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Approximating solutions of general class of variational inclusions involving generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings

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Abstract. The present research is an attempt to define a class of generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings as well as it is a study of its associated class of proximal-point mappings. The generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings is the sum of two symmetric accretive mappings and an extension of the generalized $\alpha\beta$-$H(.,.)$-accretive mapping [28]. Further the research is a discussion on graph convergence of generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings and its application includes a set-valued variational-like inclusion problem (SVLIP, in short) in semi inner product spaces. Furthermore an iterative algorithm is proposed, and an attempt is made to discuss the convergence analysis of the sequences generated from the proposed iterative algorithm. An example is constructed that demonstrate few graphics for the convergence of proximal-point mapping. Our results can be viewed as a refinement and generalization of some known results in the literature.

1. Introduction and Preliminaries

Last few years have witnessed that the researchers have focussed extensively on variational inclusions that are the generalizations of variational inequalities. Investigation of variational inclusions leads to develop an effective and feasible iterative algorithm. A number of iterative algorithms have been proposed and discussed to find out the solutions for variational inclusions. The proximal-point method is one of the most interesting and important techniques and is widely used by numerous authors to solve the variational inclusions, we refer to see the related references [3, 8–16, 18, 20, 27, 28, 31–33, 36–38, 40, 42, 43, 45–48].

Recently, Bhat and Zahoor [9] proved the existence of solutions for the system of generalized variational-like inclusions and Sahu et al. [40] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces. Their results are more general than the results obtained by Sahu et al. in [41]. In addition, they designed an iterative algorithm to approximate the solutions for the general classes of variational inclusions and their system associated with A-monotone, H-monotone and (H, φ)-η-monotone operators through the generalized resolvent operator methods. It is noted that they investigated the existence and the convergence criteria by releasing the monotonicity on

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the set-valued maps under consideration.

Most recently, Luo and Huang [34], investigated (H, φ)-η-monotone mappings in Banach spaces which produce a unifying structure for maximal monotone operators, maximal η-monotone operators, m-η-accretive operators, H-monotone operators and H, η-monotone operators. In this sequence, Ahmad et. al introduced and studied $H(.,.)$-cocoercive operators [1], $H(.,.)$-co-accretive mappings [2], $H(.,.)$-φ-η-accretive mappings [3] and $H(.,.)$-η-cocoercive operators [4, 5]. They used the proximal-point mappings (resolvent operators) and graph convergence approaches inline with above discussed mappings to established results on convergence of proposed iterative algorithms and to find out the solutions for some classes of variational inclusions. For more applications, see the references [9, 10, 13–16, 19, 20, 23–25, 27, 28, 31, 33, 34, 38, 40, 42, 43, 46].

The present study is a further insight in the ongoing works. In this paper, we have consideblue a generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping defined on a product set which is the sum of two symmetric accretive mappings. This is the generalization of generalized $\alpha\beta$-$H(.,.)$-accretive mappings [28], which is done with the idea of C_n-monotone mappings studied and analyzed by Nazemi [19]. Next, we have discussed some properties of the proximal point mapping and study the graph convergence of generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping.

As application, we have studied a set-valued variational-like inclusion problem in the semi inner product spaces and have solved it by using the proximal-point mapping associated with a generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping. In addition, we have constructed an iterative algorithm and performed the convergence of the sequences generated by the proposed algorithm. An example is constructed and shown with some graphics for the convergence of the proximal-point mapping. For detailed study, see the related references [6, 28, 45, 47, 48].

The following definitions and results are requiblue in the subsequent sections.

Definition 1.1. *"Let $\mathcal{B}$ be a vector space over the field $\mathcal{K}$ of real or complex numbers. A functional* $[.,.] : \mathcal{B}\times\mathcal{B} \to \mathcal{K}$ *is called a semi inner product if:*

(i) $[\tilde{u} + \tilde{w}, z] = [\tilde{u}, z] + [\tilde{w}, z],\ \forall \tilde{u}, \tilde{w}, z \in \mathcal{B}$;

(ii) $[\alpha\tilde{u}, z] = \alpha[\tilde{u}, z],\ \forall \alpha \in K,\ \tilde{u}, z \in \mathcal{B}$;

(iii) $[\tilde{u}, z] > 0,\ for\ \tilde{u} \neq 0$;

(iv) $|[\tilde{u}, z]|^2 \leq [\tilde{u}, \tilde{u}][z, z],\ \forall \tilde{u}, z \in \mathcal{B}$.
The pair $(\mathcal{B}, [.,.])$ *is called a semi-inner product space", [35, 40].*

"We observe that $\|\tilde{u}\| = [\tilde{u}, \tilde{u}]^{1/2}$ *is a norm on* $\mathcal{B}$*, thus each semi-inner product space is a normed linear space. There are infinitely many ways to obtain semi-inner product in a normed linear space. Giles [17] proved that the semi-inner product can be defined uniquely if assumed space* $\mathcal{B}$ *is a uniformly convex smooth Banach space. For a detailed study and fundamental results on semi-inner product spaces, see [17, 30, 35]," [9].*

Remark 1.2. *"This unique semi-inner product has the following nice properties:*

(i) $[\tilde{u}, z] = 0$ *iff* z *is orthogonal to* $\tilde{u}$*, that is iff* $\|z\| \leq \|z + \alpha\tilde{u}\|$*, for all scalars* α*;*

(ii) Generalized Riesz representation theorem: If f *is a continuous linear functional on* $\mathcal{B}$ *then there is a unique vector* $z \in \mathcal{B}$ *such that* $f(\tilde{u}) = [\tilde{u}, z]$*, for all* $\tilde{u} \in \mathcal{B}$*;*

(iii) The semi-inner product is continuous, that is for each $\tilde{u}, z \in \mathcal{B}$*, we have* $Re[z, \tilde{u} + \alpha z] \to Re[z, \tilde{u}]$ *as* $\alpha \to 0$*", [9].*

Since the sequence space $l^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces, we can define a semi-inner product on these spaces, uniquely.

Example 1.3. *"The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[\tilde{u}, z] = \frac{1}{\|z\|_p^{p-2}} \sum_j \tilde{u} z_j |z_j|^{p-2}, \ \tilde{u}, z \in l^p\text{''}, \ [40].$$

Example 1.4. *"The real Banach space $L^p(\mathcal{B}, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[g, h] = \frac{1}{\|h\|_p^{p-2}} \int_Y g(u)|h(u)|^{p-1} sgn(h(u)) d\mu, \ g, h \in L^p\text{''}, \ [17, \ 40].$$

Definition 1.5. *"Let $\mathcal{B}$ be a real Banach space, then*

(i) modulus of smoothness of $\mathcal{B}$ is a function $\rho_{\mathcal{B}} : [0, \infty) \to [0, \infty)$ is defined as

$$\rho_{\mathcal{B}}(\gamma) \ = \ \sup\left\{\frac{\|\tilde{v} + \tilde{w}\| + \|\tilde{v} - \tilde{w}\|}{2} - 1 : \ \|\tilde{v}\| = 1, \ \|\tilde{w}\| = \gamma, \gamma > 0\right\};$$

(ii) $\mathcal{B}$ be uniformly smooth if $\lim_{\gamma \to 0} \frac{\rho_{\mathcal{B}}(\gamma)}{\gamma} \ = \ 0$;

(iii) $\mathcal{B}$ be q-uniformly smooth, if there exists $c > 0$ such that $\rho_{\mathcal{B}}(\gamma) \leq c\gamma^q, q > 1$;

(iv) $\mathcal{B}$ be 2-uniformly smooth if there exists $c > 0$ such that $\rho_{\mathcal{B}}(\gamma) \leq c\gamma^2$", [40, 44].

Lemma 1.6. *"Let $p > 1$ be a real number and $\mathcal{B}$ be a smooth Banach space. Then the following statements are equivalent:*

(i) $\mathcal{B}$ is 2-uniformly smooth;

(ii) there is a constant $c > 0$ such that for every $\tilde{v}, \tilde{w} \in \mathcal{B}$, the following inequality holds

$$\|\tilde{v} + \tilde{w}\|^2 \leq \|\tilde{v}\|^2 + 2\langle \tilde{w}, f_{\tilde{v}} \rangle + c\|\tilde{w}\|^2, \tag{1}$$

where $f_{\tilde{v}} \in J(\tilde{v})$ and $J(\tilde{v}) = \{\tilde{v}^* \in \mathcal{B}^* : \langle \tilde{v}, \tilde{v}^* \rangle = \|\tilde{v}\|^2$ and $\|\tilde{v}^*\| = \|\tilde{v}\|\}$ is the normalized duality mapping", [40, 44].

"Each normed linear space $\mathcal{B}$ is a semi-inner product space (see [35]). Infact, by Hahn-Banach theorem, for each $\tilde{v} \in \mathcal{B}$, there exists at least one functional $f_{\tilde{v}} \in \mathcal{B}^*$ such that $\langle \tilde{v}, f_{\tilde{v}} \rangle = \|\tilde{v}\|^2$. Given any such mapping $f : \mathcal{B} \to \mathcal{B}^*$, we can verify that $[\tilde{w}, \tilde{v}] = \langle \tilde{w}, f_{\tilde{v}} \rangle$ defines a semi-inner product. Hence we can write the inequality (1) as

$$\|\tilde{v} + \tilde{w}\|^2 \leq \|\tilde{v}\|^2 + 2[\tilde{w}, f_{\tilde{v}}] + c\|\tilde{w}\|^2, \ \forall \tilde{v}, \tilde{w} \in \mathcal{B}. \tag{2}$$

The constant c is known as constant of smoothness of $\mathcal{B}$, is chosen with best possible minimum value", [40].

Example 1.7. *"The functional spaces $L^{\tilde{p}}$ is 2-uniformly smooth for $\tilde{p} \geq 2$ and ia $\tilde{p}$-uniformly smooth for $1 < \tilde{p} < 2$. If $2 \geq \tilde{p} < \infty$, then we have for all $\tilde{v}, \tilde{w} \in L^{\tilde{p}}$*

$$\|\tilde{v} + \tilde{w}\|^2 \leq \|\tilde{v}\|^2 + 2[\tilde{w}, f_{\tilde{v}}] + (\tilde{p} - 1)\|\tilde{w}\|^2, \ \forall \tilde{v}, \tilde{w} \in \mathcal{B}.$$

The constant $\tilde{p} - 1$ is known as constant of smoothness of $\mathcal{B}$", [40].

We consider $\mathcal{B}$ is 2-uniformly smooth Banach space and set $\mathcal{B}^p = \underbrace{\mathcal{B} \times \mathcal{B} \times ... \times \mathcal{B}}_{p \text{ times}}$ in rest of the paper unless otherwise stated. First, we recall the following concepts.

