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Degree of convergence of a function of trigonometric series in Besov spaces

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Abstract. In this paper, we study the degree of convergence of the functions of Fourier series and derived Fourier series in Besov spaces using product Hausdorff (HK) means. We also study some applications of our main results.

1. Introduction

The Besov spaces $B_q^\lambda(L^r)$ is a set of functions f from L^r spaces, which have smoothness λ and the parameter q gives finer gradation of smoothness (see (4) and (10)). This is a tool to describe the smoothness properties of a function and contains a large number of fundamental spaces such as Sobolev, Lipschitz and Hölder spaces. These spaces appear naturally in many fields of analysis. Currently, there are two definitions of Besov spaces which are in use. First uses Fourier transforms and the second uses the modulus of smoothness of function f. These two definitions are equivalent only with certain restrictions on the parameter, e.g., they are different when $r < 1$ and λ is small.
It can be noted that the Besov spaces defined by the modulus of smoothness appear more naturally in many areas of analysis including approximation theory ([17]).
Further, we observe that the degree of approximation of a function only gives the degree of polynomial with respect to the given function while the degree of convergence of a function gives convergence of the polynomial with respect to the given function. Therefore, we aim to study the degree of convergence of the functions f and f' of Fourier series and derived Fourier series respectively in Bosov norms using product Hausdorff (HK) means. Detailed objectives of this paper will be presented in section 3.
The organization of the paper is as follows: In section 2, we give preliminaries and notations related to the present work of the paper. In section 3, we propose our main results for obtaining best approximation of a function f of Besov spaces $B_q^\lambda(L^r); r \geq 1, 1 < q \leq \infty$ using product Hausdorff means of Fourier series and derived Fourier series. Applications supported by numerical results, are discussed in section-4.

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2. Preliminaries and notations

2.1. Fourier series and derived Fourier series

Let f be a 2π-periodic Lebesgue integrable function defined on $[0, 2\pi]$. Then the Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1}$$

where a_0, a_n and b_n are Fourier coefficients.
The n^{th} partial sum of (1) is given by

$$s_n(f; x) = s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi_x(t) D_n(t) dt,$$

where

$$\phi_x(t) = f(x + t) + f(x - t) - 2f(x)$$

and $D_n(t)$ (Dirichlet kernal) is defined by

$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin(\frac{t}{2})}.$$

The derived Fourier series of (1) is given by

$$f'(x) \sim \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx), \tag{2}$$

which is defined by differentiating (1) term by term.
The n^{th} partial sum of (2) is given by

$$s_n'(f'; x) = s_n'(x) - f'(x) = \frac{1}{2\pi}\int_0^{\pi} D_n(t) dg_x(t),$$

where

$$g_x(t) = f(x + t) - f(x - t) - 2tf'(x)$$

and

$$dg_x(t) = d\{f(x + t) - f(x - t)\} - 2f'(x)dt.$$

2.2. Besov spaces

Let $C_{2\pi} := C[0, 2\pi]$ denote the Banach space of all 2π-periodic continuous functions defined on $[0, 2\pi]$ under the supremum norm and

$$L^r := L^r[0, 2\pi] := \left\{f : [0, 2\pi] \to \mathbb{R}; \int_0^{2\pi} |f(x)|^r dx < \infty, r \geq 1\right\}$$

be the space of all 2π-periodic integrable functions.
The L^r-norm of a function f is defined by

$$\|f\|_r = \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r dx\right\}^{\frac{1}{r}} & \text{for } 1 \leq r < \infty, \\ \operatorname{ess\,sup}_{x\in[0,2\pi]} |f(x)| & \text{for } r = \infty. \end{cases}$$

When $r = 2$, then we have

$$\|f\|_2 = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx\Big)^{\frac{1}{2}}.$$

The modulus of continuity of a funtion $f \in L^r$ space is defined by

$$\omega(f;l) := \sup_{\substack{x,x+h\in[0,2\pi]\\ |h|<l}} |f(x+h) - f(x)|. \tag{3}$$

The j^{th} order modulus of smoothness of a function $f \in L^r$ space is defined by

$$\omega_j(f,l)_r := \sup_{0<h\leq l} \|\Delta_h^j(f,\cdot)\|_r, \quad l > 0, \tag{4}$$

where

$$\Delta_h^j(f,x) := \sum_{\rho=0}^{j} (-1)^{j-\rho}\binom{j}{\rho} f(x+\rho h), \quad j \in \mathbb{N}. \tag{5}$$

A function $f \in Lip\lambda$ class of function if

$$f(x+h) - f(x) = O(|h|^{\lambda}) \quad \text{for } 0 < \lambda \leq 1 \tag{6}$$

and $f \in Lip(\lambda, r)$ class of function if

$$\left(\int_0^{2\pi} |f(x+h) - f(x)|^r dx\right)^{\frac{1}{r}} = O(|h|^{\lambda}) \quad \text{for } 0 < \lambda \leq 1,\ r \geq 1. \tag{7}$$

Let $\lambda > 0$, $j > \lambda$ i.e. $j = [\lambda] + 1$, where j being smallest integer. For $f \in L^r$, if

$$w_j(f,l)_r = O(l^{\lambda}), \quad l > 0, \tag{8}$$

then $f \in Lip^*(\lambda, r)$ (generalized Lipschitz class) and its semi-norm is given by

$$|f|_{Lip^*(\lambda,r)} = \sup_{l>0}\left(\frac{w_j(f,l)_r}{l^{\lambda}}\right), \tag{9}$$

Remark 2.1.

(i) If $r = \infty$, $j = 0$ and f being a continuous function, then $w_j(f,l)_r$ reduces to $w(f,l)$.

(ii) If $0 < r < \infty$, $j = 1$ and f being a continuous function, then $w_j(f,l)_r$ reduces to $w_1(f,l)_r$.

(iii) If $f \in C_{2\pi}$ and $w(f,l) = O(l^{\lambda})$, then $f \in Lip\lambda$.

(iv) If $f \in L^r, 0 < r < \infty$ and $w(f,l) = O(l^{\lambda}), 0 < \lambda \leq 1$ then $f \in Lip(\lambda, r)$.

(v) If $r = \infty$, then $Lip(\lambda, r)$ class reduces to $Lip\lambda$, i.e., $Lip(\lambda) \subseteq Lip(\lambda, r)$.

(vi) From (6) and (8), $Lip(\lambda, r) \subseteq Lip^*(\lambda, r)$.

Let $\lambda > 0$ and $j > \lambda$ i.e. $j = [\lambda] + 1$. For $0 < r < q \leq \infty$, the Besov space $B_q^{\lambda}(L^r)$ is a collection of all 2π-periodic function $f \in L^r$ such that

$$\|w_j(f,\cdot)\|_{\lambda,q} = |f|_{B_q^{\lambda}(L^r)} := \begin{cases} \left\{\int_0^{\pi}\left(\frac{w_j(f,l)_r}{l^{\lambda}}\right)^q \frac{dl}{l}\right\}^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{l>0}\left(\frac{w_j(f,l)_r}{l^{\lambda}}\right), & q = \infty. \end{cases} \tag{10}$$

is finite ([15, p. 237]). It is further observed that (10) is a semi-norm if $1 \leq r, q \leq \infty$ and a quasi semi-norm in other cases ([16]).
The quasi-norm for $B_q^\lambda(L^r)$ is given by

$$\|f\|_{B_q^\lambda(L^r)} := \|f\|_r + |f|_{B_q^\lambda(L^r)} = \|f\|_r + \|w_j(f,\cdot)\|_{\lambda,q}. \tag{11}$$

For $r = 2$, the semi-norm for $B_q^\lambda(L^2)$ is given by

$$\|f\|_{B_q^\lambda(L^2)} := \|f\|_2 + |f|_{B_q^\lambda(L^2)} = \|f\|_2 + \|w_j(f,\cdot)\|_{\lambda,q}. \tag{12}$$

2.3. *Summability Means*

Let

$$u_0 + u_1 + u_2 + \cdots = \sum_{n=0}^{\infty} u_n \tag{13}$$

be an infinite series with the sequence of its n^{th} partial sum $\{s_n\}$.

2.3.1. *C^α Means*

For $\alpha > -1$, the Cesàro means $\sigma_n^\alpha(f,x)$ of order α or C^α means of (13) are defined by

$$\sigma_n^\alpha(f;x) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k(f;x),$$

where

$$A_n^\alpha = \binom{\alpha+n}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$$

for $n = 1, 2, \cdots$ and $A_0^\alpha = 1$.
If $\sigma_n^\alpha \to s$ as $n \to \infty$, then (13) is said to be summable by C^α method to a definite number s.

2.3.2. *E^a Means*

If

$$(E,a) = E_n^a = \frac{1}{(1+a)}\sum_{k=0}^{n}\binom{n}{k} a^{n-k} s_k(f;x) \to s \text{ as } n \to \infty,$$

then (13) is said to be summable by (E^a) method to a definite number s.

