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# A novel criterion for unpredictable motions

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**Abstract.** We demonstrate the extension of unpredictable motions in coupled autonomous systems with skew product structure in the case that generalized synchronization takes place. Sufficient conditions for the existence of unpredictable motions in the dynamics of the response system are provided. The theoretical results are exemplified for coupled autonomous systems in which the drive is a hybrid dynamical system and the response is a Lorenz system. The auxiliary system approach and conditional Lyapunov exponents are utilized to detect the presence of generalized synchronization.

## 1. Introduction

A special type of Poisson stable trajectory, named unpredictable, was introduced in paper [1]. An unpredictable trajectory leads to Poincaré chaos in the associated quasi-minimal set. Such trajectories take place in symbolic dynamics, logistic and Hénon maps, and the Smale horseshoe [1, 2]. One of the important features of Poincaré chaos is that it can be triggered by the presence of a single unpredictable trajectory in the dynamics. This feature is the main difference of Poincaré chaos compared to chaos in the sense of Devaney [3] and Li-Yorke [4] since a collection of motions is required to define these chaos types.

Interesting results concerning unpredictable motions as well as Poincaré chaos in topological spaces were provided in papers [5–7]. Miller [5] generalized the notion of unpredictable points to the case of semiflows with arbitrary acting abelian topological monoids, whereas Thakur and Das [6] demonstrated that at least one of the factors is Poincaré chaotic provided that the same is true for finite or countably infinite products of semiflows. Additionally, differential equations with hyperbolic linear parts exhibiting unpredictable solutions were studied in [8], and the existence of unpredictable outputs in a class of retarded cellular neural networks was investigated in [9]. Further theoretical and numerical results on Poincaré chaos can be found in the book [10].

In order to provide a larger class of differential equations possessing unpredictable trajectories, in this study we take into account coupled systems in which generalized synchronization (GS) [11] takes place.

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More precisely, we consider the systems

$$x'(t) = F(x(t)), \tag{1}$$

and

$$y'(t) = G(x(t), y(t)), \tag{2}$$

where $F : \mathbb{R}^p \to \mathbb{R}^p$ and $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ are continuous functions. Systems (1) and (2) are respectively called the drive and response. It is worth noting that the coupled system (1)-(2) has a skew product structure. Our purpose is to rigorously prove that if the drive system (1) admits an unpredictable solution, then the same is true for the response system (2) when they are synchronized in the generalized sense. Sufficient conditions to approve the unpredictability are given in Section 3.

The concept of synchronization in coupled chaotic systems was initiated by Pecora and Carroll [12] for identical ones, and it was generalized for non-identical systems by Rulkov et al. [11]. GS characterizes the state of the response system when it is driven by the output of another system, the drive. It was proved by Kocarev and Parlitz [13] that GS occurs in coupled systems of the form (1)-(2) if and only if (2) is asymptotically stable for all initial values in a neighborhood of the chaotic attractor. When GS occurs a functional relation exists between the states of the drive and response systems [11, 13]. For that reason GS allows to predict the dynamics of the response system by the dynamics of the drive [14, 15]. It was demonstrated by González-Miranda [16] that GS can take place instead of identical synchronization even in coupled systems consisting of a chaotic system driven by an identical copy of itself. Moreover, Lyapunov exponent based conditions which imply that the response state is a differentiable function of the drive state were provided by Hunt et al. [17]. Dynamics of different dimensional systems, on the other hand, was investigated by Zhang et al. [18] when the mapping between the states of drive and response systems is continuously differentiable. Furthermore, in paper [19] synchronization between coupled chaotic systems was defined by means of the existence of a synchronization manifold which is compact, diagonal-like, smooth and invariant, and the stability of the manifold was investigated through the concept of $k$-hyperbolicity.

In this study we require that the state of the response (2) is a continuous function of the state of the drive (1). This criterion can be guaranteed by the auxiliary system approach, which was suggested by Abarbanel et al. [15]. In this approach one needs to use another response system that is identical to the original one but independent of it, and monitor the final states after the transitories have died off when different initial data within the basin of attraction are utilized. GS occurs in the dynamics if the final states of the two response systems are identical [14, 15]. Mutual false nearest neighbors, conditional Lyapunov exponents, and Lyapunov functions are other techniques which can be used to determine GS [11, 13, 14, 20].

