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On the completeness of a system of Bessel functions of index $-3/2$ in weighted L^2-space

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Abstract. In this paper, we study an integral representation of some class $E_{2,+}$ of even entire functions of exponential type $\sigma \le 1$. We also obtain an analog of the Paley-Wiener theorem related to the class $E_{2,+}$. In addition, we find necessary and sufficient conditions for the completeness of a system $\left\{s_k \sqrt{xs_k} J_{-3/2}(xs_k) : k \in \mathbb{N}\right\}$ in the space $L^2((0;1); x^2 dx)$, where $J_{-3/2}$ be the Bessel function of the first kind of index $-3/2$, $(s_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers and $L^2((0;1); x^2 dx)$ be the weighted Lebesgue space of all measurable functions $f : (0;1) \to \mathbb{C}$ satisfying $\int_0^1 x^2 |f(x)|^2\, dx < +\infty$. Those results are formulated in terms of sequences of zeros of functions from the class $E_{2,+}$. We also obtain some other sufficient conditions for the completeness of the considered system of Bessel functions. Our results complement similar results on completeness of the systems of Bessel functions of index $\nu < -1$, $\nu \notin \mathbb{Z}$.

1. Introduction

Let $L^2(X)$ be the space of all measurable functions $f : X \to \mathbb{C}$ on a measurable set $X \subseteq \mathbb{R}$ endowed with a norm

$$\|f\|^2_{L^2(X)} := \int_X |f(x)|^2\, dx,$$

let $\gamma \in \mathbb{R}$ and let $L^2((0;1); x^\gamma dx)$ be the weighted Lebesgue space of all measurable functions $f : (0;1) \to \mathbb{C}$, satisfying

$$\int_0^1 x^\gamma |f(x)|^2\, dx < +\infty.$$

Let (see, for example, [2, p. 4], [17, p. 345], [27, p. 40])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \quad z = x + iy = re^{i\varphi},$$

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be the Bessel function of the first kind of index $\nu \in \mathbb{R}$, where Γ is the classical Gamma function (see, for details, [15, pp. 1505-1506]). By Hurwitz's theorem (see [2, p. 59], [27, p. 483]), for $\nu > 1$ the function $J_{-\nu}$ has an infinity of real zeros and also $2[\nu]$ pairwise conjugate complex zeros, among them two pure imaginary zeros when $[\nu]$ is an odd integer. In particular, the function $J_{-3/2}$ has an infinite set $\{s_k : k \in \mathbb{Z} \setminus \{0\}\}$ of zeros, among them s_1 and $s_{-1} = \overline{s_1} = -s_1$ are two pure imaginary zeros, the positive zeros s_k, $k \in \mathbb{N} \setminus \{1\}$, and negative zeros $s_{-k} := -s_k$, $k \in \mathbb{N} \setminus \{1\}$. Moreover (see [17, p. 350], [27, p. 55]), $\sqrt{z}J_{-3/2}(z) = -\sqrt{2/\pi}z^{-1}(\cos z + z \sin z)$ and the function $s\sqrt{xs}J_{-3/2}(xs)$ belongs to the space $L^2((0;1); x^2dx)$ for every $s \in \mathbb{C}$. A system of elements $\{e_k : k \in \mathbb{N}\}$ in a separable Hilbert space $\mathcal{H}$ is called complete (see [10, p. 131], [11, p. 4258]) if $\overline{\operatorname{span}}\{e_k : k \in \mathbb{N}\} = \mathcal{H}$.

We remark that, in survey-cum-expository review article [14], Hari M. Srivastava presented a brief introductory overview and survey of some of the recent developments on the theory and applications of the Bessel functions and other higher transcendental functions. It is in [14] in which the interested reader can find some developments involving a hybrid version of several known extensions and generalizations of the Mittag-Leffler type functions, as well as the Hurwitz-Lerch type zeta functions, together with its associated fractional integrals and fractional derivatives (see, for details, [15]). In addition, Srivastava's investigations are motivated essentially by a number of extensive developments on the familiar Laplace and Hankel transforms as well as on the extensions and generalizations of each of these integral transforms. In particular, in the article [16], Hari M. Srivastava presented several (presumably new) properties and characteristics as well as inter-relationships among each of such general families of integral transforms as Srivastava's generalized Whittaker transform, Hardy's generalized Hankel transform and Srivastava's ϵ-generalized Hankel transform. These results can indeed be appropriately specialized to deduce a large number of known or new relationships between various simpler integral transforms.