Definition 1.8. *Let $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be single-valued mappings. Then, $\mathcal{T}$ is said to be*
(i) η-accretive if

$$\Big[\mathcal{T}(w^*) - \mathcal{T}(u^*),\ \eta(w^*, u^*)\Big] \ \geq\ 0 \ \ \forall w^*,\ u^* \in \mathcal{B};$$

(ii) $\tilde{r}$-strongly η-accretive if there exists $\tilde{r} > 0$ with

$$\Big[\mathcal{T}(w^*) - \mathcal{T}(u^*),\ \eta(w^*, u^*)\Big] \ \geq\ \tilde{r}\,\|w^* - u^*\|^2 \ \ \forall w^*,\ u^* \in \mathcal{B};$$

(iii) $\tilde{s}$-relaxed η-accretive if there exists $\tilde{s} > 0$ with

$$\Big[\mathcal{T}(w^*) - \mathcal{T}(u^*),\ \eta(w^*, u^*)\Big] \ \geq\ \tilde{s}\,\|w^* - u^*\|^2 \ \ \forall w^*,\ u^* \in \mathcal{B};$$

(iv) $\tilde{t}$-Lipschitz continuous if there exists $\tilde{t} > 0$ with

$$\|\mathcal{T}(w^*) - \mathcal{T}(u^*)\| \ \leq\ \tilde{t}\,\|w^* - u^*\|,\ \ \forall w^*,\ u^* \in \mathcal{B};$$

(v) t-expansive if there exists $t > 0$ with

$$\|\mathcal{T}(w^*) - \mathcal{T}(u^*)\| \ \geq\ t\,\|w^* - u^*\|,\ \ \forall w^*,\ u^* \in \mathcal{B}.$$

Lemma 1.9. *"Let two non-negative real sequences $\{c_n\}$ and $\{e_n\}$, are satisfying $c_{n+1} \ \leq\ lc_n + e_n$ with $e_n \to 0$ and $0 < l < 1$. Then $\lim_{n\to\infty} c_n = 0$.", [32].*

Definition 1.10. *For each $i \in \{1, 2, ..., p\}$, $p \geq 3$, let $H^p : \mathcal{B}^p \to \mathcal{B}$, $\eta : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ and $A_i : \mathcal{B} \to \mathcal{B}$ be single-valued mappings. Then H^p is said to be*
(i) α_i-strongly η-accretive with A_i if there exists $\alpha_i > 0$ such that

$$\Big[H^p(v_1, ..., v_{i-1}, A_i\tilde{w}, v_{i+1}, ..., v_n) - H^p(v_1, ..., v_{i-1}, A_i\tilde{v}, v_{i+1}, ..., v_n),\ \eta(\tilde{w}, \tilde{v})\Big] \geq \alpha_i\,\|\tilde{w} - \tilde{v}\|^2,$$
$$\forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n \in \mathcal{B};$$

(ii) β_j-relaxed η-accretive with A_j if there exists $\beta_j > 0$ such that

$$\Big[H^p(v_1, ..., v_{j-1}, A_j\tilde{w}, v_{j+1}, ..., v_n) - H^p(v_1, ..., v_{j-1}, A_j\tilde{v}, v_{j+1}, ..., v_n),\ \eta(\tilde{w}, \tilde{v})\Big] \geq -\beta_j\,\|\tilde{w} - \tilde{v}\|^2,$$
$$\forall \tilde{w}, \tilde{v}, v_1, ..., v_{j-1}, v_{j+1}, ..., v_n \in \mathcal{B};$$

(iii) q_i-Lipschitz continuous with A_i if there exists $q_i > 0$ such that

$$\|H^p(v_1, ..., v_{i-1}, A_i\tilde{w}, v_{i+1}, ..., v_n) - H^p(v_1, ..., v_{i-1}, A_i\tilde{v}, v_{i+1}, ..., v_n)\| \leq q_i\,\|\tilde{w} - \tilde{v}\|,$$
$$\forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n \in \mathcal{B};$$

(iv) $\alpha_1\beta_2\alpha_3\beta_4...\alpha_{p-1}\beta_p$-symmetric η-accretive with $A_1, A_2, ..., A_p$ iff for $i \in \{1, 3, ..., p-1\}$, $H^p(..., A_i, ...)$ is α_i-strongly η-accretive with A_i and for $j \in \{2, 4, ..., p\}$, $H^p(..., A_j, ...)$ is β_j-relaxed η-accretive with A_j, where p is even, satisfying

$$\sum_{j=even} \beta_j \leq \sum_{i=odd} \alpha_i, \text{ and } \sum_{j=even} \beta_j = \sum_{i=odd} \alpha_i \text{ iff } \tilde{w} = \tilde{v};$$

(v) $\alpha_1\beta_2\alpha_3\beta_4, ...\beta_{p-1}, \alpha_p$-symmetric η-accretive with $A_1, A_2, ..., A_p$ iff for $i \in \{1, 3, ..., p\}$, $H^p(..., A_i, ...)$ is α_i-strongly η-accretive with A_i and for $j \in \{2, 4, ..., p-1\}$, $H^p(..., A_j, ...)$ is β_j-relaxed η-accretive A_j where p is odd, satisfying

$$\sum_{j=even} \beta_j \leq \sum_{i=odd} \alpha_i, \text{and } \sum_{j=even} \beta_j = \sum_{i=odd} \alpha_i \text{ iff } \tilde{w} = \tilde{v}.$$

"Let $\mathcal{M} : \mathcal{B} \to 2^{\mathcal{B}}$ be a set-valued mapping, then graph of $\mathcal{M}$ is given by $\mathrm{Gr}(\mathcal{M}) = \{(\tilde{v}, \tilde{w}) : \tilde{w} \in \mathcal{M}(\tilde{v})\}$. The domain of $\mathcal{M}$ is given by

$$\mathrm{Dom}(\mathcal{M}) = \{\tilde{v} \in \mathcal{B} : \exists \tilde{w} \in \mathcal{B} : (\tilde{v}, \tilde{w}) \in \mathrm{Gr}(\mathcal{M})\}.$$

The Range of $\mathcal{M}$ is given by

$$\mathrm{RG}(\mathcal{M}) = \{\tilde{w} \in \mathcal{B} : \exists \tilde{v} \in \mathcal{B} : (\tilde{v}, \tilde{w}) \in \mathrm{Gr}(\mathcal{M}\}).$$

The inverse of $\mathcal{M}$ is given by

$$\mathcal{M}^{-1} = \{(\tilde{w}, \tilde{v}) : (\tilde{v}, \tilde{w}) \in \mathrm{Gr}(\mathcal{M})\}.$$

For any two set-valued mappings $\mathcal{N}$ and $\mathcal{M}$, and any real number β, we define

$$\mathcal{N} + \mathcal{M} = \{(\tilde{v}, \tilde{w} + \tilde{z}) : (\tilde{v}, \tilde{w}) \in \mathrm{Gr}(\mathcal{N}), (\tilde{v}, \tilde{z}) \in \mathrm{Gr}(\mathcal{M})\},$$

$$\beta\mathcal{M} = \{(\tilde{v}, \beta\tilde{w}) : (\tilde{v}, \tilde{w}) \in \mathrm{Gr}(\mathcal{M})\}.$$

For a mapping $A : \mathcal{B} \to \mathcal{B}$ and a set-valued map $\mathcal{M} : \mathcal{B} \to 2^{\mathcal{B}}$, we define $A + \mathcal{M} = \{(\tilde{v}, \tilde{w} + \tilde{z}) : A\tilde{v} = \tilde{w}, (\tilde{v}, \tilde{z}) \in \mathrm{Gr}(\mathcal{M})\}$", [9].

Definition 1.11. *For each $i \in \{1, 2, ..., p\}, p \geq 3$, let $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a set-valued mapping and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}, g_i : \mathcal{B} \to \mathcal{B}$ be single-valued mappings . Then $\mathcal{M}$ is said to be*
(i) $\bar{\mu}_i$-strongly η-accretive with g_i if there exists $\bar{\mu}_i > 0$ such that

$$\Big[\tilde{w}_i - \tilde{v}_i,\ \eta(\tilde{w}, \tilde{v})\Big] \geq \bar{\mu}_i \, \|\tilde{w} - \tilde{v}\|^2, \ \forall \tilde{w}, \tilde{v}, v_1, ..., v_{i-1}, v_{i+1}, ..., v_p \in \mathcal{B},$$
$$\tilde{w}_i \in \mathcal{M}(v_1, ..., v_{i-1}, g_i(\tilde{w}), v_{i+1}, ..., v_n), \tilde{v}_i \in \mathcal{M}(v_1, ..., v_{i-1}, g_i(\tilde{v}), v_{i+1}, ..., v_n);$$

(ii) $\bar{\gamma}_j$-relaxed η-accretive with g_j if there exists $\bar{\gamma}_j > 0$ such that

$$\Big[\tilde{w}_j - \tilde{v}_j,\ \eta(\tilde{w}, \tilde{v})\Big] \geq -\bar{\gamma}_j \, \|\tilde{w} - \tilde{v}\|^2, \ \forall \tilde{w}, \tilde{v}, v_1, ..., v_{j-1}, v_{j+1}, ..., v_p \in \mathcal{B},$$
$$\tilde{w}_j \in \mathcal{M}(v_1, ..., v_{j-1}, g_j(\tilde{w}), v_{j+1}, ..., v_n), \tilde{v}_j \in \mathcal{M}(v_1, ..., v_{j-1}, g_j(\tilde{v}), v_{j+1}, ...v_n);$$

(iii) $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$ iff for $i \in \{1, 3, ..., p-1\}$, $\mathcal{M}(..., g_i, ...)$ is $\bar{\mu}_i$-strongly η-accretive with g_i and for $j \in \{2, 4, ..., p\}$, $\mathcal{M}(..., g_j, ...)$ is $\bar{\gamma}_j$-relaxed η-accretive with g_j, where p is even, satisfying

$$\sum_{j=even} \bar{\gamma}_j \leq \sum_{i=odd} \bar{\mu}_i, \text{ and } \sum_{j=even} \bar{\gamma}_j = \sum_{i=odd} \bar{\mu}_i \text{ iff } \tilde{w} = \tilde{v};$$

(iv) $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4, ...\bar{\mu}_p, \bar{\gamma}_{p-1}$-symmetric η-accretive with $g_1, g_2, ..., g_p$ iff for $i \in \{1, 3, ..., p\}$, $\mathcal{M}(..., g_i, ...)$ is $\bar{\mu}_i$-strongly η-accretive with g_i and for $j \in \{2, 4, ..., p-1\}$, $\mathcal{M}(..., g_j, ...)$ is $\bar{\gamma}_j$-relaxed η-accretive with g_j, where p is odd, satisfying

$$\sum_{j=even} \bar{\gamma}_j \leq \sum_{i=odd} \bar{\mu}_i, \text{ and } \sum_{j=even} \bar{\gamma}_j = \sum_{i=odd} \bar{\mu}_i \text{ iff } \tilde{w} = \tilde{v}.$$

Definition 1.12. *For each $i \in \{1, 2, ..., p\}, p \geq 3$, and let set-valued mappings $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$, $\mathcal{N}_i : \mathcal{B} \to 2^{\mathcal{B}}$ and single-valued mappings H^p, $\mathcal{K} : \mathcal{B}^p \to \mathcal{B}$, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $A_i : \mathcal{B} \to \mathcal{B}$, then $\mathcal{K}$ is said to be*

(i) $\bar{\alpha}_i$-strongly η-accretive with $\mathcal{N}_i$ and H^p in the ith-argument if there exists $\bar{\alpha}_i > 0$ such that

$$\Big[\mathcal{K}(..., w^i, ...) - \mathcal{K}(..., v^i, ...), \eta(H^p(A_1w_1, ..., A_pw_p), H^p(A_1v_1, ..., A_pv_p))\Big]$$
$$\geq \bar{\alpha}_i \, \|H^p(A_1w_1, ..., A_pw_p) - H^p(A_1v_1, ..., A_pv_p\|^2,$$
$$\forall w_1, w_2..., w_p, v_1, v_2, ..., v_p \in \mathcal{B}, w^i \in \mathcal{N}_i(w_i), v^i \in \mathcal{N}_i(v_i);$$

(ii) l_i-Lipschitz continuous in the ith-argument if there exists $l_i > 0$ such that

$$\|\mathcal{K}(v_1, .., v_{i-1}, \tilde{w}, v_{i+1}...v_p) - \mathcal{K}(v_1, .., v_{i-1}, \tilde{v}, v_{i+1}...v_p)\| \leq l_i \, \|\tilde{w} - \tilde{v}\|, \forall \tilde{w}, \tilde{v}, v_1, .., v_{i-1}, v_{i+1}...v_p \in \mathcal{B}.$$

2. Generalized $\alpha_i\beta_j$-(H^p, φ)-η-Accretive Mappings

For each $i \in \{1, 2, ..., p\}$, $p \geq 3$, and let $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a set-valued mapping and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}, H^p : \mathcal{B}^p \to \mathcal{B}, A_i, g_i : \mathcal{B} \to \mathcal{B}$ and $\varphi : \mathcal{B} \to \mathcal{B}$ be single-valued mappings. Now, we introduce and study the new class of generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings.