A numerical approach for the determination of unpredictable trajectories in continuous-time systems was proposed in paper [21]. This approach is applicable even in the case of zero largest Lyapunov exponent. The results of the present study are different compared to [21] since we provide a theoretically approved technique based on the presence of GS for the detection of unpredictable motions. Synchronization between coupled systems was not taken into account in [21]. The usability of the already existing numerical techniques such as the mutual false nearest neighbors, auxiliary system approach, and conditional Lyapunov exponents for the detection of GS is one of the advantages of this study. The papers [22] and [23], on the other hand, were concerned with delta synchronization of Poincaré chaos in gas discharge-semiconductor systems. In these studies it was demonstrated that delta synchronization occurs in the absence of GS.

In order to demonstrate the existence of unpredictable trajectories in systems of differential equations a theoretical technique was developed in [24] without making use of synchronization. There are two superiorities of the present study in comparison with [24]. The first one is the structure of the system under consideration. Except the presence of delay, the system considered in this paper is more general compared to the one studied in [24]. Secondly, in the present research the response dynamics can be taken as fully nonlinear such that the smallness of the Lipschitz constant is not required. However, the Lipschitz constant for the nonlinear term in [24] has to be sufficiently small. Besides, the proof techniques are different since the results of [24] are not related to the concept of GS. It is worth noting that since we require synchronous motions in the coupled system (1)-(2), the coupling strength should be sufficiently large. It was demonstrated by Rulkov et al. [11] that weak connections may lead to the absence of synchronization in the generalized sense. The recent results are important not only from the theoretical point of view but also for applications since GS can occur in various real world problems concerning image encryption, secure communication, lasers, electronic circuits, and neural networks [25–29].

## 2. Preliminaries

Throughout the paper we make use of the Euclidean norm for vectors. We suppose that systems (1) and (2) admit compact invariant sets $\Lambda_x \subset \mathbb{R}^p$ and $\Lambda_y \subset \mathbb{R}^q$, respectively. Accordingly, a solution of the coupled system (1)-(2) with initial condition from the set $\Lambda_x \times \Lambda_y$ remains in that set.

In what follows we consider a class of GS in which the state of the response (2) is a continuous function of the state of the drive (1). More precisely, we say that GS occurs in the dynamics of the coupled system (1)-(2) if there is a continuous transformation $\psi$ such that for each $(x_0, y_0) \in \Lambda_x \times \Lambda_y$ the relation

$$\lim_{t\to\infty} \left\| y(t) - \psi(x(t)) \right\| = 0$$

holds, where $x(t)$ and $y(t)$ are respectively the solutions of (1) and (2) with $x(0) = x_0$ and $y(0) = y_0$. In this case the synchronization manifold

$$S = \left\{ (x, y) \in \Lambda_x \times \Lambda_y : \ y = \psi(x) \right\} \tag{3}$$

is normally attracting, and in particular it is invariant under the flow of the coupled system (1)-(2) [13, 30]. Detailed information concerning normally attracting manifolds can be found in the studies [31, 32]. Moreover, the reader is referred to the book [14] and the paper [15] for further results on the continuity of the transformation $\psi$.

The definition of an unpredictable function which is utilized in the present study is as follows.

**Definition 2.1.** *([24]) A uniformly continuous function $h : \mathbb{R} \to \Lambda$, where $\Lambda$ is a compact subset of $\mathbb{R}^p$, is called unpredictable if there exist positive numbers $\epsilon_0$, $r$ and sequences $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ both of which diverge to infinity such that $\left\| h(t + \mu_n) - h(t) \right\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\left\| h(t + \mu_n) - h(t) \right\| \geq \epsilon_0$ for each $t \in [\nu_n - r, \nu_n + r]$ and $n \in \mathbb{N}$.*

The number $\epsilon_0$ in Definition 2.1 is called the unpredictability constant of the function $h(t)$ [1, 24].

## 3. The main result

The following assumptions on the response system (2) are required.

**(A1)** There exists a positive number $L_1$ such that $\left\| G(x_1, y) - G(x_2, y) \right\| \geq L_1 \left\| x_1 - x_2 \right\|$ for each $x_1, x_2 \in \Lambda_x$ and $y \in \Lambda_y$.

**(A2)** There exists a positive number $L_2$ such that $\left\| G(x, y_1) - G(x, y_2) \right\| \leq L_2 \left\| y_1 - y_2 \right\|$ for each $x \in \Lambda_x$ and $y_1, y_2 \in \Lambda_y$.

A novel criterion for the existence of an unpredictable solution in the dynamics of the response system (2) is provided in the next theorem.