Basis properties (completeness, minimality, basicity) of the systems of Bessel functions has been studied in many papers (see, for instance, [1–9, 12, 13, 17–27]). In this case, various approximation properties of the systems of Bessel functions J_ν for $\nu < -1$, $\nu \notin \mathbb{Z}$, were investigated in [5, 8, 9, 12, 13, 18, 19, 25, 26]. In particular, in [25] (see also [26]) it was proven that the system $\{s_k\sqrt{xs_k}J_{-3/2}(xs_k) : k \in \mathbb{N}\}$ is complete in the space $L^2((0;1); x^2dx)$ and the system $\{s_k\sqrt{xs_k}J_{-3/2}(xs_k) : k \in \mathbb{N} \setminus \{1\}\}$ is complete, minimal and is not a basis in this space, where $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$, $s_{-k} := -s_k$, is a sequence of zeros of the function $J_{-3/2}$. In addition, in [12] it was shown that the system $\{s_k^2\sqrt{xs_k}J_{-5/2}(xs_k) : k \in \mathbb{N} \setminus \{1; 2\}\}$ is complete and minimal in $L^2((0;1); x^4dx)$, where $(s_k)_{k\in\mathbb{N}}$ is a sequence of zeros of $J_{-5/2}$. Besides, in [13] has been established that the system $\{s_k^{\nu-1/2}\sqrt{xs_k}J_{-\nu}(xs_k) : k \in \mathbb{N} \setminus \{1; 2; \ldots; l\}\}$ is complete in $L^2((0;1); x^{2\nu-1}dx)$ if $\nu = l + 1/2$, $l \in \mathbb{N}$ and $(s_k)_{k\in\mathbb{N}}$ is a sequence of zeros of $J_{-\nu}$. However, the problem on completeness of this system in $L^2((0;1); x^{2\nu-1}dx)$ when $\nu = l + 1/2$, $l \in \mathbb{N}$ and $(s_k)_{k\in\mathbb{N}}$ is an arbitrary sequence of distinct nonzero complex numbers remains open. In this direction, in [18] the authors obtained a criterion for the completeness and minimality of a system $\{xs_k\sqrt{xs_k}J_{-3/2}(xs_k) : k \in \mathbb{N}\}$ in $L^2(0;1)$ with an arbitrary sequence of nonzero complex numbers $(s_k)_{k\in\mathbb{N}}$. Also, in [19] it was proven that the system $\{x^{-2}(s_k\sqrt{xs_k}J_{-3/2}(xs_k) - s_1\sqrt{xs_1}J_{-3/2}(xs_1)) : k \in \mathbb{N} \setminus \{1\}\}$ is complete and minimal in $L^2((0;1); x^2dx)$ where $(s_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers such that $s_k^2 \neq s_m^2$ for $k \neq m$. In addition, using methods of [5–7, 11, 18–21], in [8] were established necessary and sufficient completeness conditions of a system $\{s_k^2\sqrt{xs_k}J_{-5/2}(xs_k) : k \in \mathbb{N}\}$ in the space $L^2((0;1); x^4dx)$ with an arbitrary sequence of distinct nonzero complex numbers $(s_k)_{k\in\mathbb{N}}$ in terms of entire functions.

The aim of this paper is to obtain an analog of the Paley-Wiener theorem for some class $E_{2,+}$ of even entire functions of exponential type $\sigma \leq 1$ (see Theorem 3.1). In addition, we will find new necessary and sufficient conditions for the completeness of a system $\{\theta_k : k \in \mathbb{N}\}$ with $\theta_k(x) := s_k\sqrt{xs_k}J_{-3/2}(xs_k)$ in the space $L^2((0;1); x^2dx)$ in terms of entire functions from $E_{2,+}$ with the set of zeros coinciding with the sequence of distinct nonzero complex numbers $(s_k)_{k\in\mathbb{N}}$ (see Theorems 3.6–3.11). Also, in Theorems 3.12 and 3.13, we obtain some other sufficient conditions of the completeness of the system $\{\theta_k : k \in \mathbb{N}\}$. Our results complement the results of papers [3, 5, 8, 9, 12, 13, 18, 19, 25, 26].

