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Non-stationary dynamical systems; Shadowing theorem and some applications

Maliheh Mohtashamipour[a], Alireza Zamani Bahabadi[a,*]

[a]*Department of Pure Mathematics Ferdowsi University of Mashhad Mashhad, Iran;*

Abstract. In the present paper, we mean a sequence of maps along a sequence of spaces by a non-stationary dynamical system. We use an Anosov family as a generalization of an Anosov map, which is a sequence of diffeomorphisms along a sequence of compact Riemannian manifolds, so that the tangent bundles split into expanding and contracting subspaces, with uniform bounds for the contraction and the expansion. Also, we introduce the shadowing property on non-stationary dynamical systems. Then, we prepare the necessary conditions for the existence of the shadowing property to prove the shadowing theorem in non-stationary dynamical systems. The shadowing theorem is a known result in dynamical systems, which states that any dynamical system with a hyperbolic structure has the shadowing property. Here, we prove that the shadowing theorem is established on any invariant Anosov family in a non-stationary dynamical system. Then, as in some applications of the shadowing theorem, we check the stability of Anosov families, and also we peruse the stability of isolated invariant Anosov families in non-stationary dynamical systems.

1. Introduction

A pair (M, f), where M is a space and $f : M \to M$ is a map, is called a *dynamical system*. In a dynamical system, we analyze the behavior of trajectories of any point $p \in M$ under the iterations of f. Indeed, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space.

By non-stationary dynamical system, we mean a sequence of maps along a sequence of spaces. In this paper, we are interested in the dynamical behavior of these systems. So far, studies have been done on certain types of these systems. For example, Kawan [17] provided formulas for the metric and topological entropy of non-stationary dynamical systems given by sequences of expanding self-maps on a compact Riemannian manifold. In [27], the authors discussed the evolution of probability distributions for non-stationary dynamical systems where all maps are self-maps of a space. More examples of some (thermo)dynamical properties of non-stationary dynamical systems can be seen in [8, 16, 18–20].

As a generalization of an Anosov map on a manifold, Arnoux and Fisher [7] introduced the notion of the Anosov family. Indeed, an Anosov family is a sequence of diffeomorphisms along compact Riemannian manifolds such that the tangent bundles split into expanding and contracting subspaces with uniform bounds for the contraction and the expansion.

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* Corresponding author: Alireza Zamani Bahabadi
Email addresses: ma_mo149@alumni.um.ac.ir (Maliheh Mohtashamipour), zamany@um.ac.ir (Alireza Zamani Bahabadi)

In 1972, Sinai [32] proved the shadowing theorem for Anosov diffeomorphisms. After that, Bowen [9] presented the first formal statement of the shadowing theorem for general diffeomorphisms in 1975. Indeed, he studied the existence of the shadowing property on diffeomorphisms. To find out the significance of the shadowing theorem, we list some of its applications obtained therefore. Bowen[11, 12] and Sigmand [31] used it to prove some results about specification property. Conley [14] and Robinson [30] implied the hyperbolicity of chain recurrent sets of diffeomorphisms by this theorem. Lanford [22], Palmer [28, 29], and Anosov and Solodov [5] utilized the shadowing theorem to prove the Smale theorem in special qualifications. To prove the existence of trajectories with arbitrary itineraries, McGehee [23] used it. Also, by this theorem, Bowen [10] enriched his results about Markov partitions. Moreover, Walter [35] and Lanford [21] exerted the shadowing theorem to obtain some results on perturbations of diffeomorphisms with hyperbolic structures. In [13], the authors proved some shadowing results for both sequences of C^1-expanding self-maps of compact metric spaces and sequences of nearby C^1-Anosov diffeomorphisms.

Our main goal in this paper is to investigate the shadowing property for non-stationary dynamical systems, which are sequences of maps along different spaces.

Moser [25] and, in a different type of proof, Walters [34] showed that Anosov diffeomorphisms are semi-stable. They proved that any diffeomorphism g close enough to a given diffeomorphism f with a hyperbolic structure is semi-conjugate to f and also has a hyperbolic structure. After that, Hirsch and Pugh [15] gave some conditions to have a conjugacy between f and g. Indeed, they attained the stability of hyperbolic invariant sets for f. Moreover, Anosov [4] and Smale [33] presented different proofs for the stability of Anosov diffeomorphisms. Acevedo [3] proved the stability of Anosov families.

In this paper, we introduce the semi-stability and stability of Anosov families in non-stationary dynamical systems. We prove similar results for non-stationary dynamical systems using the shadowing theorem in different methods.

2. Preliminaries

In this section, we state the concepts and notations, which are necessary for the following section.

Definition 2.1. *Consider a sequence $(M_i)_{i\in\mathbb{Z}}$ from Riemannian manifolds (M_i, d_i), where d_i is the metric induced by a Riemannian metric on M_i.*

Assume that $\mathcal{M} := \sqcup_{i\in\mathbb{Z}} M_i$ is a disjoint union $\cup_{i\in\mathbb{Z}}(M_i \times i)$ of the sequence $(M_i)_{i\in\mathbb{Z}}$. That is, a point in $\mathcal{M}$ is a point in some M_i together with index i. So, any $p \in \mathcal{M}$ belongs to only one M_i.

