation of $(\mathfrak{M}_2(\beta))_{8k}, \quad k \in \mathbb{N}$ also are inner.

We set

$$d(e_1) = \sum_{i=1}^{s} a_i e_i, \quad d(e_2) = \sum_{i=1}^{s} b_i e_i, \quad d(x) = \sum_{i=1}^{s} \gamma_i e_i + \gamma_{1,1}x + \gamma_{2,2}y, \quad d(y) = \sum_{i=1}^{s} \tau_i e_i + \tau_{1,1}x + \tau_{2,2}y.$$

Let us take $8k \geq \max\{s,t\}$. Then we have $\overline{d(v)} = \bar{d}(\bar{v}) = ad_{\bar{c}_k}$ for some $\bar{c}_k = c_k + \mathbf{n}_2^{8k}$ and any $\bar{v} = v + \mathbf{n}_2^{8k}$.
We put

$$\begin{array}{rcl} \bar{c}_k &=& \sum\limits_{i=1}^{8k} \rho_{k,i} e_i + \lambda_k x + \mu_k y + \mathfrak{n}_2^{8k} \\ \bar{c}_k &=& \sum\limits_{i=1}^{k} (\rho_{k,8i+1} f_{8i+1} + \rho_{k,8i+2} f_{8i+2} + \rho_{k,8i+3} f_{8i+3} + \rho_{k,8i+4} f_{8i+4} + \rho_{k,8i+5} f_{8i+5} \\ && +\rho_{k,8i+6} f_{8i+6} + \rho_{k,8i+7} f_{8i+7} + \rho_{k,8i+8} f_{8i+8}) + \lambda_k x + \mu_k y + \mathfrak{n}_2^{8k}. \\ {[x, \bar{c}_k]} &=& \sum\limits_{i=1}^{k} (-\rho_{k,8i+1} f_{8i+1} + 2\rho_{k,8i+2} f_{8i+2} + \rho_{k,8i+3} f_{8i+3} - \rho_{k,8i+5} f_{8i+5} \\ && -2\rho_{k,8i+6} f_{8i+6} + \rho_{k,8i+7} f_{8i+7}) + \mathfrak{n}_2^{8k}. \end{array}$$

The equalities

$$\overline{d(e_1)} = [\bar{e}_1, \bar{c}_k], \quad \overline{d(e_2)} = [\bar{e}_2, \bar{c}_k].$$

imply

$$\sum_{i=1}^{s} a_i e_i - \sum_{i=1}^{k} (\rho_{k,8i+2} f_{8i+3} + \rho_{k,8i+3} f_{8i+4} - 3\rho_{k,8i+4} f_{8i+5} - 2\rho_{k,8i+5} f_{8i+6} + \rho_{k,8i+7} f_{8i+8} - \rho_{k,8i+8} f_{8i+9}) - \lambda_k f_1 \in \mathfrak{n}_2^{8k},$$

and

$$\sum_{i=1}^{s} b_i e_i - \sum_{i=1}^{k} (-\rho_{k,8i+1} f_{8i+3} + \rho_{k,8i+5} f_{8i+7} - \rho_{k,8i+6} f_{8i+8} + 2\rho_{k,8i+8} f_{8i+10}) + 2\lambda_k f_2 - \mu_k f_2 \in \mathfrak{n}_2^{8k},$$

Comparing the coefficient of the basis elements we derive

$$\begin{cases} \lambda_k = a_1, \quad \mu_k = 2a_1 + b_2, \quad \rho_{k,8i+1} = -b_{8i+3}, \quad \rho_{k,8i+2} = a_{8i+3}, \\ \rho_{k,8i+3} = a_{8i+4}, \quad \rho_{k,8i+4} = -\frac{1}{3} a_{8i+5}, \quad \rho_{k,8i+5} = -\frac{1}{2} a_{8i+6}, \\ \rho_{k,8i+6} = -b_{8i+8}, \quad \rho_{k,8i+7} = a_{8i+8}, \quad \rho_{k,8i+8} = -a_{8i+9}, \quad 0 \le i \le k-1. \end{cases} \tag{8}$$

