Filomat 37:18 (2023), 6089–6114
https://doi.org/10.2298/FIL2318089Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost periodic dynamical behaviors of neutral-type neural networks with discontinuous activations and mixed delays

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Abstract. This paper presents a class of neutral-type neural networks with discontinuous activations and mixed delays. By using differential inclusions theory, the non-smooth analysis theory with Lyapunov-like approach, some new sufficient criteria are given to ascertain the existence, uniqueness and globally exponential stability of the almost periodic solution for the addressed neural network system. Some recent results in the literature are generalized and improved. Finally, simulation results of two topical numerical examples are also delineated to demonstrate the effectiveness of our theoretical results.

1. Introduction

In the past decade, neural networks have attracted considerable attention because of their potential applications in associative memory, pattern recognition, optimization, model identification, signal processing, etc. Due to the complicated dynamic properties of the neural cells in the real word, it is natural to consider these complicated dynamic properties of neural cells by neutral-type neural networks. Neutral neural networks contain some information about the derivative of the past state. Due to this, neutral neural networks can be employed to characterise the properties of a neural reaction process more precisely. As was pointed out by Hale [14] that the properties of neutral operators are important for studying neutral-type functional differential equations. In recent years, many investigations have been investigated the dynamic behaviors of the neutral neural networks with delays. For instance, see [37], [38], [40], [42], [53], [56], [57]. However, almost all works in [37], [38], [40], [42], [53], [56], [57] and the references related therein on the neural networks of neutral-type have still assumed that the activation functions are continuous, Lipschitz continuous or even smooth. In fact, discontinuous behaviors of dynamical systems can be found everywhere such as impacting machines, dry friction. During the past several years, considerable efforts have been devoted to investigate the neural network systems with discontinuous activation functions, see [25], [26], [31], [35], [43], [47] and the references therein. Moreover, from the references above, we can find that the neutral terms in their systems are $x_j'(t-\tau_{ij}(t))$, $x_j'(t-\widetilde{t}_{ij}(t))$ and $x_j'(t-u)$. As was pointed out by Hale [14] that the properties of neutral operator D are important for studying neutral-type functional

2020 *Mathematics Subject Classification*. 34A36, 34C27
Keywords. Almost periodic solution, Discontinuous activations, Mixed delays, Global exponential stability.
Received: 14 February 2021; Revised: 20 May 2021; Accepted: 19 December 2021
Communicated by Marko Petković
Research supported by the National Natural Science Foundation of China (12001011), Natural Science Fund Project of the University in Anhui Province (2022AH030023).
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differential equation. Thus, neutral type neural networks with D operator have more realistic significance than non-operator-based ones in many practical applications of neural networks dynamics. Based on the complex neural reactions, neutral type neural networks with D operator can be described by the neutral-type functional differential equation, for more details, we refer the readers to [7], [8], [9], [20], [21], [22], [23], [39], [44], [45], [49], [52], [54]. For example, by using fixed point theorem, Lyapunov functional method and comparison theorem, Wang and Zhu in [49] considered the existence, global asymptotic stability and exponential stability of the unique almost periodic solution for neutral-type neural networks with delays. Based on the work of [49], be means of the Mawhin's continuation theorem of coincidence degree theory, Du et al in [9] further considered the existence and asymptotic behavior results of periodic solution for discrete-time neutral-type neural networks. However, almost all works in [7], [8], [9], [20], [21], [22], [23], [39], [49], [52], [54] and the references related therein have always assumed that the activation functions with delays are continuous, Lipschitz continuous or even smooth.

On the other hand, as pointed out in [4], [11], [29], compared with periodic effects, almost periodic effects are more frequent in many real world applications. In fact, by a recent work [2], to some extent and in the sense of category, the "amount" of almost periodic functions (not periodic) is far more than the "amount" of continuous periodic functions. That is to say, almost periodic oscillatory behavior is considered to be more accordant with reality. The almost periodic neural networks are as a natural extension of the periodic ones. In recent years, much efforts have been devoted to studying the dynamical behaviors of recurrently connected neural network with almost periodic parameters, see [18], [19], [32], [34], [51], [55] and the references therein. So, how to study the almost periodic solutions of neutral-type neural networks with discontinuous activations is important.