It is obvious that $\mathcal{M}$ can be appointed with a metric $d : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that

$$d(p,q) = \begin{cases} 1 \quad if \quad (p,q) \in M_i \times M_j, \quad i \neq j, \\ \min\{1, d_i(p,q)\} \quad if \quad p,q \in M_i. \end{cases}$$

Indeed, $\mathcal{M}$ is endowed with a Riemannian metric, which is equal to the Riemannian metric of M_i when it is restricted on M_i. The metric induced by this Riemannian metric is the same as d above. Consider a sequence $(f_i)_{i\in\mathbb{Z}}$ of diffeomorphisms f_i from M_i to M_{i+1}. We define the map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ such that $\mathcal{F}|_{M_i} = f_i$, for any $i \in \mathbb{Z}$, and we write $\mathcal{F} = (f_i)_{i\in\mathbb{Z}}$. The pair $(\mathcal{M}, \mathcal{F})$ is called a non-stationary dynamical system [7].
The nth composition of $\mathcal{F}^n$ is equal to f_i^n, for any $i \in \mathbb{Z}$, when

$$f_i^n = \begin{cases} f_{i+n-1} \circ \cdots \circ f_i : M_i \to M_{i+n}, \quad n > 0, \\ id_{M_i} : M_i \to M_i, \quad n = 0, \\ f_{i-n} \circ \cdots \circ f_{i-1} : M_i \to M_{i-n}, \quad n < 0, \end{cases}$$

where id_{M_i} is the identity map on M_i.

Take $p \in \mathcal{M}$. For some i, we have $p \in M_i$. The set $\{f_i^n(p) | n \in \mathbb{Z}\}$ is called the $\mathcal{F}$-orbit (or trajectory) of p and denoted by $O_{\mathcal{F}}(p)$.

Example 2.2. *Take a diffeomorphism f_a of a Riemannian manifold M_a. For every $i \in \mathbb{Z}$, let M_i be a copy of M_a, with the same metric, and let $f_i : M_i \to M_{i+1}$ be equal to f_a modulo this identification. Then, the pair $(\mathcal{M} := \sqcup_{i\in\mathbb{Z}} M_i, \mathcal{F} = (f_i)_{i\in\mathbb{Z}})$ is a non-stationary dynamical system. In [7], this non-stationary dynamical system is called a lift of the dynamical system (M_a, f_a). A trivial case can be a lift of identity map of M_a.*

In some papers, other names were used for a non-stationary dynamical system such as sequential or non-autonomous dynamical system [13, 18].

Definition 2.3. *A non-stationary dynamical system $(\mathcal{M}, \mathcal{F})$ is called an Anosov family [7] provided that the following conditions hold:*

i) There exists a continuous $D\mathcal{F}$-invariant splitting $T\mathcal{M} = E^s \oplus E^u$ such that for any $p \in \mathcal{M}$, $T_p\mathcal{M} = E^s_p \oplus E^u_p$, $D_p\mathcal{F}(E^s_p) = E^s_{\mathcal{F}(p)}$, and $D_p\mathcal{F}(E^u_p) = E^u_{\mathcal{F}(p)}$;

ii) There exist $c > 0$ and $0 < \lambda < 1$ such that for any $i \in \mathbb{Z}$, $p \in M_i$ and $n \in \mathbb{N}$, we have

a) if $v \in E^s_p$, then $\| D(f_i^n)_p(v) \| \leqslant c\lambda^n \| v \|$ and

b) if $v \in E^u_p$, then $\| D(f_i^{-n})_p(v) \| \leqslant c\lambda^n \| v \|$.

We call E^s_p and E^u_p, stable and unstable subspaces, respectively.

Example 2.4. *Assume that f_a is an Anosov map of a Riemannian manifold M_a and that $(\mathcal{M}, \mathcal{F})$ is a lift of (M_a, f_a). Then $(\mathcal{M}, \mathcal{F})$ is an Anosov family.*

Example 2.5. *A non-stationary dynamical system defined by a nontrivial sequence of matrices in $SL(2, N)$ acting on the two-torus is an Anosov family [7].*

In the definition of Anosov family, if $c = 1$, then we say that $(\mathcal{M}, \mathcal{F})$ is *strictly Anosov* with constant λ.

A good class of examples about non-stationary dynamical systems and Anosov families can be seen in [2, 3, 7].

Definition 2.6. *Two Riemannian metrics d and d^* are uniformly equivalent, when there exist $\beta, \beta' \in (0, \infty)$ such that $\beta d \leqslant d^* \leqslant \beta' d$.*

In [3], Acevedo proved that for any Riemannian metric d on $\mathcal{M}$, there exists a uniformly equivalent Riemannian metric d^* such that the Anosov family $(\mathcal{M}, \mathcal{F})$ with this new metric d^* has the property of angle and it is strictly Anosov.

Definition 2.7. *An Anosov family has the property of angle provided that the angle between E^s_p and E^u_p is more than zero; to wit, there exists $\alpha \in (0, 1)$ such that for any $v \in E^s_p$ and $w \in E^u_p$, we have*

$$\cos(\widehat{v, w}) \in [\alpha - 1, 1 - \alpha].$$

By an example, Muentes [26] showed that the angles between stable and unstable subspaces in Anosov families, unlike Anosov diffeomorphisms, can converge to zero along the orbit of any point of $\mathcal{M}$.

Local stable and unstable manifolds for Anosov families were presented in [1]. Since we need these notions, we state a necessary survey of [1] to clarify, as follows.

