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Sharp inequalities related to the Adamović-Mitrinović, Cusa, Wilker and Huygens results

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Abstract. In this paper, we establish sharp inequalities for trigonometric functions. For example, we consider the Wilker inequality and prove that for $0 < x < \pi/2$ and $n \geq 1$,

$$2 + \left(\sum_{j=2}^{n-1} d_{j+1} x^{2j} + \delta_n x^{2n} \right) x^3 \tan x < \left(\frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} < 2 + \left(\sum_{j=3}^{n-1} d_{j+1} x^{2j} + D_n x^{2n} \right) x^3 \tan x$$

with the best possible constants

$$\delta_n = d_n \text{ and } D_n = \frac{2\pi^6 - 168\pi^4 + 15120}{945\pi^4} \left(\frac{2}{\pi} \right)^{2n} - \sum_{j=2}^{n-1} d_{j+1} \left(\frac{2}{\pi} \right)^{2n-2j},$$

where $d_k = 2^{2k+2}\Big((4k+6)\,|B_{2k+2}| + (-1)^{k+1}\Big)/(2k+3)!$ and B_k are the Bernoulli numbers ($k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$). This improves and generalizes the results given by Mortici, Nenezić and Malešević.

1. Introduction

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \tag{1}$$

for $0 < |x| < \pi/2$. The left-hand side inequality was obtained by Adamović and Mitrinović (see [22, p. 238]), while the right-hand side inequality was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. Huygens [14] gave a rigorous proof of the right-hand side inequality, and then used it to estimate the number π. The right-hand side inequality is now known as Cusa's inequality (see [23, 32, 37, 54]). Further interesting historical facts about the right-hand side inequality can be found in [37].

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The inequalities (1) have attracted much interest of many mathematicians and have motivated a large number of research papers; see, for example, [5–7, 12, 15, 23, 28, 29, 32, 33, 41, 48–51, 54] and the references cited therein.

By using inequalities involving Schwab-Borchardt mean, Neuman [29] presented the following inequality chain:

$$(\cos x)^{1/3} < \left(\cos x \frac{\sin x}{x}\right)^{1/4} < \left(\frac{\sin x}{\operatorname{arctanh}(\sin x)}\right)^{1/2} < \left(\frac{\cos x + (\sin x)/x}{2}\right)^{1/2} < \\ < \left(\frac{1+2\cos x}{3}\right)^{1/2} < \left(\frac{1+\cos x}{2}\right)^{2/3} < \frac{\sin x}{x}, \qquad 0 < x < \frac{\pi}{2}, \tag{2}$$

which improves the first inequality in (1). Yang [49] proved that for $0 < x < \pi/2$,

$$\frac{\sin x}{x} < \left(\frac{2}{3}\cos\frac{x}{2} + \frac{1}{3}\right)^2 < \cos^3\frac{x}{3} < \frac{2+\cos x}{3}, \tag{3}$$

which improves the second inequality in (1).

Motivated by (1), in Section 3 we establish sharp inequalities for trigonometric functions. By using the obtained results, we present inequality chain and improve the double inequality (1).

Wilker [39] proposed the following two open problems:

(a) Prove that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{4}$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{5}$$

for $0 < x < \pi/2$. In [38], the inequality (4) was proved, and the following inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x, \qquad 0 < x < \frac{\pi}{2} \tag{6}$$

was also established, where the constants $(2/\pi)^4$ and $8/45$ are the best possible.

The Wilker-type inequalities (4) and (6) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs, various generalizations and improvements (cf. [4, 8, 9, 12, 13, 23–25, 27, 30–32, 34, 40, 41, 44, 45, 52–56] and the references cited therein).

A related inequality that is of interest to us is Huygens' inequality [14], which asserts that

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \qquad 0 < |x| < \frac{\pi}{2}. \tag{7}$$

Remark 1.1. *The first inequality in* (1) *can be re-written as*

$$\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x} > 1 \quad \left(\text{or } \sqrt[3]{\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x}} > 1\right) \quad \text{for all} \quad 0 < |x| < \frac{\pi}{2}. \tag{8}$$

Baricz *and* Sándor [4] *have pointed out that inequality* (8) *implies* (4) *and* (7), *by using the arithmetic-geometric mean inequality.*

Wu and Srivastava [44, Lemma 3] established Wilker-type inequality as follows:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \qquad 0 < |x| < \frac{\pi}{2}. \tag{9}$$

Neuman and Sándor [32, Theorem 2.3] proved that for $0 < |x| < \pi/2$,

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} < \frac{1}{2}\left(\frac{x}{\sin x} + \cos x\right). \tag{10}$$

By multiplying both sides of inequality (10) by $x/\sin x$, we obtain that for $0 < |x| < \pi/2$,

$$\frac{1}{2}\left(\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right) > \frac{2(x/\sin x) + x/\tan x}{3} > 1. \tag{11}$$

Chen and Sándor [12] proved the following inequality chain:

$$\begin{aligned} &\frac{(\sin x/x)^2 + \tan x/x}{2} > \left(\tfrac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right) > \frac{2\,(\sin x/x) + \tan x/x}{3} > \\ &> \left(\frac{\sin x}{x}\right)^{2/3}\left(\frac{\tan x}{x}\right)^{1/3} > \frac{1}{2}\left(\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right) > \frac{2(x/\sin x) + x/\tan x}{3} > 1 \end{aligned} \tag{12}$$

for $0 < |x| < \pi/2$.