**Theorem 3.1.** *Suppose that the assumptions $(A1)$ and $(A2)$ are fulfilled. If the drive system (1) possesses an unpredictable solution and generalized synchronization takes place in the dynamics of the coupled system (1)-(2), then the response system (2) also possesses an unpredictable solution.*

**Proof.** Let $\widetilde{x}(t)$ be an unpredictable solution of the drive system (1). Since GS occurs in the dynamics of the coupled system (1)-(2), there is a continuous transformation $\psi$ such that the synchronization manifold $S$ defined by (3) is invariant under the flow of (1)-(2). Suppose that $\widetilde{y}(t)$ is the solution of $y'(t) = G(\widetilde{x}(t), y(t))$ satisfying $\widetilde{y}(0) = \psi(\widetilde{x}(0))$ so that $(\widetilde{x}(t), \widetilde{y}(t))$ lies on the manifold $S$ for each $t \geq 0$, i.e., $\widetilde{y}(t) = \psi(\widetilde{x}(t))$. In the proof we will verify that $\widetilde{y}(t)$ is unpredictable.

Let $\mathscr{C}$ be a compact subset of $\mathbb{R}$, and fix a positive number $\epsilon$. Because the transformation $\psi$ is continuous there is a positive number $\delta$ such that if $\|x_1 - x_2\| < \delta$, then

$$\left\|\psi(x_1) - \psi(x_2)\right\| < \epsilon. \tag{4}$$

Owing to the unpredictability of $\widetilde{x}(t)$ there exist positive numbers $\epsilon_0$, $r$ and sequences $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ both of which diverge to infinity such that $\left\|\widetilde{x}(t+\mu_n) - \widetilde{x}(t)\right\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\left\|\widetilde{x}(t+\mu_n) - \widetilde{x}(t)\right\| \geq \epsilon_0$ for each $t \in [\nu_n - r, \nu_n + r]$ and $n \in \mathbb{N}$. Accordingly, there is a natural number $n_0$ such that for every $n \geq n_0$ and $t \in \mathscr{C}$ we have

$$\left\|\widetilde{x}(t+\mu_n) - \widetilde{x}(t)\right\| < \delta,$$

and therefore, the inequality

$$\left\|\psi(\widetilde{x}(t+\mu_n)) - \psi(\widetilde{x}(t))\right\| < \epsilon$$

is fulfilled by means of (4). Hence, one can confirm for $n \geq n_0$ and $t \in \mathscr{C}$ that $\left\|\widetilde{y}(t+\mu_n) - \widetilde{y}(t)\right\| < \epsilon$. For that reason, $\left\|\widetilde{y}(t+\mu_n) - \widetilde{y}(t)\right\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$.

In the remaining part of the proof, we will show the existence of positive numbers $\epsilon_1$, $r_1$ and a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ with $\theta_n \to \infty$ as $n \to \infty$ such that $\left\|\widetilde{y}(t+\mu_n) - \widetilde{y}(t)\right\| \geq \epsilon_1$ for each $t \in [\theta_n - r_1, \theta_n + r_1]$ and $n \in \mathbb{N}$.

Let us denote

$$G(x, y) = (G_1(x, y), G_2(x, y), \ldots, G_q(x, y)),$$

where $G_i(x, y)$ is a real valued function for each $1 \leq i \leq q$. Because the function $G(x, y)$ is uniformly continuous on the compact region $\Lambda_x \times \Lambda_y$ one can find a positive number $M_G$ such that $\left\|G(x, y)\right\| \leq M_G$ for each $x \in \Lambda_x, y \in \Lambda_y$. Therefore, $\widetilde{y}(t)$ is uniformly continuous on $\mathbb{R}$. Due to the uniform continuity of $\widetilde{x}(t)$, there exists a positive number $r_0$ with $r_0 \leq r$ such that

$$\left\|G(\widetilde{x}(t+\mu_n), \widetilde{y}(t)) - G(\widetilde{x}(\mu_n+\nu_n), \widetilde{y}(\nu_n))\right\| \leq \frac{L_1\epsilon_0}{4\sqrt{q}} \tag{5}$$

and

$$\left\|G(\widetilde{x}(t), \widetilde{y}(t)) - G(\widetilde{x}(\nu_n), \widetilde{y}(\nu_n))\right\| \leq \frac{L_1\epsilon_0}{4\sqrt{q}} \tag{6}$$

for each $t \in [\nu_n - r_0, \nu_n + r_0]$ and $n \in \mathbb{N}$.