2. Preliminaries

An entire function G is said to be of exponential type $\sigma \in [0;+\infty)$ ([10, p. 4], [11, p. 4262]) if for any $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that $|G(z)| \leq c(\varepsilon)\exp((\sigma+\varepsilon)|z|)$ for all $z \in \mathbb{C}$.

Denote by PW^2_σ the set of all entire functions of exponential type $\sigma \in (0;+\infty)$ whose narrowing on $\mathbb{R}$ belongs to the space $L^2(\mathbb{R})$, and by $PW^2_{\sigma,+}$ denote the class of even entire functions from PW^2_σ. By the Paley-Wiener theorem (see [10, p. 69], [11, p. 4263]), the class PW^2_σ coincides with the class of functions G admitting the representation

$$G(z) = \int_{-\sigma}^{\sigma} e^{itz} g(t)\,dt, \quad g \in L^2(-\sigma;\sigma),$$

and the class $PW^2_{\sigma,+}$ consists of the functions G representable in the form

$$G(z) = \int_0^{\sigma} \cos(tz) g(t)\,dt, \quad g \in L^2(0;\sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)} = \sqrt{2/\pi}\|G\|_{L^2(0;+\infty)}$ and

$$g(t) = \frac{2}{\pi}\int_0^{+\infty} G(z)\cos(tz)\,dz.$$

Let $\log^+ x = \max(0;\log x)$ for $x > 0$. Here and so on by $C_1, C_2, \ldots$ we denote arbitrary positive constants. To prove our main results we need the following auxiliary lemmas.

Lemma 2.1. *(see [18, p. 13], [25, p. 39]) Let an entire function Q be defined by the formula*

$$Q(z) = -\sqrt{\frac{2}{\pi}}\int_0^1 (\cos(tz) + tz\sin(tz))q(t)\,dt, \quad q \in L^2(0;1). \tag{1}$$

Then for all $z = x+iy = re^{i\varphi} \in \mathbb{C}$, we have

$$|Q(z)| \leq C_1 \frac{e^{|\operatorname{Im} z|}}{\sqrt{1+|\operatorname{Im} z|}}(1+|z|),$$

and Q is an even entire function of exponential type $\sigma \leq 1$.

Lemma 2.2. *(see [11, p. 4263]) Let Q be an entire function of exponential type $\sigma \leq 1$ for which*

$$\int_{-\infty}^{+\infty} \frac{\log^+|Q(t)|}{1+t^2}\,dt < +\infty,$$

and let $(s_k)_{k\in\mathbb{N}}$ be a sequence of nonzero roots of the function $Q(z)$. Then

$$\sum_{k\in\mathbb{N}} \left|\operatorname{Im}\frac{1}{s_k}\right| < +\infty.$$

3. Main results

Our main results are the following statements.

Theorem 3.1. *An entire function Q has the representation*

$$Q(z) = \int_0^1 z\sqrt{tz}J_{-3/2}(tz)t^2h(t)\,dt \tag{2}$$

with some function $h \in L^2((0;1);t^2dt)$ if and only if it is an even entire function of exponential type $\sigma \le 1$ such that

$$Q(0) = -\sqrt{\frac{2}{\pi}}\int_0^1 th(t)\,dt, \tag{3}$$

and the function $z^{-1}Q'(z)$ belongs to the space $PW^2_{1,+}$. If these conditions are fulfilled, then

$$h(t) = -\sqrt{\frac{2}{\pi}}\frac{1}{t^3}\int_0^{+\infty}\frac{Q'(z)}{z}\cos(tz)\,dz.$$