In analogy with (6), Chen and Cheung [9] established sharp Wilker and Huygens-type inequalities. For example, these authors proved that for $0 < x < \pi/2$,

$$2 + \frac{8}{45}x^4 + \frac{16}{315}x^5\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^4 + \left(\frac{2}{\pi}\right)^6 x^5\tan x, \tag{13}$$

where the constants $\frac{16}{315}$ and $(2/\pi)^6$ are best possible,

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \frac{2}{45}x^3\tan x, \tag{14}$$

where the constant $\frac{2}{45}$ is best possible, and

$$3 + \frac{3}{20}x^3\tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3\tan x, \tag{15}$$

where the constants $3/20$ and $(2/\pi)^4$ are best possible.

In view of (13), (14) and (15), Chen and Cheung [9] posed three conjectures. These conjectures have been proved by Chen and Paris [10, 11].

Mortici [24, Theorem 1] presented in 2014 the following double inequality:

$$\begin{aligned} 2 + \left(\frac{8}{45} - \frac{8}{945}x^2\right)x^3\tan x &< \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < \\ &< 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{16}{14175}x^4\right)x^3\tan x, \qquad 0 < x < 1. \end{aligned} \tag{16}$$

Nenezić et al. [25, Theorem 2.1] proved in 2016 that for $0 < x < \pi/2$,

$$\begin{aligned} 2 + \left(\frac{8}{45} - \frac{8}{945}x^2\right)x^3\tan x &< \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < \\ &< 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{241920 - 2688\pi^4 + 32\pi^6}{945\pi^8}x^4\right)x^3\tan x. \end{aligned} \tag{17}$$

By using power series expansions for $\sin x$ and $\cot x$, we find that

$$\begin{aligned}
\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2}{x^3\tan x} &= \frac{\sin 2x}{2x^5}+\frac{1}{x^4}-\frac{2}{x^3}\cot x\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n+1)!}x^{2n-4}+\frac{1}{x^4}-\frac{2}{x^3}\left(\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}\right)\\
&= \sum_{n=2}^{\infty}\frac{4^n\Big((-1)^n+2(2n+1)|B_{2n}|\Big)}{(2n+1)!}x^{2n-4}\\
&= \frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4+\frac{8}{467775}x^6+\frac{3184}{638512875}x^8\\
&\quad+\frac{272}{638512875}x^{10}+\frac{7264}{162820783125}x^{12}+\cdots,
\end{aligned}\tag{18}$$

where B_n ($n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!},\qquad |z|<2\pi.$$

The formula (18) led us to claim that the upper bound in (16) should be the lower bound. Chen and Paris [11] proved that for $0<x<\pi/2$,

$$\begin{aligned}
&2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{16}{14175}x^4\right)x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<\\
&\quad<2+\left(\frac{8}{45}-\frac{8}{945}x^2+\frac{241920-2688\pi^4+32\pi^6}{945\pi^8}x^4\right)x^3\tan x,
\end{aligned}\tag{19}$$

where the constants $\frac{16}{14175}$ and $\frac{241920-2688\pi^4+32\pi^6}{945\pi^8}$ are the best possible.

In Section 4, we improve and generalize the double inequalities (19) and (15).

2. Taylor's approximations

Let us consider a real function $f:(a,b)\longrightarrow\mathbb{R}$ in case when exist finite limits

$$f^{(k)}(a+)=\lim_{x\to a+}f^{(k)}(x)\ \ (\text{for } k=0,1,\ldots,n)\quad\text{and}\quad f(b-)=\lim_{x\to b-}f(x).\tag{20}$$

Then we consider first Taylor's polynomial

$$T_n^{f,a+}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a+)}{k!}(x-a)^k,\ n\in\mathbb{N}_0,\tag{21}$$

and the remainder

$$R_n^{f,a+}(x)=f(x)-T_{n-1}^{f,a+}(x).\tag{22}$$

Also, we consider the second Taylor's polynomial

$$\mathbb{T}_n^{f;a+,b-}(x)=\begin{cases}T_{n-1}^{f,a+}(x)+\dfrac{1}{(b-a)^n}R_n^{f,a+}(b-)(x-a)^n, & n\ge 1\\ f(b-), & n=0.\end{cases}\tag{23}$$

The first TAYLOR's polynomial and the second TAYLOR's polynomial we are called the *first* TAYLOR's *approximation for the function f in the right neighborhood of a*, and the *second* TAYLOR's *approximation for the function f in the right neighborhood of a*, respectively.

The next Theorem on double-sided TAYLOR's approximations from [43] is applied in the papers [42], [45], [46], [47] and considered in the papers [16], [18], [19], [20], [21], [26], [35] and [36].

Theorem 2.1. ([43], Theorem 2) *Suppose that $f(x)$ is a real function on (a,b), and that n is a positive integer such that $f^{(k)}(a+)$, for $k \in \{0,1,2,\ldots,n\}$, exist.*

Supposing that $f^{(n)}(x)$ is increasing on (a,b), then for all $x \in (a,b)$ the following inequality also holds:

$$T_n^{f,a+}(x) < f(x) < \mathbb{T}_n^{f;a+,b-}(x). \tag{24}$$

Furthermore, if $f^{(n)}(x)$ is decreasing on (a,b), then the reversed inequality of (24) holds.

The condition for the application of this theorem refers to the n-th derivative of the function and it is also close to the recent papers which refer to the n-th derivative [57], [58], [59] and [60].

