Filomat 37:19 (2023), 6355–6363
https://doi.org/10.2298/FIL2319355B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Some generaliztions of numerical radius inequalities including the off-diagonal parts of block matrices

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**Abstract.** In this paper, we introduce several numerical radius inequalities involving off-diagonal part of $2 \times 2$ positive semidefinite block matrices and their diagonal blocks. It is shown that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, then

$$w^{2r}(B) \leq \frac{1}{2}\sqrt{\left\|A^{4r\alpha} + A^{4r(1-\alpha)}\right\| \left\|C^{4r\alpha} + C^{4r(1-\alpha)}\right\|},$$

and

$$w^{2r}(B) \leq \left\|\alpha A^{\frac{r}{\alpha}} + (1-\alpha)C^{\frac{r}{1-\alpha}}\right\|,$$

for $0 < \alpha < 1$, $r \geq 1$. Moreover, we establish some numerical radius inequalities for products and sums of matrices.

## 1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called Hermitian if $A^* = A$, where $A^*$ is the complex conjugate of $A$. A Hermitian matrix is said to be positive semidefinite if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$. In this paper, we adopt $A \geq 0$ to mean $A$ is positive semidefinite.

The numerical radius $w(.)$ of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined as

$$w(A) = \max\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

One of the significant inequalities for the numerical radius is the power inequality;

$$w(A^k) \leq w^k(A) \tag{1}$$

for $A \in \mathbb{M}_n(\mathbb{C})$, $k = 1, 2, ...$[14].

2020 *Mathematics Subject Classification*. Primary 47A12; 47A30; 47A63.
*Keywords*. Positive semidefinite matrix; Block matrix; Numerical radius; Operator norm.
Received: 06 May 2022; Accepted: 11 February 2023
Communicated by Dragana Cvetković Ilić
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The numerical radius is one of the interesting norms that are defined on $\mathbb{M}_n(\mathbb{C})$ which is weakly unitarily invariant since for any unitary matrix $U$, $w(UAU^*) = w(A)$ and it is equivalent to the usual operator norm $\|.\|$. In fact, for any $A \in \mathbb{M}_n(\mathbb{C})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{2}$$

Kittaneh ([17], [18]) had established many improvements of inequality (2) by employing ingenious techniques;

$$w(A) \leq \frac{1}{2}(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}), \tag{3}$$

$$\frac{1}{4}\left\||A^*|^2 + |A|^2\right\| \leq w^2(A) \leq \frac{1}{2}\left\||A^*|^2 + |A|^2\right\|. \tag{4}$$

Several numerical radius inequalities generalizing and improving inequality (2) have been given in [1],[19], [20], and recently in [3], [4] and [5].

Authors in [10] generelized the second part of inequality (4) as follows: If $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$w^r(A) \leq \frac{1}{2}\left\||A^*|^{2\alpha r} + |A|^{2(1-\alpha)r}\right\| \tag{5}$$

and

$$w^r(A+B) \leq 2^{r-2}\left\||A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r} + |A|^{2\alpha r} + |B|^{2\alpha r}\right\| \tag{6}$$

for $0 < \alpha < 1$ and $r \geq 1$.

For other results involving the numerical radius inequalities, see [8], [9] and [11].

One of the topics that has attracted the attention of researchers in recent years is finding matrix inequalities related to positive semidefinite block matrices of the form $T = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$, where $A, B, C \in \mathbb{M}_n(\mathbb{C})$.