Proof. Necessity. Let Q has the representation (2) with some function $h \in L^2((0;1);t^2dt)$. Since

$$z\sqrt{tz}J_{-3/2}(tz) = -\sqrt{\frac{2}{\pi}}\frac{\cos(tz)+tz\sin(tz)}{t},$$

we have

$$Q(z) = -\sqrt{\frac{2}{\pi}}\int_0^1(\cos(tz)+tz\sin(tz))th(t)\,dt, \quad Q(0) = -\sqrt{\frac{2}{\pi}}\int_0^1 th(t)\,dt.$$

Therefore, by Lemma 2.1, the function Q is an even entire function of exponential type $\sigma \le 1$, and

$$Q'(z) = -\sqrt{\frac{2}{\pi}}\int_0^1 t^3z\cos(tz)h(t)\,dt, \quad Q'(0) = 0, \quad \frac{Q'(z)}{z} = -\sqrt{\frac{2}{\pi}}\int_0^1\cos(tz)t^2q(t)\,dt,$$

where $q(t) := th(t)$. Since $h \in L^2((0;1);t^2dt)$, we have $q \in L^2(0;1)$, and in accordance with the Paley-Wiener theorem, the function $z^{-1}Q'(z)$ belongs to the space $PW^2_{1,+}$. *Sufficiency.* If all the conditions of the theorem hold, then from the formula for the inverse Fourier cosine-transformation it follows that the function

$$q(t) = -\sqrt{\frac{2}{\pi}}\frac{1}{t^2}\int_0^{+\infty}\frac{Q'(z)}{z}\cos(tz)\,dz$$

belongs to the space $L^2(0;1)$, and

$$Q'(z) = -\sqrt{\frac{2}{\pi}}\int_0^1 z\cos(tz)t^3h(t)\,dt.$$

Using Fubini's theorem, we get

$$Q(z)-Q(0) = -\sqrt{\frac{2}{\pi}}\int_0^1 t^3h(t)\,dt\int_0^z w\cos(tw)\,dw = -\sqrt{\frac{2}{\pi}}\int_0^1(\cos(tz)+tz\sin(tz)-1)th(t)\,dt.$$

Further, using (3), we obtain

$$Q(z) = -\sqrt{\frac{2}{\pi}}\int_0^1(\cos(tz)+tz\sin(tz))th(t)\,dt = \int_0^1 z\sqrt{tz}J_{-3/2}(tz)t^2h(t)\,dt.$$

Thus, the theorem is proved. □

Let $\widetilde{E}_{2,+}$ be the class of entire functions Q that can be presented in the form (2) with some function $h \in L^2((0;1);t^2dt)$, and let $E_{2,+}$ be the class of even entire functions Q of exponential type $\sigma \le 1$ satisfying (3) with $h \in L^2((0;1);t^2dt)$ and the function $z^{-1}Q'(z)$ belongs to the space $PW^2_{1,+}$.

Corollary 3.2. $\widetilde{E}_{2,+} = E_{2,+}$.

Corollary 3.3. *The class $E_{2,+}$ coincides with a set of entire functions Q representing in the form* (1).

Example 3.4. *Evidently,* $\cos z \notin E_{2,+}$.

Example 3.5. *The function*

$$Q(z) = -\sqrt{\frac{2}{\pi}}\frac{\cos z}{z^2 - \pi^2/4}\left(1 - \frac{4(-\pi+2)}{\pi^3}(z^2 - \pi^2/4)\right)$$

belongs to $E_{2,+}$ with

$$h(t) = \frac{4}{\pi^3 t^3}\left(-2\cos\left(\frac{\pi}{2}t\right) - \pi t \sin\left(\frac{\pi}{2}t\right) + 2\right).$$

Theorem 3.6. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of nonzero complex numbers such that $s_k^2 \ne s_n^2$ for $k \ne n$. The system $\{\theta_k : k \in \mathbb{N}\}$ is incomplete in $L^2((0;1);x^2dx)$ if and only if a sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $s_{-k} := -s_k$, $k \in \mathbb{N}$, is a subsequence of zeros of some nonzero entire function $Q \in E_{2,+}$.*