According to assumption $(A1)$, for each natural number $n$ there is an integer $i_n$ with $1 \leq i_n \leq q$ such that the inequality

$$\left|G_{i_n}(\widetilde{x}(\mu_n+\nu_n), \widetilde{y}(\nu_n)) - G_{i_n}(\widetilde{x}(\nu_n), \widetilde{y}(\nu_n))\right| \geq \frac{L_1}{\sqrt{q}}\left\|\widetilde{x}(\mu_n+\nu_n) - \widetilde{x}(\nu_n)\right\| \geq \frac{L_1\epsilon_0}{\sqrt{q}} \tag{7}$$

is valid.

Fix a natural number $n$. For each $t \in [\nu_n - r_0, \nu_n + r_0]$, one can attain by means of the inequalities (5), (6), and (7) that

$$\begin{aligned}\left|G_{i_n}(\widetilde{x}(t+\mu_n), \widetilde{y}(t)) - G_{i_n}(\widetilde{x}(t), \widetilde{y}(t))\right| \geq\ & \left|G_{i_n}(\widetilde{x}(\mu_n+\nu_n), \widetilde{y}(\nu_n)) - G_{i_n}(\widetilde{x}(\nu_n), \widetilde{y}(\nu_n))\right| \\ & - \left|G_{i_n}(\widetilde{x}(\mu_n+\nu_n), \widetilde{y}(\nu_n)) - G_{i_n}(\widetilde{x}(t+\mu_n), \widetilde{y}(t))\right| \\ & - \left|G_{i_n}(\widetilde{x}(t), \widetilde{y}(t)) - G_{i_n}(\widetilde{x}(\nu_n), \widetilde{y}(\nu_n))\right| \\ \geq\ & \frac{L_1\epsilon_0}{2\sqrt{q}}.\end{aligned}$$

There exist points $s_1, s_2, \ldots, s_q$ in the interval $[\nu_n - r_0, \nu_n + r_0]$ such that

$$\begin{aligned}\left\|\int_{\nu_n-r_0}^{\nu_n+r_0}[G(\widetilde{x}(s+\mu_n),\widetilde{y}(s))-G(\widetilde{x}(s),\widetilde{y}(s))]\,ds\right\| &\geq 2r_0\left|G_{i_n}(\widetilde{x}(s_{i_n}+\mu_n),\widetilde{y}(s_{i_n}))-G_{i_n}(\widetilde{x}(s_{i_n}),\widetilde{y}(s_{i_n}))\right| \\ &\geq \frac{L_1 r_0 \epsilon_0}{\sqrt{q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\widetilde{y}(\mu_n+\nu_n+r_0)-\widetilde{y}(\nu_n+r_0)\right\| &\geq \left\|\int_{\nu_n-r_0}^{\nu_n+r_0}[G(\widetilde{x}(s+\mu_n),\widetilde{y}(s))-G(\widetilde{x}(s),\widetilde{y}(s))]\,ds\right\| \\ &\quad -\left\|\widetilde{y}(\mu_n+\nu_n-r_0)-\widetilde{y}(\nu_n-r_0)\right\| \\ &\quad -\int_{\nu_n-r_0}^{\nu_n+r_0}\left\|G(\widetilde{x}(s+\mu_n),\widetilde{y}(s+\mu_n))-G(\widetilde{x}(s+\mu_n),\widetilde{y}(s))\right\|ds \\ &\geq \frac{L_1 r_0\epsilon_0}{\sqrt{q}}-\left\|\widetilde{y}(\mu_n+\nu_n-r_0)-\widetilde{y}(\nu_n-r_0)\right\| \\ &\quad -\int_{\nu_n-r}^{\nu_n+r} L_2\left\|\widetilde{y}(s+\mu_n)-\widetilde{y}(s)\right\|ds.\end{aligned}$$

The last inequality implies that

$$\max_{t\in[\nu_n-r_0,\nu_n+r_0]}\left\|\widetilde{y}(t+\mu_n)-\widetilde{y}(t)\right\| \geq \frac{L_1 r_0\epsilon_0}{2\sqrt{q}(r_0L_2+1)}.$$