2.3.3. *HK Means*

Let the Hausdorff matrix $H = (h_{n,m})$ or $K = (k_{n,m})$ have the entries

$$H(or\ K) = h_{n,m}(or\ k_{n,m}) = \begin{cases} \binom{n}{m}\Delta^{n-m}\mu_m, & 0 \leq m \leq n; \\ 0, & m > n, \end{cases} \tag{14}$$

where $\{\mu_m\}$ is any real or complex sequence and for any sequence μ_m, the operator Δ is defined by $\Delta\mu_m = \mu_m - \mu_{m+1}$ and $\Delta^{m+1}\mu_m = \Delta(\Delta^m \mu_m)$.
The product Hausdorff (HK) means of $s_n(f;x)$ is defined by

$$\gamma_n^{HK}(f;x) = \sum_{m=0}^{n} h_{n,m}\left(\sum_{p=0}^{m} k_{m,p} s_p(f;x)\right).$$

If $\gamma_n^{HK} \to s$ as $n \to \infty$, then (13) is said to be summable by HK method to a definite number s.

Remark 2.2.

(i) HK means reduces to $C^\alpha C^\beta$ means if $\chi(u)\cdot\chi(v) = \prod_{k=1}^{\alpha}\prod_{j=1}^{\beta} u^k v^j, \alpha \geq 1$ and $\beta \geq 1$.

(ii) HK means reduces to $E^a E^b$ means if $h_{n,m}k_{m,p} = \binom{n}{m}\binom{m}{p}\frac{a^{n-m}}{(1+a)^n}\frac{b^{m-p}}{(1+b)^m}, 0 \leq m \leq n$ and $0 \leq p \leq m$.

2.4. Degree of Convergence

The degree of convergence of a summation method to a given function f is a measure that how fast γ_n converges to f, which is given by

$$\|f-\gamma_n\| = O\left(\frac{1}{\lambda_n}\right) \ \ ([1]),$$

where γ_n is a trigonometric polynomial of degree n and $\lambda_n \to \infty$ as $n \to \infty$.
We write

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

$$\mathcal{M}_n(t) = \frac{1}{2\pi}\sum_{m=0}^{n} h_{n,m}\left(\sum_{p=0}^{m} k_{m,p}\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}\right). \tag{15}$$

$$L_n(x) = \gamma_n^{HK}(f;x) - f(x) = \int_0^{\pi} \phi_x(t)\mathcal{M}_n(t)dt.$$

3. Main Results

3.1. Degree of Convergence of a Function of Fourier Series

The degree of approximation of a function in function spaces viz, Hölder, generalized Hölder, generalized Zygmund and Lipschitz using different means of Fourier series, has been studied by the authors [2, 3, 5–11, 13, 14] etc.
In this subsection, we study the degree of convergence of a function f of Fourier series in Besov spaces $(B_q^{\lambda}(L^r), r \geq 1; 1 < q \leq \infty)$ using HK means.
First, we establish a following theorem to obtain the degree of convergence for f of Fourier series in $B_q^{\lambda}(L^r), r \geq 1; 1 < q \leq \infty$ using HK means:

Theorem 3.1. *Let f be a 2π-periodic and Lebesgue integrable function. Then for $0 \leq \delta < \lambda < 2$, the degree of convergence of f of Fourier series in the Besov space $B_q^{\lambda}(L^r), r \geq 1; 1 < q < \infty$ using HK means, is given by*

$$\|L_n(\cdot)\|_{B_q^{\delta}(L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda-\delta-\frac{1}{q} = 1.\end{cases} \tag{16}$$

The following lemmas are required for the proof of Theorem 3.1.

Lemma 3.2. $\mathcal{M}_n(t) = O(n+1)$ *for* $0 < t \leq \frac{1}{n+1}$.

Proof. For $0 < t \leq \frac{1}{n+1}, \sin(\frac{t}{2}) \geq \frac{t}{\pi}, |\sin(p+\frac{1}{2}t)| \leq (p+\frac{1}{2}t)$, $\sup_{0<v\leq 1}|\chi'(v)| = N$ and $\sup_{0<u\leq 1}|\chi'(u)| = M$.

$$\begin{aligned}|\mathcal{M}_n(t)| &= \left|\frac{1}{2\pi}\sum_{m=0}^{n} h_{n,m}\left(\sum_{p=0}^{m} k_{m,p}\cdot\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}\right)\right|\\ &= \left|\frac{1}{2\pi}\sum_{m=0}^{n}\binom{n}{m}\int_0^1 u^m(1-u)^{n-m}|d\chi(u)|\cdot\sum_{p=0}^{m}\binom{m}{p}\int_0^1 v^p(1-v)^{m-p}|d\chi(v)|\frac{|\sin(p+\frac{1}{2})t|}{|\sin(\frac{t}{2})|}\right|\\ &\leq \frac{MN}{2\pi}\left|\sum_{m=0}^{n}\binom{n}{m}\int_0^1 u^m(1-u)^{n-m}d\chi(u)\cdot\sum_{p=0}^{m}\binom{m}{p}\int_0^1 v^p(1-v)^{m-p}dv\cdot\frac{(p+\frac{1}{2})t}{(\frac{t}{\pi})}\right|\\ &\leq \frac{MN}{4}\left|\sum_{m=0}^{n}\binom{n}{m}\int_0^1 u^m(1-u)^{n-m}d\chi(u)\cdot\sum_{p=0}^{m}\binom{m}{p}\int_0^1 v^p(1-v)^{m-p}dv\cdot(2p+1)\right|\end{aligned} \tag{17}$$

We consider

$$\sum_{p=0}^{m}\binom{m}{p}\int_0^1 v^p(1-v)^{m-p}dv\cdot(2p+1)=\int_0^1\Big[2\sum_{p=0}^{m}\binom{m}{p}v^p(1-v)^{m-p}\cdot p+\sum_{p}^{m}\binom{m}{p}v^p(1-v)^{m-p}\Big]dv$$
$$=\int_0^1(A+B)dv. \tag{18}$$

Now, we consider

$$\begin{aligned}A&=2\sum_{p=0}^{m}\binom{m}{p}v^p(1-v)^{m-p}\cdot p\\&=2(1-v)^m\sum_{p=0}^{m}\binom{m}{p}\Big(\frac{v}{1-v}\Big)^p.p\\&=2(1-v)^m\sum_{p=0}^{m}\binom{m}{p}a^p.p,\qquad\Big(\text{where } a=\frac{v}{1-v}\Big)\\&=2(1-v)^m\left\{\binom{m}{0}a^0\cdot 0+\binom{m}{1}a^1+\binom{m}{2}a^2\cdot 2+\cdots+\binom{m}{m}a^m\cdot m\right\}\\&=2(1-v)^m\left\{\binom{m}{1}a+2\binom{m}{2}a^2+\cdots+m\binom{m}{m}a^m\right\}\end{aligned} \tag{19}$$

We have

$$(1+a)^m=\binom{m}{0}1^{m-0}a^0+\binom{m}{1}1^{m-1}a^1+\binom{m}{2}1^{m-2}a^2+\cdots+\binom{m}{m}1^{m-m}a^m$$
$$\implies (1+a)^m=\binom{m}{0}+\binom{m}{1}a^1+\binom{m}{2}a^2+\cdots+\binom{m}{m}a^m \tag{20}$$

Differentiating (20) with respect to a, we get

$$m(1+a)^{m-1}=0+\binom{m}{1}+2\binom{m}{2}a+\cdots+m\binom{m}{m}a^{m-1} \tag{21}$$

Multiplying (21) by a on both the sides, we get

$$a\cdot m(1+a)^{m-1}=\binom{m}{1}a+2\binom{m}{2}a^2+\cdots+m\binom{m}{m}a^m \tag{22}$$

From (19) and (22), we have

$$\begin{aligned}A&=2(1-v)^m\left\{a\cdot m(1+a)^{m-1}\right\}\\&=2(1-v)^m\Big(\frac{v}{1-v}\Big)m\Big(1+\frac{v}{1-v}\Big)^{m-1}\\&=2(1-v)^m\Big(\frac{v}{1-v}\Big)m\Big(\frac{1}{(1-v)^{m-1}}\Big)\\&=2mv.\end{aligned} \tag{23}$$

Now, we consider

$$
\begin{aligned}
B &= \sum_{p=0}^{m} \binom{m}{p} v^p (1-v)^{m-p} \\
&= \binom{m}{0} v^0 (1-v)^m + \binom{m}{1} v^1 (1-v)^{m-1} + \cdots + \binom{m}{m} v^m (1-v)^{m-m} \\
&= (1-v+v)^m \\
&= 1. \qquad (24)
\end{aligned}
$$

Using (23) and (24) in (18), we get

$$
\sum_{p=0}^{m} \int_0^1 \binom{m}{p} v^p (1-v)^{m-p} (2p+1) dv = \int_0^1 (2mv+1) dv = (m+1). \qquad (25)
$$

Using (25) in (17), we get

$$
\begin{aligned}
|\mathcal{M}_n(t)| &\le \frac{MN}{4} \left| \sum_{m=0}^{n} (m+1) \binom{n}{m} \int_0^1 u^m (1-u)^{n-m} d\chi(u) \right| \\
&\le \frac{MN}{4} \left| \int_0^1 \left\{ \sum_{m=0}^{n} \binom{n}{m} u^m (1-u)^{n-m} m + \sum_{m=0}^{n} \binom{n}{m} u^m (1-u)^{n-m} \right\} du \right| \\
&\le \frac{MN}{4} \left| \int_0^1 (nu+1) du \right| \\
&\le \frac{MN}{4} \left(\frac{n+2}{2} \right) \\
&= O(n+1)
\end{aligned}
$$

□

Lemma 3.3. $\mathcal{M}_n(t) = O\left(\frac{1}{(n+1)t^2}\right)$ *for* $\frac{1}{n+1} < t \le \pi$.