Definition 2.1. *Let $p \geq 3$, then $\mathcal{M}$ is said to be a generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$*

(i) iff $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$ and $(H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$ $(\mathcal{B}) = \mathcal{B}$ if p is an even number;

(ii) iff $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\gamma}_{p-1}\bar{\mu}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$ and $(H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$ $(\mathcal{B}) = \mathcal{B}$ if p is an odd number .

Remark 2.2.

(i) If $\eta(\tilde{w}, \tilde{v}) = \tilde{w} - \tilde{v}$, then generalized $\alpha_i\beta_j$-$(H^p\varphi)$-η-accretive mapping blueuces to $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mapping, studied by Gupta and Khan [21, 22, 29];

(ii) If $i = 1$, $g_i = I$, and $\mathcal{M}(.,.,...) = \mathcal{M}(.)$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to (H, φ)-η-accretive mapping, studied by Luo and Huang [34] and Bhat and Zahoor [9];

(iii) If $i = 1$, $g_i = I$, $\varphi(\tilde{u}) = \rho\tilde{u}$, where $\rho > 0$, and $\mathcal{M}(.,.,...) = \mathcal{M}(.)$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to H-η-accretive mapping, studied by Fang et. al [15],

(iv) If $i = 1$, $g_i = I$, $\varphi(\tilde{u}) = \rho\tilde{u}$, where $\rho > 0$, $\mathcal{M}(.,.,...) = \mathcal{M}(.)$ and $\eta(\tilde{w}, \tilde{v}) = \tilde{w} - \tilde{v}$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to H-accretive mapping, studied by Fang and Huang [14],

(v) If $i = 1, 2$, $g_i = I$, and $\mathcal{M}(.,.,...) = \mathcal{M}(.,.)$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to $H(.,.)$-φ-η-accretive mapping, studied by Ahmad and Dilshad [3],

(vi) If $i = 1, 2$, $\varphi(\tilde{u}) = \rho\tilde{u}$, where $\rho > 0$, $\mathcal{M}(.,.,...) = \mathcal{M}(.,.)$, and $\eta(\tilde{w}, \tilde{v}) = \tilde{w} - \tilde{v}$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to generalized $\alpha\beta$-$H(.,.)$-accretive mapping, studied by Kazmi et. al [28],

(vii) If $i = 1, 2$, $g_i = I$, $\varphi(\tilde{u}) = \rho\tilde{u}$, where $\rho > 0$, $\mathcal{M}(.,.,...) = \mathcal{M}(.)$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to $(H(.,.), \eta)$-accretive mapping, studied by Wang and Ding [45],

(viii) If $i = 1, 2$, $g_i = I$, $\varphi(\tilde{u}) = \rho\tilde{u}$, where $\rho > 0$, $\mathcal{M}(.,.,...) = \mathcal{M}(.)$, and $\eta(\tilde{w}, \tilde{v}) = \tilde{w} - \tilde{v}$, then generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping blueuces to $H(.,.)$-accretive mapping, studied by Zou and Huang [47].

Let us consider the following assumptions M_1-M_5 to discus some properties of the generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings.
$\mathbf{M_1}$: If p even, H^p is $\alpha_1\beta_2\alpha_3\beta_4...\alpha_{p-1}\beta_p$-symmetric η-accretive with $A_1, A_2, ..., A_p$.
$\mathbf{M_2}$: If p odd, H^p is $\alpha_1\beta_2\alpha_3\beta_4...\beta_{p-1}\alpha_p$-symmetric η-accretive with $A_1, A_2, ..., A_p$.
$\mathbf{M_3}$: If p even, $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$.
$\mathbf{M_4}$: If p odd, $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\gamma}_{p-1}\bar{\mu}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$.
$\mathbf{M_5}$: Let η is $\hbar$-Lipschitz continuous.

Proposition 2.3. *Let assumptions M_1-M_4 be held for every $i \in \{1, 2, ...p\}$, $p \geq 3$, and let $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$, and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$, if $\Big[\tilde{x} - \tilde{y},\ \eta(u', v')\Big] \geq 0$ is satisfied for each $(v', \tilde{y}) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$, $\tilde{x} \in \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)(u')$, where $\mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)) = \{(u', \tilde{x}) : \tilde{x} \in \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)(u')\}$.*

Proof. Assume that there exists $(w_0, z_0) \notin \mathrm{Gr}\ (\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$ such that

$$\Big[z_0 - x,\ \eta(w_0, u)\Big] \geq 0,\ \forall\ (u,\ x) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)). \tag{3}$$

If p is even: Since $\mathcal{M}$ is a generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$, then, $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric η-accretive with mappings $g_1, g_2, ..., g_p$ and $(H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))(\mathcal{B}) = \mathcal{B}$, then, there exists $(w_1, z_1) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$ such that

$$H^p(A_1w_0, A_2w_0, ..., A_pw_0) + z_0 = H^p(A_1w_1, A_2w_1, ..., A_pw_1) + z_1 \in \mathcal{B}. \tag{4}$$

From (3) and (4), we have

$$z_0 - z_1 = H^p(A_1w_1, A_2w_1, ..., A_pw_1) - H^p(A_1w_0, A_2w_0, ..., A_pw_0) \in \mathcal{B},$$
$$\Big[z_0 - z_1, \eta(w_0, w_1)\Big] = \Big[H^p(A_1w_1, A_2w_1, ..., A_pw_1) - H^p(A_1w_0, A_2w_0, ..., A_pw_0), \eta(w_0, w_1)\Big].$$

Setting $(u, x) = (w_1, z_1)$ in (3) and using M_3 in (4), we obtain

$$\begin{aligned}
[(\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1}) - (\bar{\gamma}_2 + \bar{\gamma}_4 - ... + \bar{\gamma}_p)]\|w_0 - w_1\|^2 &\le \Big[z_0 - z_1, \eta(w_0, w_1)\Big] \\
&\le -\Big[H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_1, A_2w_1, ..., A_pw_1),\ \eta(w_0, w_1)\Big] \\
&= -\Big[H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_1, A_2w_0, ..., A_pw_0), \eta(w_0, w_1)\Big] \\
&\quad - \Big[H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_0, A_2w_1, ..., A_pw_0), \eta(w_0, w_1)\Big] \\
&\quad \vdots \\
&\quad - \Big[H^p(A_1w_0, A_2w_0, ..., A_pw_0) - H^p(A_1w_0, A_2w_0, ..., A_pw_1), \eta(w_0, w_1)\Big] \\
&\le -[(\alpha_1 + \alpha_3 + ... + \alpha_{p-1}) - (\beta_2 + \beta_4 + ... + \beta_p)]\|w_0 - w_1\|^2.
\end{aligned}$$

Then, we have

$$\Big[\sum \alpha_i - \sum \beta_j + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big] \|w_0 - w_1\|^2 \le 0. \tag{5}$$

Since $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$, it implies that $w_0 = w_1$. By (3), we have $z_0 = z_1$. Thus $(w_1, z_1) = (w_o, z_o) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$. Similarly, we can prove the result when p is odd.

Theorem 2.4. *Let assumptions M_1-M_4 be held for every $i \in \{1, 2, ...p\}$, $p \ge 3$, and let $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$, then $(H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}\,(g_1, g_2, ..., g_p))^{-1}$ is single-valued.*

Proof. For any given $u \in \mathcal{B}$, let $x, y \in (H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))^{-1}(u)$. It follows that

$$-H^p(A_1x, A_2x, ..., A_px) + u \in \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)x,$$
$$-H^p(A_1y, A_2y, ..., A_py) + u \in \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)y.$$

If p is even: Since $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2\bar{\mu}_3\bar{\gamma}_4...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$, we have

$$\begin{aligned}
&(\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1} - \bar{\gamma}_2 - \bar{\gamma}_4 - ... - \bar{\gamma}_p)\|x - y\|^2 \\
&\qquad \le \Big[- H^p(A_1x, A_2x, ..., A_px) + u - (-H^p(A_1y, A_2y, ..., A_py) + u), \eta(x, y)\Big] \\
\Rightarrow\ &(\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1} - \bar{\gamma}_2 - \bar{\gamma}_4 - ... - \bar{\gamma}_p)\|x - y\|^2 \\
&\qquad \le -\Big[H^p(A_1x, A_2x, ..., A_px) - H^p(A_1y, A_2y, ..., A_py),\ \eta(x, y)\Big] \\
&\qquad = -\Big[H^p(A_1x, A_2x, ..., A_px) - H^p(A_1y, A_2x, ..., A_px),\ \eta(x, y)\Big] \\
&\qquad\quad -\Big[H^p(A_1y, A_2x, ..., A_px) - H^p(A_1y, A_2y, ..., A_px),\ \eta(x, y)\Big] \\
&\qquad\quad \vdots \\
&\qquad\quad -\Big[H^p(A_1y, A_2y, ..., A_px) - H^p(A_1y, A_2y, ..., A_py),\ \eta(x, y)\Big].
\end{aligned}$$

Proceed the same as to obtain (5), we have

$$\Big[\sum \alpha_i - \sum \beta_j + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big] \|x-y\|^2 \leq 0. \tag{6}$$

Since $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$, we have $\|x-y\| \leq 0$. It implies that $x = y$. Thus $(H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))^{-1}$ is single-valued. Similarly, we can prove the result when p is odd.

Definition 2.5. *Let assumptions M_1-M_4 be held for $p \geq 3$ and $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$, and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$. A proximal-point mapping $R^{\eta, H^p(.,.,..,.)}_{\varphi, \mathcal{M}(.,.,..,.)} : \mathcal{B} \to \mathcal{B}$ is define as*

$$R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(x) \ = \ [H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)]^{-1}(x), \ \forall\, x \in \mathcal{B}. \tag{7}$$

Theorem 2.6. *Let assumptions M_1-M_5 be held for $p \geq 3$, and let $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$. Then, the proximal-point mapping $R^{\eta, H^p(.,.,..,.)}_{\varphi, \mathcal{M}(.,.,..,.)} : \mathcal{B} \to \mathcal{B}$ is Δ-Lipschitz continuous, where*

$$\Delta = \hbar \Big[\sum \alpha_i - \sum \beta_j + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big]^{-1}.$$

Proof. Let $x, y \in \mathcal{B}$ and from (7), we have

$$\begin{cases} R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(x) \ = \ (H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))^{-1}(x), \\ R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(y) \ = \ (H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))^{-1}(y). \end{cases}$$

It follows that

$$\Big(x - H^p(A_1(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(x)), A_2(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(x)), ..., A_p(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(x)))\Big) \in \varphi \circ \mathcal{M}\Big(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(x)\Big),$$
$$\Big(y - H^p(A_1(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(y)), A_2(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(y)), ..., A_p(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(y)))\Big) \in \varphi \circ \mathcal{M}\Big(R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(y)\Big).$$

Let $x^1 \ = \ R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(x)$ and $y^1 \ = \ R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(y)$.