Consider

$$\Theta_{p,q} = \limsup_{n\to\infty} \frac{1}{n} \log d(f_i^n(p), f_i^n(q));$$

and

$$\Delta_{p,q} = \limsup_{n\to\infty} \frac{1}{n} \log d(f_i^{-n}(p), f_i^{-n}(q)).$$

Given $\varepsilon > 0$, for arbitrary $p \in \mathcal{M}$, when $p \in M_i$, we set

$$\mathcal{W}^s(p,\varepsilon) := \{q \in \mathcal{M} | d(p,q) < \varepsilon, d(f_i^n(p), f_i^n(q)) < \varepsilon, \text{for any } n \in \mathbb{N} \text{ and } \Theta_{p,q} < 0\},$$

$$\mathcal{W}^u(p,\varepsilon) := \{q \in \mathcal{M} | d(p,q) < \varepsilon, d(f_i^{-n}(p), f_i^{-n}(q)) < \varepsilon, \text{for any } n \in \mathbb{N} \text{ and } \Delta_{p,q} < 0\}$$

which are called the *local stable and local unstable subsets at p*, respectively.

Let $(\mathcal{M},\mathcal{F})$ be a strictly Anosov. By [1, Prop 3.3, Ths. 6.2 and 6.3] and [3, Prop. 5.2], there exist $\delta, \xi, K^u, K^s > 0$ with the following properties:

i) $\mathcal{W}^u(p,\delta)$ and $\mathcal{W}^s(p,\delta)$ are differentiable submanifolds of $\mathcal{M}$;
ii) $T_p\mathcal{W}^u(p,\delta) = E_p^u$ and $T_p\mathcal{W}^s(p,\delta) = E_p^s$;
iii) $\mathcal{F}^{-1}(\mathcal{W}^u(p,\delta)) \subseteq \mathcal{W}^u(\mathcal{F}^{-1}(p),\delta)$ and $\mathcal{F}(\mathcal{W}^s(p,\delta)) \subseteq \mathcal{W}^s(\mathcal{F}(p),\delta)$;
iv) given $n \in \mathbb{N}$, we have
   a) $d(\mathcal{F}^{-n}(q), \mathcal{F}^{-n}(p)) \leqslant K^u \xi^n d(q,p)$, if $q \in \mathcal{W}^u(p,\delta)$, and
   b) $d(\mathcal{F}^{n}(q), \mathcal{F}^{n}(p)) \leqslant K^s \xi^n d(q,p)$, if $q \in \mathcal{W}^s(p,\delta)$.

In dynamical systems, as a field of mathematics, we study the dynamics of trajectories. So, when we have an approximate trajectory or pseudo trajectory, we try to find the next real trajectories, and for this aim, the best tool is the shadowing property. The definition of the shadowing property on dynamical systems can be seen in [24].

Now, we define the shadowing property on non-stationary dynamical systems.

Definition 2.8. *Given $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$, $x_i \in M_i$, is called a δ-pseudo trajectory for $\mathcal{F}$ if for any $i \in \mathbb{Z}$, $d(f_i(x_i), x_{i+1}) < \delta$. For $\varepsilon > 0$, this sequence is ε-shadowed by some point if there exist $i \in \mathbb{Z}$ and $y_i \in M_i$ such that for any $n \in \mathbb{Z}$,*

$$d(f_i^n(y_i), x_{i+n}) < \varepsilon.$$

Also this sequence is strong ε-shadowed by some point if there exists $y_0 \in M_0$ such that for any $n \in \mathbb{Z}$,

$$d(f_0^n(y_0), x_n) < \varepsilon.$$

We say that a non-stationary dynamical system $(\mathcal{M},\mathcal{F})$ has the (strong) shadowing property provided that for all $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo trajectory for $\mathcal{F}$ is ε-shadowed by some point of $\mathcal{M}$.

If any pseudo trajectory is ε-shadowed by a unique point, then we say that $(\mathcal{M},\mathcal{F})$ has the unique shadowing property.

If there exists a uniform constant $k > 0$ such that every $k\varepsilon$-pseudo trajectory is strong ε-shadowed by some point of M_0, then we say that $(\mathcal{M},\mathcal{F})$ has the Lipschitz shadowing property.

It is easily seen that the Lipschitz and strong shadowing properties are stronger definitions than the shadowing property that we introduced above.

In [13], the notions of strong shadowing and Lipschitz shadowing properties were presented. Then, some shadowing results were implied for both sequences of C^1-expanding maps and sequences of nearby C^1-Anosov diffeomorphisms as follows.

Example 2.9. *Let $\mathcal{F} = (f_i)_{i\in\mathbb{Z}_+}$ be a sequence of expanding maps f_i from M_i to M_{i+1}, where M_i is a locally compact metric space for every $i \in \mathbb{Z}_+$. Then $\mathcal{F}$ satisfies the Lipschitz shadowing property [13]. We know that the Lipschitz shadowing property is stronger than the shadowing property. So $\mathcal{F}$ has the shadowing property.*

Example 2.10. *Let M be a compact Riemannian manifold and let f be an Anosov C^1-diffeomorphism of M to M. By [13, Theorem 4.2], there exists a C^1-neighborhood $\mathcal{V}$ of f such that each $\mathcal{F} = (f_i)_{i\in\mathbb{Z}} \subset \mathcal{V}$ satisfies the Lipschitz shadowing property. Hence $\mathcal{F}$ has the shadowing property.*