Proof. For $\frac{1}{n+1} < t \le \pi, \sin(\frac{t}{2}) \ge \frac{t}{\pi}, \sin^2 nt \le 1,\ \sup_{0\le u\le 1} |\chi'(u)| = M$ *and* $\sup_{0\le v\le 1} |\chi'(v)| = N$.

$$
\begin{aligned}
|\mathcal{M}_n(t)| &= \left| \frac{1}{2\pi} \sum_{m=0}^{n} h_{n,m} \left(\sum_{p=0}^{m} k_{m,p} \frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})} \right) \right| \\
&= \frac{1}{2\pi} \left| \sum_{m=0}^{n} \binom{n}{m} \int_0^1 u^m (1-u)^{n-m} |d\chi(u)| \cdot \sum_{p=0}^{m} \binom{m}{m} \int_0^1 v^p (1-v)^{m-p} |d\chi(v)| \frac{\sin\left(p+\frac{1}{2}\right)t}{\sin(\frac{t}{2})} \right| \\
&\le \frac{MN}{2t} \left| \sum_{m=0}^{n} \int_0^1 \binom{n}{m} u^m (1-u)^{n-m} du \cdot \sum_{p=0}^{m} \int_0^1 \binom{m}{p} v^p (1-v)^{m-p} dv \sin\left(p+\frac{1}{2}\right)t \right| \\
&\le \frac{MN}{2t} \left| \left\{ \sum_{m=0}^{n} \int_0^1 \binom{n}{m} u^m (1-u)^{n-m} du \right\} \cdot Im \sum_{p=0}^{m} \int_0^1 \binom{m}{p} v^p (1-v)^{m-p} e^{i(p+\frac{1}{2})t} dv \right| \qquad (26)
\end{aligned}
$$

We consider

$$
\begin{aligned}
& Im\sum_{p=0}^{m}\int_0^1\binom{m}{p}v^p(1-v)^{m-p}e^{i(p+\frac{1}{2})t}dv\\
&= (1-v)^m\cdot Im\left[\sum_{p=0}^{m}\int_0^1\binom{m}{p}v^p\frac{1}{(1-v)^p}e^{ipt}e^{\frac{it}{2}}dv\right]\\
&= (1-v)^m\cdot Im\left[e^{\frac{it}{2}}\int_0^1\left\{\sum_{p=0}^{m}\binom{m}{p}\left(\frac{ve^{it}}{1-v}\right)^p\right\}dv\right]\\
&= (1-v)^m\cdot Im\left[e^{\frac{it}{2}}\int_0^1\left\{\binom{m}{0}+\binom{m}{1}\left(\frac{ve^{it}}{1-v}\right)+\cdots+\binom{m}{m}\left(\frac{ve^{it}}{1-v}\right)^m\right\}dv\right]\\
&= Im\left[e^{\frac{it}{2}}\int_0^1\left\{\binom{m}{0}(1-v)^m+\binom{m}{1}(ve^{it})(1-v)^{m-1}+\cdots+\binom{m}{m}(ve^{it})^m(1-v)^0\right\}dv\right]\\
&= Im\left[e^{\frac{it}{2}}\int_0^1(1-v+ve^{it})^m dv\right]\\
&= Im\left[e^{\frac{it}{2}}\int_0^1\{1+v(e^{it}-1)\}^m dv\right]
\end{aligned}
\tag{27}
$$

Now, solving integration

$$\int_0^1\left\{1+v(e^{it}-1)\right\}^m dv$$

Considering

$$
\begin{aligned}
\left\{1+v(e^{it}-1)\right\} &= k\\
(e^{il}-1)dv &= dk\\
dv &= \frac{dk}{(e^{it}-1)}
\end{aligned}
$$

Thus,

$$\int_0^1\frac{k^m}{(e^{it}-1)}dk=\frac{1}{(e^{it}-1)}\left[\frac{\left\{1+v(e^{it}-1)\right\}^{m+1}}{(m+1)}\right]_0^1=\left[\frac{e^{i(m+1)t}-1}{(m+1)(e^{it}-1)}\right] \tag{28}$$

Using (28) in (27), we get

$$
\begin{aligned}
Im\sum_{p=0}^{m}\int_0^1\binom{m}{p}v^p(1-v)^{m-p}e^{i(p+\frac{1}{2})t}dv &= Im\left[e^{\frac{it}{2}}\cdot\frac{e^{i(m+1)t}-1}{(m+1)(e^{it}-1)}\right]\\
&= Im\left[\frac{e^{i(m+1)t}-1}{(m+1)(e^{\frac{it}{2}}-e^{-\frac{it}{2}})}\right]\\
&= Im\left[\frac{e^{i(m+1)t}-1}{(m+1)2i\sin(\frac{t}{2})}\right]\\
&= Im\left[\frac{\cos(m+1)t+i\sin(m+1)t-1}{(m+1)2i\sin(\frac{t}{2})}\right]
\end{aligned}
$$

$$= \left[\frac{1-\cos(m+1)t}{2(m+1)\sin(\frac{t}{2})}\right]$$

$$= \left[\frac{\sin^2(m+1)\frac{t}{2}}{(m+1)\sin(\frac{t}{2})}\right] \tag{29}$$

Using (29) in (26), we have

$$\begin{aligned}
|\mathcal{M}_n(t)| &\le \frac{MN}{2t}\left|\sum_{m=0}^{n}\int_0^1 \binom{n}{m}u^m(1-u)^{n-m}du \cdot \frac{\sin^2(m+1)\frac{t}{2}}{(m+1)\sin(\frac{t}{2})}\right| \\
&\le \frac{MN}{2t}\sum_{m=0}^{n}\int_0^1 \binom{n}{m}u^m(1-u)^{n-m}du \cdot \frac{1}{(m+1)}\frac{1}{|\sin(\frac{t}{2})|} \\
&\le \frac{MN}{2t}\int_0^1\left\{\sum_{m=0}^{n}\binom{n}{m}u^m(1-u)^{n-m}\cdot\frac{\pi}{(m+1)t}\right\}du \\
&\le \frac{MN\pi}{2t^2}\int_0^1\left\{O\left(\frac{(1-u+u)^n}{n+1}\right)\right\}du \\
&= O\left(\frac{1}{(n+1)t^2}\right)
\end{aligned}$$

Hence,

$$|\mathcal{M}_n(t)| = O\left(\frac{1}{(n+1)t^2}\right).$$

□

Lemma 3.4. *([12]) Let* $1 < r \le \infty$ *and* $0 < \lambda < 2$. *If* $f \in L^r$ *then for* $0 < l,\ t \le \pi$.

(i) $\|\Phi(\cdot,l,t)\|_r \le 4w_j(f,l)_r$,

(ii) $\|\Phi(\cdot,l,t)\|_r \le 4w_j(f,t)_r$,

(iii) $\|\Phi.(t)\|_r \le 2w_j(f,t)_r$,

where $j = [\lambda]+1$.

Lemma 3.5. *Let* $0 \le \delta < \lambda < 2$. *If* $g \in B_q^\lambda(L^r), r \ge 1, 1 < q < \infty$, *then*

(i) $\int_0^\pi |\mathcal{M}_n(t)|\left(\int_0^t \frac{\|\Phi(\cdot,l,t)\|_r^q}{l^{\delta q}}\frac{dl}{l}\right)^{\frac{1}{q}} dt = O\left\{\int_0^\pi (t^{\lambda-\delta}|(\mathcal{M}_n(t)|)^{\frac{q}{q-1}}dt\right\}^{1-\frac{1}{q}}$

(ii) $\int_0^\pi |\mathcal{M}_n(t)|\left(\int_t^\pi \frac{\|\Phi(\cdot,l,t)\|_r^q}{l^{\delta q}}\frac{dl}{l}\right)^{\frac{1}{q}} dt = O\left\{\int_0^\pi (t^{\lambda-\delta+\frac{1}{q}}|\mathcal{M}_n(t)|)^{\frac{q}{q-1}}dt\right\}^{1-\frac{1}{q}}$

Proof. *This lemma can be proved along the same lines of the proof of Lemma 1 of [12].* □

Proof of Theorem 3.1

Proof. Using the integral representation of $s_p(f;x)$, we have

$$s_p(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi_x(t)\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}dt. \tag{30}$$

Denoting the product Hausdorff mean of $s_n(f;x)$ by $\gamma_n^{HK}(f;x)$, we get

$$\begin{aligned}\gamma_n^{HK}(f;x)-f(x) &= \sum_{m=0}^{n} h_{n,m}\Bigg(\sum_{p=0}^{m} k_{m,p}\left\{s_p(f;x)-f(x)\right\}\Bigg)\\ &= \sum_{m=0}^{n} h_{n,m}\Bigg(\sum_{p=0}^{m} k_{m,p}\left\{\frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}dt\right\}\Bigg)\\ &= \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\sum_{m=0}^{n} h_{n,m}\Bigg(\sum_{p=0}^{m} k_{m,p}\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}\Bigg)dt\\ &= \int_0^{\pi}\phi_x(t)\mathcal{M}_n(t)dt. \end{aligned} \tag{31}$$