From (8) we conclude that $c_k = c_{k+1}$ for any $k \ge \max\{s,t\}$. Thus, setting $c := c_k$ and $W_k = \mathrm{Span}\{x, y, e_1, \dots, e_k\}$ we get $d(z)_{|W_k} = ad_c(z)_{|W_k}$ for any $z \in \{e_1, e_2, x, y\}$ and any $k \ge \max\{s,t\}$.

Now taking into account $\bigcup_{k=1}^{\infty} W_k = \mathfrak{M}_2(\beta)$ we obtain $d = ad_c$.

□

Now we prove that the second (co)homology groups of the algebras $\mathfrak{M}_1(0)$ and $\mathfrak{M}_2(0)$ are trivial. Note that the pro-solvable extensions $\mathfrak{M}_1(0)$ and $\mathfrak{M}_2(0)$ just constructed are Borel subalgebras of centerless affine Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively

Theorem 2.12. *The second cohomology group $H^2(\mathfrak{M}_1(0), \mathfrak{M}_1(0))$ of $\mathfrak{M}_1(0)$ vanishes.*

Proof. Let $\varphi \in Z^2(\mathfrak{M}_1(0), \mathfrak{M}_1(0))$. We show that there exists a $f \in \mathrm{Hom}(\mathfrak{M}_1(0), \mathfrak{M}_1(0))$ such that $\varphi = f([x,y]) - [f(x), y] - [x, f(y)]$. An element of φ of $Z^2(\mathfrak{M}_1(0), \mathfrak{M}_1(0))$ on the basis $\{x, y, e_1, e_2, \dots\}$ is written in the form

$$\begin{aligned} \varphi(e_i, e_j) &= \sum_{k=1}^{p(i,j)} (\alpha_{i,j}^{3k-2} e_{3k-2} + \alpha_{i,j}^{3k-1} e_{3k-1} + \alpha_{i,j}^{3k} e_{3k}) + A_{i,j}^1 x + A_{i,j}^2 y, \\ \varphi(e_i, x) &= \sum_{k=1}^{s(1,i)} (\beta_{1,i}^{3k-2} e_{3k-2} + \beta_{1,i}^{3k-1} e_{3k-1} + \beta_{1,i}^{3k} e_{3k}) + B_{1,i}^1 x + B_{1,i}^2 y, \\ \varphi(e_i, y) &= \sum_{k=1}^{s(2,i)} (\beta_{2,i}^{3k-2} e_{3k-2} + \beta_{2,i}^{3k-1} e_{3k-1} + \beta_{2,i}^{3k} e_{3k}) + B_{2,i}^1 x + B_{2,i}^2 y, \\ \varphi(x, y) &= \sum_{k=1}^{p} (\gamma^{3k-2} e_{2k-2} + \gamma^{3k-1} e_{3k-1} + \gamma^{3k} e_{3k}) + C^1 x + C^2 y, \quad i, j \in \mathbb{N}, \end{aligned}$$