Hiroshima [13] proved that if the off-diagonal part $B$ is Hermitian, then

$$\left\|\begin{bmatrix} A & B \\ B & C \end{bmatrix}\right\| \leq \|A + C\|. \tag{7}$$

On the other hand, an estimation of the numerical radius of the off-diagonal block of $T$ was given by Burqan and Al-Saafin [6] as follows

$$w(B) \leq \frac{1}{2}\|A + C\|. \tag{8}$$

After that, Burqan and Abu-Rahma [7] generalized inequality (8) as follows

$$w^r(B) \leq \frac{1}{2}\|A^r + C^r\|, \text{for } r \geq 1. \tag{9}$$

An interesting generalization of inequality (9) proved by Yang [22] sserts that if $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, and $f$ is an increasing geometrically convex function, then

$$f(w(B)) \leq \frac{1}{2}\left\|f(A) + f(C)\right\|. \tag{10}$$

In this paper, we are interested in finding new upper bounds for the numerical radius of the off-diagonal block of positive semidefinite matrix based on the spectral norm of the diagonal blocks.Another generalization of inequality (9) which yields new numerical radius inequalities is introduced. More numerical radius inequalities including products and sums of matrices will be considered.

## 2. Lemmas

The following lemmas are essential to obtain and prove our results. The first lemma is a Cauchy-Schwarz inequality involving block positive semidefinite matrices [21]. The second lemma is an application of Jensen's inequality, can be found in [12]. The third lemma known as Buzano's inequality can be found in [20]. The fourth lemma follows from the spectral theorem for positive matrices and Jensen's inequality (see, e.g., [16]). The fifth lemma called Heinz inequality can be found in [15] and the last lemmas can be found in [16].

**Lemma 2.1.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. *Then*

$$\left|\langle Bx, y\rangle\right|^2 \leq \langle Ax, x\rangle \langle Cy, y\rangle, \ for\ x, y \in \mathbb{C}^n.$$

**Lemma 2.2.** *Let $a, b \geq 0$ and $0 \leq \alpha \leq 1$. Then*

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \ for\ r \geq 1.$$

**Lemma 2.3.** *Let $x, y, z \in \mathbb{C}^n$ with $\|z\| = 1$. Then*

$$\left|\langle x, z\rangle \langle z, y\rangle\right| \leq \frac{1}{2}\left(\|x\| \left\|y\right\| + \left|\langle x, y\rangle\right|\right).$$

**Lemma 2.4.** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite and $x \in \mathbb{C}^n$ with $\|x\| = 1$. Then*

$$\begin{aligned} \langle Ax, x\rangle^r &\leq \langle A^r x, x\rangle, \ for\ r \geq 1, \\ \langle A^r x, x\rangle &\leq \langle Ax, x\rangle^r, \ for\ 0 \leq r \leq 1. \end{aligned}$$

**Lemma 2.5.** *Let $A \in \mathbb{M}_n(\mathbb{C})$, $x, y \in \mathbb{C}^n$ and $0 \leq \alpha \leq 1$. Then*

$$\left|\langle Ax, y\rangle\right|^2 \leq \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle.$$

**Lemma 2.6.** *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $0 < \beta < 1$. Then*

$$\begin{bmatrix} |A^*|^{2\beta} & A^* \\ A & |A|^{2(1-\beta)} \end{bmatrix} \geq 0.$$

## 3. Main Results

In the beginning of this section, we introduce a new upper bound for the numerical radius of the off-diagonal part of positive semidefinite block matrices based on the spectral norm of the diagonal blocks.

**Theorem 3.1.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$ *and let $0 < \alpha < 1$. Then*

$$w^{2r}(B) \leq \frac{1}{2}\sqrt{\left\|A^{4r\alpha} + A^{4r(1-\alpha)}\right\| \left\|C^{4r\alpha} + C^{4r(1-\alpha)}\right\|}\ for\ r \geq 1. \tag{11}$$

*Proof.* *According to Lemma 2.1, for any unit vector $x \in \mathbb{C}^n$ we have*

$$|\langle Bx, x\rangle|^2 \leq (\langle Ax, x\rangle \langle x, Ax\rangle)^{\frac{1}{2}} (\langle Cx, x\rangle \langle x, Cx\rangle)^{\frac{1}{2}}.$$

*Since $A, C$ are positive semidefinite Lemma* 2.3, *yields that*

$$\begin{aligned} |\langle Bx, x\rangle|^2 &\leq \left(\frac{\|Ax\|^2 + \langle Ax, Ax\rangle}{2}\right)^{\frac{1}{2}} \left(\frac{\|Cx\|^2 + \langle Cx, Cx\rangle}{2}\right)^{\frac{1}{2}} \\ &= \left\langle A^2x, x\right\rangle^{\frac{1}{2}} \left\langle C^2x, x\right\rangle^{\frac{1}{2}}. \end{aligned}$$