Let

$$L_n(x)=\gamma_n^{HK}(f;x)-f(x)=\int_0^{\pi}\phi_x(t)\mathcal{M}_n(t)dt. \tag{32}$$

We know that

$$\begin{aligned}\omega_j(L_n,l)_r &= \sup_{0<h\le l}\|\Delta_h^j(L_n,\cdot)\|_r,\ \ l>0\\ &= \begin{cases}\sup_{0<h\le l}\|L_n(\cdot+h)-L_n(\cdot)\|_r & 0<\lambda<1\\ \sup_{0<h\le l}\|L_n(\cdot+h)+L_n(\cdot-h)-2L_n(\cdot)\|_r & 1\le\lambda<2,\end{cases}\\ &= \|\Upsilon_n(\cdot,l)\|_r.\end{aligned}$$

Using the definition of Besov norm given by (11), we have

$$\|L_n(\cdot)\|_{B_q^{\delta}(L^r)}=\|L_n(\cdot)\|_r+\|w_j(L_n,\cdot)\|_{\delta,q}. \tag{33}$$

Using generalized Minkowski's inequality ([4]) and Lemma 3.4(iii), we have

$$\begin{aligned}\|L_n(\cdot)\|_r &\le \int_0^{\pi}\|\Phi_{\cdot}(t)\|_r|\mathcal{M}_n(t)|dt\\ &\le 2\int_0^{\pi} w_j(f;t)_r|\mathcal{M}_n(t)|dt.\end{aligned} \tag{34}$$

Using Hölder's inequality and the definition of Besov space given in (10), we have

$$\begin{aligned}\|L_n(\cdot)\|_r &\le 2\left\{\int_0^{\pi}\left(|\mathcal{M}_n(t)|t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{q-1}}dt\right\}^{1-\frac{1}{q}}\left\{\int_0^{\pi}\left(\frac{w_j(f,t)_r}{t^{\lambda+q^{-1}}}\right)^{q}dt\right\}^{\frac{1}{q}}\\ &= O\Bigg[\left\{\int_0^{\pi}\left(|\mathcal{M}_n(t)|t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{q-1}}dt\right\}^{1-\frac{1}{q}}\Bigg]\\ &= O\Bigg[\left\{\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\int_0^{\pi}\left(|\mathcal{M}_n(t)|t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{(q-1)}}dt\right\}^{1-\frac{1}{q}}\Bigg]\\ &= O\Bigg[\left\{\int_0^{\frac{1}{n+1}}\left(|\mathcal{M}_n(t)|t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{q-1}}dt\right\}^{1-\frac{1}{q}}+\left\{\int_{\frac{1}{n+1}}^{\pi}\left(|\mathcal{M}_n(t)|t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{q-1}}dt\right\}^{1-\frac{1}{q}}\Bigg]\\ &= I_1+I_2.\end{aligned} \tag{35}$$

Using Lemma 3.2, we get

$$
\begin{aligned}
I_1 &= O\left\{\int_0^{\frac{1}{n+1}} \left(|\mathcal{M}_n(t)| t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{(q-1)}} dt\right\}^{1-\frac{1}{q}} \\
&= O\left(\int_0^{\frac{1}{n+1}} \left\{(n+1)t^{\lambda+\frac{1}{q}}\right\}^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}} \\
&= O\left[(n+1)^{\frac{q}{q-1}} \int_0^{\frac{1}{n+1}} \left\{(t)^{\lambda+\frac{1}{q}}\right\}^{\frac{q}{q-1}} dt\right]^{\frac{q-1}{q}} \\
&= O\left[(n+1)^{\frac{q}{q-1}} \int_0^{\frac{1}{n+1}} (t)^{\frac{\lambda q+1}{q-1}} dt\right]^{\frac{q-1}{q}} \\
&= O\left[(n+1)^{\frac{q}{q-1}} \cdot \left(\frac{q-1}{\lambda q+q}\right)\left(\frac{1}{n+1}\right)^{\frac{q(\lambda+1)}{q-1}}\right]^{\frac{q-1}{q}} \\
&= O\left[(n+1) \cdot \left(\frac{q-1}{\lambda q+q}\right)\frac{1}{(n+1)^{\lambda+1}}\right] \\
&= O\left(\frac{1}{(n+1)^{\lambda}}\right).
\end{aligned} \tag{36}
$$

Using Lemma 3.3, we get

$$
\begin{aligned}
I_2 &= O\left\{\int_{\frac{1}{n+1}}^{\pi} \left(|\mathcal{M}_n(t)| t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}} \\
&= O\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{(n+1)t^2} \cdot t^{\lambda+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}} \\
&= O\left\{\left(\frac{1}{n+1}\right)^{\frac{q}{q-1}} \int_{\frac{1}{n+1}}^{\pi} \left(t^{\lambda+\frac{1}{q}-2}\right)^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}} \\
&= O\begin{cases} (n+1)^{-1}, & \lambda > 1, \\ (n+1)^{-\lambda}, & \lambda < 1, \\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda = 1. \end{cases}
\end{aligned} \tag{37}
$$

Combining (35)-(37), we have

$$
\|L_n(\cdot)\|_r = O\begin{cases} (n+1)^{-1}, & \lambda > 1, \\ (n+1)^{-\lambda}, & \lambda < 1, \\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda = 1. \end{cases} \tag{38}
$$

Using generalized Minkowsli's inequality ([4]), we get

$$
\begin{aligned}
\|w_j(L_n,\cdot)\|_{\delta,q} &= \left[\int_0^{\pi} \left(\frac{w_j(L_n,l)}{l^{\delta}}\right)^q \frac{dl}{l}\right]^{\frac{1}{q}} \\
&= \left[\int_0^{\pi} \left(\frac{\|\Upsilon_n(\cdot,l)\|_r}{l^{\delta}}\right)^q \frac{dl}{l}\right]^{\frac{1}{q}}
\end{aligned}
$$

$$\leq \int_0^\pi |\mathcal{M}_n(t)| dt \left(\int_0^\pi \|\Phi(\cdot, l, t)\|_r^q \frac{dl}{l^{\delta q+1}} \right)^{\frac{1}{q}}$$

$$\leq \int_0^\pi |\mathcal{M}_n(t)| \left[\int_0^t \frac{\|\Phi(\cdot, l, t)\|_r^q}{l^{\delta q}} \cdot \frac{dl}{l} \right]^{\frac{1}{q}} dt + \int_0^\pi |\mathcal{M}_n(t)| \left[\int_t^\pi \frac{\|\Phi(\cdot, l, t)\|_r^q}{l^{\delta q}} \cdot \frac{dl}{l} \right]^{\frac{1}{q}} dt$$

Using Lemma 3.5, we get

$$\|w_j(L_n, \cdot)\|_{\delta,q} = O\left[\int_0^\pi \left\{ |\mathcal{M}_n(t)| t^{\lambda-\delta} \right\}^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}} + O\left[\int_0^\pi \left\{ |\mathcal{M}_n(t)| t^{\lambda-\delta+\frac{1}{q}} \right\}^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= O(I_3 + I_4) \tag{39}$$

Since $(x+y)^r \leq x^r + y^r$ for positive x, y and $0 < r \leq 1$ for $r = 1 - \frac{1}{q} < 1$ then

$$I_3 = \left[\int_0^\pi \left(|\mathcal{M}_n(t)| t^{\lambda-\delta} \right)^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= \left[\int_0^{\frac{1}{n+1}} \left(|\mathcal{M}_n(t)| t^{\lambda-\delta} \right)^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}} + \left[\int_{\frac{1}{n+1}}^\pi \left(|\mathcal{M}_n(t)| t^{\lambda-\delta} \right)^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= I_{31} + I_{32} \tag{40}$$

Using Lemma 3.2, we have

$$I_{31} = \left[\int_0^{\frac{1}{n+1}} \left(|\mathcal{M}_n(t)| t^{\lambda-\delta} \right)^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= O\left[\int_0^{\frac{1}{n+1}} \left\{ t^{\lambda-\delta}(n+1) \right\}^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= O\left[(n+1)^{-\lambda+\delta+\frac{1}{q}} \right]. \tag{41}$$

Using Lemma 3.3, we have

$$I_{32} = \left[\int_{\frac{1}{n+1}}^\pi \left(|\mathcal{M}_n(t)| t^{\lambda-\delta} \right)^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= O\left[\int_{\frac{1}{n+1}}^\pi \left\{ \left(\frac{1}{(n+1)t^2} \right) t^{(\lambda-\delta)} \right\}^{\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= O\left(\frac{1}{n+1} \right) \cdot \left[\int_{\frac{1}{n+1}}^\pi t^{(\lambda-\delta-2)\frac{q}{q-1}} dt \right]^{1-\frac{1}{q}}$$
$$= O \begin{cases} (n+1)^{-1}, & \lambda - \delta - \frac{1}{q} > 1, \\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda - \delta - \frac{1}{q} < 1, \\ (n+1)^{-1} \{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda - \delta - \frac{1}{q} = 1. \end{cases} \tag{42}$$

Combining (40)-(42), we have

$$I_3 = O \begin{cases} (n+1)^{-1}, & \lambda - \delta - \frac{1}{q} > 1, \\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda - \delta - \frac{1}{q} < 1, \\ (n+1)^{-1} \{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda - \delta - \frac{1}{q} = 1. \end{cases} \tag{43}$$