Actually, time delays, especially the time-varying delays may turn expected dynamics of the proposed neural network into some undesired complex dynamical behaviors. In general, the results of the stability analysis and synchronization analysis for delayed neural networks contain delay-dependent and delay-independent criteria. However, the former can derive less conservativeness and take more advantages in the practical applications. In view of the way it occurs, time delay should have two types: discrete delay and distributed delay. In reality, discrete (time-varying) delay and distributed delay always occur simultaneously. For more knowledge about the practical design and application of time-delayed neural networks with discontinuous activation functions, we refer to [5], [6], [17], [33], [36], [41], [43], [48], [50].

Compared with the neural networks with continuous activations and delays, little attention has been devoted to the study of the almost periodic dynamic behavior of the neural networks with discontinuous activations and delays so far, see [46], [50]. In addition, as far as we know, there is no works concerning on the neutral-type neural networks with discontinuous activations and no results concerning on the almost periodic dynamic behavior of the neutral-type neural networks with discontinuous activations and mixed delays. The major difficulty may contain the following three aspects:

- In order to obtain almost periodic dynamic behavior, the new framework dealing with the neutral terms, discontinuous activation and mixed time varying delays should be established.

- Since the almost periodic solution has special properties, the corresponding stability analysis with become more complicated if the discontinuous activations and the neutral terms exist. Thus, some useful mathematical analysis techniques are required to solve this influence.

Therefore, it is not only of theoretically interesting, but also practical significance to study the almost periodic dynamic behavior of the neutral-type neural networks with discontinuous activations and mixed delays. In order to solve the difficulties listed above and motivated by the previous works, in the present paper, we are concerned with the following neutral-type neural network system with discontinuous activations and mixed delays:

$$\begin{aligned}(A_i x_i)'(t) = &- d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) \\ &+ \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} f_j(x_j(s))ds + I_i(t), \quad i = 1, 2, ..., n,\end{aligned} \tag{1.1}$$

where A_i is a difference operator defined by

$$A_i x_i(t) = x_i(t) - \sum_{j=1}^{n} h_{ij}(t) x_i(t - \delta_{ij}(t)), \quad i = 1, 2, ..., n,$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^\top \in \mathbb{R}^n$ and $x_i(t)$ denotes the state variable of the potential of the ith neuron; $d_i(t)$ represents the self-inhibition with which the ith neuron will reset its potential to the resting state in isolations when disconnected from the network; $a_{ij}(t)$ denotes the connection strength of the jth neuron on the ith neuron; $b_{ij}(t)$ and $c_{ij}(t)$ are the delayed feedbacks of the jth neuron on the ith neuron, with time-varying delay and distributed delay, respectively; $\tau_{ij}(t)$ and $\delta_{ij}(t)$ denote the discrete time-varying delay, $\sigma_{ij}(t)$ denotes the distributed time-varying delay; $f_i(x_i(t))$ represents the activation function of the ith neuron; $I_i(t)$ denotes the external input to the ith neuron.

The remainder part of this paper is organized as follows. In Section 2 , some basic definitions and preliminary lemmas are introduced. In Section 3 , some new criteria are given to establish the existence result of almost periodic solutions. In Section 4 , we give the sufficient conditions to guarantee the uniqueness and global exponential stability of the almost periodic solutions. In Section 5 , we provide two numerical examples to demonstrate the theoretical results. Finally, some conclusions are stated in Section 6 .

2. Essential Definitions and Lemmas

In this section, we state some definitions and preliminary lemmas, which will be used throughout this paper. Firstly, let us recall some basic notations and facts concerning set-valued maps.

Define

$$\begin{aligned}&d_i^M = \max_{1\le i\le n}\sup_{t\in\mathbb{R}} |d_i(t)|, I_i^M = \max_{1\le i\le n}\sup_{t\in\mathbb{R}} |I_i(t)|, a_{ij}^M = \max_{1\le i,j\le n}\sup_{t\in\mathbb{R}} |a_{ij}(t)|,\\ &b_{ij}^M = \max_{1\le i,j\le n}\sup_{t\in\mathbb{R}} |b_{ij}(t)|, c_{ij}^M = \max_{1\le i,j\le n}\sup_{t\in\mathbb{R}} |c_{ij}(t)|, h_{ij}^M = \max_{1\le i,j\le n}\sup_{t\in\mathbb{R}} |h_{ij}(t)|.\end{aligned}$$