If p is even: Since $\varphi \circ \mathcal{M}$ is $\bar{\mu}_1\bar{\gamma}_2...\bar{\mu}_{p-1}\bar{\gamma}_p$-symmetric η-accretive with $g_1, g_2, ..., g_p$, we have

$$\begin{aligned}
&\Big[(x - H^p(A_1(x^1), A_2(x^1), ..., A_p(x^1))) - (y - H^p(A_1(y^1), A_2(y^1), ..., A_p(y^1))), \eta(x^1, y^1)\Big] \\
&\qquad\qquad \geq (\bar{\mu}_1 - \bar{\gamma}_2 + \bar{\mu}_3 - \bar{\gamma}_4 + ... + \bar{\mu}_{p-1} - \bar{\gamma}_p)\|x^1 - y^1\|^2, \\
&\Big[x - y - (H^p(A_1(x^1), A_2(x^1), ..., A_p(x^1)) - H^p(A_1(y^1), A_2(y^1), ..., A_p(y^1))), \eta(x^1, y^1)\Big] \\
&\qquad\qquad \geq (\bar{\mu}_1 + \bar{\mu}_3 + ... + \bar{\mu}_{p-1} - (\bar{\gamma}_2 + \bar{\gamma}_4 - ... + \bar{\gamma}_p))\|x^1 - y^1\|^2.
\end{aligned}$$

We have

$$\begin{aligned}
\Big\|x - y\Big\|\, \eta\Big(x^1, y^1\Big) \ \geq \ & \Big[x - y,\ \eta\Big(x^1, y^1\Big)\Big] \\
\geq \ & \Big[H^p(A_1(x^1), A_2(x^1), ..., A_p(x^1)) - H^p(A_1(y^1), A_2(y^1), ..., A_p(y^1)), \\
& \eta\Big(x^1, y^1\Big)\Big] + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big) \Big\|x^1 - y^1\Big\|^2 \\
\geq \ & \alpha_1 \Big\|x^1 - y^1\Big\|^2 - \beta_2 \Big\|x^1 - y^1\Big\|^2 + \alpha_3 \Big\|x^1 - y^1\Big\|^2 - ...\beta_p \Big\|x^1 - y^1\Big\|^2 \\
& + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big) \Big\|x^1 - y^1\Big\|^2 \\
= \ & \Big[\sum \alpha_i - \sum \beta_j + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big] \ \Big\|x^1 - y^1\Big\|^2.
\end{aligned}$$

Hence

$$\left\|x-y\right\| \eta\left(x^1, y^1\right) \geq \left[\sum \alpha_i - \sum \beta_j + \left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right] \left\|x^1 - y^1\right\|^2,$$
$$\left\|x-y\right\| \hbar \left\|x^1 - y^1\right\| \geq \left[\sum \alpha_i - \sum \beta_j + \left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right] \left\|x^1 - y^1\right\|^2,$$

that is,

$$\left\|R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(x) - R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(y)\right\| \leq \Delta \left\|x-y\right\|, \forall\, x, y \in \mathcal{B},$$

where $\Delta = \hbar\left[\sum \alpha_i - \sum \beta_j + \left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right]^{-1}$. Similarly, we can prove the result when p is odd.

3. Graph Convergence for $\alpha_i\beta_j$-(H^p, φ)-η-Accretive Mappings

Graph convergence has a significant role in the study of vibrational problems, approximation theory and optimization problems etc. For a deep study on graph convergence, see Aubin and Frankowska [7], Rockafellar [39] and Sahu et.al., [41].

Definition 3.1. *Let $\mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be a set-valued mapping, then graph of $\mathcal{M}$ given as:*

$$\mathrm{Gr}(\mathcal{M}(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_p)) = \{((\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_p), x^*) : x^* \in \mathcal{M}(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_p)\}.$$

Now, we will discuss the graph convergence of generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings.

Definition 3.2. For $n = 0, 1, 2, ...$, let $\mathcal{M}_n,\ \mathcal{M} :\ \mathcal{B}^p \to 2^{\mathcal{B}}$ be set-valued mappings such that $\mathcal{M},\ \mathcal{M}_n$ are generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$. Graph convergence of sequence $\{\varphi \circ \mathcal{M}_n\}$ to $\varphi \circ \mathcal{M}$ expressed as $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$, if for each $(g_1(x), g_2(x), ..., g_p(x)), y) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$, there exists a sequence
$((g_1(x_n), g_2(x_n), ..., g_p(x_n)), y_n) \in\ \mathrm{Gr}(\varphi \circ \mathcal{M}_n(g_1, g_2, ..., g_p))$ such that

$$g_1(x_n) \to g_1(x), g_2(x_n) \to g_2(x), ..., g_p(x_n) \to g_p(x),\ y_n \to\ y\ \text{ as } n \to\ \infty.$$

Theorem 3.3. *Let us consider the assumptions M_1-M_5 hold good. For $n = 0, 1, 2, ...$, $\mathcal{M}_n, \mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be generalized $\alpha_i\beta_j$-(H^p, φ)-η-accretive mappings with mappings $(A_1, A_2, ..., A_p)$ and $(g_1, g_2, ..., g_p)$ and $\sum \bar{\mu}_i > \sum \bar{\gamma}_j, \sum \alpha_i > \sum \beta_j$. For each $i \in \{1, 2, ..., p\}$, $p \geq 3$, we assume that*
(i) *H^p. is q_i-Lipschitz continuous with respect to A_i;*
(ii) *g_i is r_i-expansive in the ith-argument.*
Then $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$ if and only if

$$R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}(x) \to R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(x),\ \forall x \in \mathcal{B},$$

where $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}(x) = (H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}_n(g_1, g_2, ..., g_p))^{-1}(x),$
$R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(x) = (H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))^{-1}(x).$

Proof. From Theorem 2.6, we know that $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}$ and $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}$ are Δ-Lipschitz continuous.

If part: Assume that $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$.
Given for any $x \in \mathcal{B}$, let $z_n = R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}_n(.,.,...)}(x)$, $z = R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}(x)$.
Then $\left[x - H^p(A_1 z, A_2 z, ..., A_p z)\right] \in \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)(z)$

or $\Big[z,\ [x - H^p(A_1z, A_2z, ..., A_pz)]\Big] \in \ \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$.
By the definition of $\mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$, there exists a sequence $\{g_1(\tilde{z}_n), g_2(\tilde{z}_n), ..., g_p(\tilde{z}_n), \tilde{y}_n\}$ such that

$$g_1(\tilde{z}_n) \to g_1(z), g_2(\tilde{z_n}) \to g_2(z), ..., g_p(\tilde{z}_n) \to g_p(z), \tilde{y}_n \to [x - H^p(A_1z, A_2z, ..., A_pz)] \tag{8}$$

as $n \to \infty$. Since $\tilde{y}_n \in \varphi \circ \mathcal{M}_n(g_1(\tilde{z}_n), g_2(\tilde{z}_n), ..., g_p(\tilde{z}_n))$, we have

$$H^p(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n) + \tilde{y}_n \ \in \ [H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}_n(g_1, g_2, ..., g_p)](\tilde{z}_n).$$

Therefore, $\tilde{z_n} = R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}_n(.,.,...)}[H^p(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n) + \tilde{y}_n]$.
Using the Δ-Lipschitz continuity of $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}$, we have

$$\begin{aligned}
\|z_n - z\| \ &\leq \ \|z_n - \tilde{z}_n\| \ + \ \|\tilde{z}_n - z\| \\
&= \ \|R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}(x) - R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}[H^p(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n) + \tilde{y}_n]\| \ + \ \|z'_n - z\| \\
&\leq \ \Delta\, \|x - H^p(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n) - \tilde{y}_n\| \ + \ \|z'_n - z\| \\
&\leq \ \Delta\, [\|x - H^p(A_1z, A_2z, ..., A_pz) - \tilde{y}_n\| \\
&\quad + \|H^p(A_1z, A_2z, ..., A_pz) - H^p(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n)\|] \ + \ \|\tilde{z}_n - z\|.
\end{aligned} \tag{9}$$

Using the q_i-Lipschitz continuity of H^p, we have

$$\|H^p(A_1z, A_2z, ..., A_pz) - H^p(A_1\tilde{z}_n, A_2\tilde{z}_n, ..., A_p\tilde{z}_n)\| \leq (q_1 + q_2 + ... + q_p)\|\tilde{z}_n - z\|. \tag{10}$$

Using (9) and (10), we have

$$\|z_n - z\| \leq \Delta\|x - H^p(A_1z, A_2z, ..., A_pz) - \tilde{y}_n\| + \Big[1 + \Delta(q_1 + q_2 + ... + q_p)\Big]\|\tilde{z}_n - z\|. \tag{11}$$

As g_i is r_i-expansive, then we have

$$\|g_i(\tilde{z}_n) - g_i(z)\| \geq r_i\|\tilde{z}_n - z\| \geq 0. \tag{12}$$

We have $g_i(\tilde{z}_n) \to g_i(z)$ as $n \to \infty$. Using (9), (12) and let $n \to \infty$ we get $\tilde{z}_n \to z$ and

$$\Big\|[x - H^p(A_1z, A_2z, ..., A_pz) - \tilde{y}_n]\Big\| \to \ 0.$$

By (11), we have $\|z_n - z\| \to 0$ as $n \to \infty$, that is,

$$R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}(u) \to R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(u).$$

Only if part: Suppose that $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)} \to R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}$, $\forall\, u \in \mathcal{B},\ \rho > 0$. For any given $(g_1(x), g_2(x), ..., g_p(x), y) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)$, we have $y \in \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)$

$$H^p(A_1x, A_2x, ..., A_px) + y \ \in \ [H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)](x)$$

Therefore, $x \ = \ R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}[H^p(A_1x, A_2x, ..., A_px) + y]$. Let

$$x_n \ = R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}_n(.,.,...)}[H^p(A_1x, A_2x, ..., A_px) + y].$$

Then,

$$\begin{aligned}
&[H^p(A_1x, A_2x, ..., A_px) - H^p(A_1x_n, A_2x_n, ..., A_px_n) + y] \\
&\qquad \in \varphi \circ \mathcal{M}_n(g_1(x_n), g_2(x_n), ..., g_p(x_n)).
\end{aligned}$$

Let $y_n = [H^p(A_1x, A_2x, ..., A_px) - H^p(A_1x_n, A_2x_n, ..., A_px_n) + y]$.
Now, we evaluate

$$\begin{aligned}
\|y_n - y\| &= \left\|[H^p(A_1x, A_2x, ..., A_px) - H^p(A_1x_n, A_2x_n, ..., A_px_n) + y] - y\right\| \\
&= \|H^p(A_1x, A_2x, ..., A_px) - H^p(A_1x_n, A_2x_n, ..., A_px_n)\| \\
&\leq \ (q_1 + q_2 + ... + q_p)\ \|x_n - x\| \qquad (13) \\
&= \ q\ \|x_n - x\|, \text{ where } q = (q_1 + q_2 + ... + q_p). \qquad (14)
\end{aligned}$$

As $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)} \to R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}}$ for given any $u \in \mathcal{B}$, we have $\|x_n - x\| \to 0$. Let $n \to \infty$, equation (13) gives $y_n \to y$. Therefore, $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$. This completes the proof.

Now, we are providing the following consolidated example in support of $\alpha_i\beta_j$-(H^p, φ)-η-accretive mapping, graph convergence of $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$ and $R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}_n(.,.,...)} \to R^{\eta, H^p(.,.,...)}_{\varphi, \mathcal{M}(.,.,...)}$ by using MATLAB programming.

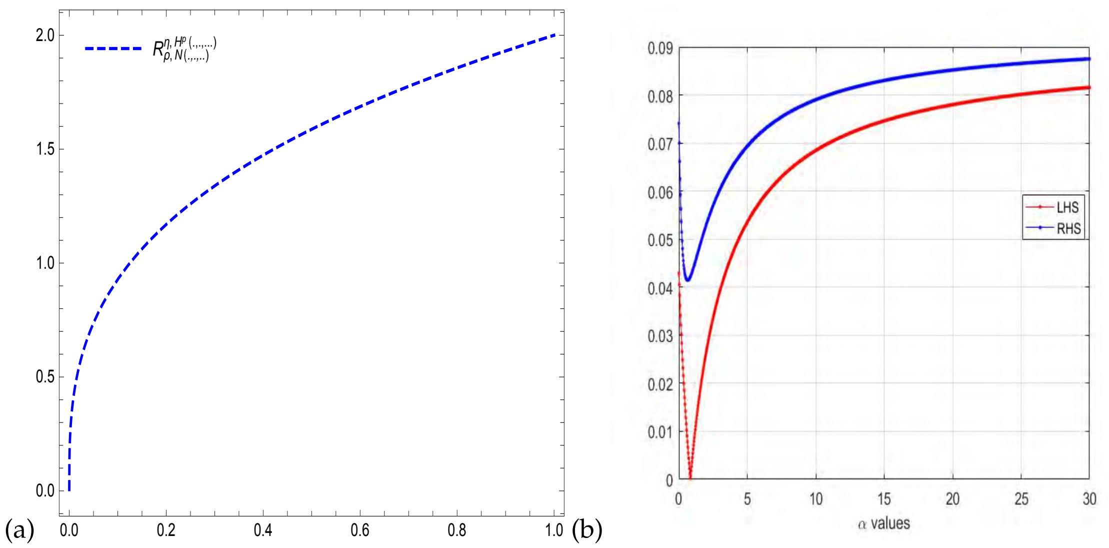

Figure 1: (a) show the graph of $R^{\eta, H^p(.,.,...)}_{\rho,\ \mathcal{M}(.,.,...)}$ for $p = 10$, where $(\varphi \circ \mathcal{M})(z) = \frac{p}{2}\left[\frac{2z}{5} - \frac{z}{15}\right]$
(b) show the convergence of $R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)} \to R^{\eta, H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}$ as $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$ for $p = 10$, where $(\varphi \circ \mathcal{M}_n)(z) = \frac{p}{2}\left[\frac{2z}{5} - \frac{z}{15}\right] + \frac{1+n}{n^3}$ and $(\varphi \circ \mathcal{M})(z) = \frac{p}{2}\left[\frac{2z}{5} - \frac{z}{15}\right]$.