*Thus, for any $r \geq 1$, we have*

$$|\langle Bx, x\rangle|^{2r} \leq \left\langle A^2x, x\right\rangle^{\frac{r}{2}} \left\langle C^2x, x\right\rangle^{\frac{r}{2}}$$

*Applying Lemma* 2.5 *to get*

$$\left\langle A^2x, x\right\rangle^{\frac{r}{2}} \leq \left(\left\langle A^{4\alpha}x, x\right\rangle^{\frac{r}{2}} \left\langle A^{4(1-\alpha)}x, x\right\rangle^{\frac{r}{2}}\right)^{\frac{1}{2}}$$

*By the arithmetic geometric mean inequality, we get*

$$\left\langle A^2x, x\right\rangle^{\frac{r}{2}} \leq \left(\frac{\left\langle A^{4\alpha}x, x\right\rangle^{r} + \left\langle A^{4(1-\alpha)}x, x\right\rangle^{r}}{2}\right)^{\frac{1}{2}}.$$

*Lemma* 2.4 *implies that*

$$\left\langle A^2x, x\right\rangle^{\frac{r}{2}} \leq \left(\frac{\left\langle A^{4r\alpha} + A^{4r(1-\alpha)}x, x\right\rangle}{2}\right)^{\frac{1}{2}}.$$

*Similarly*

$$\left\langle C^2x, x\right\rangle^{\frac{r}{2}} \leq \left(\frac{\left\langle C^{4r\alpha} + C^{4r(1-\alpha)}x, x\right\rangle}{2}\right)^{\frac{1}{2}}.$$

*Theefore,*

$$\begin{aligned} |\langle Bx, x\rangle|^{2r} &\leq \left(\frac{\left\langle A^{4r\alpha} + A^{4r(1-\alpha)}x, x\right\rangle}{2}\right)^{\frac{1}{2}} \left(\frac{\left\langle C^{4r\alpha} + C^{4r(1-\alpha)}x, x\right\rangle}{2}\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\sqrt{(\langle A^{4r\alpha} + A^{4r(1-\alpha)}x, x\rangle)(\langle C^{4r\alpha} + C^{4r(1-\alpha)}x, x\rangle)}. \end{aligned}$$

*and so*

$$\begin{aligned} w^{2r}(B) &= \max\left\{|\langle Bx, x\rangle|^{2r} : x \in \mathbb{C}^n, \|x\| = 1\right\} \\ &\leq \max\left\{\frac{1}{2}\sqrt{(\langle A^{4r\alpha} + A^{4r(1-\alpha)}x, x\rangle)(\langle C^{4r\alpha} + C^{4r(1-\alpha)}x, x\rangle)} : x \in \mathbb{C}^n, \|x\| = 1\right\} \\ &\leq \frac{1}{2}\sqrt{\left\|A^{4r\alpha} + A^{4r(1-\alpha)}\right\| \left\|C^{4r\alpha} + C^{4r(1-\alpha)}\right\|}. \end{aligned}$$

*This completes the proof.* □

In particular, considering $r = 1$ and $\alpha = \frac{1}{2}$ in the previus theorem, we get the following known result.

**Corollary 3.2.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$.Then*

$$w^2(B) \leq \|A\| \|C\|. \tag{12}$$

Another bound for the numerical radius of the off-diagonal block of positive semidefinite block matrices is the following one:

**Theorem 3.3.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$ and let $0 < \alpha < 1$.Then*

$$w^2(B) \leq \frac{1}{2}\sqrt{\left(\frac{1}{2}\left\|A^{4\alpha} + A^{4(1-\alpha)}\right\| + \left\|A^2\right\|\right)\left(\frac{1}{2}\left\|C^{4\alpha} + C^{4(1-\alpha)}\right\| + \left\|C^2\right\|\right)}. \tag{13}$$