Choose $f \in \mathrm{Hom}(\mathfrak{M}_1(0), \mathfrak{M}_1(0))$ as follows

$$\begin{aligned} f(e_{3i-2}) &= -\sum_{k=1}^{s(2,3i-2)} (\beta_{2,3i-2}^{3k-1} e_{3k-1} + \beta_{2,3i-2}^{3k} e_{3k}) + a^{3i-2} e_{3i-2} + \sum_{k=1, k\neq i}^{s(1,3i-2)} \tfrac{1}{i-k} \beta_{1,3i-2}^{3k-2} e_{3k-2} \\ &\quad - B_{2,3i-2}^1 x - B_{2,3i-2}^2 y, \\ f(e_{3i-1}) &= \sum_{k=1}^{s(2,3i-1)} (\tfrac{1}{2} \beta_{2,3i-1}^{3k-2} e_{3k-2} + \beta_{2,3i-1}^{3k} e_{3k}) + a^{3i-1} e_{3i-1} + \sum_{k=1, k\neq i}^{s(1,3i-1)} \tfrac{1}{i-k} \beta_{1,3i-1}^{3k-1} e_{3k-1} \\ &\quad + B_{2,3i-1}^1 x + B_{2,3i-1}^2 y, \\ f(e_{3i}) &= \sum_{k=1}^{s(2,3i)} (\beta_{2,3i}^{3k-2} e_{3k-2} - \beta_{2,3i}^{3k-1} e_{3k-1}) + a^{3i} e_{3i} + \sum_{k=1, k\neq i}^{s(1,3i)} \tfrac{1}{i-k} \beta_{1,3i}^{3k} e_{3k} + \tfrac{1}{i} B_{1,3i}^1 x + \tfrac{1}{i} B_{1,3i}^2 y, \\ f(x) &= \sum_{k=1}^{p} (\gamma^{3k-2} e_{3k-2} - \gamma^{3k-1} e_{3k-1}) + F_{1,1} x + F_{1,2} y, \\ f(y) &= \sum_{k=1}^{p} \tfrac{1}{k} \gamma^{3k} e_{3k} + F_{2,1} x + F_{2,2} y, \quad i, j \in \mathbb{N}, \end{aligned}$$

with the constraints

$$\begin{array}{rcl} a^{3i}+a^{3i-1}-a^{3(i+j)-1} & = & \alpha^{3(i+j)-1}_{3i-1,3j}, \\ a^{3j}+a^{3i-2}-a^{3(i+j)-2} & = & -\alpha^{3(i+j)-2}_{3i-2,3j}, \\ a^{3i-1}+a^{3j-2}-a^{3(i+j-1)} & = & -\alpha^{3(i+j-1)}_{3i-1,3j-2}. \end{array}$$

Let us consider $\chi = \varphi - \psi \in Z^2(\mathfrak{M}_1(0), \mathfrak{M}_1(0))$, where $\psi(x, y) = f([x, y]) - [f(x), y] - [x, f(y)]$ and show that χ is trivial. First of all by the definition for χ one has

$$\begin{array}{rcl} \chi(e_{3i-2}, y) & = & \sum\limits_{k=1}^{s(2,3i-2)} \beta^{3k-2}_{2,3i-2} e_{3k-2} + (iF_{2,1} - F_{2,2})e_{3i-2}, \\ \chi(e_{3i-1}, y) & = & \sum\limits_{k=1}^{s(2,3i-1)} \beta^{3k-1}_{2,3i-1} e_{3k-1} + ((i-1)F_{2,1} + F_{2,2})e_{3i-1}, \\ \chi(e_{3i}, y) & = & \sum\limits_{k=1}^{s(2,3i)} \beta^{3k}_{2,3i} e_{3k} + iF_{2,1}e_{3i} + B^1_{2,3i}x + B^2_{2,3i}y, \\ \chi(e_{3i-2}, x) & = & \sum\limits_{k=1}^{s(1,3i-2)} (\beta^{3k-1}_{1,3i-2} e_{3k-1} + \beta^{3k}_{1,3i-2} e_{3k}) + (iF_{1,1} - F_{1,2} + \beta^{3i-2}_{1,3i-2})e_{3i-2} + B^1_{1,3i-2}x + B^2_{1,3i-2}y, \\ \chi(e_{3i-1}, x) & = & \sum\limits_{k=1}^{s(1,3i-1)} (\beta^{3k-2}_{1,3i-1} e_{3k-2} + \beta^{3k}_{1,3i-1} e_{3k}) + ((i-1)F_{1,1} + F_{1,2} + \beta^{3i-1}_{1,3i-1})e_{3i-1} + B^1_{1,3i-1}x + B^2_{1,3i-1}y, \\ \chi(e_{3i}, x) & = & \sum\limits_{k=1}^{s(1,3i)} (\beta^{3k-2}_{1,3i} e_{3k-2} + \beta^{3k-1}_{1,3i} e_{3k-1}) + (iF_{1,1} + \beta^{3i}_{1,3i})e_{3i}, \\ \chi(x, y) & = & C^1x + C^2y. \quad \alpha^{3(i+j)-1}_{3i-1,3j} = 0, \ \alpha^{3(i+j)-2}_{3i-2,3j} = 0, \ \alpha^{3(i+j-1)}_{3i-1,3j-2} = 0. \end{array}$$