Remark 2.2. *In the previous inequality*

$$T_{n-1}^{f,a+}(x) + \frac{f^{(n)}(a+)}{n!}(x-a)^n < f(x) < T_{n-1}^{f,a+}(x) + \frac{1}{(b-a)^n}\left(f(b-) - T_{n-1}^{f,a+}(b-)\right)(x-a)^n, \tag{25}$$

the coefficients

$$\frac{f^{(n)}(a+)}{n!} \quad \text{and} \quad \frac{1}{(b-a)^n}\Big(f(b-) - T_{n-1}^{f,a+}(b-)\Big) \tag{26}$$

are the best possible constants.

In this paper we use

Theorem 2.3. ([20], Theorem 4) *Consider the real analytic functions $f : (a,b) \longrightarrow \mathbb{R}$:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, \tag{27}$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for all $k \in \mathbb{N}_0$. Then,

$$T_0^{f,a+}(x) \leq \ldots \leq T_n^{f,a+}(x) \leq T_{n+1}^{f,a+}(x) \leq \ldots \leq f(x) \leq \ldots \leq \mathbb{T}_{m+1}^{f;a+,b-}(x) \leq \mathbb{T}_m^{f;a+,b-}(x) \leq \ldots \leq \mathbb{T}_0^{f;a+,b-}(x), \tag{28}$$

for all $x \in (a,b)$.

Elementary power series expansions. The following elementary power series expansions are useful in our investigation.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty, \tag{29}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty, \tag{30}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1}, \quad |x| < \frac{\pi}{2}, \tag{31}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1}, \quad 0 < |x| < \pi, \tag{32}$$

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!} x^{2n-1}, \quad |x| < \pi, \tag{33}$$

where B_n $(n = 0, 1, 2, \ldots)$ are Bernoulli numbers.

3. Sharp inequalities inspired by (1)

The first inequality in (1) is equivalent to

$$\frac{x}{\tan x} < \left(\frac{\sin x}{x}\right)^2, \qquad 0 < x < \frac{\pi}{2}. \tag{34}$$

Let us consider the following function with power series

$$\begin{aligned} f_1(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} &= \frac{1-\cos 2x}{2x^2} - x \cot x \\ &= \sum_{n=2}^{\infty} \left(\frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!} \right) x^{2n} \\ &= \frac{1}{15}x^4 - \frac{1}{945}x^6 + \frac{1}{2835}x^8 + \frac{8}{467775}x^{10} + \cdots \end{aligned} \tag{35}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!}, \qquad n = 2, 3, 4, \ldots. \tag{36}$$

We use the next auxiliary statement.

Lemma 3.1. *The following are true*:

$$a_2 = \frac{1}{15} > 0, \; a_3 = -\frac{1}{945} < 0 \tag{37}$$

and

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!} > 0, \tag{38}$$

for integers $n \geq 4$.

Proof. By direct computation we obtained:

$$\begin{aligned} a_2 &= \frac{1}{15} = \lim_{x\to 0} \frac{f_1(x)}{x^4} > 0, \\ a_3 &= -\frac{1}{945} = \lim_{x\to 0} \frac{f_1(x) - \frac{1}{15}x^4}{x^6} < 0. \end{aligned} \tag{39}$$

Next, we consider the following inequalities [1, p. 805]

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \left(\frac{1}{1-2^{1-2n}}\right), \qquad n \geq 1. \tag{40}$$

Using the first inequality in (40), we obtain that for $n \geq 4$,

$$\frac{2^{2n}|B_{2n}|}{(2n)!} - \frac{2^{2n+1}}{(2n+2)!} > \frac{2^{2n}}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}} - \frac{2^{2n+1}}{(2n+2)!} = \frac{2^{2n+1}\Big((2n+2)! - (2\pi)^{2n}\Big)}{(2\pi)^{2n}\cdot(2n+2)!}.$$

By induction on n, it is easy to see that

$$(2n+2)! > (2\pi)^{2n}, \qquad n \geq 4.$$

Hence, we have

$$a_n = \frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!} > 0, \qquad n \geq 4.$$

□

Let's specify a list of Taylor's approximations

k	$T_k^{f_1,0+}(x)$	$\mathbb{T}_k^{f_1;0+,\pi/2-}(x)$
0	0	$\frac{4}{\pi^2}$
1	0	$\frac{8}{\pi^3}x$
2	0	$\frac{16}{\pi^4}x^2$
3	0	$\frac{32}{\pi^5}x^3$
4	$\frac{1}{15}x^4$	$\frac{64}{\pi^6}x^4$
5	$\frac{1}{15}x^4$	$T_4^{f_1,0+}(x)+\frac{-2\pi^6+1920}{15\pi^7}x^5$
6	$\frac{1}{15}x^4-\frac{1}{945}x^6$	$T_5^{f_1,0+}(x)+\frac{-4\pi^6+384}{15\pi^8}x^6$
7	$\frac{1}{15}x^4-\frac{1}{945}x^6$	$T_6^{f_1,0+}(x)+\frac{2\pi^8-504\pi^6+483840}{945\pi^9}x^7$
8	$\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8$	$T_7^{f_1,0+}(x)+\frac{4\pi^8-1008\pi^6+967680}{945\pi^{10}}x^8$
9	$\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8$	$T_8^{f_1,0+}(x)+\frac{-2\pi^{10}+24\pi^8-6048\pi^6+5806080}{945\pi^{11}}x^9$
10	$\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8+\frac{8}{467775}x^{10}$	$T_9^{f_1,0+}(x)+\frac{-4\pi^{10}+48\pi^8-12096\pi^6+11612160}{2835\pi^{12}}x^{10}$