Example 3.4. *Let $\mathcal{B}$ be 2-uniformly smooth Banach space and $\mathcal{B} = \mathbb{R}$. Let p is an even number and $A_i : \mathbb{R} \to \mathbb{R}$ for each $i \in \{1, 2, ..., p\}$, is given by*

$$A_1(z) = \frac{z^3}{27},\ A_3(z) = \frac{z^3}{27}, ..., A_{p-1}(z) = \frac{z^3}{27},$$
$$A_2(z) = \frac{z}{3},\ A_4(z) = \frac{z}{3}, ..., A_p(z) = \frac{z}{3},$$

such that the inequality $yz + y^2 + z^2 \geq 1$ is satisfied for all $y, z \in \mathbb{R}$.
Let $g_i : \mathbb{R} \to \mathbb{R}$ for each $i \in \{1, 2, ..., p\}$, is given by

$$g_1(z) = \frac{2z}{5},\ g_3(z) = \frac{2z}{5}, ..., g_{p-1}(z) = \frac{2z}{5},$$
$$g_2(z) = \frac{z}{15},\ g_4(z) = \frac{z}{15}, ..., g_p(z) = \frac{z}{15}.$$

Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $\eta(y,z) = y - z$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(z) = \rho z$ (let $\rho = 1$) with $\varphi(y + z) = \varphi(y) + \varphi(z)$.

Assume that $H^p : \mathbb{R}^p \to \mathbb{R}$ is defined by

$$H(A_1(z), A_2(z), ..., A_{p-1}(z), A_p(z)) \; = \; A_1(z) - A_2(z) + ... + A_{p-1}(z) - A_p(z).$$

Assume that set-valued mappings $\mathcal{M}_n, \mathcal{M} : \mathbb{R}^p \to 2^{\mathbb{R}}$ are defined by

$$\begin{aligned}\mathcal{M}_n(g_1(z), g_2(z), ..., g_{p-1}(z), g_p(z)) &= g_1(z) - g_2(z) + ... + g_{p-1}(z) - g_p(z) + \frac{1+n}{n^3},\\ \mathcal{M}(g_1(z), g_2(z), ..., g_{p-1}(z), g_p(z)) &= g_1(z) - g_2(z) + ... + g_{p-1}(z) - g_p(z).\end{aligned}$$

Let for any $u_2, u_3, ...u_p \in \mathbb{R}$,

$$\begin{aligned}&\Big[H^p(A_1(y), u_2, ..., u_{p-1}) - H^p(A_1(z), u_2, ..., u_{p-1}),\ \eta(y,z)\Big]\\ &\qquad = \Big[A_1(y) - A_1(z),\ y - z\Big]\\ &\qquad = \Big[\frac{y^3}{27} - \frac{z^3}{27},\ y - z\Big]\\ &\qquad = \frac{1}{27}\,(y-z)^2(y^2 + z^2 + yz)\\ &\qquad \geq \frac{1}{27}\,(y-z)^2 = \frac{1}{27}\,\|y - z\|^2.\end{aligned}$$

Thus, H^p is $\frac{1}{27}$-strongly η- accretive with A_1. In the similar way, we can show that H^p is $\frac{1}{27}$-strongly η-accretive with A_i for all $i \in \{1, 3, ..., p-1\}$.

Let for any $u_1, u_3, ...u_p \in \mathbb{R}$,

$$\begin{aligned}&\Big[H^p(u_1, A_2(y), ..., u_{p-1}) - H^p(u_1 A_2(z), ..., u_{p-1}),\ \eta(y,z)\Big]\\ &\qquad = -\Big[A_2(y) - A_2(z),\ y - z\Big]\\ &\qquad = -\Big[\frac{y}{3} - \frac{z}{3},\ y - z\Big]\\ &\qquad = -\frac{1}{3}\,(y-z)^2\\ &\qquad \geq -\frac{4}{3}\,(y-z)^2 = -\frac{4}{3}\,\|y - z\|^2.\end{aligned}$$

Thus, H^p is $\frac{4}{3}$-relaxed η- accretive with A_2. In the similar way, we can show that H^p is $\frac{4}{3}$-relaxed η- accretive with A_i for all $i \in \{2, 4, ..., p\}$.

Let for any $v_2, v_3, ...v_p \in \mathbb{R}$,

$$\begin{aligned}&\Big[\varphi \circ \mathcal{M}_n(g_1(y), v_2, ..., v_{p-1}) - \varphi \circ \mathcal{M}_n(g_1(z), v_2, ..., v_{p-1}),\ \eta(y,z)\Big]\\ &\qquad = \Big[g_1(y) + \frac{1+n}{n^3} - g_1(z) - \frac{1+n}{n^3},\ y - z\Big]\\ &\qquad = \Big[\frac{2y}{5} - \frac{2z}{5},\ y - z\Big]\\ &\qquad = \frac{2}{5}\,(y-z)^2\\ &\qquad \geq \frac{2}{6}\,(y-z)^2 = \frac{1}{3}\,\|y - z\|^2.\end{aligned}$$

Thus, $\varphi \circ \mathcal{M}_n$ is $\frac{1}{3}$-strongly η-accretive with g_1. In the similar way, we can show that $\varphi \circ \mathcal{M}_n$ is $\frac{1}{3}$-strongly η-accretive with g_i for all $i \in \{1, 3, ..., p-1\}$.

Let for any $v_1, v_3, ...v_p \in \mathbb{R}$,

$$\begin{aligned}
&\Big[\varphi \circ \mathcal{M}_n(v_1, g_2(y), ..., v_{p-1}) - \varphi \circ \mathcal{M}_n(v_1, g_2(z), ..., v_{p-1}),\ \eta(y,z)\Big] \\
&\qquad = \Big[g_2(y) + \frac{1+n}{n^3} - g_2(z) - \frac{1+n}{n^3},\ y - z\Big] \\
&\qquad = -\Big[\frac{y}{15} - \frac{z}{15},\ y - z\Big] \\
&\qquad = -\frac{1}{15}\,(y-z)^2 \\
&\qquad \geq -\frac{16}{15}\,(y-z)^2 = -\frac{16}{15}\,\|y-z\|^2.
\end{aligned}$$

Thus, $\varphi \circ \mathcal{M}_n$ is $\frac{16}{15}$-relaxed η-accretive with g_2. In the similar way, we can show that $\varphi \circ \mathcal{M}_n$ is $\frac{16}{15}$-relaxed η-accretive with g_i for all $i \in \{2, 4, ..., p\}$.
Similarly, we can show that $\varphi \circ \mathcal{M}$ is $\frac{1}{3}$-strongly η-accretive with g_i for all $i \in \{1, 2, ..., p-1\}$ and $\varphi \circ \mathcal{M}$ is $\frac{16}{15}$-relaxed η-accretive with g_i for all $i \in \{2, 4, ..., p\}$.

One can easily verify the following for $\rho = 1$:

$$[H^p(A_1, A_2, ..., A_p) + \varphi \circ \mathcal{M}(g_1, g_2, ..., g_p)](\mathbb{R}) = \mathbb{R}.$$

Now, we will show that $\varphi \circ \mathcal{M}_n \xrightarrow{G} \varphi \circ \mathcal{M}$, if for each $(g_1(z), g_2(z), ..., g_p(z)), y) \in \mathrm{Gr}(\varphi \circ \mathcal{M}(g_1, g_2, ..., g_p))$, there exists a sequence
$((g_1(z_n), g_2(z_n), ..., g_p(z_n), y_n) \in \mathrm{Gr}(\varphi \circ \mathcal{M}_n(g_1, g_2, ..., g_p))$ such that $g_1(z_n) \to g_1(z)$, $g_2(z_n) \to g_2(z), ..., g_p(z_n) \to g_p(z)$, $y_n \to y$ as $n \to \infty$. For this, we consider

$$\begin{aligned}
&z_n = \Big(1 + \frac{1}{n^2}\Big) z, \\
&g_1(z_n) = g_3(z_n) = ... = g_{p-1}(z_n) = \frac{2z_n}{5}, \\
&g_2(z_n) = g_4(z_n) = ... = g_p(z_n) = \frac{z_n}{15},\ n \in \mathbb{N}.
\end{aligned}$$

Since, $\lim_n z_n = \lim_n \Big(1 + \frac{1}{n^2}\Big) z = z$ Thus, we have $z_n \to z$ as $n \to \infty$.

Now $\lim\limits_n g_1(z_n) \to g_1(z), \lim\limits_n g_3(z_n) \to g_3(z), ..., \lim\limits_n g_{p-1}(z_n) \to g_{p-1}(z)$,
$\lim\limits_n g_2(z_n) \to g_2(z), \lim\limits_n g_4(z_n) \to g_4(z), ..., \lim\limits_n g_p(z_n) \to g_p(z)$.

Since

$$\begin{aligned}
y_n &= \varphi \circ \mathcal{M}(g_1(z_n), g_2(z_n), ..., g_p(z_n)) \\
&= \varphi\Big(g_1(z_n) - g_2(z_n) + ... + g_{p-1}(z_n) - g_p(z_n) + \frac{1+n}{n^3}\Big) \\
&= \underbrace{\Big(\frac{2z_n}{5} - \frac{z_n}{15} + ... + \frac{2z_n}{5} - \frac{z_n}{15}\Big)}_{p \text{ terms}} + \frac{1+n}{n^3} \\
&= \underbrace{\Big(\frac{2z}{5} + \frac{2z}{5} + ... + \frac{2z}{5}\Big)}_{\frac{p}{2} \text{ terms}} - \underbrace{\Big(\frac{z}{15} + \frac{z}{15} + ... + \frac{z}{15}\Big)}_{\frac{p}{2} \text{ terms}}. \\
&= \Big[\frac{p}{2} \times \frac{2z_n}{5} - \frac{p}{2} \times \frac{z_n}{15}\Big] + \frac{1+n}{n^3}.
\end{aligned}$$

Now we compute

$$\lim_n y_n = \lim_n \left[\frac{p}{2}\times\frac{2z_n}{5} - \frac{p}{2}\times\frac{z_n}{15}\right] + \frac{1+n}{n^3} = \left[\frac{p}{2}\times\frac{2z}{5} - \frac{p}{2}\times\frac{z}{15}\right].$$
$$= \underbrace{\left(\frac{2z}{5}+\frac{2z}{5}+\ldots+\frac{2z}{5}\right)}_{\frac{p}{2}\ terms} - \underbrace{\left(\frac{z}{15}+\frac{z}{15}+\ldots+\frac{z}{15}\right)}_{\frac{p}{2}\ terms}.$$
$$= \Big(g_1(z)+g_3(z)+\ldots+g_{p-1}(z)\Big) - \Big(g_2(z)+g_4(z)+\ldots+g_p(z)\Big).$$
$$= g_1(z)-g_2(z)+g_3(z)-g_4(z)+\ldots+g_{p-1}(z)-g_p(z).$$
$$= \varphi\circ\mathcal{M}(z) = y.$$

Therefore, $y_n \to y$ as $n\to\infty$ and hence, $\varphi\circ\mathcal{M}_n \xrightarrow{G} \varphi\circ\mathcal{M}$. Next, we will show that $R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)} \to R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}$ as $\varphi\circ\mathcal{M}_n \xrightarrow{G} \varphi\circ\mathcal{M}$. The proximal-point mappings for $\rho=1$, are given by

$$R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}(z) = [H^p(A_1,A_2,...,A_p)+\varphi\circ\mathcal{M}_n(g_1,g_2,...,g_p)]^{-1}(z) = 3\left(z-\frac{1+n}{n^3}\right)^{\frac{1}{3}}$$
$$R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(z) = [H^p(A_1,A_2,...,A_p)+\varphi\circ\mathcal{M}(g_1,g_2,...,g_p)]^{-1}(z) = 3z^{\frac{1}{3}}$$

We evaluate $\left\|R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)}(x) - R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}(z)\right\| = \left\|3\left(z-\frac{1+n}{n^3}\right)^{\frac{1}{3}} - 3z^{\frac{1}{3}}\right\|$, which shows that $\left\|R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)} - R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}\right\| \to 0$ as $n\to\infty$. i.e. $R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}_n(.,.,...)} \to R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}$ as $\varphi\circ\mathcal{M}_n \xrightarrow{G} \varphi\circ\mathcal{M}$.