Secondly, if we impose to ψ the cocycle identities $Z = 0$ (see (3)) we derive the following set of constraints.

2-cocyle identity		Constraints
$Z(e_{3i}, x, y) = 0, \ i \geq 1$	$\Rightarrow$	$\begin{cases} C^1 = 0, \ \beta^{3k}_{2,3i} = 0, \ \ k \neq i, \ B^1_{2,3i} = B^2_{2,3i} = 0, \\ \beta^{3k-2}_{1,3i} = \beta^{3k-1}_{1,3i} = 0, \ \ k \geq 1, \end{cases}$
$Z(e_{3i-2}, x, y) = 0, \ i \geq 1$	$\Rightarrow$	$\begin{cases} C^2 = 0, \ \beta^{3k-2}_{2,3i-2} = 0, \ k \neq i, \ B^1_{1,3i-2} = B^2_{1,3i-2} = 0, \\ \beta^{3k-1}_{1,3i-2} = \beta^{3k}_{1,3i-2} = 0, \ \ k \geq 1, \end{cases}$
$Z(e_{3i-1}, x, y) = 0, \ i \geq 1$	$\Rightarrow$	$\begin{cases} \beta^{3k-1}_{2,3i-1} = 0, \ \ k \neq i, \ \ B^1_{1,3i-1} = B^2_{1,3i-1} = 0, \\ \beta^{3k-2}_{1,3i-1} = \beta^{3k}_{1,3i-1} = 0, \ \ k \geq 16. \end{cases}$
$Z(e_{3i-1}, e_{3j}, y) = 0, \ i, j \geq 1,$	$\Rightarrow$	$\begin{cases} \alpha^{3k-2}_{3i-1,3j} = \alpha^{3k}_{3i-1,3j} = A^1_{3i-1,3j} = A^2_{3i-1,3j} = 0, \ k \geq 1, \\ \beta^{3(i+j)-1}_{2,3(i+j)-1} = \beta^{3j}_{2,3j} + \beta^{3i-1}_{2,3i-1}, \Rightarrow \beta^{3i-1}_{2,3i-1} = (i-1)\beta^3_{2,3} + \beta^2_{2,2}, \end{cases}$
$Z(e_{3i-2}, e_{3j}, y) = 0, \ i, j \geq 1,$	$\Rightarrow$	$\begin{cases} \alpha^{3k-1}_{3i-2,3j} = \alpha^{3k}_{3i-2,3j} = A^1_{3i-2,3j} = A^2_{3i-2,3j} = 0, \ k \geq 1, \\ \beta^{3(i+j)-2}_{2,3(i+j)-2} = \beta^{3j}_{2,3j} + \beta^{3i-2}_{2,3i-2}, \Rightarrow \beta^{3i-2}_{2,3i-2} = (i-1)\beta^3_{2,3} + \beta^1_{2,1}, \end{cases}$
$Z(e_{3i-1}, e_{3j-2}, y) = 0, \ i, j \geq 1,$	$\Rightarrow$	$\begin{cases} \alpha^{3k-2}_{3i-1,3j-2} = \alpha^{3k-1}_{3i-1,3j-2} = 0, \ k \geq 1, \\ \beta^{3i}_{2,3i} = (i-1)\beta^3_{2,3} + \beta^2_{2,2} + \beta^1_{2,1}, \Rightarrow \beta^3_{2,3} = \beta^2_{2,2} + \beta^1_{2,1}. \end{cases}$