Proof. According to the Hahn-Banach theorem (see [10, p. 131], [11, p. 4258]), the system $\{\theta_k : k \in \mathbb{N}\}$ is incomplete in $L^2((0;1);x^2dx)$ if and only if there exists a nonzero function $h \in L^2((0;1);x^2dx)$ such that

$$\int_0^1 s_k\sqrt{ts_k}J_{-3/2}(ts_k)t^2h(t)\,dt = 0$$

for all $k \in \mathbb{N}$. Hence, taking into account Theorem 3.1, we obtain the required proposition. Theorem 3.6 is proved. □

Theorem 3.7. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $|\operatorname{Im} s_k| \ge \delta|s_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. If a system $\{\theta_k : k \in \mathbb{N}\}$ is complete in $L^2((0;1);x^2dx)$, then*

$$\sum_{k=1}^{\infty}\frac{1}{|s_k|} = +\infty. \tag{4}$$

Proof. Suppose, to the contrary, that the system $\{\theta_k : k \in \mathbb{N}\}$ is not complete in the space $L^2((0;1);x^2dx)$. Then, by Theorem 3.6, there exists a nonzero entire function $Q \in E_{2,+}$ for which the sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. By virtue of Corollary 3.3, the function Q is of the kind (1). Due to Lemma 2.1, we have $|Q(x)| \le C_1(1+|x|)$ for all $x \in \mathbb{R}$. This implies

$$\int_{-\infty}^{+\infty}\frac{\log^+|Q(x)|}{1+x^2}\,dx < +\infty.$$

Therefore, by Lemma 2.2, we get

$$\sum_{k\in\mathbb{N}}\left|\operatorname{Im}\frac{1}{s_k}\right| < +\infty.$$

Since $|\operatorname{Im} s_k| \geq \delta |s_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$, and

$$\left|\operatorname{Im} \frac{1}{s_k}\right| = \frac{|\operatorname{Im} s_k|}{|s_k|^2} \geq \frac{\delta}{|s_k|},$$

we have

$$\sum_{k=1}^{\infty} \frac{1}{|s_k|} < +\infty.$$

This contradicts condition (4). Thus, the theorem is proved. □

Theorem 3.8. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $s_k^2 \neq s_m^2$ for $k \neq m$. Let a sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $s_{-k} := -s_k$, be a sequence of zeros of some even entire function G of exponential type $\sigma \leq 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, we have $|G(z)| \geq C_2(1+|z|)e^{|\operatorname{Im} z|}$. Then the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$.*

Proof. Assume the converse. Then, according to Theorem 3.6, there exists a nonzero even entire function $Q \in E_{2,+}$ for which the sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Let $V(z) = Q(z)/G(z)$. Then V is an even entire function of order $\tau \leq 1$, for which by Corollary 3.3 and Lemma 2.1, we obtain

$$|V(z)| \leq C_3 \frac{1}{\sqrt{1+|\operatorname{Im} z|}}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Therefore, according to the Phragmén-Lindelöf theorem (see [10, p. 38], [11, p. 4263]), we get $V(z) \equiv 0$. Hence $Q(z) \equiv 0$. This contradiction proves the theorem. □

Corollary 3.9. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of zeros of the function $J_{-3/2}$. Then the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$.*

Proof. Indeed, a sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $s_{-k} = -s_k$, is a sequence of zeros of an entire function $G(z) = \cos z + z\sin z$, and this function satisfies the conditions of Theorem 3.8. Therefore, a system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$. Corollary 3.9 is proved. □

Theorem 3.10. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $s_k^2 \neq s_m^2$ for $k \neq m$. Let a sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $s_{-k} := -s_k$, be a sequence of zeros of some even entire function $G \notin E_{2,+}$ of exponential type $\sigma \leq 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, the inequality $|G(z)| \geq C_4(1+|z|)^{-\alpha}e^{|\operatorname{Im} z|}$ holds with $\alpha < 3/2$. Then the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$.*