Suppose that

$$\max_{t\in[\nu_n-r_0,\nu_n+r_0]}\left\|\widetilde{y}(t+\mu_n)-\widetilde{y}(t)\right\| = \left\|\widetilde{y}(\mu_n+\zeta_n)-\widetilde{y}(\zeta_n)\right\|$$

for some $\zeta_n \in [\nu_n - r_0, \nu_n + r_0]$. Let us denote

$$\widetilde{r} = \min\left\{\frac{L_1 r_0\epsilon_0}{8M_G\sqrt{q}(r_0L_2+1)}, r_0\right\}.$$

It can be confirmed for $t \in [\zeta_n - \widetilde{r}, \zeta_n + \widetilde{r}]$ that

$$\begin{aligned}\left\|\widetilde{y}(t+\mu_n)-\widetilde{y}(t)\right\| &\geq \left\|\widetilde{y}(\mu_n+\zeta_n)-\widetilde{y}(\zeta_n)\right\|-\left|\int_{\zeta_n}^{t}\left\|G(\widetilde{x}(s+\mu_n),\widetilde{y}(s+\mu_n))-G(\widetilde{x}(s),\widetilde{y}(s))\right\|ds\right| \\ &\geq \frac{L_1 r_0\epsilon_0}{2\sqrt{q}(r_0L_2+1)}-2\widetilde{r}M_G \\ &= \frac{L_1 r_0\epsilon_0}{4\sqrt{q}(r_0L_2+1)}.\end{aligned}$$

Now, we define $\theta_n = \zeta_n + \widetilde{r}/2$ if $\zeta_n \in [\nu_n - r_0, \nu_n]$ and $\theta_n = \zeta_n - \widetilde{r}/2$ if $\zeta_n \in (\nu_n, \nu_n + r_0]$. The sequence $\{\theta_n\}_{n\in\mathbb{N}}$ diverges to infinity since the same is true for the sequence $\{\nu_n\}_{n\in\mathbb{N}}$. The inequality $\left\|\widetilde{y}(t+\mu_n)-\widetilde{y}(t)\right\| \geq \epsilon_1$ holds for each $t \in [\theta_n - r_1, \theta_n + r_1]$ and $n \in \mathbb{N}$, where $r_1 = \widetilde{r}/2$ and the unpredictability constant $\epsilon_1$ of $\widetilde{y}(t)$ is defined by

$$\epsilon_1 = \frac{L_1 r_0\epsilon_0}{4\sqrt{q}(r_0L_2+1)}. \tag{8}$$

Thus, $\widetilde{y}(t)$ is unpredictable. □

**Remark 3.2.** *In the proof of Theorem 3.1 the sequence $\{\theta_n\}_{n\in\mathbb{N}}$ is constructed in such a way that the interval $[\theta_n - r_1, \theta_n + r_1]$ is a subset of $[\nu_n - r, \nu_n + r]$ for each $n \in \mathbb{N}$. Therefore, if $\widetilde{x}(t)$ is an unpredictable solution of (1), then the solution $z(t) = (\widetilde{x}(t), \widetilde{y}(t))$ of the coupled system (1)-(2) lying on the synchronization manifold S defined by (3) is also unpredictable such that $\left\|z(t+\mu_n) - z(t)\right\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\left\|z(t+\mu_n) - z(t)\right\| \geq (\epsilon_0^2 + \epsilon_1^2)^{1/2}$ for each $t \in [\theta_n - r_1, \theta_n + r_1]$ and $n \in \mathbb{N}$, in which $\epsilon_0$ and $\epsilon_1$ are respectively the unpredictability constants of $\widetilde{x}(t)$ and $\widetilde{y}(t)$. Hence, one can conclude under the assumptions of Theorem 3.1 that the coupled system (1)-(2) possesses an unpredictable solution. Moreover, the unpredictable solution $\widetilde{y}(t)$ of (2) is asymptotically stable by the results of paper [13].*

In the next section we demonstrate the extension of unpredictable solutions among coupled systems in which the drive is an autonomous hybrid system and the response is a Lorenz system.

## 4. An example

According to the results of paper [2], the logistic map

$$\eta_{i+1} = \mu\eta_i(1-\eta_i), \tag{9}$$

where $i \in \mathbb{Z}$, possesses an unpredictable orbit for the values of the real parameter $\mu$ between $3 + (2/3)^{1/2}$ and 4. Moreover, for such values of $\mu$ the unit interval $[0,1]$ is invariant under the iterations of (9) [33].