Let

$$F_{2,1} = -(\beta^2_{2,2} + \beta^1_{2,1}), \ \ F_{2,2} = \beta^2_{2,2}$$

then

$$\chi(e_{3i-1}, y) = 0, \quad \chi(e_{3i-2}, y) = 0, \quad \chi(e_{3i}, y) = 0.$$

Here are the constraints obtained by applying a few more identities

2-cocyle identity		Constraints
$Z(e_{3i-1}, e_{3j}, x) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\alpha_{3i-1,3j}^{3k-1} = 0,\ k \geq i + j,\ \beta_{1,3(i+j)-1}^{3(i+j)-1} = \beta_{1,3j}^{3j} + \beta_{1,3i-1}^{3i-1},$
$Z(e_{3i-2}, e_{3j}, x) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\alpha_{3i-1,3j}^{3k-2} = 0,\ k \geq i + j,\ \beta_{1,3(i+j)-2}^{3(i+j)-2} = \beta_{1,3j}^{3j} + \beta_{1,3i-2}^{3i-2},$
$Z(e_{3i-1}, e_{3j-2}, x) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\begin{cases} \alpha_{3i-1,3j-2}^{3k} = 0,\ \ k \neq i + j - 1,\ A_{3i-1,3j-2}^{1} = A_{3i-1,3j-2}^{2} = 0, \\ \beta_{1,3(i+j-1)}^{3(i+j-1)} = \beta_{1,3j-2}^{3j-2} + \beta_{1,3i-1}^{3i-1}, \end{cases}$
$Z(e_{3i}, e_{3j}, y) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\alpha_{3i,3j}^{3k-1} = \alpha_{3i,3j}^{3k-2} = 0,\ k \geq 1,$
$Z(e_{3i-1}, e_{3j-1}, y) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\alpha_{3i-1,3j-1}^{k} = 0,\ k \geq 1,$
$Z(e_{3i-2}, e_{3j-2}, y) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\alpha_{3i-2,3j-2}^{k} = 0,\ k \geq 1,$
$Z(e_{3i}, e_{3j}, e_{3k-1}) = 0,\ i, j \geq 1,$	$\Rightarrow$	$\alpha_{3i,3j}^{3k} = 0,\ k \geq 1.$

As a result we obtain

$$\beta_{1,3i-1}^{3i-1} = (i-1)\beta_{1,3}^{3} + \beta_{1,2}^{2},\ \ \beta_{1,3i-2}^{3i-2} = i\beta_{1,3}^{3} - \beta_{1,2}^{2},\ \beta_{1,3i}^{3i} = i\beta_{1,3}^{3}.$$

Finally, by setting

$$F_{1,1} = -\beta_{1,3}^{3},\ \ F_{1,2} = -\beta_{1,2}^{2}$$

we get

$$\chi(e_{3i-1}, x) = 0, \quad \chi(e_{3i-2}, x) = 0, \quad \chi(e_{3i}, x) = 0.$$

Therefore, $\chi \equiv 0$. □

Theorem 2.13. *The second cohomology group $H^2(\mathfrak{M}_2(0), \mathfrak{M}_2(0))$ of $\mathfrak{M}_2(0)$ vanishes.*

Proof. The proof of the theorem is similar to that of Theorem 2.12. □

Conjecture 2.14 (B.A.Omirov). *The second cohomology groups of $\mathfrak{M}_1(\alpha)$ for $\alpha \neq 0$ and $\mathfrak{M}_2(\beta)$ for $\beta \neq 0$ also are trivial.*

References

[1] K.K. Abdurasulov, B.A. Omirov, I.S. Rakhimov, G.O. Solijanova, *Residually solvable extensions of an infinite dimensional filiform Leibniz algebra*, Journal of Algebra **585** (2021), 697–722.