Proof. Assume the converse. Then, according to Theorem 3.6, there exists a nonzero even entire function $Q \in E_{2,+}$ for which the sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. Let $V(z) = Q(z)/G(z)$. Then V is an even entire function of order $\tau \leq 1$, for which by Corollary 3.3 and Lemma 2.1, we get

$$|V(z)| \leq C_5 \frac{(1+|z|)^{\alpha+1}}{\sqrt{1+|\operatorname{Im} z|}}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Since $\alpha + 1 < 5/2$, according to the Phragmén-Lindelöf theorem, the function V is a polynomial of degree $\zeta < 2$. However, V is an even entire function, and therefore the function V is a constant. Hence, $Q(z) = C_6G(z)$ and $Q \notin E_{2,+}$. Thus, we have a contradiction and the proof of the theorem is completed. □

Theorem 3.11. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $s_k^2 \neq s_m^2$ for $k \neq m$. Let a sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $s_{-k} := -s_k$, be a sequence of zeros of some even entire function $F \notin E_{2,+}$ of exponential type $\sigma \le 1$ such that for some $\alpha < 1$ and $\eta \in \mathbb{R}$*

$$|F(x+i\eta)| \ge \delta|x|^{-\alpha}, \quad \delta > 0, \quad |x| > 1. \tag{5}$$

Then the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$.

Proof. Let $F \notin E_{2,+}$ and the inequality (5) is true. Suppose, to the contrary, that the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is not complete in $L^2((0;1);x^2dx)$. Then, by Theorem 3.6, there exists a nonzero even entire function $Q \in E_{2,+}$ which vanishes at the points s_k. However, the sequence $(s_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a sequence of zeros of an even entire function $F(z) \notin E_{2,+}$ of exponential type $\sigma \le 1$. Therefore, $T(z) = Q(z)/F(z)$ is an even entire function of order $\tau \le 1$. Since $Q \in E_{2,+}$, taking into account Corollary 3.3 and Lemma 2.1, we obtain

$$|Q(x+i\eta)| \le C_7 \frac{e^{|\eta|}}{\sqrt{1+|\eta|}}\left(1+\sqrt{x^2+\eta^2}\right), \quad x \in \mathbb{R}.$$

Using (5), we get $|T(x+i\eta)| \le C_8(1+|x|)^{1+\alpha}$, $x \in \mathbb{R}$. It is easy to see that $T(z)$ is a polynomial of degree $\zeta < 2$. Further, since T is an even entire function, we have $T(z) = C_9$. Furthermore, $F(z) = C_{10}Q(z)$ and $F(z) \in E_{2,+}$. This contradiction concludes the proof of the theorem. □

Let $n(t)$ be the number of points of the sequence $(s_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ satisfying the inequality $|s_k| \le t$, i.e., $n(t) := \sum_{|s_k|\le t} 1$, and let

$$N(r) := \int_0^r \frac{n(t)}{t}\,dt, \quad r > 0.$$

Theorem 3.12. *Let $(s_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of distinct nonzero complex numbers. If*

$$\limsup_{r\to+\infty}\left(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - \log(1+r)\right) = +\infty,$$

then the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$.

Proof. It suffices to assume the incompleteness of the system $\left\{\theta_k : k \in \mathbb{N}\right\}$ and prove that

$$\limsup_{r\to+\infty}\left(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - \log(1+r)\right) < +\infty. \tag{6}$$

Due to Theorem 3.6, there exists a nonzero even entire function $Q \in E_{2,+}$ of exponential type $\sigma \le 1$ for which the sequence $(s_k)_{k\in\mathbb{N}}$ is a subsequence of zeros. We may consider that $Q(0) = 1$. Then, consecutively applying the Jensen formula (see [10, p. 10], [11, p. 4316]), Corollary 3.3 and Lemma 2.1, we obtain

$$\begin{aligned} N(r) &\le \frac{1}{2\pi}\int_0^{2\pi} \log|Q(re^{i\varphi})|\,d\varphi \\ &\le C_{11} + \frac{1}{2\pi}\int_0^{2\pi}\left(r|\sin\varphi| - \frac{1}{2}\log(1+r|\sin\varphi|) + \log(1+r)\right)d\varphi \\ &\le C_{11} + \frac{1}{2\pi}\int_0^{2\pi}\left(r|\sin\varphi| - \frac{1}{2}\log r - \frac{1}{2}\log|\sin\varphi| + \log(1+r)\right)d\varphi \\ &= \frac{2r}{\pi} - \frac{1}{2}\log r + \log(1+r) + C_{12}, \quad r > 0, \end{aligned}$$