Based on a method from [3] and [17] we have

Theorem 3.2. *For the function*

$$f_1(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} = \sum_{n=2}^{\infty}\left(\frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!}\right)x^{2n} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$\begin{aligned} &T_0^{f_1,0+}(x) = T_1^{f_1,0+}(x) = T_2^{f_1,0+}(x) = T_3^{f_1,0+}(x) = 0 < \\ &< T_6^{f_1,0+}(x) = T_7^{f_1,0+}(x) < T_8^{f_1,0+}(x) = T_9^{f_1,0+}(x) < \\ &< T_{10}^{f_1,0+}(x) < f_1(x) < T_4^{f_1,0+}(x) = T_5^{f_1,0+}(x) \end{aligned}$$

and

$$\begin{aligned} &f_1(x) < \mathbb{T}_{10}^{f_1;0+,\pi/2-}(x) < \mathbb{T}_9^{f_1;0+,\pi/2-}(x) < \mathbb{T}_8^{f_1;0+,\pi/2-}(x) < \mathbb{T}_7^{f_1;0+,\pi/2-}(x) < \\ &< \mathbb{T}_4^{f_1;0+,\pi/2-}(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x) < \mathbb{T}_6^{f_1;0+,\pi/2-}(x) < \\ &< \mathbb{T}_3^{f_1;0+,\pi/2-}(x) < \mathbb{T}_2^{f_1;0+,\pi/2-}(x) < \mathbb{T}_1^{f_1;0+,\pi/2-}(x) < \mathbb{T}_0^{f_1;0+,\pi/2-}(x), \end{aligned}$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Let us emphasize that some Taylor's approximatins $T_i^{f_1,0+}(x)$ and $\mathbb{T}_j^{f_1;0+,\pi/2-}(x)$ have intersections over interval $(0,\pi/2) = (0, 1.570796326...)$ in exactly one point $c_{i,j} \in (0,\pi/2)$ for $i, j \in \{0, 1, \ldots, 10\}$. All that cases are given by the following two tables:

i, j	$f_i(x) < T_i^{f_1,0+}(x) < \mathbb{T}_j^{f_1;0+,\pi/2-}(x),\ x \in \left(0, c_{i,j}\right)$	$f_i(x) < \mathbb{T}_j^{f_1;0+,\pi/2-}(x) < T_i^{f_1,0+}(x), x \in \left(c_{i,j}, \frac{\pi}{2}\right)$	$c_{i,j}$
0, 4	$f_1(x) < T_0^{f_1,0+}(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{0,4})$	$f_1(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x) < T_0^{f_1,0+}(x), x \in \left(c_{0,4}, \frac{\pi}{2}\right)$	1.570228574...
0, 5	$f_1(x) < T_0^{f_1,0+}(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{0,5})$	$f_1(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x) < T_0^{f_1,0+}(x), x \in \left(c_{0,5}, \frac{\pi}{2}\right)$	1.570228574...
1, 4	$f_1(x) < T_1^{f_1,0+}(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{1,4})$	$f_1(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x) < T_1^{f_1,0+}(x), x \in \left(c_{1,4}, \frac{\pi}{2}\right)$	1.570039369...
1, 5	$f_1(x) < T_1^{f_1,0+}(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{1,5})$	$f_1(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x) < T_1^{f_1,0+}(x), x \in \left(c_{1,5}, \frac{\pi}{2}\right)$	1.570039369...
2, 4	$f_1(x) < T_2^{f_1,0+}(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{2,4})$	$f_1(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x) < T_2^{f_1,0+}(x), x \in \left(c_{2,4}, \frac{\pi}{2}\right)$	1.569661027...
2, 5	$f_1(x) < T_2^{f_1,0+}(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{2,5})$	$f_1(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x) < T_2^{f_1,0+}(x), x \in \left(c_{2,5}, \frac{\pi}{2}\right)$	1.569661027...
3, 4	$f_1(x) < T_3^{f_1,0+}(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{3,4})$	$f_1(x) < \mathbb{T}_4^{f_1;0+,\pi/2-}(x) < T_3^{f_1,0+}(x), x \in \left(c_{3,4}, \frac{\pi}{2}\right)$	1.568526547...
3, 5	$f_1(x) < T_3^{f_1,0+}(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x),\ x \in (0, c_{3,5})$	$f_1(x) < \mathbb{T}_5^{f_1;0+,\pi/2-}(x) < T_3^{f_1,0+}(x), x \in \left(c_{3,5}, \frac{\pi}{2}\right)$	1.568526547...