[2] K.K. Abdurasulov, B.A. Omirov, G.O. Solijanova, *On pro-solvable Lie algebras with maximal pro-nilpotent ideals* $\mathbf{m}_0$ *and* $\mathbf{m}_2$, (2020), arXiv:2001.06621v1 [math.RA], https://doi.org/10.48550/arXiv.2001.06621

[3] K.K. Abdurasulov, G.O. Solijanova, *Maximal pro-solvable Lie algebras with maximal positively graded ideals of length* $\frac{3}{2}$, Bulletin of the Institute of Mathematics **5** (2020), 25–32.

[4] J.M. Ancochea Bermudez, R. Campoamor-Stursberg, *Cohomologically rigid solvable Lie algebras with a nilradical of arbitrary characteristic sequence*, Linear Algebra and Its Applications **488** (2016), 135–147.

[5] Sh. Ayupov, B. Yusupov, *2-Local derivations of infinite-dimensional Lie algebras*, Journal of Algebra and Its Applications **19**(5) (2020), 2050100 (12 pages) DOI: 10.1142/S0219498820501005.

[6] L.M. Camacho, R.M. Navarro, B.A. Omirov, *Residually solvable extensions of pro-nilpotent Leibniz superalgebras*, J. Geom. Phys. **172** (2022), Paper No. 104414, 13 pp.

[7] R. Campoamor-Stursberg, *Solvable Lie algebras with an* $\mathbb{N}$*-graded nilradical of maximum nilpotency degree and their invariants*, J. Physics A. **43**(14) (2010), 145–202.

[8] A. Fialowski, *Deformations of nilpotent Kac-Moody algebras*, Studia Sci. Math. Hung. **19** (1984), 19, 465 – 483.

[9] A. Fialowski, *Classification of Graded Lie Algebras with Two Generators*, Vestn. Mosk. Univ., Ser. 1: Mat., Mekh. **38**(2) (1983), 62–64 [Moscow Univ. Math. Bull. **38**(2) (1983), 76–79].

[10] H. Garland, *Dedekind's* η*-function and the cohomology of infinite-dimensional Lie algebras*, Proc. Nat. Acad. Sc. (USA) **72** (1975), 2493 – 2495.

[11] V. Kac, *Infinite-Dimensional Lie Algebras*, (3rd edition), Cambridge Univ. Press, Cambridge, 1990.

[12] K. Khakimdjanova, Yu. Khakimdjanov, *Sur une classe d'algebres de Lie de dimension infinie*, Communications in Algebra **29**(1) (2001), 177–191 (in French).

[13] Kh.A. Khalkulova, K.K. Abdurasulov, *Solvable Lie algebras with maximal dimension of complementary space to nilradical*, Uzbek Math. J. **1** (2018), 90–98, DOI: 10.29229/uzmj.2018-1-8.

[14] G. Leger, E. Luks, *Cohomology theorems for Borel-like solvable Lie algebras in arbitrary characteristic*, Canadian J. Math. **24** (1972), 1019–1026.

[15] J. Lepowsky, *Generalized Verma modules, loop spaces cohomology and Mac-Donald-type identities*, Ann. Sc. Ecole Norm. Sup. (4) **12**(2) (1979), 169 – 234.

[16] J. Lepowsky, S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. Math. **29** (1978), 15 – 59.

[17] D.V. Millionshchikov, *Naturally Graded Lie Algebras (Carnot Algebra) of Slow Growth*, arXiv:1705.07494v2 (2017).