Definition 2.11. *Two non-stationary dynamical systems* $(\sqcup_{i\in\mathbb{Z}} M_i, (f_i)_{i\in\mathbb{Z}})$ *and* $(\sqcup_{i\in\mathbb{Z}} M_i, (g_i)_{i\in\mathbb{Z}})$ *are said to be semi-conjugate if there exists a sequence* $(h_i)_{i\in\mathbb{Z}}$ *such that* h_i *is a continuous and onto map from* M_i *to* M_i*, and we have* $f_i \circ h_i = h_{i+1} \circ g_i$*, for all* $i \in \mathbb{Z}$*.*
We say that a non-stationary dynamical system $(\sqcup_{i\in\mathbb{Z}} M_i, (f_i)_{i\in\mathbb{Z}})$ *is semi-stable if there exists* $\varepsilon > 0$ *such that any non-stationary dynamical system* $(\sqcup_{i\in\mathbb{Z}} M_i, (g_i)_{i\in\mathbb{Z}})$*,* ε*-close to* $((M_i)_{i\in\mathbb{Z}}, (f_i)_{i\in\mathbb{Z}})$*, is semi-conjugate to* $(\sqcup_{i\in\mathbb{Z}} M_i, (f_i)_{i\in\mathbb{Z}})$*.*
If for any $i \in \mathbb{Z}$*,* h_i *is also one-to-one, then* $h := (h_i)_{i\in\mathbb{Z}}$ *is called a conjugacy, and* $(\sqcup_{i\in\mathbb{Z}} M_i, (f_i)_{i\in\mathbb{Z}})$*, or briefly* $(f_i)_{i\in\mathbb{Z}}$*, is said to be stable.*

Other conjugacies such as sequential conjugacy and almost conjugacy can be seen in [13].

Definition 2.12. *We say that* Λ *is a subset of* $\mathcal{M}$ *if there exists a family* $(\Lambda_i)_{i\in\mathbb{Z}}$ *such that* $\Lambda = \sqcup_{i\in\mathbb{Z}}\Lambda_i$ *and, for any* $i \in \mathbb{Z}$*,* Λ_i *is a subset of* M_i*.*

Definition 2.13. *A subset* $\Lambda = \sqcup_{i\in\mathbb{Z}}\Lambda_i$ *of* $\mathcal{M}$ *is called* $\mathcal{F}$*-invariant if, for any* $i \in \mathbb{Z}$*,* $f_i(\Lambda_i) \subseteq \Lambda_{i+1}$*.*

Definition 2.14. *In a non-stationary dynamical system* $(\mathcal{M} = \sqcup_{i\in\mathbb{Z}} M_i, \mathcal{F} = (f_i)_{i\in\mathbb{Z}})$*, a subset* $\Lambda = \sqcup_{i\in\mathbb{Z}}\Lambda_i$ *of* $\mathcal{M} = \sqcup_{i\in\mathbb{Z}} M_i$ *is called isolated, if there exists an open neighborhood* $\mathcal{U} = \sqcup_{i\in\mathbb{Z}} U_i$ *of* Λ *in* $\mathcal{M}$ *such that, for any* $i \in \mathbb{Z}$*,* U_i *is an open neighborhood of* Λ_i *in* M_i *and also* $\Lambda_i = \bigcap_{n=-\infty}^{\infty} (f_i^n)^{-1}(U_{i+n})$*. The neighborhood* $\mathcal{U}$ *is called an isolated neighborhood.*

3. Results

Along this section, we assume that $\mathcal{M} := \sqcup_{i\in\mathbb{Z}} M_i$ is a disjoint union $\cup_{i\in\mathbb{Z}}(M_i \times i)$ of a sequence $(M_i)_{i\in\mathbb{Z}}$ from Riemannian manifolds (M_i, d_i), where d_i is the metric induced by a Riemannian metric on M_i, for all $i \in \mathbb{Z}$. Also, we assume that $(f_i)_{i\in\mathbb{Z}}$ is a sequence of diffeomorphisms from M_i to M_{i+1} and that $\mathcal{F} = (f_i)_{i\in\mathbb{Z}}$.

Before we state main theorems, we list some required propositions of [2], as follows.

Proposition 3.1. *[2] Let* $(\mathcal{M}, \mathcal{F})$ *be an Anosov family, and let* $T_p\mathcal{M} = E^s_p \oplus E^u_p$*. Then,* E^s_p *and* E^u_p *depend continuously on* p*.*

Proposition 3.2. *[2] Let* d *and* d^* *be two uniformly equivalent Riemannian metrics on* $\mathcal{M}$*. A non-stationary dynamical system* $(\mathcal{M}, d, \mathcal{F})$ *is an Anosov family if and only if* $(\mathcal{M}, d^*, \mathcal{F})$ *is an Anosov family.*

Proposition 3.3. *[2] There exists a* C^∞*-Riemannian metric* d^* *on* $(\mathcal{M}, d)$ *that is uniformly equivalent to* d *on each* $\mathcal{M}_i$*, such that* $(\mathcal{M}, d^*, \mathcal{F})$ *is a strictly Anosov. Furthermore,* $(\mathcal{M}, d^*, \mathcal{F})$ *satisfies the property of angle.*

for simplicity, in this section, we consider that $(\mathcal{M}, \mathcal{F})$ is an Anosov family with the property of angle.

Remark 3.4. *Take* $\varepsilon > 0$ *and* $p \in \mathcal{M}$*. Let* $B_p(0, \varepsilon) := E^u_p(\varepsilon) \times E^s_p(\varepsilon)$ *be an* ε*-box in* T_pM_i *and let the map* $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ *be the exponential map on* $T_p\mathcal{M}$*. Then* $B(p, \varepsilon) := \exp_p(B_p(0, \varepsilon))$ *is a neighborhood of* p *in* $\mathcal{M}$*.*

We prepare necessary conditions to obtain the shadowing property, as follows.