whence it follows (6). The theorem is proved. □

Theorem 3.13. *Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. Let $|s_k| \le \Delta k+\beta+\alpha_k$ for $0<\Delta<\pi/2$, $-\Delta<\beta<1-\Delta(2+\pi)/\pi$, and the sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\alpha_k \ge 0$, $\alpha_k = O(1)$ as $k\to+\infty$ and*

$$\sum_{k=1}^{\infty} |\alpha_{k+1}-\alpha_k| < +\infty, \quad \sum_{k=1}^{\infty} \frac{\alpha_k}{k} < +\infty.$$

Then the system $\left\{\theta_k : k\in\mathbb{N}\right\}$ is complete in $L^2((0;1);x^2dx)$.

Proof. Let $\mu_k = \Delta k+\beta+\alpha_k$, $k\in\mathbb{N}$, and

$$n_1(t) = \sum_{\mu_k\le t} 1, \quad N_1(r) = \int_0^r \frac{n_1(t)}{t}\,dt, \quad r>0.$$

Then $n(t)\ge n_1(t)$, $N(r)\ge N_1(r)$ and $n_1(t)=m$ for $\Delta m+\beta+\alpha_m \le t < \Delta(m+1)+\beta+\alpha_{m+1}$ ($n_1(t)=0$ on $(0;\mu_1)$). Let $r\in[\mu_j;\mu_{j+1})$. Then $j=r/\Delta+O(1)$ as $r\to+\infty$. Therefore, under the assumptions of the theorem, by analogy with [7, p. 894] (see also [6, p. 9]), we obtain

$$\begin{aligned} N_1(r) &\ge \sum_{k=1}^{j-1} k\log\frac{\Delta(k+1)+\beta}{\Delta k+\beta} - \left|\sum_{k=1}^{j-1} k\left(\log\frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} - \log\frac{\Delta(k+1)+\beta}{\Delta k+\beta}\right)\right| + O(1) \\ &\ge \frac{r}{\Delta} - \left(\frac12+\frac{\beta}{\Delta}\right)\log r - C_{13}\sum_{k=1}^{\infty}\left(|\alpha_{k+1}-\alpha_k| + \frac{\alpha_k}{k}\right) + O(1) \\ &\ge \frac{r}{\Delta} - \left(\frac12+\frac{\beta}{\Delta}\right)\log r + O(1), \quad r\to+\infty. \end{aligned}$$

In view of this, for $0<\Delta<\pi/2$ and $-\Delta<\beta<1-\Delta(2+\pi)/\pi$, we get

$$\begin{aligned} &\limsup_{r\to+\infty}\left(N(r)-\frac{2r}{\pi}+\frac12\log r-\log(1+r)\right) \\ &\quad\ge \limsup_{r\to+\infty}\left(N_1(r)-\frac{2r}{\pi}+\frac12\log r-\log(1+r)\right) \\ &\quad\ge \limsup_{r\to+\infty}\left(\frac{r}{\Delta}-\left(\frac12+\frac{\beta}{\Delta}\right)\log r-\frac{2r}{\pi}+\frac12\log r-\log(1+r)+O(1)\right) \\ &\quad\ge \limsup_{r\to+\infty}\left(r\left(\frac{1}{\Delta}-\frac{2}{\pi}\right)-\left(\frac{\beta}{\Delta}+1\right)\log(1+r)+O(1)\right) \\ &\quad\ge \limsup_{r\to+\infty}\left(r\left(\frac{1}{\Delta}-\frac{2}{\pi}-\frac{\beta}{\Delta}-1\right)+O(1)\right) = +\infty. \end{aligned}$$

Finally, according to Theorem 3.12, we obtain the required proposition. The proof of theorem is completed. □

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