[18] D.V. Millionshchikov, *Lie Algebras of Slow Growth and Klein-Gordon PDE*, Algebras and Representation Theory **21**(5) (2018), 1037–1069

[19] V.V. Morozov, *Classification of nilpotent Lie algebras of sixth order*, Izv. Vysš. Učebn. Zaved. Matematika **5**(4) (1958), 161–171 (in Russian).

[20] G.M. Mubaryakzyanov, *The classification of real structures of Lie algebra of order five*, Izv. VUZ. Math. **3** (1963), 99 – 106.

[21] G.M. Mubarakzjanov, *On solvable Lie algebras*, Izv. Vysš. Učebn. Zaved. Matematika **32**(1) (1963), 114–123 (in Russian).

[22] G.M. Mubarakzjanov, *Classification of real structures of Lie algebras of fifth order*, Izv. Vysš. Učebn. Zaved. Matematika, **34**(3) (1963), 99–106 (in Russian).

[23] G.M. Mubarakzjanov, *Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element*, Izv. Vysš. Učebn. Zaved. Matematika **35**(4) (1963), 104–116 (in Russian).

[24] G.M. Mubarakzjanov, *Certain theorems on solvable Lie algebras*, Izv. Vysš. Učebn. Zaved. Matematika **55**(6) (1966), 95–98 (in Russian).

[25] J.-C. Ndogmo and P. Winternitz, *Generalized Casimir operators of solvable Lie algebras with abelian nilradicals*, J. Phys. **A 27**(8) (1994), 2787–2800.

[26] J.-C. Ndogmo and P. Winternitz, *Solvable Lie algebras with abelian nilradicals*, J. Phys. **A 27**(2) (1994), 405–423.

[27] J. L. Rubin and P. Winternitz, *Solvable Lie algebras with Heisenberg ideals*, J. Phys. **A 26**(5) (1993), 1123–1138.

[28] A. Shabanskaya, *Solvable indecomposable extensions of two nilpotent Lie algebras*, Communications in Algebra **44**(8) (2016), 3626–3667.

[29] A. Shabanskaya, G. Thompson, *Six-dimensional Lie algebras with a five-dimensional nilradical*, J. Lie Theory **23**(2) (2013), 313–355.

[30] A. Shabanskaya, G. Thompson, *Solvable extensions of a special class of nilpotent Lie algebras*, Arch. Math. (Brno) **49**(3) (2013, 63–81.

[31] A. Shalev, E.I. Zel'manov, *Narrow Lie algebras: A coclass theory and a characterization of the Witt algebra*, J. Algebra **189** (1997), 294 – 331.

[32] A. Shalev, E.I. Zelmanov, *Narrow algebras and groups*, J. of Math. Sciences **93**(6) (1999), 951 – 963.

[33] Šnobl and P. Winternitz. *Classification and Identification of Lie Algebras*, CRM Monograph Series, AMS **33** (2014), 306 pp.

[34] L. Šnobl and P. Winternitz, *A class of solvable Lie algebras and their Casimir invariants*, J. Phys. **A 38**(12) (2005), 2687–2700.

[35] L. Šnobl and P. Winternitz, *All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency* $n-1$, J. Phys. **A 42**(10) (2009), 105201.

[36] L. Šnobl and P. Winternitz, *Solvable Lie algebras with Borel nilradicals*, J. Phys. **A 45**(9) (2012), 095202.

[37] L. Šnobl, *On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras*, J. Phys. A, Math. Theor. **43** (2010), Article ID 505202.

[38] M. Vergne, *Cohomologie des algébres de Lie nilpotentes. Application á l'étude de la variété des algébres de Lie nilpotentes*, Bull. Soc. Math. France **98** (1970), 81–116 (in French).

[39] B.Ju. Weisfeiler, *Infinite-dimensional filtered Lie algebras and their connection with graded Lie algebras*, Funkt. Analis i ego Prilozh. **2**(1) (1968), 94–95. (English translation: Funct. Anal. Appl. **2** (1968), 88–89).