*Proof.* *According to Lemma 2.1, for any unit vector $x \in \mathbb{C}^n$ we have*

$$|\langle Bx, x\rangle|^2 \leq \langle Ax, x\rangle \langle Cx, x\rangle.$$

*Since $A, C$ are positive semidefinite, Lemma 2.5 yields that*

$$\begin{aligned} |\langle Bx, x\rangle|^2 &\leq \left(\left\langle A^{2\alpha}x, x\right\rangle \left\langle A^{2(1-\alpha)}x, x\right\rangle\right)^{\frac{1}{2}} \left(\left\langle C^{2\alpha}x, x\right\rangle \left\langle C^{2(1-\alpha)}x, x\right\rangle\right)^{\frac{1}{2}} \\ &= \left(\left\langle A^{2\alpha}x, x\right\rangle \left\langle x, A^{2(1-\alpha)}x\right\rangle\right)^{\frac{1}{2}} \left(\left\langle C^{2\alpha}x, x\right\rangle \left\langle x, C^{2(1-\alpha)}x\right\rangle\right)^{\frac{1}{2}}. \end{aligned}$$

*Applying Lemma 2.3, we have*

$$\begin{aligned} |\langle Bx, x\rangle|^2 &\leq \frac{1}{2}\sqrt{\left(\left\|A^{2\alpha}x\right\| \left\|A^{2(1-\alpha)}x\right\| + \left\langle A^{2\alpha}x, A^{2(1-\alpha)}x\right\rangle\right)\left(\left\|C^{2\alpha}x\right\| \left\|C^{2(1-\alpha)}x\right\| + \left\langle C^{2\alpha}x, C^{2(1-\alpha)}x\right\rangle\right)} \\ &= \frac{1}{2}\sqrt{\left(\langle A^{4\alpha}x, x\rangle^{\frac{1}{2}} \left\langle A^{4(1-\alpha)}x, x\right\rangle^{\frac{1}{2}} + \left\langle A^{2\alpha}x, A^{2(1-\alpha)}x\right\rangle\right)\left(\langle C^{4\alpha}x, x\rangle^{\frac{1}{2}} \left\langle C^{4(1-\alpha)}x, x\right\rangle^{\frac{1}{2}} + \left\langle C^{2\alpha}x, C^{2(1-\alpha)}x\right\rangle\right)}. \end{aligned}$$

*By the arithmetic geometric mean inequality, we get*

$$|\langle Bx, x\rangle|^2 \leq \sqrt{\left(\frac{1}{4}\left\langle\left(A^{4\alpha} + A^{4(1-\alpha)}\right)x, x\right\rangle + \frac{1}{2}\left\langle A^{2\alpha}x, A^{2(1-\alpha)}x\right\rangle\right)\left(\frac{1}{4}\left\langle\left(C^{4\alpha} + C^{4(1-\alpha)}\right)x, x\right\rangle + \frac{1}{2}\left\langle C^{2\alpha}x, C^{2(1-\alpha)}x\right\rangle\right)}.$$

*Which is equivalent to*

$$|\langle Bx, x\rangle|^2 \leq \sqrt{\left(\frac{1}{4}\left\langle\left(A^{4\alpha} + A^{4(1-\alpha)}\right)x, x\right\rangle + \frac{1}{2}\langle A^2x, x\rangle\right)\left(\frac{1}{4}\left\langle\left(C^{4\alpha} + C^{4(1-\alpha)}\right)x, x\right\rangle + \frac{1}{2}\langle C^2x, x\rangle\right)}.$$

