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Strong laws for weighted sums of some dependent random variables and applications

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying a generalized Rosenthal type inequality and stochastically dominated by a random variable X. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. We study the Marcinkiewicz-Zygmund type strong laws for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ under the condition that the exponential moment of the random variable X exists. These results are the interesting supplements for some known results. As statistical applications, we provide the strong consistency of LS estimators in simple linear EV regression models with widely orthant dependent random errors.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and define

$$S_n = \sum_{i=1}^{n} a_{ni}X_i$$

where the weights $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a triangular array of constants or random variables independent of $\{X_n, n \geq 1\}$. Many useful linear statistics have the expression S_n, such as, least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. The almost sure limiting behavior of the weighted sums S_n has been studied by many authors. The classical Marcinkiewicz-Zygmund strong law of large numbers states that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^p < \infty$ for some $1 \leq p < 2$, then

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i \to 0 \quad a.s.$$

Bai and Cheng [1] obtained a Marcinkiewicz-Zygmund strong law of large numbers for weighted sums of i.i.d. random variables as follows.

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Theorem 1.1. ([1, Theorem 2.1]) *Assume that*

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^{\alpha}\right)^{1/\alpha}<\infty$$

for some $1<p<2$ *and* $1<\alpha,\beta<\infty$ *such that* $1/p=1/\alpha+1/\beta$. *Let* $\mathbb{E}X_1=0$ *and* $\mathbb{E}|X_1|^{\beta}<\infty$, *then we have*

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}a_{ni}X_i\to 0\quad a.s.$$

The strong law of Bai and Cheng [1] has been generalized and extended in several directions. Ko and Kim [7] considered the case that negatively orthant dependent random variables with finite moment generating function, and gave a strong law of large numbers for weighted sums. Cai [2] established the strong laws for weighted sums of a sequence of negatively associated random variables. Sung [19] studied the weighted sums of negatively associated random variables and improved the result of Cai [2]. Shen et al. [18] obtained the almost sure convergence and the strong stability for the weighted sums of the negatively superadditive dependent random variables. Huang et al. [6] studied φ-mixing random variables under a mixing rate condition $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$. Wu et al. [27] considered a class of random variables which satisfies a Rosenthal type inequality. Yi et al. [28] improved the results of Wu et al. [27] by weakening the moment condition and applied their result to widely orthant dependent sequence.

We recall the concept of stochastically dominated sequence. A sequence of random variables $\{X_n, n\geq 1\}$ is said to be stochastically dominated by a random variable X, if there exists a positive constant C such that

$$\mathbb{P}(|X_n|>x)\leq C\mathbb{P}(|X|>x)$$

for all $x\geq 0$ and $n\geq 1$. The following theorem is obtained by Yi et al. [28].

Theorem 1.2. ([28, Theorem 2.1 and Theorem 2.2]) *Let* $1\leq p<2$ *and* $\alpha,\beta>0$ *with* $1/p=1/\alpha+1/\beta$. *Assume that* $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ *is an array of constants satisfying* $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$, *and* $\{X_n, n\geq 1\}$ *is a sequence of mean zero random variables stochastically dominated by a random variable X satisfying* $\mathbb{E}|X|^{\beta}<\infty$.

(1) For any $n\geq 1$ *and* $s\geq 2$, *if the sequence* $\{X_n, n\geq 1\}$ *satisfies*

$$\begin{aligned}&\mathbb{E}\left(\left|\sum_{k=1}^{n}(f_{nk}(X_k)-\mathbb{E}f_{nk}(X_k))\right|^{s}\right)\\&\quad\leq C_s\sum_{k=1}^{n}\mathbb{E}|f_{nk}(X_k)|^{s}+g(n,s)\left(\sum_{k=1}^{n}\mathbb{E}(f_{nk}(X_k))^{2}\right)^{s/2},\end{aligned}$$

we have

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}a_{ni}X_i\to 0\quad a.s.$$

(2) For any $n\geq 1$ *and* $s\geq 2$, *if the sequence* $\{X_n, n\geq 1\}$ *satisfies*

$$\begin{aligned}&\mathbb{E}\left(\max_{1\leq m\leq n}\left|\sum_{k=1}^{m}(f_{nk}(X_k)-\mathbb{E}f_{nk}(X_k))\right|^{s}\right)\\&\quad\leq C_s\sum_{k=1}^{n}\mathbb{E}|f_{nk}(X_k)|^{s}+g(n,s)\left(\sum_{k=1}^{n}\mathbb{E}(f_{nk}(X_k))^{2}\right)^{s/2},\end{aligned}$$

we have

$$\frac{1}{n^{1/p}}\max_{1\leq m\leq n}\sum_{i=1}^{m}a_{ni}X_i\to 0\quad a.s.$$

Here C_s is a positive constant depending only on s, $g(n,s)$ is a positive function and $\{f_{nk}(x), 1 \le k \le n, n \ge 1\}$ is an array of nondecreasing functions.

Intuitively, when $\beta \to \infty$ in Theorem 1.2, which mean that $\mathbb{E}|X|^s < \infty$ for any $s > 0$, then $\alpha \to p$. Hence the aim of the present paper is to study the case: if $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$, then under what conditions, we have

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} a_{ni} X_i \to 0 \quad a.s.$$

or

$$\frac{1}{n^{1/\alpha}} \max_{1 \le m \le n} \sum_{i=1}^{m} a_{ni} X_i \to 0 \quad a.s.$$

Hence these results are the interesting supplements for the work of Yi et al. [28].

The rest of this paper is organized as follows. The main results are stated in Section 2 and their proofs are given in Section 3. In Section 4, as an application of the strong laws, we provide the strong consistency of the least squares estimators in simple linear errors-in-variables regression models with widely orthant dependent random errors. Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance.

2. Main results

We start to consider the case $0 < \alpha < 2$ for the condition $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$.