Again using the inequality $(x+y)^r \leq x^r + y^r$ for positive x, y and $0 < r \leq 1$ for $r = 1 - \frac{1}{q} < 1$ then

$$I_4 = O\left[\int_0^{\pi} \left\{|\mathcal{M}_n(t)| t^{\lambda-\delta+\frac{1}{q}}\right\}^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= \left[\int_0^{\frac{1}{n+1}} \left(|\mathcal{M}_n(t)| t^{\lambda-\delta+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}} + \left[\int_{\frac{1}{n+1}}^{\pi} \left(|\mathcal{M}_n(t)| t^{\lambda-\delta+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= I_{41} + I_{42}. \tag{44}$$

Using Lemma 3.2, we have

$$I_{41} = \left[\int_0^{\frac{1}{n+1}} \left(|\mathcal{M}_n(t)| t^{\lambda-\delta+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= O\left[\int_0^{\frac{1}{n+1}} \left(t^{\lambda-\delta+\frac{1}{q}}(n+1)\right)^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= O\left(\frac{1}{(n+1)^{\lambda-\delta}}\right) \tag{45}$$

Using Lemma 3.3, we have

$$I_{42} = \left[\int_{\frac{1}{n+1}}^{\pi} \left(|\mathcal{M}_n(t)| t^{\lambda-\delta+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{(n+1)t^2} \cdot t^{\lambda-\delta+\frac{1}{q}}\right)^{\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= O\left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\pi} t^{(\lambda-\delta+\frac{1}{q})\frac{q}{q-1}} dt\right]^{1-\frac{1}{q}}$$
$$= O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda-\delta-\frac{1}{q} = 1.\end{cases} \tag{46}$$

Combining (44)-(46), we have

$$I_4 = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda-\delta-\frac{1}{q} = 1.\end{cases} \tag{47}$$

Now, combining (39), (43) and (47), we have

$$\|w_j(L_n,\cdot)\|_{\delta,q} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda-\delta-\frac{1}{q} = 1.\end{cases} \tag{48}$$

Combining (33), (38) and (48), we have

$$\|L_n(\cdot)\|_{B_q^{\delta}(L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}} & \lambda-\delta-\frac{1}{q} = 1.\end{cases} \tag{49}$$

□

Remark 3.6. *When $q = \infty$, then Besov space $B_q^\lambda(L^r), r \geq 1, \lambda \geq 0$ reduces to generalized Lipschitz class $Lip^*(\lambda, r)$ and the corresponding norm $\|\cdot\|_{B_\infty^\lambda(L^r)}$ is given by*

$$\|f\|_{B_\infty^\lambda(L^r)} = \|f\|_{Lip^*(\lambda,r)} = \|f\|_r + \sup_{l>0} \frac{w_j(f,l)_r}{l^\lambda}. \tag{50}$$

Thus, in view of Remark 3.6, we establish the following theorem to obtain degree of convergence for $f \in Lip^*(\lambda, r), r \geq 1, q = \infty$:

Theorem 3.7. *Let f be a 2π-periodic and Lebesgue integrable function belonging to generalized Lipschitz class $Lip^*(\lambda, L^r), r \geq 1; q = \infty$. Then for $0 \leq \delta < \lambda < 2$, the degree of convergence of a function f of Fourier series using HK means, is given by*

$$\|L_n(\cdot)\|_{Lip^*(\delta,L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta > 1,\\ (n+1)^{-\lambda+\delta}, & \lambda-\delta < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\} & \lambda-\delta = 1.\end{cases} \tag{51}$$

The following Lemma is required for the proof of Theorem 3.7.

Lemma 3.8. *([12]) Let $0 \leq \delta < \lambda < 2$. If $f \in B_q^\lambda(L^r), r \geq 1, q = \infty$, then*

$$\sup_{0<l,t\leq\pi}\left(\frac{\|\Phi(\cdot,l,t)\|_r}{l^\delta}\right) = O(t^{\lambda-\delta}). \tag{52}$$

Proof of Theorem 3.7

Proof. By the definition of the Besov norm given in (50), we have

$$\|L_n(\cdot)\|_{B_\infty^\delta(L^r)} = \|L_n(\cdot)\|_r + \|w_j(L_n,\cdot)\|_{\delta,\infty}. \tag{53}$$

Using (8) in (34), we get

$$\begin{aligned}\|L_n(\cdot)\|_r &\leq 2\int_0^\pi w_j(f,t)_r|\mathcal{M}_n(t)|dt\\ &= O\left[\int_0^{\frac{1}{n+1}} t^\lambda|\mathcal{M}_n(t)|dt + \int_{\frac{1}{n+1}}^\pi t^\lambda|\mathcal{M}_n(t)|dt\right]\\ &= O(J_1+J_2).\end{aligned} \tag{54}$$

Using Lemma 3.2, we get

$$\begin{aligned}J_1 &= \int_0^{\frac{1}{n+1}} t^\lambda|\mathcal{M}_n(t)|dt\\ &= O(n+1)\int_0^{\frac{1}{n+1}} t^\lambda dt\\ &= O(n+1)^{-\delta}.\end{aligned} \tag{55}$$

Using Lemma 3.3, we get

$$
\begin{aligned}
J_2 &= \int_{\frac{1}{n+1}}^{\pi} t^{\lambda} |\mathcal{M}_n(t)| dt \\
&= \int_{\frac{1}{n+1}}^{\pi} t^{\lambda} \frac{1}{(n+1)t^2} dt \\
&= \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} t^{\lambda-2} dt \\
&= \begin{cases} (n+1)^{-1}, & \lambda > 1, \\ (n+1)^{-\lambda}, & \lambda < 1, \\ (n+1)^{-1}\{\log(n+1)\pi\} & \lambda = 1. \end{cases}
\end{aligned} \tag{56}
$$

Combining (54)-(56), we get

$$
\|L_n(\cdot)\|_r = \begin{cases} (n+1)^{-1}, & \lambda > 1, \\ (n+1)^{-\lambda}, & \lambda < 1, \\ (n+1)^{-1}\{\log(n+1)\pi\} & \lambda = 1. \end{cases} \tag{57}
$$

Using the generalized Minkowsli's inequality ([4]), we get

$$
\begin{aligned}
\|w_j(L_n, \cdot)\|_{\delta,\infty} &= \sup_{l>0} \left(\frac{w_j(L_n, l)_r}{l^{\delta}} \right) \\
&= \sup_{l>0} \left(\frac{\|\Upsilon_n(\cdot, l)\|_r}{l^{\delta}} \right) \\
&= \sup_{l>0} \left[\frac{1}{l^{\delta}} \left(\frac{1}{2\pi} \int_0^{2\pi} |\Upsilon_n(x, l)|^r dx \right)^{\frac{1}{r}} \right] \\
&= \sup_{l>0} \left[\frac{1}{l^{\delta}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{\pi} \Phi(x, l, t) \mathcal{M}_n(t) dt \right|^r dx \right)^{\frac{1}{r}} \right] \\
&\leq \sup_{l>0} \left[\frac{1}{l^{\delta}} \left(\frac{1}{2\pi} \right)^{\frac{1}{r}} \int_0^{\pi} \left\{ \int_0^{2\pi} |\Phi(x, l, t)|^r \cdot |\mathcal{M}_n(t)|^r dx \right\}^{\frac{1}{r}} dt \right] \\
&= \sup_{l>0} \left[\frac{1}{l^{\delta}} \int_0^{\pi} \|\Phi(\cdot, l, t)\|_r |\mathcal{M}_n(t)| dt \right] \\
&\leq \int_0^{\pi} \left(\sup_{l>0} \frac{\|\Phi(\cdot, l, t)\|_r}{l^{\delta}} \right) |\mathcal{M}_n(t)| dt
\end{aligned}
$$

Using Lemma 3.8, we get

$$
\begin{aligned}
\|w_j(L_n, \cdot)\|_{\delta,\infty} &= O \int_0^{\pi} t^{\lambda-\delta} |\mathcal{M}_n(t)| dt \\
&= O \left[\int_0^{\frac{1}{n+1}} t^{\lambda-\delta} |\mathcal{M}_n(t)| dt + \int_{\frac{1}{n+1}}^{\pi} t^{\lambda-\delta} |\mathcal{M}_n(t)| dt \right] \\
&= O(R_1 + R_2).
\end{aligned} \tag{58}
$$

Using Lemma 3.2, we get

$$\begin{aligned} R_1 &= \left[\int_0^{\frac{1}{n+1}} t^{\lambda}|\mathcal{M}_n(t)|dt\right] \\ &= O(n+1)\int_0^{\frac{1}{n+1}} t^{\lambda-\delta}dt \\ &= O\left(\frac{1}{(n+1)^{\lambda-\delta}}\right) \end{aligned} \tag{59}$$

Using Lemma 3.3, we get

$$\begin{aligned} R_2 &= \left[\int_{\frac{1}{n+1}}^{\pi} t^{\lambda-\delta}|\mathcal{M}_n(t)|dt\right] \\ &= \left[\int_{\frac{1}{n+1}}^{\pi} t^{\lambda-\delta}\cdot\frac{1}{(n+1)t^2}dt\right] \\ &= O\left(\frac{1}{n+1}\right)\int_0^{\frac{1}{n+1}} t^{\lambda-\delta-2}dt \\ &= O\begin{cases}(n+1)^{-1}, & \lambda-\delta>1,\\ (n+1)^{-\lambda+\delta}, & \lambda-\delta<1,\\ (n+1)^{-1}\{\log(n+1)\pi\}, & \lambda-\delta=1.\end{cases} \end{aligned} \tag{60}$$

Combining (58)-(60), we get

$$\|w_j(L_n,\cdot)\|_{\delta,\infty} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta>1,\\ (n+1)^{-\lambda+\delta}, & \lambda-\delta<1,\\ (n+1)^{-1}\{\log(n+1)\pi\}, & \lambda-\delta=1.\end{cases} \tag{61}$$

Combining (53), (57) and (61), we have

$$\|L_n(\cdot)\|_{B^{\delta}_{\infty}(L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta>1,\\ (n+1)^{-\lambda+\delta}, & \lambda-\delta<1,\\ (n+1)^{-1}\{\log(n+1)\pi\}, & \lambda-\delta=1.\end{cases} \tag{62}$$

□

Corollaries

The following corollaries are deduced from Theorem 3.1.