*Therefore,*

$$\begin{aligned} w^2(B) &= \max\left\{|\langle Bx, x\rangle|^2 : x \in \mathbb{C}^n, \|x\| = 1\right\} \\ &\leq \max\left\{\frac{1}{2}\sqrt{\left(\frac{1}{2}\left\langle\left(A^{4\alpha} + A^{4(1-\alpha)}\right)x, x\right\rangle + \langle A^2x, x\rangle\right)\left(\frac{1}{2}\left\langle\left(C^{4\alpha} + C^{4(1-\alpha)}\right)x, x\right\rangle + \langle C^2x, x\rangle\right)} : x \in \mathbb{C}^n, \|x\| = 1\right\} \\ &\leq \frac{1}{2}\sqrt{\left(\frac{1}{2}\left\|A^{4\alpha} + A^{4(1-\alpha)}\right\| + \left\|A^2\right\|\right)\left(\frac{1}{2}\left\|C^{4\alpha} + C^{4(1-\alpha)}\right\| + \left\|C^2\right\|\right)}. \end{aligned}$$

□

It should be mentioned here that for $r = 1$,inequality (11) becomes

$$w^2(B) \leq \frac{1}{2}\sqrt{\left\|A^{4\alpha} + A^{4(1-\alpha)}\right\| \left\|C^{4\alpha} + C^{4(1-\alpha)}\right\|}$$

and since $\left\|P^2\right\| \leq \frac{1}{2}\left\|P^{4\alpha} + P^{4(1-\alpha)}\right\|$ for any positive semidefinite matrix $P \in \mathbb{M}_n(\mathbb{C})$, inequality (13) is stronger than inequality (11) for $r = 1$.

In whats follow, we introduce a generalization of inequality (9), which yields interesting new numerical radius inequalities.

**Theorem 3.4.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$ *and let* $0 < \alpha < 1$.*Then*

$$w^{2r}(B) \leq \left\| \alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}} \right\| \text{ for } r \geq 1. \tag{14}$$

*Proof.* Since $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, for every unit vector $x \in \mathbb{C}^n$ Lemma 2.1 implies

$$\begin{aligned} |\langle Bx, x\rangle|^2 &\leq \langle Ax, x\rangle \langle Cx, x\rangle \\ &= \left\langle \left(A^{\frac{1}{\alpha}}\right)^{\alpha} x, x \right\rangle \left\langle \left(C^{\frac{1}{1-\alpha}}\right)^{1-\alpha} x, x \right\rangle \end{aligned}$$

Since $0 < \alpha < 1$, Lemma 2.4 yields that

$$|\langle Bx, x\rangle|^2 \leq \left\langle A^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \left\langle C^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha}$$

Therefore, by using Lemma 2.2, we have

$$|\langle Bx, x\rangle|^2 \leq \left( \alpha \left\langle A^{\frac{1}{\alpha}} x, x \right\rangle^{r} + (1-\alpha) \left\langle C^{\frac{1}{1-\alpha}} x, x \right\rangle^{r} \right)^{\frac{1}{r}}$$

Again, applying Lemma 2.4 for $r \geq 1$ to get

$$|\langle Bx, x\rangle|^2 \leq \left( \alpha \left\langle A^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle C^{\frac{r}{1-\alpha}} x, x \right\rangle \right)^{\frac{1}{r}}.$$

Thus,

$$|\langle Bx, x\rangle|^{2r} \leq \left\langle \left( \alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}} \right) x, x \right\rangle.$$

and so

$$\begin{aligned} w^{2r}(B) &= \max \left\{ |\langle Bx, x\rangle|^{2r} : x \in \mathbb{C}^n, \|x\| = 1 \right\} \\ &\leq \max \left\{ \left\langle \left( \alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}} \right) x, x \right\rangle : x \in \mathbb{C}^n, \|x\| = 1 \right\} \\ &= \left\| \alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}} \right\|, \end{aligned}$$

as required. □

Using the fact that if $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, then $\begin{bmatrix} X^*AX & X^*B^*Y^* \\ YBX & YCY^* \end{bmatrix} \geq 0$ for any $X, Y \in \mathbb{M}_n(\mathbb{C})$, we have the following corollary.

**Corollary 3.5.** *Let $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$ *and let* $0 < \alpha < 1$. *Then*

$$w^{2r}(YBX) \leq \left\| \alpha \left(X^*AX\right)^{\frac{r}{\alpha}} + (1-\alpha) \left(YCY^*\right)^{\frac{r}{1-\alpha}} \right\| \text{ for } r \geq 1. \tag{15}$$

## 4. Inequalities for Sums and Products of Matrices

Interesting inequalities for numerical radius of products and sums of matrices are introduced. The following result generalize inequality (6).