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero random variables stochastically dominated by a random variable X and satisfying*

$$\begin{aligned}
&\mathbb{E}\left(\max_{1 \le m \le n} \left|\sum_{k=1}^{m} (f_{nk}(X_k) - \mathbb{E} f_{nk}(X_k))\right|^s\right) \\
&\le g_1(n,s) \sum_{k=1}^{n} \mathbb{E}|f_{nk}(X_k)|^s + g_2(n,s) \left(\sum_{k=1}^{n} \mathbb{E}(f_{nk}(X_k))^2\right)^{s/2},
\end{aligned} \tag{2.1}$$

where $n \ge 1, s \ge 2$, $g_1(n,s), g_2(n,s)$ are two positive functions and $\{f_{nk}(x), 1 \le k \le n, n \ge 1\}$ is an array of nondecreasing functions. Assume that $\mathbb{E}\exp(h|X|^\gamma) < \infty$ for some $h > 0$ and $\gamma > 0$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \;\; \text{and} \;\; \max_{1 \le i \le n} |a_{ni}|^\alpha = O(n^t) \tag{2.2}$$

for some $0 < \alpha < 2$ and $0 < t < 1$. Furthermore, suppose that there exists a constant $s > 2$ such that

$$\sum_{n=1}^{\infty} \frac{g_1(n,s)}{n^{(s/\alpha-1)(1-t)}} < \infty, \quad \sum_{n=1}^{\infty} \frac{g_2(n,s)}{n^{s(1-t)(1/\alpha-1/2)}} < \infty. \tag{2.3}$$

Then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \le j \le n} \left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty.$$

In particular, we have

$$\frac{1}{n^{1/\alpha}} \max_{1 \le j \le n} \sum_{i=1}^{j} a_{ni} X_i \to 0 \quad a.s.$$

Remark 2.2. *From Lemma 3.4 in Section 3, the condition* $\mathbb{E}\exp(h|X|^\gamma) < \infty$ *implies that* $\mathbb{E}|X|^s < \infty$ *for all* $s > 0$.

Remark 2.3. *The additional condition* $\max_{1\le i\le n} |a_{ni}|^\alpha = O(n^t)$ *is to exclude some extreme cases. For example, if the weights* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *satisfy*

$$|a_{ni}|^\alpha = \begin{cases} 0, & \text{for } 1 \le i \le n-1 \\ n, & \text{for } i = n \end{cases},$$

then the results in Theorem 2.1 can not be obtained.

Remark 2.4. *Usually the function* $f_{nk}(x)$ *can be taken as the following forms:*

$$x^+,\ \ x^-,\ \ xI(|x| < b)\ \text{ or }\ xI(|x| < b) - bI(x < -b) + bI(x > b),$$

for any $b > 0$.

Remark 2.5. *There are many random sequences which satisfy the Rosenthal type inequality (2.1). For the independent case, (2.1) holds by combining the Rosenthal inequality [16] with Doob inequality. For the dependent cases, we refer to Shao [17] for the negatively associated random sequence; Utev and Peligrad [20] for* ρ^**-mixing random sequence; Wang et al. [23] for* φ*-mixing random variables with* $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$*; Ding et al. [4] for widely orthant dependent sequence; Yuan and An [30] for an asymptotically almost negatively associated sequence; Hu [5] and Wang et al. [22] for negatively super-additive dependent sequence; Wu [26] for negatively dependent sequence.*

The second condition in (2.3) show that the parameter α in (2.2) can not be taken 2 (since $g_2(n,s)$ is greater than or equal to some constant usually), so it is necessary to consider the case $\alpha = 2$.

Theorem 2.6. *Assume that the weights* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *in Theorem 2.1 satisfy*

$$\sum_{i=1}^{n} |a_{ni}|^2 = O(n^\rho) \tag{2.4}$$

for some $0 \le \rho < 1$. *Furthermore, suppose that there exists a constant* $s > 2$ *such that*

$$\sum_{n=1}^{\infty} \frac{g_1(n,s)}{n^{s(1/\alpha-\rho/2)}} < \infty, \quad \sum_{n=1}^{\infty} \frac{g_2(n,s)}{n^{s(1/\alpha-\rho/2)}} < \infty. \tag{2.5}$$

Then for any α *satisfying (2.5), we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty \ \text{ for } \ \varepsilon > 0.$$

In particular, we have

$$\frac{1}{n^{1/\alpha}} \max_{1\le j\le n} \sum_{i=1}^{j} a_{ni}X_i \to 0 \quad a.s.$$

Remark 2.7. *The condition (2.4) implies that* $\max_{1\le i\le n} |a_{ni}|^2 = O(n^\rho)$.

Remark 2.8. *Under the conditions in Theorem 2.1 and Theorem 2.6, if the Rosenthal type inequality (2.1) is replaced by*

$$\begin{aligned}&\mathbb{E}\left|\sum_{k=1}^{n}(f_{nk}(X_k) - \mathbb{E}f_{nk}(X_k))\right|^s \\ &\le g_1(n,s)\sum_{k=1}^{n}\mathbb{E}|f_{nk}(X_k)|^s + g_2(n,s)\left(\sum_{k=1}^{n}\mathbb{E}(f_{nk}(X_k))^2\right)^{s/2},\end{aligned}$$

then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty.$$

In particular, we have

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} a_{ni} X_i \to 0 \quad a.s.$$

Next we present the strong law for weighted sums of widely orthant dependent random variables. The concept of widely orthant dependent sequence was introduced by Wang et al. [21] as follows. For the random variables $\{X_n, n \geq 1\}$, if for each $n \geq 1$, there exists a positive real number $g_U(n)$ such that for all $x_i \in \mathbb{R}, 1 \leq i \leq n$

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} \mathbb{P}(X_i > x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely upper orthant dependent. If for each $n \geq 1$, there exists a positive real number $g_L(n)$ such that for all $x_i \in \mathbb{R}$, $1 \leq i \leq n$

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely lower orthant dependent. If they are both widely upper orthant dependent and widely lower orthant dependent sequences, then we say that the $\{X_n, n \geq 1\}$ are widely orthant dependent. In the above definition, $g_U(n)$ and $g_L(n)$, $n \geq 1$, are called dominating coefficients. If for all $n \geq 1$, $g_U(n) = g_L(n) = C$ for some positive constant C, then $\{X_n, n \geq 1\}$ are said to be extended negatively dependent. In particular, if $C = 1$, then $\{X_n, n \geq 1\}$ are said to be negatively orthant dependent or negatively dependent.