and

i, j	$\mathbb{T}_j^{f_1;0+,\pi/2-}(x) < T_i^{f_1,0+}(x) < f_1(x),\ x \in \left(0, c_{i,j}\right)$	$T_i^{f_1,0+}(x) < \mathbb{T}_j^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{i,j}, \frac{\pi}{2}\right)$	$c_{i,j}$
4, 6	$\mathbb{T}_6^{f_1;0+,\pi/2-}(x) < T_4^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{4,6})$	$T_4^{f_1,0+}(x) < \mathbb{T}_6^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{4,6}, \frac{\pi}{2}\right)$	0.3017187013...
4, 7	$\mathbb{T}_7^{f_1;0+,\pi/2-}(x) < T_4^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{4,7})$	$T_4^{f_1,0+}(x) < \mathbb{T}_7^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{4,7}, \frac{\pi}{2}\right)$	0.3017187013...
4, 8	$\mathbb{T}_8^{f_1;0+,\pi/2-}(x) < T_4^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{4,8})$	$T_4^{f_1,0+}(x) < \mathbb{T}_8^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{4,8}, \frac{\pi}{2}\right)$	0.3065585396...
4, 9	$\mathbb{T}_9^{f_1;0+,\pi/2-}(x) < T_4^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{4,9})$	$T_4^{f_1,0+}(x) < \mathbb{T}_9^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{4,9}, \frac{\pi}{2}\right)$	0.3065585396...
4, 10	$\mathbb{T}_{10}^{f_1;0+,\pi/2-}(x) < T_4^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{4,10})$	$T_4^{f_1,0+}(x) < \mathbb{T}_{10}^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{4,10}, \frac{\pi}{2}\right)$	0.3065818906...
5, 6	$\mathbb{T}_6^{f_1;0+,\pi/2-}(x) < T_5^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{5,6})$	$T_5^{f_1,0+}(x) < \mathbb{T}_6^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{5,6}, \frac{\pi}{2}\right)$	0.05795414341...
5, 7	$\mathbb{T}_7^{f_1;0+,\pi/2-}(x) < T_5^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{5,7})$	$T_5^{f_1,0+}(x) < \mathbb{T}_7^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{5,7}, \frac{\pi}{2}\right)$	0.05795414341...
5, 8	$\mathbb{T}_8^{f_1;0+,\pi/2-}(x) < T_5^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{5,8})$	$T_5^{f_1,0+}(x) < \mathbb{T}_8^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{5,8}, \frac{\pi}{2}\right)$	0.05801924550...
5, 9	$\mathbb{T}_9^{f_1;0+,\pi/2-}(x) < T_5^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{5,9})$	$T_5^{f_1,0+}(x) < \mathbb{T}_9^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{5,9}, \frac{\pi}{2}\right)$	0.05801924550...
5, 10	$\mathbb{T}_{10}^{f_1;0+,\pi/2-}(x) < T_5^{f_1,0+}(x) < f_1(x),\ x \in (0, c_{5,10})$	$T_5^{f_1,0+}(x) < \mathbb{T}_{10}^{f_1;0+,\pi/2-}(x) < f_1(x),\ x \in \left(c_{5,10}, \frac{\pi}{2}\right)$	0.05801925617...

All other Taylor's approximatins have no intersections.

Based on Theorem 2.3 we have

Theorem 3.3. *For the function*

$$f_1(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} = \sum_{n=2}^{\infty} \left(\frac{2^{2n}|B_{2n}|}{(2n)!} + \frac{(-1)^n 2^{2n+1}}{(2n+2)!}\right) x^{2n} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$T_6^{f_1,0+}(x) \leq \ldots \leq T_n^{f_1,0+}(x) \leq T_{n+1}^{f_1,0+}(x) \leq \ldots \leq f_1(x) \leq \ldots \leq \mathbb{T}_{m+1}^{f_1;0+,\pi/2-}(x) \leq \mathbb{T}_m^{f_1;0+,\pi/2-}(x) \leq \ldots \leq \mathbb{T}_7^{f_1;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$ *and* $n \geq 6,\ m \geq 7$.

Let us consider an empty sum as zero (elsewhere throughout this paper).

We propose the following conjecture.

Conjecture 3.4. *For $0 < x < \pi/2$ and $n \geq 2$, we have*

$$\sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^j2^{2j+1}}{(2j+2)!}\right)x^{2j}+a_nx^{2n-1}\sin x<\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}<\sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^j2^{2j+1}}{(2j+2)!}\right)x^{2j}+\Theta_nx^{2n-1}\sin x, \quad (41)$$

with the best possible constants

$$a_n=\frac{2^{2n}|B_{2n}|}{(2n)!}+\frac{(-1)^n2^{2n+1}}{(2n+2)!} \quad (42)$$

and

$$\Theta_n=\left(\frac{2}{\pi}\right)^{2n+1}-\sum_{j=2}^{n-1}a_j\left(\frac{2}{\pi}\right)^{2n-2j-1}. \quad (43)$$

Remark 3.5. *In fact, we can prove the first inequality in* (41). *We then obtain for $0 < x < \pi/2$ and $n \geq 2$,*

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x} &> \sum_{j=2}^{n}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^j2^{2j+1}}{(2j+2)!}\right)x^{2j}\\ &= \sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^j2^{2j+1}}{(2j+2)!}\right)x^{2j}+a_nx^{2n}\\ &> \sum_{j=2}^{n-1}\left(\frac{2^{2j}|B_{2j}|}{(2j)!}+\frac{(-1)^j2^{2j+1}}{(2j+2)!}\right)x^{2j}+a_nx^{2n-1}\sin x.\end{aligned} \quad (44)$$

Hence, the first inequality in (41) *holds for all $n \geq 2$.*

Let us remark that the function $g_1(x) = f_1(x) - \frac{1}{15}x^4 + \frac{1}{945}x^6$ has power series with positive coefficients. Then, based on the previous Theorem we have:

Statement 3.6. *For $0 < x < \pi/2$ and $n \geq 2$,*

$$\sum_{j=2}^{n-1}a_jx^{2j}+\alpha_nx^{2n}<\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}<\sum_{j=2}^{n-1}a_jx^{2j}+A_nx^{2n} \quad (45)$$

with the best possible constants

$$\alpha_n=a_n \quad \text{and} \quad A_n=\left(\frac{2}{\pi}\right)^{2n+2}-\sum_{j=2}^{n-1}a_j\left(\frac{2}{\pi}\right)^{2n-2j}. \quad (46)$$

Next, we consider the following function

$$
\begin{aligned}
f_2(x) &= \frac{\left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x}}{x^3 \sin x} \\
&= \frac{1}{x^5}\sin x + \frac{1}{x^2}\left(-\frac{\cos x}{\sin^2 x}\right) \\
&= \frac{1}{x^5}\sin x + \frac{1}{x^2}(\csc x)' \\
&= \sum_{n=2}^{\infty} \frac{2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!} x^{2n-4} \qquad \text{(see (29), (33))} \\
&= \frac{1}{15} + \frac{19}{1890}x^2 + \frac{167}{113400}x^4 + \frac{479}{2494800}x^6 + \dots \qquad (47)
\end{aligned}
$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$
b_n = \frac{2(4n^2-1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!}, \qquad n = 2, 3, 4, \dots. \qquad (48)
$$

We use the next auxiliary statement.

Lemma 3.7. *The following are true*:

$$
b_n = \frac{2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!} > 0, \qquad (49)
$$

for integers $n \geq 2$.

Proof. Using the first inequality in (40), we obtain that for $n \geq 2$,

$$
2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| > \frac{4(2n-1)(2^{2n-1}-1)\cdot(2n+1)!}{(2\pi)^{2n}} > 1 \qquad (50)
$$

(The second inequality in (50) can be shown by induction on n, we omit it), which implies

$$
b_n > 0, \qquad n \geq 2.
$$

□

Let's specify a list of Taylor's approximations for the function $f_2(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_2,0+}(x)$	$\mathbb{T}_k^{f_2;0+,\pi/2-}(x)$
0	$\frac{1}{15}$	$\frac{32}{\pi^5}$
1	$\frac{1}{15}$	$\frac{1}{15} + \frac{-2\pi^5+960}{15\pi^6}x$
2	$\frac{1}{15} + \frac{19}{1890}x^2$	$\frac{1}{15} + \frac{-4\pi^5+1920}{15\pi^7}x^2$
3	$\frac{1}{15} + \frac{19}{1890}x^2$	$\frac{1}{15} + \frac{19}{1890}x^2 + \frac{-19\pi^7-504\pi^5+241920}{945\pi^8}x^3$
4	$\frac{1}{15} + \frac{19}{1890}x^2 + \frac{167}{113400}x^4$	$\frac{1}{15} + \frac{19}{1890}x^2 + \frac{-38\pi^7-1008\pi^5+483840}{945\pi^9}x^4$

Based on Theorem 2.3 we have

Theorem 3.8. *For the function*

$$f_2(x) = \frac{\left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x}}{x^3 \sin x}$$
$$= \sum_{n=2}^{\infty} \left(\frac{2(2n-1)(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!} \right) x^{2n-4} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$T_0^{f_2,0+}(x) \leq \ldots \leq T_k^{f_2,0+}(x) \leq T_{k+1}^{f_2,0+}(x) \leq \ldots \leq f_2(x) \leq \ldots \leq \mathbb{T}_{k+1}^{f_2;0+,\pi/2-}(x) \leq \mathbb{T}_k^{f_2;0+,\pi/2-}(x) \leq \ldots \leq \mathbb{T}_0^{f_2;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Then, based on the previous Theorem we have

Statement 3.9. *For* $0 < x < \pi/2$ *and* $n \geq 0$,

$$\left(\sum_{j=0}^{n-1} b_{j+2} x^{2j} + \beta_n x^{2n} \right) x^3 \sin x < \left(\frac{\sin x}{x}\right)^2 - \frac{x}{\tan x} < \left(\sum_{j=0}^{n-1} b_{j+2} x^{2j} + B_n x^{2n} \right) x^3 \sin x \tag{51}$$

with the best possible constants

$$\beta_n = b_{n+2} \quad \textit{and} \quad B_n = \left(\frac{2}{\pi}\right)^{2n+5} - \sum_{j=2}^{n-1} b_{j+2} \left(\frac{2}{\pi}\right)^{2n-2j}. \tag{52}$$

Finaly, we consider the following function

$$f_3(x) = \frac{\frac{2+\cos x}{3} - \frac{\sin x}{x}}{x^3 \sin x}$$
$$= \frac{2}{3x^3} \csc x + \frac{1}{3x^3} \cot x \frac{1}{x^4}$$
$$= \sum_{n=2}^{\infty} \left(\frac{2^{2n} - 4}{3 \cdot (2n)!} |B_{2n}| \right) x^{2n-4} \qquad \text{(see (30), (33))}$$
$$= \frac{1}{180} + \frac{1}{1512} x^2 + \frac{1}{14400} x^4 + \frac{17}{2395008} x^6 + \ldots \tag{53}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$c_n = \frac{2^{2n} - 4}{3 \cdot (2n)!} |B_{2n}|, \qquad n = 0, 1, 2, \ldots. \tag{54}$$

The next auxiliary statement is obvious.

Lemma 3.10. *The following are true*:

$$c_n = \frac{2^{2n} - 4}{3 \cdot (2n)!} |B_{2n}| > 0, \tag{55}$$

for integers $n \geq 0$.