**Theorem 4.1.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$, $0 < \alpha < 1$ and $0 < \beta < 1$. Then*

$$w^{2r}(A+B) \leq \left\| \alpha \left( |A^*|^{2\beta} + |B^*|^{2\beta} \right)^{\frac{r}{\alpha}} + (1-\alpha) \left( |A|^{2(1-\beta)} + |B|^{2(1-\beta)} \right)^{\frac{r}{1-\alpha}} \right\|, \text{ for } r \geq 1. \tag{16}$$

*Proof.* Since the sum of positive semidefinite matrices is also positive semidefinite and by applying Lemma 2.6, we have

$$\begin{bmatrix} |A^*|^{2\beta} + |B^*|^{2\beta} & A^* + B^* \\ A + B & |A|^{2(1-\beta)} + |B|^{2(1-\beta)} \end{bmatrix} \geq 0.$$

Theorem 3.4 implies that

$$w^{2r}(A+B) \leq \left\| \alpha \left( |A^*|^{2\beta} + |B^*|^{2\beta} \right)^{\frac{r}{\alpha}} + (1-\alpha) \left( |A|^{2(1-\beta)} + |B|^{2(1-\beta)} \right)^{\frac{r}{1-\alpha}} \right\|.$$

This completes the proof. □

For $\beta = \frac{1}{2}$ in inequality (16), we get the following power numerical radius inequality for sum matrices.

$$w^{2r}(A+B) \leq \left\| \alpha(|A^*| + |B^*|)^{\frac{r}{\alpha}} + (1-\alpha)(|A| + |B|)^{\frac{r}{1-\alpha}} \right\|, \text{ for } r \geq 1. \tag{17}$$

Now, an estimate for the numerical radius of commutators is produced based on the following result.

**Theorem 4.2.** *Let* $A, B, C, D, X, Y \in \mathbb{M}_n(\mathbb{C})$, $0 < \alpha < 1$. *Then*

$$w^{2r}(Y(AC^* + BD^*)X) \leq \left\| \alpha(X^*(AA^* + BB^*)X)^{\frac{r}{\alpha}} + (1-\alpha)(Y(CC^* + DD^*)Y^*)^{\frac{r}{1-\alpha}} \right\|, \textit{for } r \geq 1 \tag{18}$$

*Proof.* We know that

$$\begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \geq 0,$$

for any $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. So by Corollary 3.5, we have

$$w^{2r}(Y(AC^* + BD^*)X) \leq \left\| \alpha(X^*(AA^* + BB^*)X)^{\frac{r}{\alpha}} + (1-\alpha)(Y(CC^* + DD^*)Y^*)^{\frac{r}{1-\alpha}} \right\|,$$

for any $X, Y \in \mathbb{M}_n(\mathbb{C})$. □

In view of the inequality (18) by letting $X = Y = I$, $C^* = B$ and $D^* = \pm A$, we get the following numerical radius inequality for commutators.

$$w^{2r}(AB \pm BA) \leq \left\| \alpha(AA^* + BB^*)^{\frac{r}{\alpha}} + (1-\alpha)(A^*A + B^*B)^{\frac{r}{1-\alpha}} \right\|, \text{ for } 0 < \alpha < 1 \text{ and } r \geq 1. \tag{19}$$

and letting $X = Y = I$, $C = B$ and $D = B = 0$, we get

$$w^{2r}(AB^*) \leq \left\| \alpha(AA^*)^{\frac{r}{\alpha}} + (1-\alpha)(BB^*)^{\frac{r}{1-\alpha}} \right\|, \text{ for } 0 < \alpha < 1 \text{ and } r \geq 1. \tag{20}$$

## 5. Generalization of Hiroshima's inequality

For a positive semidefinite block matrix $T = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$, where $A, B, C \in \mathbb{M}_n(\mathbb{C})$, it is well-known that

$$\|T\| = \left\| \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \right\| \leq \|A\| + \|C\|. \tag{21}$$

However, if $B$ is Hermitian, Hiroshima ([13]) obtained a refinement, more precisely

$$\|T\| = \left\| \begin{bmatrix} A & B \\ B & C \end{bmatrix} \right\| \leq \|A + C\|. \tag{22}$$

In this section we obtain an improvement of the inequality (22). In order to do this, we need the following useful decomposition achieved by Bourin et al. in Corollaries 2.1 and 2.2 in [2].