Let $\left\{\eta_i^*\right\}_{i\in\mathbb{Z}}$ be an unpredictable orbit of (9) with $\mu = 3.94$, which belongs to the unit interval $[0,1]$, and suppose that $\gamma : \mathbb{R} \to [0,1]$ is the piecewise constant function defined by $\gamma(t) = \eta_i^*$ for $t \in (i, i+1]$, $i \in \mathbb{Z}$. The function $\gamma(t)$ is the solution of the impulsive system

$$\gamma'(t) = 0, \;\; \Delta\gamma|_{t=i} = \eta_i^* - \eta_{i-1}^* \tag{10}$$

satisfying the initial condition $\gamma(0) = \eta_{-1}^*$, where $\Delta\gamma|_{t=i} = \gamma(i+) - \gamma(i)$ and $\gamma(i+) = \lim_{t\to i^+}\gamma(t)$. The impulse moments of (10) coincide with the ones of the solution of the discontinuous dynamical system

$$s'(t) = -1, \;\; \Delta s|_{s=0} = 1 \tag{11}$$

with $s(0) = 0$.

It was demonstrated in study [8] that the differential equation

$$\phi'(t) = -\phi(t) + \gamma(t), \tag{12}$$

admits the unique uniformly continuous unpredictable solution

$$\phi(t) = \int_{-\infty}^{t} e^{-(t-s)}\gamma(s)ds,$$

which is globally exponentially stable. Theorem 5.2 [2], on the other hand, implies that the function

$$\widetilde{\phi}(t) = \Big(2\phi(t) + 0.1\sin(\phi(t)),\ 3\phi(t),\ 2.5\phi(t) + 0.2\cos(\phi(t))\Big)$$

is also unpredictable. For that reason, the system

$$\begin{aligned} x_1'(t) &= -0.5x_1(t) + 2\phi(t) + 0.1\sin(\phi(t)) \\ x_2'(t) &= -0.3x_2(t) + 0.2\arctan(x_1(t)) + 3\phi(t) \\ x_3'(t) &= -0.4x_3(t) + 2.5\phi(t) + 0.2\cos(\phi(t)) \end{aligned} \tag{13}$$

possesses a unique unpredictable solution by Theorem 4.1 [8]. Accordingly, the autonomous hybrid system (9)-(10)-(11)-(12)-(13) has a unique unpredictable solution. Figure 1 shows the time-series of the $x_1$, $x_2$, and $x_3$-coordinates of the solution of this hybrid system corresponding to the initial data $\eta_0 = 0.76$, $\phi(0) = 0.41$, $x_1(0) = 2.26$, $x_2(0) = 6.48$, $x_3(0) = 3.89$, and value of the parameter $\mu = 3.94$. The irregularity of each of the time-series confirms the presence of an unpredictable solution.

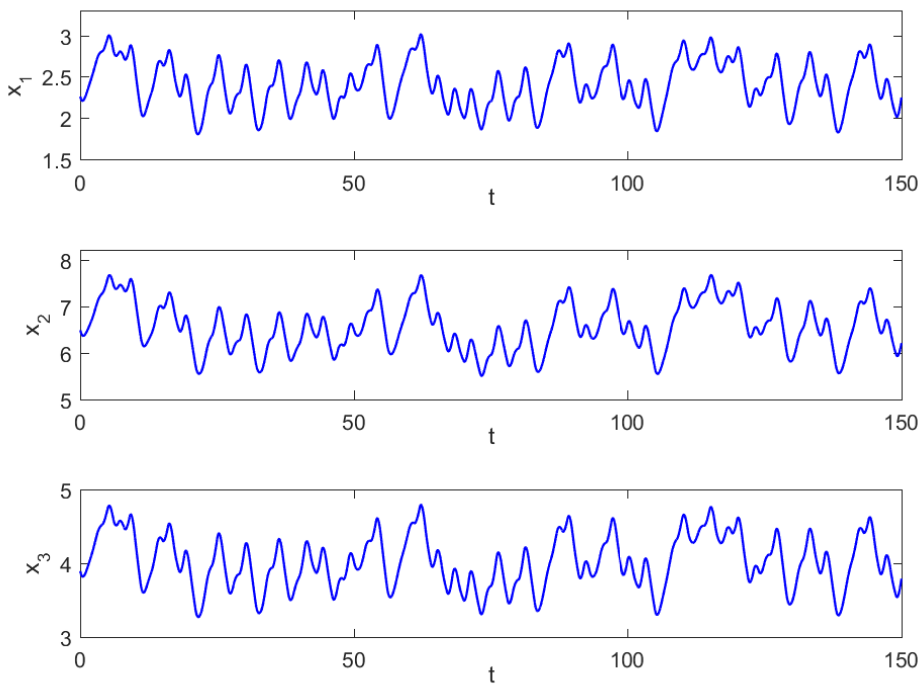


Figure 1: Time-series of the $x_1$, $x_2$, and $x_3$-coordinates of the autonomous hybrid system (9)-(10)-(11)-(12)-(13). Each time-series is irregular, and this confirms the presence of an unpredictable solution in the dynamics.