Theorem 3.5 (Shadowing theorem). *Let* $(\mathcal{M}, \mathcal{F})$ *be a non-stationary dynamical system. Let* $\Lambda = \sqcup_{i\in\mathbb{Z}}\Lambda_i$ *be an* $\mathcal{F}$*-invariant subset of* $\mathcal{M} := \sqcup_{i\in\mathbb{Z}} M_i$*, and let the pair* $(\Lambda, \mathcal{F}|_\Lambda)$ *be an Anosov family in* $(\mathcal{M}, \mathcal{F})$*, where* $\mathcal{F}|_\Lambda := (f_i|_{\Lambda_i})_{i\in\mathbb{Z}}$*. Then, there exists a neighborhood* $\mathcal{V}$ *of* Λ *such that* $(\mathcal{M}, \mathcal{F})$ *has the unique shadowing property on* $\mathcal{V}$*. Also, if* Λ *is an isolated* $\mathcal{F}$*-invariant subset of* $\mathcal{M}$ *and* $(\Lambda, \mathcal{F}|_\Lambda)$ *is an Anosov family in* $(\mathcal{M}, \mathcal{F})$*, then* $(\mathcal{M}, \mathcal{F})$ *has the unique shadowing property on* Λ*.*

Proof. Take $\varepsilon > 0$, $0 < \alpha < \varepsilon$ and $p \in \mathcal{M}$, arbitrarily. By Proposition 3.1, we extend the splitting $T_p\mathcal{M} = E^u_p \oplus E^s_p$ to a neighborhood $\mathcal{V} = \sqcup_{i\in\mathbb{Z}}\mathcal{V}_i$ of Λ in $\mathcal{M}$. We take a sequence $\xi = (\xi_i)_{i\in\mathbb{Z}}$ of positive real numbers such that, for any $i \in \mathbb{Z}$, $\mathcal{V}_i$ contains ξ_i-neighborhood O_i of Λ_i. Consider $B_p(0, \alpha)$ and $B(p, \alpha)$ as in Remark 3.4. If ξ_i is small enough for any $i \in \mathbb{Z}$, then we can choose $\delta > 0$ such that for any δ-pseudo trajectory $(x_i)_{i\in\mathbb{Z}}$ in $O = \sqcup_{i\in\mathbb{Z}} O_i$, there is $\ell > \lambda$ such that $f_i(B(x_i, \alpha)) \subset B(x_{i+1}, \ell\alpha)$ for any $i \in \mathbb{Z}$. Note that λ is the same as in the definition of Anosov family. For any $i \in \mathbb{Z}$, consider the map $F_i : B_{x_i}(0, \alpha) \to B_{x_{i+1}}(0, \ell\alpha)$, by

$F_i = \exp_{x_{i+1}}^{-1} \circ f_i \circ \exp_{x_i}$. We know that $\exp_{x_i}(B_{x_i}(0,\alpha)) = B(x_i,\alpha)$ and that $f_i(B(x_i,\alpha)) \subset B(x_{i+1},\ell\alpha)$. So we have $F_i(B_{x_i}(0,\alpha)) \subset B_{x_{i+1}}(0,\ell\alpha)$. Therefore,

$$\begin{aligned} Set\ \Gamma_0^u(\alpha) &= \bigcap_{t=0}^{\infty} \Big(F_{-1}o\cdots oF_{-t}\Big)\Big(B_{x_{-t}}(o,\alpha)\Big), \\ \Gamma_0^s(\alpha) &= \bigcap_{t=-\infty}^{0} \Big(F_0^{-1}o\cdots oF_{-t-1}^{-1}\Big)\Big(B_{x_{-t}}(o,\alpha)\Big), \\ \bar{\Gamma}_0^u(\alpha) &= \exp_{x_0}(\Gamma_0^u(\varepsilon))\ and \\ \bar{\Gamma}_0^s(\alpha) &= \exp_{x_0}(\Gamma_0^s(\varepsilon)). \end{aligned}$$

Obviously, $\Gamma_0^u(\alpha)$ and $\bar{\Gamma}_0^u(\alpha)$ are unstable disks in $B_{x_0}(0,\alpha)$ and M_0 near x_0, respectively. Similarly, it is clear that $\Gamma_0^s(\alpha)$ is a stable disk in $B_{x_0}(0,r)$ and that $\bar{\Gamma}_0^s(\alpha)$ is a stable disk in M_0 near x_0.
If $\bar{y} \in \bar{\Gamma}_0^u(\alpha)$, then $(F_{-1}o\cdots oF_{-t})^{-1}(\bar{y}) \in B_{x_{-t}}(o,\alpha)$, and if $y \in \Gamma_0^u(\alpha)$, then $f_0^{-t}(y) \in B(x_{-t},\alpha)$, that is, $d(f_0^{-t}(y),x_{-t}) < \alpha$, for any $t \in [0,\infty)$.
Also, when $\bar{z} \in \bar{\Gamma}_0^s(\alpha)$, then $(F_0^{-1}o\cdots oF_{-t-1}^{-1})^{-1}(\bar{z}) \in B_{x_{-t}}(0,\alpha)$, and if $z \in \Gamma_0^s(\alpha)$, then $f_0^{-t}(z) \in B(x_{-t},\alpha)$, which means $d(f_0^{-t}(z),x_{-t}) < \alpha$ for any $t \in (-\infty,0]$.
As we said in the first of this section, $(\mathcal{M},\mathcal{F})$ has the property of angle, so there exists a unique point $\bar{q} \in \bar{\Gamma}_0^u(\alpha) \bigcap \bar{\Gamma}_0^s(\alpha)$ and naturally, a unique point $q \in \Gamma_0^u(\alpha) \bigcap \Gamma_0^s(\alpha)$ such that for every $t \in (-\infty,\infty)$, we have $d(f_0^t(q),x_t) < \alpha$. It implies that δ-pseudo trajectory $(x_i)_{i\in\mathbb{Z}}$ is ε-shadowed with unique point $q \in \mathcal{O}$. Hence $(\mathcal{M},\mathcal{F})$ has the unique shadowing property on ξ-neighborhood $\mathcal{O}$ of $\Lambda \in \mathcal{M}$.
Now, let Λ is an isolated invariant set with isolating neighborhood $\mathcal{U}$. If α and ξ_i, for any $i \in \mathbb{Z}$, are small enough such that $B(p,\alpha) \subset \mathcal{U}$ for all $p \in \mathcal{O}$, then $f_0^j(q) \in B(x_j,\alpha) \subset \mathcal{U}$, for all $j \in \mathbb{Z}$. It implies that $q \in \Lambda$. Hence if Λ is an isolated invariant set in $\mathcal{M}$, then $(\mathcal{M},\mathcal{F})$ has the unique shadowing property on Λ. □