Let's specify a list of Taylor's approximations for the function $f_3(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_3,0+}(x)$	$\mathbb{T}_k^{f_3;0+,\pi/2-}(x)$
0	$\frac{1}{180}$	$\frac{-48+16\pi}{3\pi^4}$
1	$\frac{1}{180}$	$\frac{1}{180}+\frac{-\pi^4+960\pi-2880}{90\pi^5}x$
2	$\frac{1}{180}+\frac{1}{1512}x^2$	$\frac{1}{180}+\frac{-\pi^4+960\pi-2880}{45\pi^6}x^2$
3	$\frac{1}{180}+\frac{1}{1512}x^2$	$\frac{1}{180}+\frac{1}{1512}x^2+\frac{-5\pi^6-168\pi^4+161280\pi-483840}{3870\pi^7}x^3$
4	$\frac{1}{180}+\frac{1}{1512}x^2+\frac{1}{14400}x^4$	$\frac{1}{180}+\frac{1}{1512}x^2+\frac{-5\pi^6-168\pi^4+161280\pi-483840}{1890\pi^8}x^4$

Based on Theorem 2.3 we have

Theorem 3.11. *For the function*

$$f_3(x) = \frac{\frac{2+\cos x}{3}-\frac{\sin x}{x}}{x^3\sin x} = \sum_{n=2}^{\infty}\left(\frac{2^{2n}-4}{3\cdot(2n)!}|B_{2n}|\right)x^{2n-4} : \left(0,\frac{\pi}{2}\right)\longrightarrow R$$

we have

$$T_0^{f_3,0+}(x)\le\ldots\le T_k^{f_3,0+}(x)\le T_{k+1}^{f_3,0+}(x)\le\ldots\le f_3(x)\le\ldots\le \mathbb{T}_{k+1}^{f_3;0+,\pi/2-}(x)\le\mathbb{T}_k^{f_3;0+,\pi/2-}(x)\le\ldots\le\mathbb{T}_0^{f_3;0+,\pi/2-}(x),$$

for all $x\in\left(0,\frac{\pi}{2}\right)$.

Then, based on the previous Theorem we have

Statement 3.12. *For* $0<x<\pi/2$ *and* $n\ge 0$,

$$\left(\sum_{j=0}^{n-1}c_{j+2}x^{2j}+\gamma_n x^{2n}\right)x^3\sin x<\frac{2+\cos x}{3}-\frac{\sin x}{x}<\left(\sum_{j=0}^{n-1}c_{j+2}x^{2j}+C_n x^{2n}\right)x^3\sin x \tag{56}$$

with the best possible constants

$$\gamma_n=c_n \quad \text{and} \quad C_n=\frac{\pi-3}{3}\left(\frac{2}{\pi}\right)^{2n+4}-\sum_{j=2}^{n-1}c_{j+2}\left(\frac{2}{\pi}\right)^{2n-2j}. \tag{57}$$

4. Sharp Wilker and Huygens inequalities

In purpose to generalize of the double inequality (19) we consider the following function

$$f_4(x)=\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2}{x^3\tan x}-\frac{8}{45}+\frac{8}{945}x^2 = \frac{1}{x^4}+\frac{\sin 2x}{2x^5}-\frac{2\cot x}{x^3}-\frac{8}{45}+\frac{8}{945}x^2 = \frac{f(x)}{x^3}:\left(0,\frac{\pi}{2}\right)\longrightarrow R, \tag{58}$$

where the function

$$f(x) = \frac{1}{x} + \frac{\sin 2x}{2x^2} - 2\cot x - \frac{8}{45}x^3 + \frac{8}{945}x^5 : \left(0, \frac{\pi}{2}\right) \longrightarrow R \tag{59}$$

is considered in the paper [35]. Therefore

$$\begin{aligned} f_4(x) &= \sum_{n=3}^{\infty} \frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!} x^{2n-2} \\ &= \frac{16}{14175}x^4 + \frac{8}{467775}x^6 + \frac{3184}{638512875}x^8 + \frac{272}{638512875}x^{10} + \ldots \end{aligned} \tag{60}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$d_n = \frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!}, \qquad n = 3, 4, 5, \ldots. \tag{61}$$

The next auxiliary statement is obvious.

Lemma 4.1. *The following are true*:

$$d_n = \frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!} > 0, \tag{62}$$

for integers $n \geq 3$.

Let's specify a list of Taylor's approximations for the function $f_4(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_4,0+}(x)$	$\mathbb{T}_k^{f_4;0+,\pi/2-}(x)$
0	0	$\frac{2\pi^6 - 168\pi^4 + 15120}{945\pi^4}$
1	0	$\frac{4\pi^6 - 336\pi^4 + 30240}{945\pi^5}x$
2	0	$\frac{8\pi^6 - 672\pi^4 + 60480}{945\pi^6}x^2$
3	0	$\frac{16\pi^6 - 1344\pi^4 + 120960}{945\pi^7}x^3$
4	$\frac{16}{14175}x^4$	$\frac{32\pi^6 - 2688\pi^4 + 241920}{945\pi^8}x^4$