Theorem 2.9. *Let $\{X_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables stochastically dominated by a random variable X with $\mathbb{E}X_n = 0$ for $n \geq 1$. Define $g(n) = \max\{g_U(n), g_L(n)\}$, where $g_U(n)$ and $g_L(n)$ are the dominating coefficients of $\{X_n, n \geq 1\}$. Assume that $\mathbb{E}\exp(h|X|^\gamma) < \infty$ for some $h > 0$ and $\gamma > 0$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \ \ and \ \ \max_{1 \leq i \leq n} |a_{ni}|^\alpha = O(n^t)$$

for some $0 < \alpha < 2$ and $0 < t < 1$. Then for any $\tau \geq 0$, $g(n) = O(n^\tau)$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq j \leq n}\left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty, \ \ for \ \ \varepsilon > 0.$$

In particular, we have

$$\frac{1}{n^{1/\alpha}} \max_{1 \leq j \leq n} \sum_{i=1}^{j} a_{ni} X_i \to 0 \quad a.s.$$

If all weights a_{ni} in Theorem 2.9 have the same value, then $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for any $\alpha > 0$. Hence we have the following corollary.

Corollary 2.10. *Let $\{X_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables stochastically dominated by a random variable X with $\mathbb{E}X_n = 0$ for $n \geq 1$. Define $g(n) = \max\{g_U(n), g_L(n)\}$, where $g_U(n)$ and $g_L(n)$ are the* *dominating coefficients of* $\{X_n, n \geq 1\}$*. Assume that* $\mathbb{E}\exp(h|X|^\gamma) < \infty$ *for some* $h > 0$ *and* $\gamma > 0$*. Then for any* $\tau \geq 0$, $g(n) = O(n^\tau)$*, and any* $0 < \alpha < 2$*, we have*

$$\frac{1}{n^{1/\alpha}} \max_{1\leq j\leq n} \sum_{i=1}^{j} X_i \to 0 \quad a.s.$$

Theorem 2.11. *Assume that* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *in Theorem 2.9 satisfy*

$$\sum_{i=1}^{n} |a_{ni}|^2 = O(n^\rho)$$

for some $0 \leq \rho < 1$*. Then for any* $\tau \geq 0$, $g(n) = O(n^\tau)$*, and any* $0 < \alpha < 2/\rho$*, we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty, \quad for \quad \varepsilon > 0.$$

In particular, we have

$$\frac{1}{n^{1/\alpha}} \max_{1\leq j\leq n} \sum_{i=1}^{j} a_{ni}X_i \to 0 \quad a.s.$$

Remark 2.12. *In Theorem 2.11, if* $\rho = 0$*, i.e.,*

$$\sum_{i=1}^{n} |a_{ni}|^2 = O(1).$$

then for any $\tau \geq 0$, $g(n) = O(n^\tau)$*, and any* $\alpha > 0$*, we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\right) < \infty, \quad for \quad \varepsilon > 0.$$

In particular, we have

$$\frac{1}{n^{1/\alpha}} \max_{1\leq j\leq n} \sum_{i=1}^{j} a_{ni}X_i \to 0 \quad a.s.$$

3. Proofs of main results

In order to prove the main results, the following lemmas are needed.

Lemma 3.1. ([25, Lemma 1.7]) Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $s > 0$ and $b > 0$, we have

$$\mathbb{E}|X_n|^s I(|X_n| \leq b) \leq C_1 \left(\mathbb{E}|X|^s I(|X| \leq b) + b^s \mathbb{P}(|X| > b)\right)$$

and

$$\mathbb{E}|X_n|^s I(|X_n| > b) \leq C_2 \mathbb{E}|X|^s I(|X| > b)$$

where C_1 and C_2 are positive constants. Consequently, $\mathbb{E}|X_n|^s \leq C\mathbb{E}|X|^s$.

Lemma 3.2. ([4, Lemma 3.2]) Let $\{X_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with $\mathbb{E}X_n = 0$ and $\mathbb{E}|X_n|^r < \infty$ for some $r \geq 2$ and all $n \geq 1$. Then there exists a positive constant C_r depending only on r such that for all $n \geq 1$,

$$\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^r\right) \leq C_r (\log n)^r \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^r + g(n)\left(\sum_{i=1}^{n} \mathbb{E}|X_i|^2\right)^{\frac{r}{2}}\right)$$

Lemma 3.3. ([24, Corollary 2.1]) Let $\{X_n, n \geq 1\}$ be a sequence of widely orthant dependent random variables with dominating coefficients $g(n) = \max\{g_U(n), g_L(n)\}, n \geq 1$. If $\{f_n, n \geq 1\}$ is a sequence of real nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of widely orthant dependent random variables with the same dominating coefficients $g(n)$.

Lemma 3.4. Let X be a random variable and $\mathbb{E}\exp(h|X|^\gamma) < \infty$ for some $h > 0$ and $\gamma > 0$. Then we have $\mathbb{E}|X|^s < \infty$ for any $s > 0$.

Proof. By using Fubini's theorem and Markov's inequality, we have

$$\begin{aligned}\mathbb{E}|X|^s =& \int_0^\infty \mathbb{P}(|X|^s > x)dx \\ \leq& C\int_0^\infty e^{-hx^{\gamma/s}}\mathbb{E}\exp(h|X|^\gamma)dx < \infty.\end{aligned}$$

□

Proof. **[Proof of Theorem 2.1]** Without loss of generality, we can assume that $a_{ni} \geq 0$ for all $1 \leq i \leq n, n \geq 1$. For every $1 \leq i \leq n$ and $n \geq 1$, define

$$X'_{ni} = -n^{1/\alpha}I(X_i < -n^{1/\alpha}) + X_iI(|X_i| \leq n^{1/\alpha}) + n^{1/\alpha}I(X_i > n^{1/\alpha}). \tag{3.1}$$

By the Hölder's inequality, for $1 \leq s < \alpha$, we get

$$\sum_{i=1}^n |a_{ni}|^s \leq \left(\sum_{i=1}^n |a_{ni}|^{s\frac{\alpha}{s}}\right)^{\frac{s}{\alpha}}\left(\sum_{i=1}^n 1\right)^{\frac{\alpha-s}{\alpha}} \leq Cn \tag{3.2}$$

and for $s \geq \alpha$, we get

$$\sum_{i=1}^n |a_{ni}|^s \leq \sum_{i=1}^n |a_{ni}|^\alpha |a_{ni}|^{s-\alpha} \leq C\sum_{i=1}^n |a_{ni}|^\alpha \left(n^t\right)^{\frac{s-\alpha}{\alpha}} \leq Cn^{\frac{ts}{\alpha}+(1-t)}. \tag{3.3}$$