In the following result, we imply stability and, clearly, semi-stability of Anosov families, as some applications of the shadowing theorem.

Theorem 3.6. *Every Anosov family is stable.*

Proof. Let $(\mathcal{M},\mathcal{F})$ be an Anosov family. By Theorem 3.5, for given $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo trajectory for $\mathcal{F}$ in $\mathcal{M}$ is uniquely ε-shadowed by some point of $\mathcal{M}$.
Let $(\mathcal{M},\mathcal{G} := (g_i)_{i\in\mathbb{Z}})$ be a non-stationary dynamical system, δ-close to $(\mathcal{M},\mathcal{F})$. Take $x \in \mathcal{M}$. There exists $i \in \mathbb{Z}$ such that $x \in M_i$. The sequence $\{g_i^j(x)\}_{j\in\mathbb{Z}}$ is a δ-chain for $\mathcal{F}$ in $\mathcal{M}$ because, for any $i \in \mathbb{Z}$, we have

$$d(f_{i+j}(g_i^j(x)), g_i^{j+1}(x)) = d(f_{i+j}(g_i^j(x)), g_{i+j}(g_i^j(x)) < \delta.$$

By Theorem 3.5, there exists a unique point $y \in M_i$ such that the sequence $\{g_i^j(x)\}_{j\in\mathbb{Z}} \in \mathcal{M}$ is ε-shadowed by y. We set $h_i(x) := y$. So we have

$$d(f_i^j(h_i(x)), g_i^j(x)) < \varepsilon.$$

Since the point x is arbitrary, we can consider the sequence $\mathcal{H} := (h_i)_{i\in\mathbb{Z}}$ such that, for every $i \in \mathbb{Z}$, h_i is a map from M_i to M_i as above. We claim that $\mathcal{H} := (h_i)_{i\in\mathbb{Z}}$ is a conjugacy from $\mathcal{G}$ to$\mathcal{F}$. To imply this point, we prove the following steps.

1) Each h_i is well-defined because, for any $x \in M_i$, the sequence $\{g_i^j(x)\}$ is ε-shadowed by a unique point y of M_i.

2) Each h_i is continuous. Indeed, similar to proof of Theorem 3.5, we have

$$h_i(x) = \bigcap_{j=-\infty}^{\infty} f_i^j\Big(B((g_i^j)^{-1}(x),\alpha)\Big).$$

Note that α is the same as in Theorem 3.5. We can find a positive number $s \in \mathbb{Z}$ such that

$$d(h_i(x), \bigcap_{j=-s}^{s} f_i^j(B((g_i^j)^{-1}(x), \alpha)) < \frac{\varepsilon}{2},$$

and obviously, for z close enough to x in M_i, we have

$$d(h_i(x), \bigcap_{j=-s}^{s} f_i^j(B((g_i^j)^{-1}(z), \alpha)) < \varepsilon.$$

On the other hand,

$$h_i(z) = \bigcap_{j=-\infty}^{\infty} f_i^j(B((g_i^j)^{-1}(z), \alpha)),$$

which is a subset of the set $\bigcap_{j=-N}^{N} f_i^j(B((g_i^j)^{-1}(z), \alpha))$. Hence, we have

$$d(h_i(x), h_i(y)) < \varepsilon.$$

This implies that h_i is continuous, for any $i \in \mathbb{Z}$.

3) For all $i \in \mathbb{Z}$, we have $f_i \circ h_i = h_{i+1} \circ g_i$. By the beginning of the proof, for given $g_i(x) \in M_{i+1}$, the $\mathcal{F}$-orbit of the point $h_{i+1}(g_i(x))$ uniquely ε-shadows the sequence $\{g_{i+1}^j(g_i(x))\}_{j\in\mathbb{Z}}$. It means that, for any $i \in \mathbb{Z}$, we have

$$d(f_{i+1}^j(h_{i+1}(g_i(x))), g_{i+1}^j(g_i(x))) < \varepsilon.$$

On the other hand, we have

$$\varepsilon > d(f_i^j(h_i(x)), g_i^j(x)) = d(f_{i+1}^j \circ f_i(h_i(x)), g_{i+1}^j(g_i(x))).$$