Corollary 3.9. *The error estimation of a function $f \in B^{\delta}_q(L^r), r \geq 1, 1 < q \leq \infty$ by ΔHC^{α} means of its Fourier series is given by*

$$\|L_n(\cdot)\|_{B^{\delta}_q(L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q}>1,\\ (n+1)^{-\lambda+\delta+\frac{1}{q}}, & \lambda-\delta-\frac{1}{q}<1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}}, & \lambda-\delta-\frac{1}{q}=1.\end{cases}$$

Corollary 3.10. *The error estimation of a function $f \in B_q^{\delta}(L^r), r \geq 1, 1 < q \leq \infty$ by ΔHE^q means of its Fourier series is given by*

$$\|L_n(\cdot)\|_{B_q^{\delta}(L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta-\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} = 1.\end{cases}$$

Corollary 3.11. *The error estimation of a function $f \in B_q^{\delta}(L^r), r \geq 1, 1 < q \leq \infty$ by $C^{\alpha}E^q$ means of its Fourier series is given by*

$$\|L_n(\cdot)\|_{B_q^{\delta}(L^r)} = O\begin{cases}(n+1)^{-1}, & \lambda-\delta-\frac{1}{q} > 1,\\ (n+1)^{-\lambda+\delta-\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} < 1,\\ (n+1)^{-1}\{\log(n+1)\pi\}^{1-\frac{1}{q}}, & \lambda-\delta-\frac{1}{q} = 1.\end{cases}$$

3.2. Degree of Convergence of a Function of Derived Fourier Series

In this subsection, we study the degree of convergence of a function in Besov space using HK means of derived Fourier series and establish the following theorem. Also we observe that the result obtained in a following theorem provides best approximation of the function f' in Besov norm.

Remark 3.12. *Since the derived Fourier series converges uniformly in L^2-norm, we find the degree of convergence of derived Fourier series in L^2-norm.*

Theorem 3.13. *Let f' be a 2π-periodic and integrable function belonging to Besov space $B_q^{\lambda}(L^2), 1 < q < \infty$. Then for $0 \leq \delta < \lambda < 2$, the degree of convergence of a function f' of derived Fourier series using HK mean, is given by*

$$\begin{aligned}\|L'_n(\cdot)\|_{B_q^{\delta}(L^2)} &= O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} \frac{|dg_x(t)|}{t^2}\right)\\ &\quad + O\left((n+1)\int_0^{\frac{1}{n+1}} \left(t^{-\delta-\frac{1}{q}}\right)|dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\left(t^{-\delta-\frac{1}{q}-2}\right)|dg_x(t)|\right).\end{aligned} \tag{63}$$

Proof of Theorem 3.13

Proof. Using the integral representation of $s'_p(f';x)$, we have

$$s'_p(f;x) - f'(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})} dg_x(t). \tag{64}$$

Let $\gamma_n'^{HK}(f';x)$ represents a product Hausdorff mean of $s'_n(f';x)$, then we write

$$\begin{aligned}\gamma_n'^{HK}(f';x) - f'(x) &= \sum_{m=0}^{n} h_{n,m}\left(\sum_{p=0}^{m} k_{m,p}\left\{s'_p(f';x) - f'(x)\right\}\right)\\ &= \sum_{m=0}^{n} h_{n,m}\left(\sum_{p=0}^{m} k_{m,p}\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}dg_x(t)\right\}\right)\\ &= \frac{1}{2\pi}\int_0^{\pi}\sum_{m=0}^{n} h_{n,m}\left(\sum_{p=0}^{m} k_{m,p}\frac{\sin(p+\frac{1}{2})t}{\sin(\frac{t}{2})}\right)dg_x(t)\\ &= \int_0^{\pi}\mathcal{M}_n(t)dg_x(t).\end{aligned}$$

Let

$$L'_n(x) = \gamma'^{HK}_n(f';x) - f'(x) = \int_0^\pi \mathcal{M}_n(t)dg_x(t). \tag{65}$$

Using the definition of Besov norm given by (12), we have

$$||L'_n(\cdot)||_{B^\delta_q(L^2)} = ||L'_n(\cdot)||_2 + ||w_j(L'_n,\cdot)_2||_{\delta,q}. \tag{66}$$

Using generalized Minkowski's inequality ([4]), we have

$$\begin{aligned} ||L'_n(\cdot)||_2 &\le \int_0^\pi |\mathcal{M}_n(t)||dg_x(t)| \\ &\le \int_0^{\frac{1}{n+1}} |\mathcal{M}_n(t)||dg_x(t)| + \int_{\frac{1}{n+1}}^\pi |\mathcal{M}_n(t)||dg_x(t)| \end{aligned} \tag{67}$$

Now, using Lemmas 3.2 and 3.3, we have

$$||L'_n(\cdot)||_2 = O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi \frac{|dg_x(t)|}{t^2}\right). \tag{68}$$

Now, using the definition of Besov space, we have

$$||w_j(L'_n,\cdot)_2||_{\delta,q} \le \left\{\int_0^\pi \left(\frac{||L_n(\cdot,l)||_2}{l^\delta}\right)^q \frac{dl}{l}\right\}^{\frac{1}{q}} \tag{69}$$

Using generalized Minkowski's inequality ([4]), we have

$$\begin{aligned} ||w_j(L'_n,\cdot)_2||_{\delta,q} &\le \left\{\int_0^\pi \Big(|\mathcal{M}_n(t)|\cdot|dg_x(t)|\Big)^q \frac{dl}{l^{\delta q+1}}\right\}^{\frac{1}{q}} \\ &\le \left\{\int_0^\pi |\mathcal{M}_n(t)|\cdot|dg_x(t)|\left(\int_0^\pi \frac{dl}{l^{\delta q+1}}\right)^{\frac{1}{q}}\right\} \\ &\le \left\{\int_0^\pi |\mathcal{M}_n(t)|\cdot|dg_x(t)|\left(\int_0^t \frac{dl}{l^{\delta q+1}}\right)^{\frac{1}{q}}\right\} + \left\{\int_0^\pi |\mathcal{M}_n(t)|\cdot|dg_x(t)|\left(\int_t^\pi \frac{dl}{l^{\delta q+1}}\right)^{\frac{1}{q}}\right\} \end{aligned} \tag{70}$$

Using the second mean value theorem, we have

$$\begin{aligned} ||w_j(L'_n,\cdot)_2||_{\delta,q} &\le \left\{\int_0^\pi |\mathcal{M}_n(t)|\cdot|dg_x(t)|(t^{-\delta-\frac{1}{q}})\right\} \\ &\le \left\{\int_0^{\frac{1}{n+1}} |\mathcal{M}_n(t)|(t^{-\delta-\frac{1}{q}})|dg_x(t)|\right\} + \left\{\int_{\frac{1}{n+1}}^\pi |\mathcal{M}_n(t)|(t^{-\delta-\frac{1}{q}})|dg_x(t)|\right\}. \end{aligned} \tag{71}$$

Using Lemmas 3.2 and 3.3, we have

$$||w_j(L'_n,\cdot)_2||_{\delta,q} = O\left((n+1)\int_0^{\frac{1}{n+1}} \left(t^{-\delta-\frac{1}{q}}\right)|dg_x(t)| + \frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi \left(t^{-\delta-\frac{1}{q}-2}\right)|dg_x(t)|\right). \tag{72}$$

Combining (66), (68) and (72), we have

$$\begin{aligned} ||L'_n(\cdot)_2||_{B^\delta_q(L_2)} &= O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi \frac{|dg_x(t)|}{t^2}\right) \\ &\quad + O\left((n+1)\int_0^{\frac{1}{n+1}} \left(t^{-\delta-\frac{1}{q}}\right)|dg_x(t)| + \frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi \left(t^{-\delta-\frac{1}{q}-2}\right)|dg_x(t)|\right). \end{aligned} \tag{73}$$