4. Set-valued Variational-like Inclusions

Let $\mathcal{B}$ be a 2-uniformly smooth Banach space. For each $i\in\{1,2,...,p\}$, $p\geq 3$, let $H^p,\mathcal{K}:\mathcal{B}^p\to\mathcal{B}$, $\eta:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$, and $A_i,\varphi,g_i:\mathcal{B}\to\mathcal{B}$ be single-valued mappings, $\mathcal{N}_i:\mathcal{B}\to 2^{\mathcal{B}}$ be set-valued mappings. Let $\mathcal{M}:\mathcal{B}^p\to 2^{\mathcal{B}}$ be a generalized (H^p,φ)-η-accretive mapping with mappings $(A_1,A_2,...,A_p)$ and $(g_1,g_2,...,g_p)$.

Now, the problem is to find $\tilde{x}\in\mathcal{B}$, $\tilde{u}_1\in\mathcal{N}_1(\tilde{x}),\tilde{u}_2\in\mathcal{N}_2(\tilde{x}),...,\tilde{u}_p\in\mathcal{N}_p(\tilde{x})$ such that

$$\Theta\in\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)+\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}),...,g_p(\tilde{x})). \tag{15}$$

Problem (15) is called the set-valued variational-like inclusions (SVLIP, in short).
Special cases:
(i) If $\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)=\mathcal{K}(\tilde{u}_1,\tilde{u}_2)$, $\eta(\tilde{u}_1,\tilde{u}_2)=\tilde{u}_1-\tilde{u}_2$ and $\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}),...,g_p(\tilde{x}))=\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}))$, then SVLIP (15) blueuced to find $\tilde{x}\in\mathcal{B}$, $\tilde{u}_1\in\mathcal{N}_1(\tilde{x}),\tilde{u}_2\in\mathcal{N}_2(\tilde{x})$ such that

$$\Theta\in\mathcal{K}(\tilde{u}_1,\tilde{u}_2)+\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x})). \tag{16}$$

(ii) If $g_1=g_2=g$, $\mathcal{N}_1=\mathcal{N}_2=\mathcal{N}$, $\eta(\tilde{u}_1,\tilde{u}_2)=\tilde{u}_1-\tilde{u}_2$ and $\mathcal{M}(.,.)=\mathcal{M}(.)$, then problem (16) blueuced to find $\tilde{x}\in\mathcal{B}$, $\tilde{u}\in\mathcal{M}(\tilde{x})$ such that

$$\Theta\in\tilde{u}+\mathcal{M}(g(\tilde{x})). \tag{17}$$

Problem (17) studied by Huang [26] when $\mathcal{M}$ is m-accretive mapping.

(iii) If $\mathcal{K}(\tilde{u}_1,\tilde{u}_2)=\mathcal{K}(\tilde{x})$, $\eta(\tilde{u}_1,\tilde{u}_2)=\tilde{u}_1-\tilde{u}_2$ and $\mathcal{N}$ is single-valued mapping, then problem (16) blueuced to find $\tilde{x}\in\mathcal{B}$ such that

$$\Theta\in\mathcal{K}(\tilde{x})+\mathcal{M}(\tilde{x}). \tag{18}$$

Problem (18) studied by Zou and Huang [47] when $\mathcal{M}$ is $H(.,.)$-accretive mapping. For generalized m-accretive mapping, Problem (18) studied by Bi et al. [11].

Definition 4.1. A set-valued mapping $\mathcal{N}:\mathcal{B}\to CB(\mathcal{B})$ is said to be *$\tilde{\mathcal{D}}$-Lipschitz continuous* with $\zeta>0$, if

$$\tilde{\mathcal{D}}(\mathcal{N}\tilde{y},\mathcal{N}\tilde{z})\leq\ \zeta\,\|\tilde{y}-\tilde{z}\|,\ \ \forall\tilde{y},\ \tilde{z}\in\mathcal{B}.$$

Lemma 4.2. *Let us consider SVLIP (15) with mapping $\varphi:\mathcal{B}\to\mathcal{B}$ such that $\varphi(\tilde{v}+\tilde{w})=\varphi(\tilde{v})+\varphi(\tilde{w})$ and $Ker(\varphi)=\{0\}$, where $Ker(\varphi)=\{\tilde{v}\in\mathcal{B}:\ \varphi(\tilde{v})=0\}$. If $(\tilde{x},\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)$, where $\tilde{x}\in\mathcal{B}$, $\tilde{u}_1\in\mathcal{N}_1(\tilde{x}),\tilde{u}_2\in\mathcal{N}_2(\tilde{x}),...,\tilde{u}_p\in\mathcal{N}_p(\tilde{x})$ is a solution of SVLIP (15) if and only if $(\tilde{x},\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)$ satisfies the following relation:*

$$\tilde{x}\ =\ R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}(.,.,...)}\,[H^p(A_1,A_2,...,A_p)(\tilde{x})-\varphi\circ\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)].\tag{19}$$

Proof. Let $(\tilde{x},\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)$ be a solution of SVLIP (15), then $(\tilde{x},\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)$ satisfy the following condition

$$\tilde{x}\ =\ R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}(.,.,...)}\,[H^p(A_1,A_2,...,A_p)(\tilde{x})-\varphi\circ\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)].\tag{20}$$

$$\begin{aligned}
&\tilde{x}=[H^p(A_1,A_2,...,A_p)+\varphi\circ\mathcal{M}(g_1,g_2,...,g_p)]^{-1}[H^p(A_1,A_2,...,A_p)(\tilde{x})-\varphi\circ\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)]\\
&\Leftrightarrow[H^p(A_1,A_2,...,A_p)(\tilde{x})-\varphi\circ\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)]\in[H^p(A_1,A_2,...,A_p)(\tilde{x})-\varphi\circ\mathcal{M}(g_1,g_2,...,g_p)\tilde{x}]\\
&\Leftrightarrow 0\in\varphi\circ(\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)+\varphi\circ\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}),...,g_p(\tilde{x}))\\
&\Leftrightarrow 0\in\varphi\circ[\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)+\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}),...,g_p(\tilde{x}))]\\
&\Leftrightarrow\varphi^{-1}(0)\in\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)+\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}),...,g_p(\tilde{x}))\\
&\Leftrightarrow\Theta\in\mathcal{K}(\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_p)+\mathcal{M}(g_1(\tilde{x}),g_2(\tilde{x}),...,g_p(\tilde{x})).
\end{aligned}$$

Now, we establish the result in context of uniqueness for the solution of SVLIP (15).

Theorem 4.3. *Let SVLIP (15) hold in assumptions M_1-M_5 with mapping $\varphi:\mathcal{B}\to\mathcal{B}$ such that $\varphi(\tilde{v}+\tilde{w})=\varphi(\tilde{v})+\varphi(\tilde{w})$ and $Ker(\varphi)=\{0\}$, where $Ker(\varphi)=\{\tilde{v}\in\mathcal{B}:\ \varphi(\tilde{v})=0\}$, and $\mathcal{M}_n,\mathcal{M}:\mathcal{B}^p\to 2^{\mathcal{B}}$ be generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mappings with $\sum\bar{\mu}_i>\sum\bar{\gamma}_j,\ \sum\alpha_i>\sum\beta_j$. For each $i\in\{1,2,...,p\}$, we assume the following:*
(i) $\mathcal{N}_i$ is ζ_i-$\mathcal{D}$-Lipschitz continuous;
(ii) H^p is q_i-Lipschitz continuous with A_i;
(iii) $\varphi\circ\mathcal{K}$ is $\bar{\alpha}_i$-strongly η-accretive with g_i and $H^p(A_1,A_2,..,A_p)$ in the ith-argument;
(iv) $\varphi\circ\mathcal{K}$ is λ_i-Lipschitz continuous in ith-argument;
(v) in addition, the following condition

$$\Delta\sqrt{q^2-2\bar{\alpha}q^2+c\tilde{\lambda}^2}<1\tag{21}$$

is satisfied, where $\Delta=\left[\sum\alpha_i-\sum\beta_j+\left(\sum\bar{\mu}_i-\sum\bar{\gamma}_j\right)\right]^{-1}$.

Then, the general nonlinear operator equation (15) based on generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mapping framework has a unique solution $(\tilde{x}^1,\tilde{u}^1,\tilde{u}^2,..,\tilde{u}^p)$ in $\mathcal{B}$.

Proof. Let us consider the mapping $\mathcal{T}:\mathcal{B}\to\mathcal{B}$, given by

$$\mathcal{T}(\tilde{x}^1)=R^{\eta,H^p(.,.,...)}_{\varphi,\ \mathcal{M}(.,.,...)}\left[H^p(A_1\tilde{x}^1,A_2\tilde{x}^1,...,A_p\tilde{x}^1)-\varphi\circ\mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)\right],\ \forall\,\tilde{x}^1,\tilde{u}^1,\tilde{u}^2,..,\tilde{u}^p\in\mathcal{B}.\tag{22}$$

Using (22) and Theorem 3.3, we have

$$\begin{aligned}
\|\mathcal{T}(\tilde{x}^1)-\mathcal{T}(\tilde{y}^1)\|\ &=\ \|R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}(.,.,...)}\,[H^p(A_1,A_2,...,A_p)(\tilde{x}^1)-\varphi\circ\mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\\
&\quad -R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}(.,.,...)}\,[H^p(A_1,A_2,...,A_p)(\tilde{y}^1)-\varphi\circ\mathcal{K}(\tilde{v}^1,\tilde{v}^2,...,\tilde{v}^p)]\|\\
&\leq\ \Delta\|H^p(A_1,A_2,...,A_p)(\tilde{x}^1)-\varphi\circ\mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)\\
&\quad -(H^p(A_1,A_2,...,A_p)(\tilde{y}^1)-\varphi\circ\mathcal{K}(\tilde{v}^1,\tilde{v}^2,...,\tilde{v}^p))\|
\end{aligned}\tag{23}$$

By Lemma 1.6, we have

$$
\begin{aligned}
&\|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) \\
&\qquad -(H^p(A_1, A_2, ..., A_p)(\tilde{y}^1) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p))\|^2 \\
&\qquad \le \|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - H^p(A_1, A_2, ..., A_p)(\tilde{y}^1)\|^2 \\
&\qquad -2\Big[\varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p), \\
&\qquad \eta(H^p(A_1, A_2, ..., A_p)(\tilde{x}^1), H^p(A_1, A_2, ..., A_p)(\tilde{y}^1))\Big] \\
&\qquad +c\|\varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p)\|^2. \qquad (24)
\end{aligned}
$$