**Lemma 5.1.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. *Then, there exist unitary matrices $U, V, W, Z \in \mathbb{M}_{2n}(\mathbb{C})$ such that*

$$\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} = U \begin{bmatrix} \frac{A+C}{2} + Re(B^*) & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+C}{2} - Re(B^*) \end{bmatrix} V^*, \tag{23}$$

*and*

$$\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} = W \begin{bmatrix} \frac{A+C}{2} + Im(B^*) & 0 \\ 0 & 0 \end{bmatrix} W^* + Z \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+C}{2} - Im(B^*) \end{bmatrix} Z^*. \tag{24}$$

Here $Re(X)$ and $Im(X)$ denote the real and imaginary part of the matrix $X \in \mathbb{M}_n(\mathbb{C})$, i.e.

$$Re(X) = \frac{X^+ X^*}{2} \qquad \text{and} \qquad Im(X) = \frac{X^- X^*}{2i}. \tag{25}$$

We now derive a generalization of Hiroshima's inequality.

**Theorem 5.2.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. *Then,*

$$\left\| \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \right\| \leq \min\{\alpha, \beta\}, \tag{26}$$

*where*

$$\alpha = \left\| \frac{A+C}{2} + Re(B^*) \right\| + \left\| \frac{A+C}{2} - Re(B^*) \right\|,$$

*and*

$$\beta = \left\| \frac{A+C}{2} + Im(B^*) \right\| + \left\| \frac{A+C}{2} - Im(B^*) \right\|.$$

*Proof.* By (23) and the fact that $\|\cdot\|$ is unitarily invariant we have

$$\left\| \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \frac{A+C}{2} + Re(B^*) & 0 \\ 0 & 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+C}{2} - Re(B^*) \end{bmatrix} \right\|.$$

Since $\left\| \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} \right\| = \max\{\|R\|, \|S\|\}$ for any $R, S \in \mathbb{M}_n(\mathbb{C})$, we have

$$\left\| \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \right\| \leq \left\| \frac{A+C}{2} + Re(B^*) \right\| + \left\| \frac{A+C}{2} - Re(B^*) \right\|.$$

The proof of the other inequality is similar using the imaginary part. □

From the previously obtained statement, we obtain the following inequality.

**Proposition 5.3.** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. *Then,*

$$\left\| \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix} \right\| \leq \min\{\alpha, \beta\}, \tag{27}$$

*where*

$$\alpha = \left\| \frac{A+C}{2} + Re(B^*) \right\| + \left\| \frac{A+C}{2} - Re(B^*) \right\|,$$

*and*

$$\beta = \left\| \frac{A+C}{2} + Im(B^*) \right\| + \left\| \frac{A+C}{2} - Im(B^*) \right\|.$$

*In particular,*

$$w(B) \leq \|B\| \leq \min\{\alpha, \beta\}. \tag{28}$$

Finally, as consequence of Theorem 5.2 we obtain the Hiroshima's Theorem.

**Corollary 5.4.** *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$ *with* $B$ *Hermitian. Then,*

$$\left\| \begin{bmatrix} A & B \\ B & C \end{bmatrix} \right\| \leq \|A + C\|. \tag{29}$$

*Proof.* As $B$ is Hermitian, then $Im(B) = 0$. Thus, from Theorem 5.2, we have

$$\left\| \begin{bmatrix} A & B \\ B & C \end{bmatrix} \right\| \leq w(A + C) = \|A + C\|. \tag{30}$$

□

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