Since the point $h_{i+1}(g_i(x)) \in M_{i+1}$ is the only point which ε-shadows the sequence $\{g_{i+1}^j(g_i(x))\}_{i\in\mathbb{Z}}$, we have $f_i \circ h_i(x) = h_{i+1} \circ g_i(x)$. The point $x \in \mathcal{M}$ is arbitrary, so, for all $i \in \mathbb{Z}$, we have

$$f_i \circ h_i = h_{i+1} \circ g_i.$$

4) Each h_i is onto. To prove this claim, we note that each h_i is ε-close to id_{M_i} or the identity map on M_i, because, by step 3), for $j = -i + 1$, we have

$$\varepsilon > d(f_i^{-i+1}(h_i(x)), g_i^{-i+1}(x)) = d(h_i(x), id_{M_i}(x)).$$

Also, h_i is homotopic to id_{M_i}, and so it induces an isomorphism on top homology group of M_i. Hence h_i is onto.

These properties implies that $\mathcal{H} := (h_i)_{i\in\mathbb{Z}}$ is a semi-conjugacy from $\mathcal{G}$ to $\mathcal{F}$, and so the Anosov family $(\mathcal{M}, \mathcal{F})$ is semi-stable. To prove that $\mathcal{H} := (h_i)_{i\in\mathbb{Z}}$ is a conjugacy from $\mathcal{G}$ to$\mathcal{F}$, we need the following step.

5) Each h_i is one to one. To this aim, let $h_i(x) = h_i(y)$. Since, by step 3), $h_{i+1} \circ g_i(x) = f_i \circ h_i(x)$, it is easily seen that $h_{i+j} \circ g_i^j(x) = f_i^j \circ h_i(x)$, and also, $h_{i+j} \circ g_i^j(y) = f_i^j \circ h_i(y)$. Therefore,

$$h_{i+j} \circ g_i^j(x) = h_{i+j} \circ g_i^j(y).$$

Again, we know that two sequences $\{g_{i+j}^k(g_i^j(x))\}_{k\in\mathbb{Z}}$ and $\{g_{i+j}^k(g_i^j(y))\}_{k\in\mathbb{Z}}$, are uniquely ε-shadowed by $h_{i+j} \circ g_i^j(x)$ and $h_{i+j} \circ g_i^j(y)$, respectively. Hence these two sequences are equal and so $x = y$.
□

Hereunder, we state necessary conditions to obtain the stability of an Anosov family in a non-stationary dynamical system.

Theorem 3.7. *Let $(\mathcal{M}, \mathcal{F})$ be a non-stationary dynamical system, and let $\Lambda = \sqcup_{i\in\mathbb{Z}}\Lambda_i$ be an isolated invariant subset of $\mathcal{M}$ with isolating neighborhood $\mathcal{U} = \sqcup_{i\in\mathbb{Z}}U_i$. If $(\Lambda, \mathcal{F}|_\Lambda)$ is an Anosov family, then $\mathcal{F}|_\Lambda$ is stable. That is, there exists $\varepsilon > 0$ such that for every non-stationary dynamical system $(\mathcal{M}, \mathcal{G})$, ε-close to $(\mathcal{M}, \mathcal{F})$, there exists an isolated invariant subset $\Gamma = \sqcup_{i\in\mathbb{Z}}\Gamma_i$ in $\mathcal{U}$ such that $\Gamma_i = \bigcap_{n=-\infty}^{\infty}(g_i^n)^{-1}(U_{i+1})$, and $(\Gamma, \mathcal{G}|_\Gamma)$ is an Anosov family conjugated to $(\Lambda, \mathcal{F}|_\Lambda)$.*

Proof. By the assumptions and the second part of Theorem 3.5, for given $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo trajectory in δ-neighborhood of Λ is uniquely ε-shadowed by some point in Λ. Since Λ is an isolated invariant set for $\mathcal{F}$ in $\mathcal{M}$, we can take a positive integer k such that the set $\bigcap_{d=-k}^{k}(f_i^j)^{-1}(\mathcal{U}_{i+j})$ is in $\frac{\delta}{2}$-neighborhood of Λ.
For some enough small C^0-neighborhood $\mathcal{V}_i$ of f_i, the set $\bigcap_{j=-k}^{k}(g_i^j)^{-1}(\mathcal{U}_{i+j})$ is a subset of $\frac{\delta}{2}$-neighborhood of Λ, for all $g_i \in \mathcal{V}_i$. Given $x \in \bigcap_{j=-k}^{k}(g_i^j)^{-1}(\mathcal{U}_{i+j})$, the sequence $\{g_i^j(x)\}_{j=-\infty}^{\infty}$ is a δ-pseudo trajectory for $\mathcal{F}$ in $\frac{\delta}{2}$-neighborhood of Λ. Consider $\Gamma_i = \bigcap_{j=-\infty}^{\infty}(g_i^j)^{-1}(\mathcal{U}_{i+j})$.
Each $\mathcal{V}_i$ can be taken so small such that for any $\mathcal{G} = (g_i)_{i\in\mathbb{Z}} \in V = (\mathcal{V}_i)_{i\in\mathbb{Z}}$, $(\Gamma, \mathcal{G})$ is an Anosov family.
Take $\mathcal{G} = (g_i)_{i\in\mathbb{Z}} \in V$. For any $x \in \Gamma_i$, the sequence $\{g_i^j(x)\}_{j\in\mathbb{Z}}$ is a δ-pseudo trajectory for $\mathcal{F}$. By Theorem 3.5 and similar to the proof of Theorem 3.6, there exists a sequence $\mathcal{H} = (h_i)_{i\in\mathbb{Z}}$ of continuous maps $h_i : \Gamma_i \to \Lambda_i$ such that for any $x \in \Gamma_i$, there exists a unique point $y = h_i(x) \in \Lambda_i$ satisfying $d(f_i^j \circ h_i(x), g_i^j(x)) < \varepsilon$, for all $j \in \mathbb{Z}$. Also, we have