By using q_i-Lipschitz continuity of H^p, we have

$$
\begin{aligned}
&\|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - H^p(A_1, A_2, ..., A_p)(\tilde{y}^1)\| \\
&\qquad \le \|H^p(A_1\tilde{x}^1, A_2\tilde{x}^1, ..., A_p\tilde{x}^1)) - H^p(A_1\tilde{y}^1, A_2\tilde{x}^1, ..., A_p\tilde{x}^1)\| \\
&\qquad\quad +\|H^p(A_1\tilde{y}^1, A_2\tilde{x}^1, ..., A_p\tilde{x}^1) - H^p(A_1\tilde{y}^1, A_2\tilde{y}^1, ..., A_p\tilde{x}^1)\| \\
&\qquad\qquad \vdots \\
&\qquad\qquad \vdots \\
&\qquad\quad +\|H^p(A_1\tilde{y}^1, A_2\tilde{y}^1, ..., A_p\tilde{x}^1) - H^p(A_1\tilde{y}^1, A_2\tilde{y}^1, ..., A_p\tilde{y}^1)\| \\
&\qquad \le (q_1 + q_2 + ... + q_p)\|\tilde{x}^1 - \tilde{y}^1\| \\
&\qquad = q\|\tilde{x}^1 - \tilde{y}^1\|, \text{ where } \mathrm{q} = (\mathrm{q}_1 + \mathrm{q}_2 + ... + \mathrm{q}_\mathrm{p}). \qquad (25)
\end{aligned}
$$

Now, we compute the following:

$$
\begin{aligned}
&\Big[\varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p), \eta(H^p(A_1, A_2, ..., A_p)(\tilde{x}^1), H^p(A_1, A_2, ..., A_p)(\tilde{y}^1))\Big] \\
&\qquad = \Big[\varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{u}^2, ..., \tilde{u}^p), \\
&\qquad\qquad \eta(H^p(A_1, A_2, ..., A_p)(\tilde{x}^1), H^p(A_1, A_2, ..., A_p)(\tilde{y}^1))\Big] \\
&\qquad\quad +\Big[\varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{u}^p), \\
&\qquad\qquad \eta(H^p(A_1, A_2, ..., A_p)(\tilde{x}^1), H^p(A_1, A_2, ..., A_p)(\tilde{y}^1))\Big] \\
&\qquad\quad \vdots \\
&\qquad\quad \vdots \\
&\qquad\quad +\Big[\varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p), \\
&\qquad\qquad \eta(H^p(A_1, A_2, ..., A_p)(\tilde{x}^1), H^p(A_1, A_2, ..., A_p)(\tilde{y}^1))\Big] \\
&\qquad \ge \bar{\alpha}_1\|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - H^p(A_1, A_2, ..., A_p)(\tilde{y}^1)\|^2 \\
&\qquad\quad +\bar{\alpha}_2\|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - H^p(A_1, A_2, ..., A_p)(\tilde{y}^1)\|^2 \\
&\qquad\quad \vdots \\
&\qquad\quad \vdots \\
&\qquad\quad +\bar{\alpha}_p\|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - H^p(A_1, A_2, ..., A_p)(\tilde{y}^1)\|^2 \\
&\qquad \ge (\bar{\alpha}_1 + \bar{\alpha}_2 + ... + \bar{\alpha}_p)\|H^p(A_1, A_2, ..., A_p)(\tilde{x}^1) - H^p(A_1, A_2, ..., A_p)(\tilde{y}^1))\|^2 \\
&\qquad \ge (\bar{\alpha}_1 + \bar{\alpha}_2 + ... + \bar{\alpha}_p)(q_1 + q_2 + ... + q_p)^2\|\tilde{x}^1 - \tilde{y}^1\|^2 \\
&\qquad = \bar{\alpha}q^2\|\tilde{x}^1 - \tilde{y}^1\|^2, \text{ where } \bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2 + ... + \bar{\alpha}_p. \qquad (26)
\end{aligned}
$$

By using the λ_i-Lipschitz continuity of $\varphi \circ \mathcal{K}$ and ζ_i-$\mathcal{D}$-Lipschitz continuity of $\mathcal{N}_i$, we have

$$\begin{aligned}
&\|\varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p)\| \\
&\quad \leq \|\varphi \circ \mathcal{K}(\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{u}^2, ..., \tilde{u}^p)\| \\
&\qquad + \|\varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{u}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{u}^p)\| \\
&\qquad : \\
&\qquad : \\
&\qquad + \|\varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{u}^p) - \varphi \circ \mathcal{K}(\tilde{v}^1, \tilde{v}^2, ..., \tilde{v}^p)\| \\
&\quad \leq \lambda_1 \|\tilde{u}^1 - \tilde{v}^1\| + \lambda_2 \|\tilde{u}^2 - \tilde{v}^2\| + ... + \lambda_p \|\tilde{u}^p - \tilde{v}^p\|\| \\
&\quad \leq \lambda_1 \tilde{\mathcal{D}}(\mathcal{N}_1(\tilde{x}^1), \mathcal{N}_1(\tilde{y}^1)) + \lambda_2 \tilde{\mathcal{D}}(\mathcal{N}_2(\tilde{x}^1), \mathcal{N}_2(\tilde{y}^1)) + ... + \lambda_p \tilde{\mathcal{D}}(\mathcal{N}_p(\tilde{x}^p), \mathcal{N}_p(\tilde{y}^p))
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\leq \Big(\lambda_1 \zeta_1 + \lambda_2 \zeta_2 + ... + \lambda_p \zeta_p\Big) \|\tilde{x}^1 - \tilde{y}^1\| \\
&= \tilde{\lambda} \|\tilde{x}^1 - \tilde{y}^1\|, \text{ where } \tilde{\lambda} = \lambda_1 \zeta_1 + \lambda_2 \zeta_2 + ... + \lambda_p \zeta_p.
\end{aligned} \tag{28}$$

Using equation (24)-(28) in equation (23), we have

$$\|\mathcal{T}(\tilde{x}^1) - \mathcal{T}(\tilde{y}^1)\| \leq \Delta \Big[q^2 + c\tilde{\lambda}^2 - 2\bar{\alpha} q^2\Big]^{\frac{1}{2}} \|x^1 - y^1\|.$$

Let

$$\|\mathcal{T}(\tilde{x}^1) - \mathcal{T}(\tilde{y}^1)\| \leq \text{Ł} \|\tilde{x}^1 - \tilde{y}^1\|, \text{ where Ł} = \Delta \Big[q^2 + c\tilde{\lambda}^2 - 2\bar{\alpha} q^2\Big]^{\frac{1}{2}}. \tag{29}$$

We have, $\Delta \sqrt{q^2 - 2\bar{\alpha} q^2 + c\tilde{\lambda}^2} < 1$. From condition (21), we have $0 \leq \text{Ł} < 1$, so (29) implies that

$$\mathcal{T} = R^{\eta, H^p(.,.,...)}_{\varphi,\, \mathcal{M}(.,.,...)} \Big[H^p(A_1, A_2, ..., A_p) - \varphi \circ \mathcal{K}\Big],$$

is a contraction mapping and has a unique fixed point $\tilde{x}^1$ in $\mathcal{B}$. Hence $\tilde{x}^1$ is a unique solution of SVLIP (15).

If $\mathcal{B} = L^{\tilde{p}}$, $2 \leq \tilde{p} < \infty$, then Theorem 4.3 blueuces to the following result:

Corollary 4.4. *Let SVLIP (15) hold in assumptions* M_1*-*M_5 *with mapping* $\varphi : L^{\tilde{p}} \to L^{\tilde{p}}$ *such that* $\varphi(\tilde{v} + \tilde{w}) = \varphi(\tilde{v}) + \varphi(\tilde{w})$ *and* $Ker(\varphi) = \{0\}$*, where* $Ker(\varphi) = \{\tilde{v} \in L^{\tilde{p}} : \varphi(\tilde{v}) = 0\}$*, and* $\mathcal{M}_n, \mathcal{M} : L^{\tilde{p}p} \to 2^{L^{\tilde{p}}}$ *be generalized* $\alpha_i\beta_j$*-*(H^p, φ)*-*η*-accretive mappings with* $\sum \bar{\mu}_i > \sum \bar{\gamma}_j$, $\sum \alpha_i > \sum \beta_j$*. For each* $i \in \{1, 2, ..., p\}$*, we assume the following:*
(i) $\mathcal{N}_i$ *is* ζ_i*-*$\mathcal{D}$*-Lipschitz continuous;*
(ii) H^p *is* q_i*-Lipschitz continuous with* A_i*;*
(iii) $\varphi \circ \mathcal{K}$ *is* $\bar{\alpha}_i$*-strongly* η*-accretive with* g_i *and* $H^p(A_1, A_2, .., A_p)$ *in the ith-argument;*
(iv) $\varphi \circ \mathcal{K}$ *is* λ_i*-Lipschitz continuous in ith-argument;*
(v) in addition, condition $\Delta \sqrt{q^2 - 2\bar{\alpha} q^2 + (\tilde{p} - 1)\tilde{\lambda}^2} < 1$ *is satisfied, where* $\tilde{p} - 1$ *is the constant of smoothness and* $\Delta = \Big[\sum \alpha_i - \sum \beta_j + \Big(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\Big)\Big]^{-1}$.

Then, the general nonlinear operator equation (15) based on generalized $\alpha_i\beta_j$*-*(H^p, φ)*-*η*-accretive mapping framework has a unique solution* $(\tilde{x}^1, \tilde{u}^1, \tilde{u}^2, .., \tilde{u}^p)$ *in* $L^{\tilde{p}}$.

Now, we construct the following iterative algorithm for finding the approximate solution of SGVLI (15):

Algorithm 4.5. *For any given $\tilde{x}_0^1 \in \mathcal{B}$, select $\tilde{u}_0^1 \in \mathcal{N}_1(\tilde{x}_0^1), \tilde{u}_0^2 \in \mathcal{N}_2(\tilde{x}_0^1), ..., \tilde{u}_0^p \in \mathcal{N}_p(\tilde{x}_0^1)$ and obtain $\{\tilde{x}_n^1\}$, $\{\tilde{u}_n^1\}$, $\{\tilde{u}_n^2\}$,..., $\{\tilde{u}_n^p\}$, by the following iterative algorithm*

$$\begin{aligned}
\tilde{x}_{n+1}^1 &= R_{\varphi,\mathcal{M}(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}_n^1) - \varphi \circ K(\tilde{u}_n^1,\tilde{u}_n^2,...,\tilde{u}_n^p)],\\
\tilde{u}_n^1 \in \mathcal{N}_1(\tilde{x}_n^1): \ \|\tilde{u}_{n+1}^1 - \tilde{u}_n^1\| &\leq \left[1+\frac{1}{n+1}\right]\tilde{\mathcal{D}}(\mathcal{N}_1(\tilde{x}_{n+1}^1),\mathcal{N}_1(\tilde{x}_n^1)),\\
\tilde{u}_n^2 \in \mathcal{N}_2(\tilde{x}_n^1): \ \|\tilde{u}_{n+1}^2 - \tilde{u}_n^2\| &\leq \left[1+\frac{1}{n+1}\right]\tilde{\mathcal{D}}(\mathcal{N}_2(\tilde{x}_{n+1}^1),\mathcal{N}_2(\tilde{x}_n^1)),\\
&\vdots\\
&\vdots\\
\tilde{u}_n^p \in \mathcal{N}_p(\tilde{x}_n^1): \ \|\tilde{u}_{n+1}^p - \tilde{u}_n^p\| &\leq \left[1+\frac{1}{n+1}\right]\tilde{\mathcal{D}}(\mathcal{N}_p(\tilde{x}_{n+1}^1),\mathcal{N}_p(\tilde{x}_n^1)),
\end{aligned}$$

$n = 0,1,2,....$ and $\tilde{\mathcal{D}}(.,.)$ is the Hausdorff metric on $CB(\mathcal{B})$.

Now, we establish the convergence result for the solution of SVLIP (15).