□

Remark 3.14. *When $q = \infty$ the Besov space $B^{\delta}_{q}(L^r), r \geq 1, \delta \geq 0$ reduces to generalized Lipschitz class $Lip^*(\delta, r)$ and the corresponding norm $\|\cdot\|_{B^{\delta}_{\infty}(L^r)}$ is given by*

$$\|f\|_{B^{\delta}_{\infty}(L_r)} = \|f\|_{Lip^*(\delta,r)} = \|f\|_r + \sup_{l>0} \frac{w_j(f,l)_r}{l^{\delta}}. \tag{74}$$

Thus, in view of Remark 3.14, we estiblish the following theorem to obtain error estimation for $f' \in Lip^*(\delta, 2), r = 2, q = \infty$:

Theorem 3.15. *Let f' be a 2π-periodic and integrable function belonging to generalized Lipschitz class $Lip^*(\lambda, L^2), q = \infty$. Then for $0 \leq \delta < \lambda < 2$, the degree of convergence of a function f' of derived Fourier series using HK means, is given by*

$$\begin{aligned}\|L'_n(\cdot)\|_{Lip^*(\delta,L^2)} &= O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{|dg_x(t)|}{t^2}\right) \\ &\quad + O(n+1)\sup_{0\leq l,t\leq\pi}\left(\int_0^{\frac{1}{n+1}} (l^{-\delta})|dg_x(t)|\right) + \left(\frac{1}{n+1}\right)\sup_{0\leq l,t\leq\pi}\left(\int_{\frac{1}{n+1}}^{\pi} (l^{-\delta-2})|dg_x(t)|\right).\end{aligned} \tag{75}$$

Proof of Theorem 3.15

Proof. By the definition of the Besov norm given in (74), we have

$$\|L'_n(\cdot)\|_{Lip^*(\delta,L_2)} = \|L'_n(\cdot)\|_2 + \|w_j(L'_n, \cdot)_2\|_{\delta,\infty}. \tag{76}$$

Using the generalized Minkowski's inequality ([4]), we have

$$\begin{aligned}\|w_j(L'_n, \cdot)_2\|_{\delta,\infty} &= \sup_{0\leq l,t\leq\pi}\left(\frac{w_j(L'_n, l)_2}{l^{\delta}}\right) \\ &= \sup_{0\leq l,t\leq\pi}\left(\frac{\|\Upsilon'_n(\cdot, l)\|_2}{l^{\delta}}\right) \\ &\leq \sup_{0\leq l,t\leq\pi}\left(\frac{1}{l^{\delta}}\int_0^{\pi} |\mathcal{M}_n(t)|\cdot|dg_x(t)|\right) \\ &\leq \sup_{0\leq l,t\leq\pi}\left(\frac{1}{l^{\delta}}\int_0^{\frac{1}{n+1}} |\mathcal{M}_n(t)|\cdot|dg_x(t)| + \frac{1}{l^{\delta}}\int_{\frac{1}{n+1}}^{\pi} |\mathcal{M}_n(t)|\cdot|dg_x(t)|\right)\end{aligned}$$

Using Lemma 3.8, we get

$$\|w_j(L'_n, \cdot)_2\|_{\delta,\infty} = O(n+1)\sup_{0\leq l,t\leq\pi}\left(\int_0^{\frac{1}{n+1}} (l^{-\delta})|dg_x(t)|\right) + O\left(\frac{1}{n+1}\right)\sup_{0\leq l,t\leq\pi}\left(\int_{\frac{1}{n+1}}^{\pi} (l^{-\delta-2})|dg_x(t)|\right). \tag{77}$$

From (68), (76) and (77), we have

$$\begin{aligned}\|L'_n(\cdot)\|_{Lip^*(\delta,L^2)} &= O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{|dg_x(t)|}{t^2}\right) \\ &\quad + O(n+1)\sup_{0\leq l,t\leq\pi}\left(\int_0^{\frac{1}{n+1}} (l^{-\delta})|dg_x(t)|\right) + O\left(\frac{1}{n+1}\right)\sup_{0\leq l,t\leq\pi}\left(\int_{\frac{1}{n+1}}^{\pi} (l^{-\delta-2})|dg_x(t)|\right).\end{aligned} \tag{78}$$

□

Corollaries

The following corollaries are deduced from Theorem 3.13:

Corollary 3.16. *The error estimation of a function* $f \in B_q^{\delta}(L^2), 1 < q \le \infty$ *by* ΔHC^{α} *means of its derived Fourier series is given by*

$$\|L_n'(\cdot)_2\|_{B_q^{\delta}(L_2)} = O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{|dg_x(t)|}{t^2}\right) + O\left((n+1)\int_0^{\frac{1}{n+1}} \left(t^{-\delta-\frac{1}{q}}\right)|dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \left(t^{-\delta-\frac{1}{q}-2}\right)|dg_x(t)|\right).$$

Corollary 3.17. *The error estimation of a function* $f \in B_q^{\delta}(L^2), 1 < q \le \infty$ *by* ΔHE^{q} *means of its derived Fourier series is given by*

$$\|L_n'(\cdot)_2\|_{B_q^{\delta}(L_2)} = O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{|dg_x(t)|}{t^2}\right) + O\left((n+1)\int_0^{\frac{1}{n+1}} \left(t^{-\delta-\frac{1}{q}}\right)|dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \left(t^{-\delta-\frac{1}{q}-2}\right)|dg_x(t)|\right).$$

Corollary 3.18. *The error estimation of a function* $f \in B_q^{\delta}(L^2), 1 < q \le \infty$ *by* $C^{\alpha}E^{q}$ *means of its derived Fourier series is given by*

$$\|L_n'(\cdot)_2\|_{B_q^{\delta}(L_2)} = O\left((n+1)\int_0^{\frac{1}{n+1}} |dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{|dg_x(t)|}{t^2}\right) + O\left((n+1)\int_0^{\frac{1}{n+1}} \left(t^{-\delta-\frac{1}{q}}\right)|dg_x(t)| + \frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \left(t^{-\delta-\frac{1}{q}-2}\right)|dg_x(t)|\right).$$

4. Applications

In this section, we study some applications of our main results.

4.1. Degree of Convergence of Fourier Series

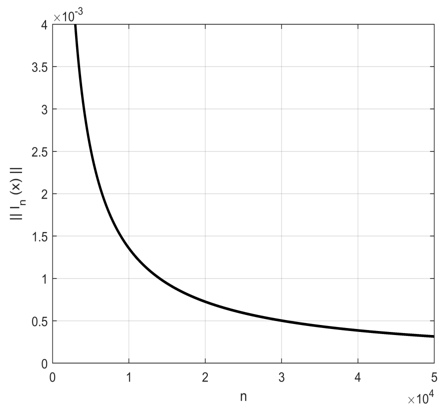
Consider a function $f(x) = \sin x$ and $h_{n,m} = \frac{\binom{n-m+1}{n-m}}{\binom{n+2}{n}}$ for $n \ge m$ and $h_{n,m} = 0$ for $n < m$.
We have
$K_n^{HK}(t) = O(n+1)$ for $0 < t \le \frac{1}{n+1}$ and $K_n^{HK}(t) = O\left(\frac{1}{(n+1)t^2}\right)$ for $\frac{1}{n+1} < t \le \pi$
Taking $\lambda = 1, \delta = 0$ and $q = \infty$. Since $|\sin t| \le 1$ and $\sin(\frac{t}{2}) \ge \frac{t}{\pi}$, for $0 < t \le \pi$, we have
$\|L_n(\cdot)\|_r = O\left(\frac{1}{n+1}\right)$ and $\|w_j(L_n, \cdot)_r\|_{\delta,q} = O\left(\frac{\log(n+1)\pi}{n+1}\right)$.
Thus, the error estimation of $f(x) = \sin x$ is obtained by

$$\|L_n(\cdot)_r\|_{B_q^{\delta}(L_2)} = \|L_n(\cdot)\|_r + \|w_j(L_n, \cdot)_r\|_{\delta,q} = O\left(\frac{1 + \log(n+1)\pi}{n+1}\right).$$

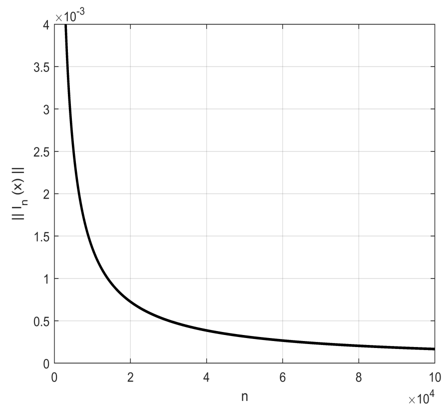
Now, we construct a following table for different values of n:

n	$\|\|L_n(x)\|\| = \frac{1+\log(n+1)\pi}{(n+1)}$
100	0.0669292119
1000	0.0090444402
10000	0.0011354035
50000	0.0002592854
100000	0.0001365752
500000	0.0000305341
1000000	0.0000159602
.	.
.	.
.	.
∞	0

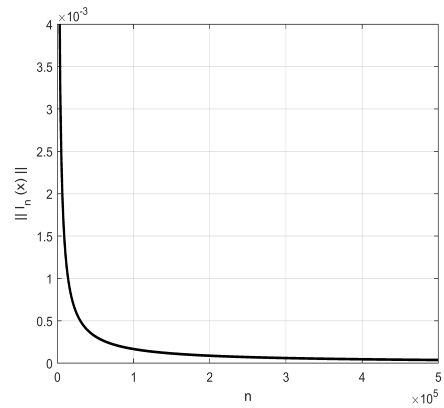
Table 1: Degree of convergence of function $f(x) = \sin(x)$



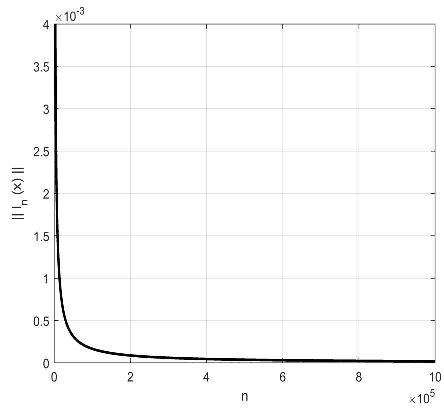
(a) For $n = 50000$



(b) For $n = 100000$



(c) For $n = 500000$



(d) For $n = 1000000$

Figure 1: Degree of convergence of Fourier series $f(x) = \sin(x)$.