$$h_{i+1} \circ g_i = f_i \circ h_i, \qquad i \in \mathbb{Z}.$$

Similarly, the sequence $\{f_i^j(z)\}_{j\in\mathbb{Z}}$ is a δ-pseudo trajectory for $\mathcal{G}$ that ε-shadowed by the unique point $w = \mathcal{K}_i(z)$. This defines a sequence $\mathcal{K} = (\mathcal{K}_i)_{i\in\mathbb{Z}}$ of continuous maps $\mathcal{K}_i$ from Λ_i to $\Gamma_i, i \in \mathbb{Z}$. Moreover, we have

$$\mathcal{K}_{i+1} \circ f_i = g_i \circ \mathcal{K}_i, \qquad i \in \mathbb{Z}.$$

Now, we prove that, for any i, h_i is one to one. Indeed, we prove that, for any $i \in \mathbb{Z}$, $\mathcal{K}_i \circ h_i, = id_{\Gamma_i}$.
Take $i \in \mathbb{Z}$, as we said above, for given $x \in \Gamma_i$, we have $d(f_i^j \circ h_i(x), g_i^j(x)) < \varepsilon$. This equality shows that the sequence $\{f_i^j(h_i(x))\}_{i\in\mathbb{Z}}$ is ε-shadowed by $\mathcal{G}$-orbit of x. So, we have $k_i \circ h_i(x) = x$. Now, if $h_i(x_1) = h_i(x_2)$, then $x_1 = k_i \circ h_i(x_1) = k_i \circ h_i(x_2) = x_2$. It implies that each h_i is one to one.
To prove that each h_i is onto, take i, and consider $y \in \Lambda_i$. We know that for any $j \in \mathbb{Z}$,
$d(g_i^j(k_i(y)), f_i^j(y)) < \varepsilon$. Hence the sequence $\{g_i^j(k_i(y))\}$ is ε-shadowed by $\mathcal{F}$-orbit of y, and, so $h_i(k_i(y)) = y$. It implies that h_i is onto, for any $i \in \mathbb{Z}$. □

Example 3.8. *Assume that f_a is an Anosov map of a Riemannian manifold M_a and that $(\mathcal{M}, \mathcal{F})$ is a lift of (M_a, f_a). It is known that every Anosov map has the shadowing property, and also, having the shadowing property of $(\mathcal{M}, \mathcal{F})$ is equivalent to having the shadowing property of f_a [6]. So, $(\mathcal{M}, \mathcal{F})$ is an Anosov family and also has the shadowing property.*

Example 3.9. *Let f_a be an Anosov map of a Riemannian manifold M_a and let $(\mathcal{M} := \sqcup_{i\in\mathbb{Z}}M_i, \mathcal{F} = (f_i)_{i\in\mathbb{Z}})$ be a lift of (M_a, f_a). Assume that $g_i : M_i \to M_{i+1}$, $i \in \mathbb{Z}$, is an arbitrary sequence from an α-neighborhood of f_a in the C^{1+1}-norm, for sufficiently small α. Then $(\mathcal{M}, \mathcal{G} = (g_i)_{i\in\mathbb{Z}})$ is an Anosov family [7]. Also, by Theorem 3.5, $(\mathcal{M}, \mathcal{F})$ has the shadowing property.*

Example 3.10. *Let $S = [0, 1] \times [0, 1]$ be the unit square, let A be the semidisk of radius $1/2$ on the bottom of S, and let B be the semidisk of radius $1/2$ on the top of S. Set $N = S \cup A \cup B$. We define the map f_a from N to itself such that first, f_a stretches N out to be over twice as tall and less than half as wide; second, it blends this longer and thinner region in the middle and puts it down. Hence it crosses S twice such that we have $f_a(N) \subset N$ and $f_a(B) \subset A$. Finally, we extend f_a to S^2 such that it takes the point at infinity to itself as a source for f_a on S^2. This diffeomorphism $f_a : S^2 \to S^2$ has the geometric (Smale) horseshoe, Λ_a, introduced by Smale in [33]. Indeed, Λ_a is an isolated invariant Cantor set in*

S^2. Let $(\mathcal{M}, \mathcal{F})$ be a lift of (S^2, f_a). Let Λ_i be a copy of Λ_a in M_i. Then $\Lambda = \sqcup_{i\in\mathbb{Z}}\Lambda_i$ is an isolated invariant subset of $\mathcal{M}$ and $(\Lambda, \mathcal{F}|_\Lambda)$ is an Anosov family. By Theorem 3.5, $(\Lambda, \mathcal{F}|_\Lambda)$ has the shadowing property. Moreover, Theorems 3.6 and 3.7 imply that the Anosov family $(\Lambda, \mathcal{F}|_\Lambda)$ is stable.

Remark 3.11. *For subsequent studies on the field of non-stationary dynamical systems, the Anosov closing lemma is an interesting subject. Chaos can also be a significant issue. Furthermore, we intend to study other types of shadowing properties in non-stationary dynamical systems, for example, limit shadowing and average shadowing properties.*

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