Theorem 4.6. *Let SVLIP (15) hold in assumptions M_1-M_5 with mapping $\varphi : \mathcal{B} \to \mathcal{B}$ such that $\varphi(\tilde{v}+\tilde{w}) = \varphi(\tilde{v}) + \varphi(\tilde{w})$ and $Ker(\varphi) = \{0\}$, where $Ker(\varphi) = \{\tilde{v} \in \mathcal{B} : \ \varphi(\tilde{v}) = 0\}$, and $\mathcal{M}_n, \mathcal{M} : \mathcal{B}^p \to 2^{\mathcal{B}}$ be generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mappings with $\sum \bar{\mu}_i > \sum \bar{\gamma}_j, \ \sum \alpha_i > \sum \beta_j$. For each $i \in \{1,2,...,p\}$, we assume the following:*
(i) $\mathcal{N}_i$ is ζ_i-$\mathcal{D}$-Lipschitz continuous;
(ii) H^p is q_i-Lipschitz continuous with A_i;
(iii) $\varphi \circ \mathcal{K}$ is $\bar{\alpha}_i$-strongly η-accretive with g_i and $H^p(A_1,A_2,..,A_p)$ in the ith-argument;
(iv) $\varphi \circ \mathcal{K}$ is λ_i-Lipschitz continuous in ith-argument;
(v) in addition, the following condition

$$\Delta\sqrt{q^2 - 2\bar{\alpha}q^2 + c\tilde{\lambda}^2} < 1 \tag{30}$$

is satisfied, where $\Delta = \left[\sum \alpha_i - \sum \beta_j + \left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right]^{-1}$.

Then iterative sequences $(\{\tilde{x}_n^1\},\{\tilde{u}_n^1\},\{\tilde{u}_n^2\},....,\{\tilde{u}_n^p\})$ developed by Algorithm 4.5 converge strongly to $(\tilde{x}^1,\tilde{u}^1,\tilde{u}^2,..,\tilde{u}^p)$ a solution of SVLIP (15).

Proof. Now, we prove that $\tilde{x}_n^1 \longrightarrow \tilde{x}^1$ as $n \to \infty$. Infact, it follows from Theorem 3.3 and Algorithm 4.5 that

$$\begin{aligned}
\|\tilde{x}_{n+1}^1 - \tilde{x}^1\| &= \|R_{\varphi,\mathcal{M}_n(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}_n^1) - \varphi \circ \mathcal{K}(\tilde{u}_n^1,\tilde{u}_n^2,...,\tilde{u}_n^p)]\\
&\quad - R_{\varphi,\mathcal{M}(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(x^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\|\\
&\leq \|R_{\varphi,\mathcal{M}_n(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}_n^1) - \varphi \circ \mathcal{K}(\tilde{u}_n^1,\tilde{u}_n^2,...,\tilde{u}_n^p)]\\
&\quad - R_{\varphi,\mathcal{M}_n(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\|\\
&\quad + \|R_{\varphi,\mathcal{M}_n(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\\
&\quad - R_{\varphi,\mathcal{M}(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\|
\end{aligned} \tag{31}$$

By Theorem 3.3, we have

$$\begin{aligned}
&R_{\varphi,\mathcal{M}_n(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p) \to\\
&\qquad R_{\varphi,\mathcal{M}(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)].
\end{aligned} \tag{32}$$

Let

$$\begin{aligned}
\theta_n = &\|R_{\varphi,\mathcal{M}_n(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\\
&\quad - R_{\varphi,\mathcal{M}(.,.,...)}^{\eta,H^p(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1) - \varphi \circ \mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\| \to 0.
\end{aligned} \tag{33}$$

In the light of equations (22)-(25), one can obtain

$$\|R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}_n(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1_n)-\varphi\circ\mathcal{K}(\tilde{u}^1_n,\tilde{u}^2_n,...,\tilde{u}^p_n)]$$
$$-R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}_n(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1)-\varphi\circ\mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)]\|\leq Ł_n\|\tilde{x}^1_n-\tilde{x}^1\|, \tag{34}$$

$$\text{where}\quad Ł_n=\Delta\sqrt{q^2-2\bar{\alpha}q^2+c\tilde{\lambda}^2\left(1+\frac{1}{n}\right)^2}.$$

Using (32)-(34) in (31), we get

$$\|\tilde{x}^1_{n+1}-\tilde{x}^1\| = Ł_n\|\tilde{x}^1_n-\tilde{x}^1\|+\theta_n,\ \text{where}\ Ł=\Delta\sqrt{q^2-2\bar{\alpha}q^2+c\tilde{\lambda}^2}. \tag{35}$$

From (30), we have

$$\Delta\sqrt{q^2-2\bar{\alpha}q^2+c\tilde{\lambda}^2}<1.$$

Thus, we have $\lim_{n\to\infty} Ł_n \to Ł$ with $0\leq Ł<1$, and from (33) $\lim_{n\to\infty}\theta_n\to 0$. From Lemma 1.9, $\lim_{n\to\infty}\tilde{x}^1_n\to\tilde{x}^1$. By $\mathcal{D}$-Lipschitz continuity of $\mathcal{N}_1,\mathcal{N}_2,...,\mathcal{N}_p$ and Algorithm 4.5, we have

$$\begin{aligned}
\|\tilde{u}^1_{n+1}-\tilde{u}^1_n\| &\leq \left[1+\frac{1}{n+1}\right]\tilde{\mathcal{D}}(\mathcal{N}_1(\tilde{x}^1_{n+1}),\mathcal{N}_1(\tilde{x}^1_n))\leq\left[1+\frac{1}{n+1}\right]\zeta_1\|\tilde{x}^1_{n+1}-\tilde{x}^1_n\|,\\
\|\tilde{u}^2_{n+1}-\tilde{u}^2_n\| &\leq \left[1+\frac{1}{n+1}\right]\tilde{\mathcal{D}}(\mathcal{N}_2(\tilde{x}^1_{n+1}),\mathcal{N}_2(\tilde{x}^1_n))\leq\left[1+\frac{1}{n+1}\right]\zeta_2\|\tilde{x}^1_{n+1}-\tilde{x}^1_n\|,\\
&\vdots\\
&\vdots\\
\|\tilde{u}^p_{n+1}-\tilde{u}^p_n\| &\leq \left[1+\frac{1}{n+1}\right]\tilde{\mathcal{D}}(\mathcal{N}_p(\tilde{x}^1_{n+1}),\mathcal{N}_p(\tilde{x}^1_n))\leq\left[1+\frac{1}{n+1}\right]\zeta_p\|\tilde{x}^1_{n+1}-\tilde{x}^1_n\|.
\end{aligned}$$

It shows that $\{\tilde{u}^1_n\},\{\tilde{u}^2_n\},....,\{\tilde{u}^p_n\}$ are Cauchy sequences, then there exists $\tilde{u}^1,\tilde{u}^2,...\tilde{u}^p$ such that $\tilde{u}^1_n\to\tilde{u}^1,\tilde{u}^2_n\to\tilde{u}^2,...,\tilde{u}^p_n\to\tilde{u}^p$, as $n\to\infty$. Now, we show that $\tilde{u}^1\in\mathcal{N}_1(\tilde{x}^1)$. Since $\tilde{u}^1_n\in\mathcal{N}_1(\tilde{x}^1)$, we have

$$\begin{aligned}
d(\tilde{u}^1,\mathcal{N}_1(\tilde{x}^1)) &\leq \|\tilde{u}^1-\tilde{u}^1_n\|+d(\tilde{u}^1_n,\mathcal{N}_1(\tilde{x}^1))\\
&\leq \|\tilde{u}^1-\tilde{u}^1_n\|+\tilde{\mathcal{D}}(\mathcal{N}_1(x^1_{\tilde{n}}),\mathcal{N}_1(\tilde{x}^1))\\
&\leq \|\tilde{u}^1-\tilde{u}^1_n\|+\zeta_1\|\tilde{x}^1_n-\tilde{x}^1\|.
\end{aligned}$$

Since $\mathcal{N}_1(\tilde{x}^1)$ is closed, thus $\tilde{u}^1\in\mathcal{N}_1(\tilde{x}^1)$. Similarly, we can prove $\tilde{u}^2\in\mathcal{N}_2(\tilde{x}^1),\tilde{u}^3\in S_3(\tilde{x}^1),...,\tilde{u}^p\in\mathcal{N}_p(\tilde{x}^1)$. By continuity of $H^p,A_i,\varphi\circ\mathcal{K}$, and $R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}(.,.,...)}$, we know that $(\tilde{x}^1,\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)$ is satisfying the following relation:

$$\tilde{x}^1 = R^{\eta,H^p(.,.,...)}_{\varphi,\mathcal{M}(.,.,...)}[H^p(A_1,A_2,...,A_p)(\tilde{x}^1)-\varphi\circ\mathcal{K}(\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)].$$

By Theorem 4.3, SVLIP (15) have a solution $(\tilde{x}^1,\tilde{u}^1,\tilde{u}^2,...,\tilde{u}^p)$.

If $\mathcal{B}=L^{\tilde{p}}$, $2\leq\tilde{p}<\infty$, then Theorem 4.6 blueuces to the following result:

Corollary 4.7. *Let SVLIP (15) hold in assumptions M_1-M_5 with mapping $\varphi: L^{\tilde{p}}\to L^{\tilde{p}}$ such that $\varphi(\tilde{v}+\tilde{w})=\varphi(\tilde{v})+\varphi(\tilde{w})$ and $Ker(\varphi)=\{0\}$, where $Ker(\varphi)=\{\tilde{v}\in L^{\tilde{p}}:\ \varphi(\tilde{v})=0\}$, and $\mathcal{M}_n,\mathcal{M}: L^{\tilde{p}^p}\to 2^{L^{\tilde{p}}}$ be generalized $\alpha_i\beta_j$-$H^p(.,.,...)$-accretive mappings with $\sum\bar{\mu}_i>\sum\bar{\gamma}_j$, $\sum\alpha_i>\sum\beta_j$. For each $i\in\{1,2,...,p\}$, we assume the following:*
(i) $\mathcal{N}_i$ is ζ_i-$\mathcal{D}$-Lipschitz continuous;
(ii) H^p is q_i-Lipschitz continuous with A_i;
(iii) $\varphi\circ\mathcal{K}$ is $\bar{\alpha}_i$-strongly η-accretive with g_i and $H^p(A_1,A_2,..,A_p)$ in the ith-argument;
(iv) $\varphi\circ\mathcal{K}$ is λ_i-Lipschitz continuous in ith-argument;

(v) in addition, condition $\Delta\sqrt{q^2 - 2\bar{\alpha}q^2 + (\tilde{p}-1)\tilde{\lambda}^2} < 1$ *is satisfied, where* $\tilde{p}-1$ *is the constant of smoothness and* $\Delta = \left[\sum \alpha_i - \sum \beta_j + \left(\sum \bar{\mu}_i - \sum \bar{\gamma}_j\right)\right]^{-1}$.

Then iterative sequences $(\{\tilde{x}^1_n\}, \{\tilde{u}^1_n\}, \{\tilde{u}^2_n\}, ..., \{\tilde{u}^p_n\})$ *developed by Algorithm 4.5 converge strongly to* $(\tilde{x}^1, \tilde{u}^1, \tilde{u}^2, .., \tilde{u}^p)$ *a solution of SVLIP (15).*

5. Conclusions

This article is a discussion on generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mappings which consist of $(H(.,.),\eta)$-accretive mappings, the generalized $\alpha\beta$-$H(.,.)$-accretive mappings, $H(.,.)$-accretive mappings, etc. as special cases. Since variational inclusions, generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mappings, and proximal-point mappings have applications in physics, economics and management sciences, we consideblue and studied a SVLIP (15) including a generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mapping. We also discussed the uniqueness and existence of solution of SVVLIP (15) in 2-uniformly smooth Banach spaces. The results that are obtained for the proximal-point mapping inline with the generalized $\alpha_i\beta_j$-(H^p,φ)-η-accretive mappings conferblue in this article can be continued in future to the Yosida inclusion problems in the setting of semi-inner product spaces.

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