Firstly, we show that

$$n^{-1/\alpha}\max_{1\leq m\leq n}\left|\sum_{i=1}^m \mathbb{E}a_{ni}X'_{ni}\right| \to 0. \tag{3.4}$$

For the case $1 < \alpha < 2$, by using (3.2), Markov's inequality, the condition $\mathbb{E}X_n = 0$, Lemma 3.1 and Lemma 3.4 (by taking s such that $s/\alpha > 1 - 1/\alpha$), we have for all n large enough

$$\begin{aligned}&n^{-1/\alpha}\max_{1\leq m\leq n}\left|\sum_{i=1}^m \mathbb{E}a_{ni}X'_{ni}\right| \\ \leq& n^{-1/\alpha}\sum_{i=1}^n |a_{ni}|\mathbb{E}|X_i|I(|X_i| > n^{1/\alpha}) + \sum_{i=1}^n |a_{ni}|\mathbb{P}(|X_i| > n^{1/\alpha}) \\ \leq& Cn^{1-1/\alpha}\mathbb{E}|X|I(|X| > n^{1/\alpha}) + Cn\mathbb{P}(|X| > n^{1/\alpha}) \\ \leq& Cn^{1-1/\alpha}\mathbb{E}|X|I(|X| > n^{1/\alpha}) \\ \leq& Cn^{1-s/\alpha-1/\alpha}\mathbb{E}|X|^{s+1} \to 0.\end{aligned}$$

For the case $0 < \alpha \leq 1$, by using (3.3), Markov's inequality, the condition $\mathbb{E}X_n = 0$, Lemma 3.1 and Lemma 3.4, we have for all n large enough

$$n^{-1/\alpha}\max_{1\leq m\leq n}\left|\sum_{i=1}^m \mathbb{E}a_{ni}X'_{ni}\right|$$

$$
\begin{aligned}
&\leq n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\mathbb{E}|X_i|I(|X_i|>n^{1/\alpha})+\sum_{i=1}^{n}|a_{ni}|\mathbb{P}(|X_i|>n^{1/\alpha})\\
&\leq Cn^{(1-1/\alpha)(1-t)}\mathbb{E}|X|I(|X|>n^{1/\alpha})+Cn^{t/\alpha+(1-t)}\mathbb{P}(|X|>n^{1/\alpha})\\
&\leq Cn^{(1-1/\alpha)(1-t)}\mathbb{E}|X|I(|X|>n^{1/\alpha})\to 0.
\end{aligned}
$$

Hence we have (3.4). Now for any $\varepsilon > 0$, from (3.4), we have

$$
\begin{aligned}
&\mathbb{P}\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}X_i\right|>2n^{1/\alpha}\varepsilon\right)\\
&\leq\mathbb{P}\left(\max_{1\leq m\leq n}|X_m|>n^{1/\alpha}\right)\\
&\quad+\mathbb{P}\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}(X_{ni}^{'}-\mathbb{E}X_{ni}^{'})\right|>2n^{1/\alpha}\varepsilon-\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}\mathbb{E}a_{ni}X_{ni}^{'}\right|\right)\\
&\leq n\mathbb{P}\left(|X|>n^{1/\alpha}\right)+\mathbb{P}\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}(X_{ni}^{'}-\mathbb{E}X_{ni}^{'})\right|>n^{1/\alpha}\varepsilon\right).
\end{aligned}
\tag{3.5}
$$

By using Markov's inequality and Lemma 3.4 (by taking s such that $s/\alpha > 2$), we have

$$
\sum_{n=1}^{\infty}n\mathbb{P}\left(|X|>n^{1/\alpha}\right)\leq\sum_{n=1}^{\infty}n^{1-s/\alpha}\mathbb{E}|X|^s<\infty. \tag{3.6}
$$

From the condition (2.1), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}(X_{ni}^{'}-\mathbb{E}X_{ni}^{'})\right|>n^{1/\alpha}\varepsilon\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{s/\alpha}}\sum_{i=1}^{n}|a_{ni}|^s\mathbb{E}|X_{ni}^{'}|^s\\
&\quad+C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s/\alpha}}\left(\sum_{i=1}^{n}a_{ni}^2\mathbb{E}(X_{ni}^{'})^2\right)^{s/2}\\
&=:I_1+I_2.
\end{aligned}
\tag{3.7}
$$

From the inequality (3.3) and by taking s satisfying (2.3), we have

$$
\begin{aligned}
I_1&\leq C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{s/\alpha}}\sum_{i=1}^{n}|a_{ni}|^s\left(\mathbb{E}|X_i|^sI(|X_i|\leq n^{1/\alpha})+n^{s/\alpha}\mathbb{P}(|X_i|>n^{1/\alpha})\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{s/\alpha}}n^{\frac{ts}{\alpha}+(1-t)}\mathbb{E}|X|^s\\
&\leq C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{(s/\alpha-1)(1-t)}}<\infty
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
I_2 \le & C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s/\alpha}}\left(\sum_{i=1}^{n}a_{ni}^2\left(\mathbb{E}|X_i|^2 I(|X_i|\le n^{1/\alpha})+n^{2/\alpha}\mathbb{P}(|X_i|>n^{1/\alpha})\right)\right)^{s/2}\\
\le & C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s/\alpha}}\left(n^{\frac{2t}{\alpha}+(1-t)}\mathbb{E}|X|^2\right)^{s/2}\\
\le & C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s(1-t)(1/\alpha-1/2)}}<\infty.
\end{aligned}
\tag{3.9}
$$