Based on Theorem 2.3 we have

Theorem 4.2. *For the function*

$$\begin{aligned} f_4(x) &= \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x} - \frac{8}{45} + \frac{8}{945}x^2 \\ &= \sum_{n=3}^{\infty} \left(\frac{2^{2n+2}\left((4n+6)|B_{2n+2}| + (-1)^{n+1}\right)}{(2n+3)!} \right) x^{2n-2} : \left(0, \frac{\pi}{2}\right) \longrightarrow R \end{aligned}$$

we have

$$T_0^{f_4,0+}(x) \leq \ldots \leq T_k^{f_4,0+}(x) \leq T_{k+1}^{f_4,0+}(x) \leq \ldots \leq f_4(x) \leq \ldots \leq \mathbb{T}_{k+1}^{f_4;0+,\pi/2-}(x) \leq \mathbb{T}_k^{f_4;0+,\pi/2-}(x) \leq \ldots \leq \mathbb{T}_0^{f_4;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Let us remark that the function $g_4(x) = f_4(x) - \frac{8}{45} + \frac{8}{945}x^2$ has power series with positive coefficients. Then, based on the previous Theorem we have:

Statement 4.3. *For* $0 < x < \pi/2$ *and* $n \geq 4$,

$$2 + \left(\sum_{j=2}^{n-1} d_{j+1}x^{2j} + \delta_n x^{2n}\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\sum_{j=2}^{n-1} d_{j+1}x^{2j} + D_n x^{2n}\right)x^3 \tan x \tag{63}$$

with the best possible constants

$$\delta_n = d_n \quad \textit{and} \quad D_n = \frac{2\pi^6 - 168\pi^4 + 15120}{945\pi^4}\left(\frac{2}{\pi}\right)^{2n} - \sum_{j=2}^{n-1} d_{j+1}\left(\frac{2}{\pi}\right)^{2n-2j}. \tag{64}$$

Finaly, we consider the following function

$$\begin{aligned} f_5(x) &= \frac{2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} - 3}{x^3 \tan x} \\ &= \frac{2}{x^4}\cos x + \frac{1}{x^4} - \frac{3}{x^3}\cot x \\ &= \sum_{n=2}^{\infty} 2\frac{(-1)^n + 3\cdot 2^{2n+3}|B_{2n+4}|}{(2n+4)!}x^{2n} \qquad \text{(see (30), (32))} \\ &= \frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4 + \frac{47}{739200}x^6 + \ldots \end{aligned} \tag{65}$$

over interval $\left(0, \frac{\pi}{2}\right)$. Let us denote

$$e_n = 2\frac{3\cdot 2^{2n+3}|B_{2n+4}| + (-1)^n}{(2n+4)!} \qquad n = 0, 1, 2, \ldots. \tag{66}$$

The next auxiliary statement is obvious.

Lemma 4.4. *The following are true*:

$$e_n = 2\frac{3\cdot 2^{2n+3}|B_{2n+4}| + (-1)^n}{(2n+4)!} > 0, \qquad n = 0, 1, 2, \ldots, \tag{67}$$

for integers $n \geq 0$.

Let's specify a list of Taylor's approximations for the function $f_3(x)$ over interval $(0, \pi/2)$:

k	$T_k^{f_5,0+}(x)$	$\mathbb{T}_k^{f_5;0+,\pi/2-}(x)$
0	$\frac{3}{20}$	$\frac{16}{\pi^4}$
1	$\frac{3}{20}$	$\frac{3}{20} + \frac{-3\pi^4 + 320}{10\pi^5}x$
2	$\frac{3}{20} + \frac{1}{280}x^2$	$\frac{3}{20} + \frac{-3\pi^4 + 320}{5\pi^6}x^2$
3	$\frac{3}{20} + \frac{1}{280}x^2$	$\frac{3}{20} + \frac{1}{280}x^2 + \frac{-\pi^6 - 168\pi^4 + 17920}{140\pi^7}x^3$
4	$\frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4$	$\frac{3}{20} + \frac{1}{280}x^2 + \frac{-\pi^6 - 168\pi^4 + 17920}{70\pi^8}x^4$

Based on Theorem 2.3 we have

Theorem 4.5. *For the function*

$$f_5(x) = \frac{2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} - 3}{x^3 \tan x} = \sum_{n=2}^{\infty} 2\frac{(-1)^n + 3\cdot 2^{2n+3}|B_{2n+4}|}{(2n+4)!}\,x^{2n} : \left(0, \frac{\pi}{2}\right) \longrightarrow R$$

we have

$$T_0^{f_5,0+}(x) \le \ldots \le T_k^{f_5,0+}(x) \le T_{k+1}^{f_5,0+}(x) \le \ldots \le f_5(x) \le \ldots \le \mathbb{T}_{k+1}^{f_5;0+,\pi/2-}(x) \le \mathbb{T}_k^{f_5;0+,\pi/2-}(x) \le \ldots \le \mathbb{T}_0^{f_5;0+,\pi/2-}(x),$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Then, based on the previous Theorem we have

Statement 4.6. *For* $0 < x < \pi/2$ *and* $n \ge 0$,

$$3 + \left(\sum_{j=2}^{n-1} e_j x^{2j} + \eta_n x^{2n}\right) x^3 \tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\sum_{j=2}^{n-1} e_j x^{2j} + E_n x^{2n}\right) x^3 \tan x \tag{68}$$

with the best possible constants

$$\eta_n = e_n \quad and \quad E_n = \left(\frac{2}{\pi}\right)^{2n+4} - \sum_{j=2}^{n-1} e_j \left(\frac{2}{\pi}\right)^{2n-2j}. \tag{69}$$

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