Remark 4.1. *From Table 1 and Fig.* 1(*a*) *to* 1(*d*), *we observe that the results obtained in Theorem 3.1 and Theorem 3.7 together for* $1 < q \leq \infty$ *provide the best approximation of the function* f.

4.2. Degree of Convergence of Derived Fourier Series

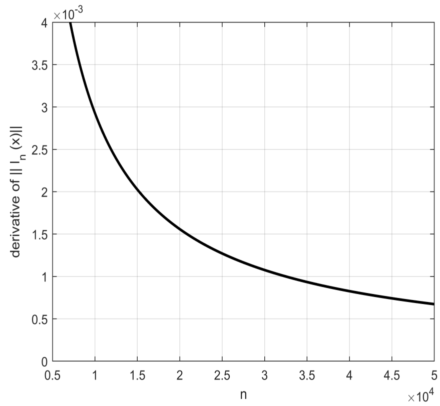
Consider a function $f'(x) = \sin x$ and $h_{n,m} = \frac{\binom{n-m+1}{n-m}}{\binom{n+2}{n}}$ for $n \geq m$ and $h_{n,m} = 0$ for $n < m$.
Thus, $dg_x(t) = -4 \sin x (\sin^2 \frac{t}{2}) dt$.
We have
$K_n^{HK}(t) = O(n+1)$ for $0 < t \leq \frac{1}{n+1}$ and $K_n^{HK}(t) = O\left(\frac{1}{(n+1)t^2}\right)$ for $\frac{1}{n+1} < t \leq \pi$.
Taking $\lambda = 1, \delta = 0$ and $r = \infty$. Since $|\sin t| \leq 1$ and $\sin(\frac{t}{2}) \geq \frac{t}{\pi}$ for $0 < t \leq \pi$, we have
$\|l'_n(\cdot)\|_2 = O\left(\frac{\pi}{n+1}\right)$ and $\|w_j(L'_n, \cdot)_2\|_{\delta,q} = O\left(\frac{1+2\log(n+1)\pi}{n+1}\right)$.
Thus, the error estimation of $f'(x) = \sin x$ is obtained by

$$\|L'_n(\cdot)\|_{B_q^\delta(L_2)} = \|L'_n(\cdot)_2\|_2 + \|w_j(L'_n, \cdot)_2\|_{\delta,q} = O\left(\frac{1 + \pi + 2\log(n+1)\pi}{n+1}\right).$$

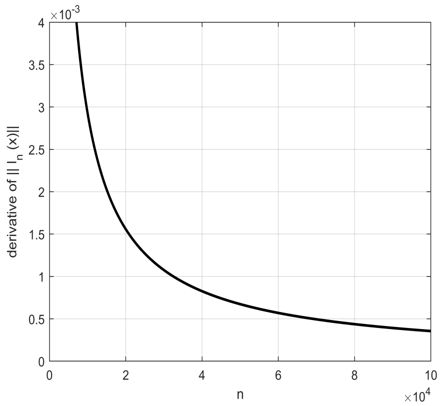
Now, we construct a following table for different values of n:

n	$\|L'_n(x)\| = \frac{1+\pi+2\log(n+1)\pi}{(n+1)}$
100	0.1550623115
1000	0.0202283337
10000	0.0024849448
50000	0.0005614018
100000	0.0002945663
500000	0.0000653514
1000000	0.000034062
.	.
.	.
∞	0

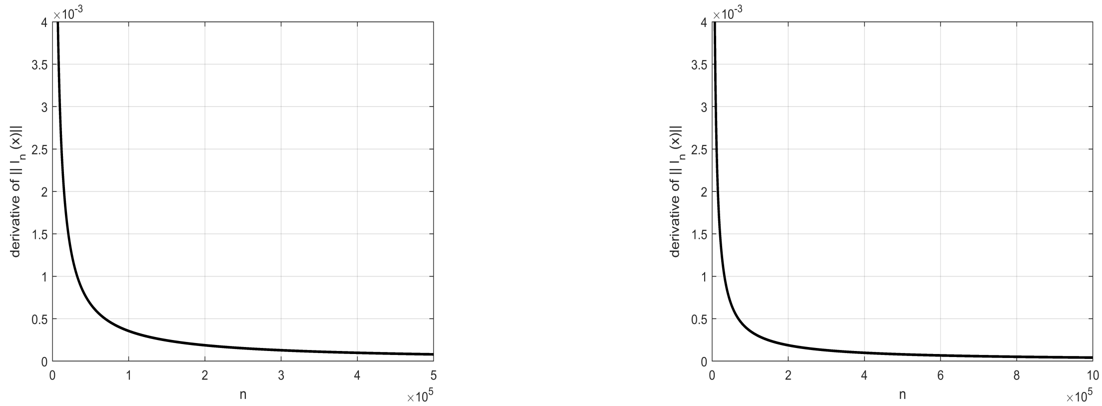
Table 2: Degree of convergence of derived Fourier series of $f'(x) = \sin(x)$.



(a) For $n = 50000$



(b) For $n = 100000$



(c) For $n = 500000$ (d) For $n = 1000000$

Figure 2: Degree of convergence of derived Fourier series of $f'(x) = \sin(x)$.

Remark 4.2. *From Table 2 and Fig.* 2(*a*) *to* 2(*d*)*, we observe that the results obtained in Theorem 3.13 and Theorem 3.15 together for* $1 < q \leq \infty$ *provide the best approximation of the function* f'.

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References

[1] B. A. London, Degree of approximation of Hölder continuous functions, Electronic Theses and Dissertations, University of Central Florida, (2008).
[2] B. E. Rhoades, The degree of approximation of functions and their conjugates belonging to several general Lipschitz classes by Hausdorff matrix means of the Fourier series and conjugate series of a Fourier series, Tamkang J. Math., 45(4) (2014), 389–395.
[3] B. E. Rhoades, On the degree of approximation of functions belonging to the weighted $(L^p, \xi(t))$ class by Hausdorff means, Tamkang J. Math., 32 (2001), 305–314.
[4] C. K. Chui, An introduction to wavelets (Wavelet analysis and its applications), Academic Press, USA, 1, (1992).
[5] H. K. Nigam, Degree of approximation of a function belonging to weighted $(L_r, \xi(t))$ class by $(C, 1)(E, q)$ means, Tamkang J. Math., 42 (2011), 31–37.
[6] H. K. Nigam, On approximation in generalized Zygmund Class, Demonstr. Math., 52 (2019), 370-387.
[7] H. K. Nigam and Hadish, Approximation of a function in Hölder class using double Karamata $(K^{\lambda,\mu})$ method, Eur. J. Pure Appl. Math., 13 (2020), 567-578.
[8] H. K. Nigam and Hadish, Best approximation of functions in generalized Hölder class, J. Inequal. Appl., 276 (2018), 1–15.
[9] H. K. Nigam, M. Mursaleen and S. Rani, Approximation of function using generalized Zygmund class, Adv. Differ. Equ., 34 (2021), 1-22.
[10] H. K. Nigam, M. Mursaleen and S. Rani, Approximation of functions in generalized Zygmund class by double Hausdorff matrix, Adv. Differ. Equ., 317 (2020), 1-19.
[11] H. K. Nigam and Md. Hadish, On approximation of function in generalized Zygmund class using $C^{\eta}T$ operator, J. Math. Inequalities, 14 (2020), 273–289.
[12] H. Mohanty, G. Das and B. K. Ray, Degree of approximation of Fourier series of function in Besov space by (N, p_n) mean, J. Orissa Math. Soc., 30 (2011), 13–34.
[13] J. K. Kushwaha and Dhakal, Approximation of a function belonging to $Lip(\alpha, r)$ class by $N_{p,q}C_1$ summability method of its Fourier series, Nepal J. Sci. Technol., 14 (2013), 117-122.
[14] M. V. Singh, M. L. Mittak and B. E. Rhoades, Approximation of functions in the generalized Zygmund class using Hausdorff means, J. Inequal. Appl., 101 (2017), 1-11.

[15] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Mathematical Society Student texts, Cambridge University Press, New York, 37 (1997).
[16] R. A. Devore and G. Lorentz, Constructive Approximation, Springer-Verlay, Berlin Heidelberg, New York, (1993).
[17] R. A. Devore and V. A. Popov, Interpolation of Besov space, Trans. Am. Math. Soc., 305 (1988).