Hence, from (3.5)-(3.9), we get the desired results. □

Proof. **[Proof of Theorem 2.6]** The proof is similar as Theorem 2.1. We assume that $a_{ni}\ge 0$ for all $1\le i\le n, n\ge 1$. For every $i\ge 1$ and $n\ge 1$, define X'_{ni} as in (3.1). By the Hölder's inequality, for $1\le s<2$, we get

$$
\sum_{i=1}^{n}|a_{ni}|^s\le\left(\sum_{i=1}^{n}|a_{ni}|^{s\frac{2}{s}}\right)^{\frac{s}{2}}\left(\sum_{i=1}^{n}1\right)^{\frac{2-s}{2}}\le Cn^{1-\frac{s}{2}(1-\rho)}
\tag{3.10}
$$

and for $s\ge 2$, we get

$$
\sum_{i=1}^{n}|a_{ni}|^s\le\sum_{i=1}^{n}|a_{ni}|^2|a_{ni}|^{s-2}\le C\sum_{i=1}^{n}|a_{ni}|^2\left(n^{\rho}\right)^{\frac{s-2}{2}}\le Cn^{\frac{\rho s}{2}}.
\tag{3.11}
$$

Firstly, we show that

$$
n^{-1/\alpha}\max_{1\le m\le n}\left|\sum_{i=1}^{m}\mathbb{E}a_{ni}X'_{ni}\right|\to 0.
\tag{3.12}
$$

By using (3.10), Markov's inequality, the condition $\mathbb{E}X_n=0$, Lemma 3.1 and Lemma 3.4 (by taking s such that $s>\frac{1}{2}\alpha(1+\rho)-1$), we have for all n large enough

$$
\begin{aligned}
& n^{-1/\alpha}\max_{1\le m\le n}\left|\sum_{i=1}^{m}\mathbb{E}a_{ni}X'_{ni}\right|\\
\le & n^{-1/\alpha}\sum_{i=1}^{n}|a_{ni}|\mathbb{E}|X_i|I(|X_i|>n^{1/\alpha})+\sum_{i=1}^{n}|a_{ni}|\mathbb{P}(|X_i|>n^{1/\alpha})\\
\le & Cn^{\frac{1}{2}(1+\rho)-\frac{1}{\alpha}}\mathbb{E}|X|I(|X|>n^{1/\alpha})+Cn^{\frac{1}{2}(1+\rho)}\mathbb{P}(|X|>n^{1/\alpha})\\
\le & Cn^{\frac{1}{2}(1+\rho)-\frac{1}{\alpha}}\mathbb{E}|X|I(|X|>n^{1/\alpha})\\
\le & Cn^{\frac{1}{2}(1+\rho)-\frac{1}{\alpha}-\frac{s}{\alpha}}\mathbb{E}|X|^{s+1}\to 0.
\end{aligned}
$$

Hence we have (3.12). Now for any $\varepsilon>0$, from (3.12), we have

$$
\begin{aligned}
& \mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}a_{ni}X_i\right|>2n^{1/\alpha}\varepsilon\right)\\
\le & \mathbb{P}\left(\max_{1\le m\le n}|X_m|>n^{1/\alpha}\right)\\
& +\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}a_{ni}(X'_{ni}-\mathbb{E}X'_{ni})\right|>2n^{1/\alpha}\varepsilon-\max_{1\le m\le n}\left|\sum_{i=1}^{m}\mathbb{E}a_{ni}X'_{ni}\right|\right)\\
\le & n\mathbb{P}\left(|X|>n^{1/\alpha}\right)+\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}a_{ni}(X'_{ni}-\mathbb{E}X'_{ni})\right|>n^{1/\alpha}\varepsilon\right).
\end{aligned}
\tag{3.13}
$$

By using Markov's inequality and Lemma 3.4 (by taking s such that $s/\alpha > 2$), we have

$$\sum_{n=1}^{\infty} n\mathbb{P}\left(|X| > n^{1/\alpha}\right) \leq \sum_{n=1}^{\infty} n^{1-s/\alpha}\mathbb{E}|X|^s < \infty. \tag{3.14}$$

From the condition (2.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} a_{ni}(X'_{ni} - \mathbb{E}X'_{ni})\right| > n^{1/\alpha}\varepsilon\right)\\
\leq &C\sum_{n=1}^{\infty} \frac{g_1(n,s)}{n^{s/\alpha}}\sum_{i=1}^{n}|a_{ni}|^s\mathbb{E}|X'_{ni}|^s\\
&\quad + C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s/\alpha}}\left(\sum_{i=1}^{n} a_{ni}^2\mathbb{E}(X'_{ni})^2\right)^{s/2}\\
=:&I_1 + I_2.
\end{aligned} \tag{3.15}$$

From the inequality (3.11) and by taking s satisfying (2.5), we have

$$\begin{aligned}
I_1 \leq &C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{s/\alpha}}\sum_{i=1}^{n}|a_{ni}|^s\left(\mathbb{E}|X_i|^sI(|X_i|\leq n^{1/\alpha}) + n^{s/\alpha}\mathbb{P}(|X_i| > n^{1/\alpha})\right)\\
\leq &C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{s/\alpha}}n^{\rho s/2}\mathbb{E}|X|^s\\
\leq &C\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{s(1/\alpha-\rho/2)}} < \infty
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
I_2 \leq &C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s/\alpha}}\left(\sum_{i=1}^{n}a_{ni}^2\left(\mathbb{E}|X_i|^2I(|X_i|\leq n^{1/\alpha}) + n^{2/\alpha}\mathbb{P}(|X_i| > n^{1/\alpha})\right)\right)^{s/2}\\
\leq &C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s/\alpha}}\left(n^{\rho}\mathbb{E}|X|^2\right)^{s/2}\\
\leq &C\sum_{n=1}^{\infty}\frac{g_2(n,s)}{n^{s(1/\alpha-\rho/2)}} < \infty.
\end{aligned} \tag{3.17}$$