Next, let us take into account the Lorenz system [34]

$$\begin{aligned} y_1'(t) &= -20y_1(t) + 20y_2(t) \\ y_2'(t) &= -y_1(t)y_3(t) + 41.05y_1(t) - y_2(t) \\ y_3'(t) &= y_1(t)y_2(t) - 3y_3(t). \end{aligned} \tag{14}$$

We use system (13) as the drive, and establish unidirectional coupling between (13) and (14) by setting up the response system

$$\begin{aligned} y_1'(t) &= -20y_1(t) + 20y_2(t) + 2.9x_1(t) \\ y_2'(t) &= -y_1(t)y_3(t) + 41.05y_1(t) - y_2(t) + 2.6x_2(t) \\ y_3'(t) &= y_1(t)y_2(t) - 3y_3(t) + 2.4x_3(t), \end{aligned} \tag{15}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))$ is a solution of (13). It can be verified that the assumptions $(A1)$ and $(A2)$ hold for system (15).

Using the solution of the drive system (13) whose coordinates are depicted in Figure 1, the trajectory of the response system (15) shown in Figure 2 is obtained. In the simulation, the initial data $y_1(0) = 10.91$, $y_2(0) = 10.58$, and $y_3(0) = 41.62$ are utilized. The irregular behavior of this trajectory reveals the presence of an unpredictable solution.

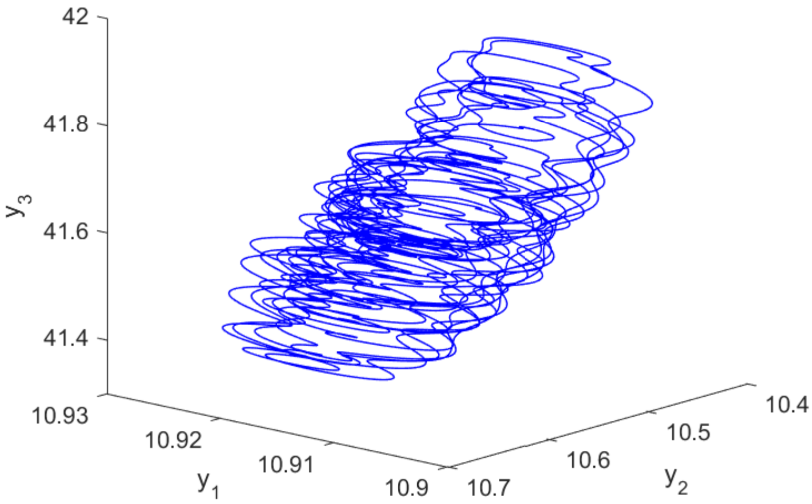


Figure 2: The projection of the trajectory of autonomous hybrid system (9)-(10)-(11)-(12)-(13)-(15) on the $y_1 - y_2 - y_3$ space. The irregularity observed in the figure approves that system (15) admits an unpredictable solution.

Now, we will make use of the auxiliary system approach [15] to show the presence of GS. We consider the auxiliary system

$$\begin{aligned} z_1'(t) &= -20z_1(t) + 20z_2(t) + 2.9x_1(t) \\ z_2'(t) &= -z_1(t)z_3(t) + 41.05z_1(t) - z_2(t) + 2.6x_2(t) \\ z_3'(t) &= z_1(t)z_2(t) - 3z_3(t) + 2.4x_3(t), \end{aligned} \tag{16}$$