Hence, from (3.13)-(3.17), we get the desired results. □

Proof. **[Proof of Theorem 2.9]** For every $i \geq 1$ and $n \geq 1$, define X'_{ni} as in (3.1). From Lemma 3.3, we know that for each $n > 1$, $\{X'_{ni}, 1 \leq i \leq n\}$ is a sequence of widely orthant dependent random variables and by using Lemma 3.2, the Rosenthal type inequality (2.1) holds. In addition, we have

$$g_1(n,s) = (\log n)^s \quad \text{and} \quad g_2(n,s) = (\log n)^s g(n).$$

For given $\tau > 0$, we can take

$$s > \max\left\{\frac{(2-t)\alpha}{1-t}, \frac{1+\tau}{(1-t)(1/\alpha - 1/2)}\right\}$$

such that

$$\sum_{n=1}^{\infty}\frac{g_1(n,s)}{n^{(s/\alpha-1)(1-t)}} = \sum_{n=1}^{\infty}\frac{(\log n)^s}{n^{(s/\alpha-1)(1-t)}} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{g_2(n,s)}{n^{s(1-t)(1/\alpha-1/2)}} \leq \sum_{n=1}^{\infty} \frac{(\log n)^s n^{\tau}}{n^{s(1-t)(1/\alpha-1/2)}} < \infty.$$

Hence the conditions in Theorem 2.1 hold, and the desired results can be obtained. □

Proof. **[Proof of Theorem 2.11]** The proof is similar as Theorem 2.9, so we omit it. □

4. Simple linear errors-in-variables regression model

In this section, we consider the following simple linear errors-in-variables (EV) regression model:

$$\eta_i = \theta + \beta x_i + \varepsilon_i, \ \ \xi_i = x_i + \delta_i, \ \ 1 \leq i \leq n, \tag{4.1}$$

where $\theta, \beta, x_1, x_2, \cdots$ are unknown constants (parameters), $(\varepsilon_1, \delta_1), (\varepsilon_2, \delta_2), \cdots$ are random variables and ξ_i, η_i, $i = 1, 2, \cdots$ are observable. From (4.1) we have

$$\eta_i = \theta + \beta \xi_i + \nu_i, \ \ \nu_i = \varepsilon_i - \beta\delta_i, \ \ 1 \leq i \leq n. \tag{4.2}$$

Consider formally (4.2) as a usual regression model of η_i on ξ_i, we get the least squares (LS) estimators of θ and β as

$$\hat{\beta}_n = \frac{\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)(\eta_i - \bar{\eta}_n)}{\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2}, \quad \hat{\theta}_n = \bar{\eta}_n - \hat{\beta}_n \bar{\xi}_n,$$

where $\bar{\xi}_n = n^{-1}\sum_{i=1}^{n} \xi_i$, and other similar notations, such as $\bar{\eta}_n$, $\bar{\delta}_n$, $\bar{x}_n$ are defined in the same way.

Due to the simple form and wide applicability, the EV model (4.1) has been studied by many authors in the past three decades. Under the case that the errors are sequences of independent random variables, Cui [3] proved the asymptotic normality of M-estimates in the EV model. Liu and Chen [8] gave the consistency of the LS estimator for the linear EV regression model, and obtained that both weak and strong consistency of the estimator are equivalent, but it is not so for quadratic-mean consistency. Miao et al. [14] and Miao and Yang [13] gave the central limit theorem and the law of iterated logarithm for the LS estimators $\hat{\beta}_n$ and $\hat{\theta}_n$ in the simple linear EV regression model (4.1). In [12], Miao et al. obtained the consistency and asymptotic normality for the LS estimators $\hat{\beta}_n$ and $\hat{\theta}_n$, which weaken some known conditions and improve some known results (see [8, 14]). Moreover, they proved the large deviation principle for $\hat{\beta}_n$ and $\hat{\theta}_n$ under the assumptions that $(\varepsilon_i, \delta_i)_{i\geq 1}$ possess normal distributions. Miao and Liu [11] obtained the exponential convergence rate (moderate deviation) for the estimators $\hat{\beta}_n$ and $\hat{\theta}_n$ under the weaker moment assumptions and the stronger moderate deviation scale conditions. Miao [10] proved another moderate deviation principle for the estimators $\hat{\beta}_n$ and $\hat{\theta}_n$ under the different conditions from the works in [11]. Miao et al. [15] established the asymptotic normality for the LS estimators of the unknown parameters β and θ under the assumptions that the errors are m-dependent, martingale differences, ϕ-mixing, ρ-mixing and α-mixing. Recently, Liu et al. [9] obtained a necessary and sufficient condition for the convergence rate of the strong consistency for each of the unknown parameters. Yi et al. [29] improved and extended the works in Liu et al. [9] from independent case to widely orthant dependent random errors.

Theorem 4.1. *([29, Theorem 1.1]) Let $\{\varepsilon, \varepsilon_n, n \geq 1\}$ and $\{\delta, \delta_n, n \geq 1\}$ be two sequences of identically distributed widely orthant dependent random variables with dominating coefficients $g_L(n)$ and $g_U(n)$, $g_L^{'}(n)$ and $g_U^{'}(n)$ for $n \geq 1$, respectively. Suppose that $\mathbb{E}\varepsilon = \mathbb{E}\delta = 0$, $0 < \mathbb{E}|\varepsilon|^{2tp/(2t-p)} < \infty$, $0 < \mathbb{E}|\delta|^{2tp/(2t-p)} < \infty$ for some $1 < p < 2$ and $1 \leq t < 2p/(4-2p)$, and there exist a positive function $g(x)$ for $x \geq 0$ and a nonnegative constant $0 \leq \tau < \infty$ such that $g(x) = O(x^{\tau})$, and $\max\{g_L(n), g_U(n), g_L^{'}(n), g_U^{'}(n)\} \leq g(n)$ for $n \geq 1$. Then the following statements hold:*

(i) If $n^{2-1/t}/s_n = O(1)$ and $n^{2-1/p}/s_n \to 0$, then

$$n^{1-1/p}(\hat{\beta}_n - \beta) \to 0 \quad a.s.$$

(ii) If $\sup_{n\geq 1} \min\{n, s_n\} n^{1-1/t} \bar{x}_n^2 / s_n^* < \infty$ *and* $n^{2-1/p} \bar{x}_n / s_n^* \to 0$*, then*

$$n^{1-1/p}(\hat{\theta}_n - \theta) \to 0 \quad a.s.$$

where $s_n = \sum_{i=1}^n (x_i - \bar{x}_n)^2$ *and* $s_n^* = \max\{n, s_n\}$.

Now we consider the case that the exponential moments of the random errors exist.

Theorem 4.2. *Let* $\{\varepsilon, \varepsilon_n, n \geq 1\}$ *and* $\{\delta, \delta_n, n \geq 1\}$ *be two sequences of identically distributed widely orthant dependent random variables with* $\mathbb{E}\varepsilon = \mathbb{E}\delta = 0$*. Define* $g(n) = \max\{g_L(n), g_U(n), g_L^{'}(n), g_U^{'}(n)\}$ *for* $n \geq 1$*, where* $\{g_U(n), g_L(n)\}$ *and* $\{g_U^{'}(n), g_L^{'}(n)\}$ *are the dominating coefficients of* $\{\varepsilon, \varepsilon_n, n \geq 1\}$ *and* $\{\delta, \delta_n, n \geq 1\}$ *respectively. Assume that* $\mathbb{E}\exp(h|\varepsilon|^\gamma) < \infty$, $\mathbb{E}\exp(h|\delta|^\gamma) < \infty$ *for some* $h > 0$ *and* $\gamma > 0$*. Then for any* $\tau \geq 0$, $g(n) = O(n^\tau)$*, the following statements hold:*

(i) Let $n^{2-1/p}/s_n \to 0$ *for some* $p > 0$ *and assume that there exists a constant* $\alpha > 0$ *such that* $n^{2+2/\alpha-2/p}/s_n = O(1)$*, then*

$$n^{1-1/p}(\hat{\beta}_n - \beta) \to 0 \quad a.s.$$

(ii) Let $n^{2-1/p}\bar{x}_n/s_n^* \to 0$ *for some* $1 < p < 2$ *and assume that there exists a constant* $\alpha > 0$ *such that* $n^{2+2/\alpha-2/p}\bar{x}_n^2/s_n^* = O(1)$*, then*

$$n^{1-1/p}(\hat{\theta}_n - \theta) \to 0 \quad a.s.$$

where $s_n = \sum_{i=1}^n (x_i - \bar{x}_n)^2$ *and* $s_n^* = \max\{n, s_n\}$.

Remark 4.3. *In Theorem 4.1, the convergent rate of* $\hat{\beta}_n - \beta$ *and* $\hat{\theta}_n - \theta$ *is between* $o(1)$ *and* $o(n^{-1/2})$*. In Theorem 4.2, we give the better convergent rate of* $\hat{\beta}_n - \beta$ *by taking* $p > 2$*. In addition, if* $\min\{n, s_n\} = n$ *and* $t > p$*, we can take* $\alpha > 2p$ *such that*

$$\min\{n, s_n\} n^{1-1/t}\bar{x}_n^2/s_n^* \geq n^{2-1/p}\bar{x}_n^2/s_n^* \geq n^{2+2/\alpha-2/p}\bar{x}_n^2/s_n^*.$$

Hence for this case, the conditions in (ii) of Theorem 4.2 is weaker than ones in Theorem 4.1.

Proof. **[Proof of Theorem 4.2]** From Lemma 3.3, we know that $\{(\varepsilon_n^+)^2, n \geq 1\}$ is a sequence of widely orthant dependent random variables with dominating coefficients $g_U(n), g_L(n)$ for $n \geq 1$, and $\{(\delta_n^+)^2, n \geq 1\}$ is a sequence of widely orthant dependent random variables with dominating coefficients $g_U(n)$, $g_L(n)$ for $n \geq 1$. By using Corollary 2.10, we have

$$\frac{1}{n}\sum_{i=1}^n (\varepsilon_i^+)^2 \xrightarrow{a.s.} \mathbb{E}(\varepsilon^+)^2, \quad \frac{1}{n}\sum_{i=1}^n (\varepsilon_i^-)^2 \xrightarrow{a.s.} \mathbb{E}(\varepsilon^-)^2$$

and

$$\frac{1}{n}\sum_{i=1}^n (\delta_i^+)^2 \xrightarrow{a.s.} \mathbb{E}(\delta^+)^2, \quad \frac{1}{n}\sum_{i=1}^n (\delta_i^-)^2 \xrightarrow{a.s.} \mathbb{E}(\delta^-)^2$$

which implies

$$\frac{1}{n}\sum_{i=1}^n \varepsilon_i^2 \xrightarrow{a.s.} \mathbb{E}\varepsilon^2 \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^n \delta_i^2 \xrightarrow{a.s.} \mathbb{E}\delta^2. \tag{4.3}$$

By simple calculation, we have

$$\begin{aligned}\hat{\beta}_n - \beta =& \frac{\sum_{i=1}^n(\xi_i - \bar{\xi}_n)\varepsilon_i - \beta\sum_{i=1}^n(x_i - \bar{x}_n)\delta_i - \beta\sum_{i=1}^n(\delta_i - \bar{\delta}_n)^2}{\sum_{i=1}^n(\xi_i - \bar{\xi}_n)^2} \\ =& \frac{\sum_{i=1}^n(\delta_i - \bar{\delta}_n)\varepsilon_i + \sum_{i=1}^n(x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) - \beta\sum_{i=1}^n(\delta_i - \bar{\delta}_n)^2}{\sum_{i=1}^n(\xi_i - \bar{\xi}_n)^2}\end{aligned} \tag{4.4}$$

and

$$\hat{\theta}_n - \theta = (\beta - \hat{\beta}_n)\bar{x}_n + (\beta - \hat{\beta}_n)\bar{\delta}_n - \beta\bar{\delta}_n + \bar{\varepsilon}_n. \tag{4.5}$$

(i) From (4.3) and the condition $n^{2-1/p}/s_n \to 0$, we have

$$n^{1-1/p}\frac{1}{s_n}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \leq \frac{n^{2-1/p}}{s_n}\frac{1}{n}\sum_{i=1}^{n}\delta_i^2 \to 0, \ \ a.s. \tag{4.6}$$

By the same reason, it follows that

$$n^{1-1/p}\frac{1}{s_n}\left|\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)\varepsilon_i\right| \leq \frac{n^{1-1/p}}{2s_n}\sum_{i=1}^{n}\left((\delta_i - \bar{\delta}_n)^2 + (\varepsilon_i - \bar{\varepsilon}_n)^2\right) \xrightarrow{a.s.} 0. \tag{4.7}$$