which is an identical copy of the response system (15). Again using the solution of (13) shown in Figure 1 and the initial data $y_1(0) = 10.91$, $y_2(0) = 10.58$, $y_3(0) = 41.62$, $z_1(0) = 10.52$, $z_2(0) = 10.03$, and $z_3(0) = 41.96$, we represent in Figure 3 the projection of the stroboscopic plot of the hybrid system (9)-(10)-(11)-(12)-(13)-(15)-(16) on the $y_1 - z_1$ plane. The plot is obtained by omitting the first 200 iterations in order to eliminate the transients. Because it takes place on the line $z_1 = y_1$, one can confirm that GS occurs, and therefore, the state of (15) is a continuous function of the state of (13). More precisely, there exists a continuous transformation $\psi$, which has no explicit time dependence, taking the trajectories $x(t) = (x_1(t), x_2(t), x_3(t))$ of (13) into the trajectories $y(t) = (y_1(t), y_2(t), y_3(t))$ of (15) such that $y(t) = \psi(x(t))$ on the corresponding synchronization manifold [14, 15]. Hence, in accordance with our theoretical results, the response system (15) possesses an unpredictable solution. Additionally, an unpredictable motion takes place also in the dynamics of the coupled system (13)-(15) in accordance with Remark 3.2.

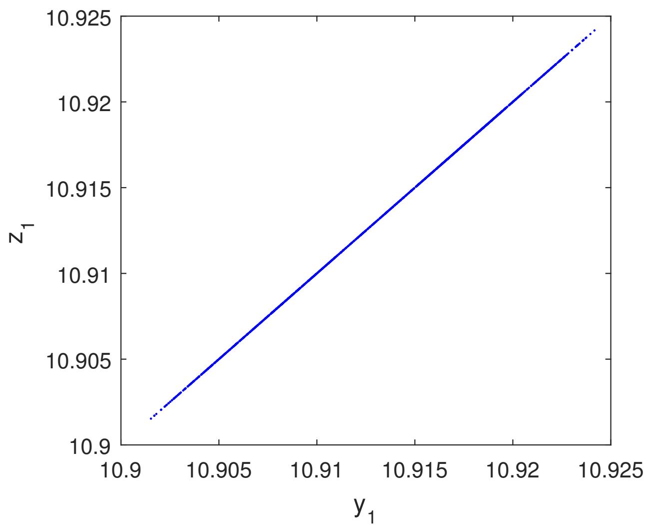


Figure 3: The result of the auxiliary system approach applied to the coupled system (13)-(15). The figure confirms that (13) and (15) are synchronized in the generalized sense and there is an unpredictable solution of the response system (15).

To approve the presence of GS one more time, let us evaluate the conditional Lyapunov exponents of the response (15). For that purpose, we take into account the corresponding variational system

$$\begin{aligned} w_1'(t) &= -20w_1(t) + 20w_2(t) \\ w_2'(t) &= (-y_3(t) + 41.05)w_1(t) - w_2(t) - y_1(t)w_3(t) \\ w_3'(t) &= y_2(t)w_1(t) + y_1(t)w_2(t) - 3w_3(t). \end{aligned} \tag{17}$$

When the solution $y(t) = (y_1(t), y_2(t), y_3(t))$ of (15) with $\eta_0 = 0.76$, $\phi(0) = 0.41$, $x_1(0) = 2.26$, $x_2(0) = 6.48$, $x_3(0) = 3.89$, $y_1(0) = 10.91$, $y_2(0) = 10.58$, $y_3(0) = 41.62$, and $\mu = 3.94$ is utilized, the largest Lyapunov exponent of system (17) is calculated as $-0.2371$, i.e., all conditional Lyapunov exponents of system (15) are negative. This confirms that (13) and (15) are synchronized in the generalized sense [13, 14], and accordingly, the response system (15) as well as the coupled system (13)-(15) have unpredictable solutions.

## 5. Concluding remarks

Synchronization is one of the phenomena that can occur in coupled chaotic systems [14]. This phenomenon can be observed in various fields such as image encryption, secure communication, lasers, electronic circuits, and neural networks [25–29]. The presence of an unpredictable trajectory makes the corresponding dynamics exhibit chaotic behavior [1]. The main novelty of this paper is the usage of GS to verify the existence of an unpredictable trajectory in the dynamics of not only the response (2) but also the coupled system (1)-(2). In other words, it is rigorously proved that if the drive system (1) possesses an unpredictable solution, then the systems (2) as well as (1)-(2) also have the same property under the conditions mentioned in Section 3. The proposed technique makes it possible to obtain high dimensional systems possessing unpredictable trajectories. Equation (8), which is obtained in the proof of Theorem 3.1, implies that the unpredictability constant for the unpredictable solution of (2) is proportional to the one of (1), and it becomes smaller if a higher dimensional response system is taken into account. In the future, our approach can be used to detect unpredictable trajectories in unidirectionally or mutually coupled time-delayed systems [35].

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