Since

$$\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 = \sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + 2\sum_{i=1}^{n}(x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n) + \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \tag{4.8}$$

then, from Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{s_n}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 - 1\right| \\
\leq &\frac{1}{s_n}\left(2\sum_{i=1}^{n}\left|(x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n)\right| + \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2\right) \\
\leq &\frac{1}{s_n}\left(2\sqrt{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}\sqrt{\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2} + \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2\right) \\
= &2\sqrt{\frac{1}{s_n}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2} + \frac{1}{s_n}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \xrightarrow{a.s.} 0
\end{aligned}$$

where we used the limit (4.6). Hence, we have

$$\frac{1}{s_n}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 \xrightarrow{a.s.} 1. \tag{4.9}$$

By letting

$$a_{ni} = \frac{(x_i - \bar{x}_n)}{\sqrt{s_n}},$$

we have

$$\sum_{i=1}^{n}a_{ni}^2 = 1,$$

which, by using Remark 2.12, yields,

$$\begin{aligned}
n^{1-1/p}\frac{1}{s_n}\sum_{i=1}^{n}(x_i - \bar{x}_n)\varepsilon_i &= \frac{n^{1-1/p+1/\alpha}}{\sqrt{s_n}}\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n}a_{ni}\varepsilon_i \xrightarrow{a.s.} 0 \\
n^{1-1/p}\frac{1}{s_n}\sum_{i=1}^{n}(x_i - \bar{x}_n)\delta_i &= \frac{n^{1-1/p+1/\alpha}}{\sqrt{s_n}}\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n}a_{ni}\delta_i \xrightarrow{a.s.} 0.
\end{aligned} \tag{4.10}$$

Hence, by (4.4), (4.6), (4.7), (4.9) and (4.10), we have

$$n^{1-1/p}(\hat{\beta}_n - \beta) \to 0, \ \ a.s. \tag{4.11}$$

(ii) From Corollary 2.10, we have

$$n^{1-1/p}\bar{\varepsilon}_n \xrightarrow{a.s.} 0 \ \ \text{and} \ \ n^{1-1/p}\bar{\delta}_n \xrightarrow{a.s.} 0. \tag{4.12}$$

By letting

$$a_{ni} = \frac{x_i - \bar{x}_n}{\sqrt{s_n^*}}, \tag{4.13}$$

and from the definition of s_n^*, we have

$$\sum_{i=1}^n a_{ni}^2 = O(1),$$

which, by using Remark 2.12, yields

$$\frac{1}{s_n^*}\sum_{i=1}^n (x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n) = \frac{1}{s_n^*}\sum_{i=1}^n (x_i - \bar{x}_n)\delta_i = \frac{1}{\sqrt{s_n^*}}\sum_{i=1}^n a_{ni}\delta_i \xrightarrow{a.s.} 0. \tag{4.14}$$

From (4.3) and Corollary 2.10, we have

$$\frac{1}{n}\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2 = \frac{1}{n}\sum_{i=1}^n \delta_i^2 - (\bar{\delta}_n)^2 \xrightarrow{a.s.} \mathbb{E}\delta^2 \tag{4.15}$$

and

$$\frac{1}{n}\left|\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\varepsilon_i\right| \leq \frac{1}{2n}\sum_{i=1}^n \left((\delta_i - \bar{\delta}_n)^2 + (\varepsilon_i - \bar{\varepsilon}_n)^2\right) \xrightarrow{a.s.} \frac{1}{2}(\mathbb{E}\delta^2 + \mathbb{E}\varepsilon^2). \tag{4.16}$$

From (4.8), (4.14) and (4.15), we have

$$\begin{aligned}
\min\{1, \mathbb{E}\delta^2\} &\leq \liminf_{n\to\infty}\left(\frac{s_n}{s_n^*} + \frac{1}{s_n^*}\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2\right) \\
&\leq \liminf_{n\to\infty}\left(\frac{1}{s_n^*}\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2\right) \\
&\leq \limsup_{n\to\infty}\left(\frac{1}{s_n^*}\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2\right) \\
&\leq \limsup_{n\to\infty}\left(\frac{s_n}{s_n^*} + \frac{1}{s_n^*}\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2\right) \leq 1 + \mathbb{E}\delta^2 \ \ a.s.
\end{aligned} \tag{4.17}$$

As the same proof as (4.14), we have

$$\frac{1}{s_n^*}\sum_{i=1}^n (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) \xrightarrow{a.s.} 0. \tag{4.18}$$

From (4.4), (4.15), (4.16), (4.17) and (4.18), we have

$$\limsup_{n\to\infty} |\hat{\beta}_n - \beta| < \infty \ \ a.s.$$

which implies

$$n^{1-1/p}(\beta - \hat{\beta}_n)\bar{\delta}_n \xrightarrow{a.s.} 0. \tag{4.19}$$

Now from the condition $n^{2-1/p}\bar{x}_n/s_n^* \to 0$, we get

$$\frac{n^{1-1/p}\bar{x}_n}{s_n^*}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \xrightarrow{a.s.} 0 \tag{4.20}$$

and

$$\frac{n^{1-1/p}\bar{x}_n}{s_n^*}\left|\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)\varepsilon_i\right| \xrightarrow{a.s.} 0. \tag{4.21}$$

From (4.13), by using Theorem 2.9, we have

$$\begin{aligned} \frac{n^{1-1/p}\bar{x}_n}{s_n^*}\sum_{i=1}^{n}(x_i - \bar{x}_n)\varepsilon_i &= \frac{n^{1-1/p+1/\alpha}\bar{x}_n}{\sqrt{s_n^*}}\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n}a_{ni}\varepsilon_i \xrightarrow{a.s.} 0 \\ \frac{n^{1-1/p}\bar{x}_n}{s_n^*}\sum_{i=1}^{n}(x_i - \bar{x}_n)\delta_i &= \frac{n^{1-1/p+1/\alpha}\bar{x}_n}{\sqrt{s_n^*}}\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n}a_{ni}\delta_i \xrightarrow{a.s.} 0. \end{aligned} \tag{4.22}$$

From (4.20), (4.21) and (4.22), we get

$$n^{1-1/p}(\beta - \hat{\beta}_n)\bar{x}_n \xrightarrow{a.s.} 0. \tag{4.23}$$

Hence from (4.5), (4.12), (4.19) and (4.23), we have

$$n^{1-1/p}(\hat{\theta}_n - \theta) \to 0, \quad a.s.$$

□

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