Let $K(\mathbb{R}^n)$ denote the collection of all nonempty compact subsets of $\mathbb{R}^n$ with the Hausdorff metric ρ defined by

$$\rho(A, B) = \max\{\beta(A, B), \beta(B, A)\}, \;\; A, B \in K(\mathbb{R}^n),$$

where

$$\beta(A, B) = \sup\{\mathrm{dist}(x, B) : x \in A\}, \; \beta(B, A) = \sup\{\mathrm{dist}(y, A) : y \in b\}.$$

Obviously, with metric ρ, $K(\mathbb{R}^n)$ is a complete metric space. Let

$$Kv(\mathbb{R}^n) = \{A \in K(\mathbb{R}^n) : A \text{ is convex}\}.$$

Definition 2.1. *Suppose $E \subset \mathbb{R}^m$. Then $x \mapsto F(x)$ is called a setvalued map from $E \hookrightarrow \mathbb{R}^n$ if to each point x of a set $E \subset \mathbb{R}^m$, there corresponds a nonempty set $F(x) \subset \mathbb{R}^n$. A set-valued map $F : E \to K(\mathbb{R}^n)$ is said to be upper semicontinuous (USC) at $x_0 \in E$, if $\beta(F(x), F(x_0)) \to 0$ as $x \to x_0$. $F(x)$ is said to have a closed (convex, compact) image if for each $x \in E$, $F(x)$ is closed (convex, compact). Graph$(F(E)) = \{(x, y) : x \in E,$ and $y \in F(x)\}$, where E is subset of $\mathbb{R}^n$.*

Now we introduce the concept of Filippov solution (see Filippovn [10]). Consider the following differential system in vector notation:

$$\frac{dx}{dt} = f(t, x), \tag{2.1}$$

where $f(t, x)$ is discontinuous in x.

Definition 2.2. *Consider the set-valued map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defined as*

$$F(t, x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}[f(t, \mathfrak{B}(x, \delta) \setminus N)], \tag{2.2}$$

where $\mathfrak{B}(x, \delta)$ is the ball of center x and radius δ; $\overline{co}(E)$ is the closure of the convex hull of set E; intersection is taken over all sets N of measure zero and over all $\delta > 0$; $\mu(N)$ is Lebesgue measure of set N. A vector-value function $x(t)$ defined on a nondegenerate interval $I \subseteq \mathbb{R}$ is called a Filippov solution of (2.1)*, if it is absolutely continuous on any subinterval $[t_1, t_2]$ of I and for almost all $t \in I$, $x(t)$ satisfies the differential inclusion*

$$\frac{dx}{dt} \in F(t, x). \tag{2.3}$$

Lemma 2.3. *([24]) If $\sum_{j=1}^{n} h_{ij}^M < 1$, then the inverse of difference operator A_i, denoted by A_i^{-1}, exists and*

$$|A_i^{-1}| = \sup_{t\in[0,\omega]} |A_i^{-1}(t)| \le 1\Big/\Big(1 - \sum_{j=1}^{n} h_{ij}^M\Big), \ \ i = 1, 2, ..., n.$$

From Lemma 2.3 , we can see that the inverse of difference operator A_i, denoted by A_i^{-1}, exits. Let $(A_i x_i)(t) = u_i(t)$, then $x_i(t) = (A_i^{-1} u_i)(t)$. Thus, system (1.1) transforms to the following system, for convenience, we still use $x_i(t)$ to denote the state solution, that is,

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n} h_{ij}(t)(A_i^{-1}x_i)(t - \delta_{ij}(t)) + \sum_{j=1}^{n} a_{ij}(t) f_j\Big((A_j^{-1}x_j)(t)\Big) \\ &+ \sum_{j=1}^{n} b_{ij}(t) f_j\Big((A_j^{-1}x_j)(t - \tau_{ij}(t))\Big) + \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t} f_j\Big((A_j^{-1}x_j)(s)\Big)ds + I_i(t), \ i = 1, 2, ..., n. \end{aligned} \tag{2.4}$$

Next, let us consider the differential equation system (2.4). Since dropping the assumption of continuity on the activation functions, we need to specify what is meant by a solution of the equation system (2.4) with discontinuous right-hand sides. Moreover, we need to introduce the concept of an output solution associated with a solution of (2.4). For this purpose, we extend the concept of the Filippov solution to the differential equation system (2.4) as follows.

Definition 2.4. *A vector function $x = (x_1, x_2, ..., x_n)^\top : [-\varsigma, b) \to \mathbb{R}^n$, $\varsigma = \max_{1\le i,j\le n}\{\tau_{ij}^M, \sigma_{ij}^M\}$ and $b \in (0, +\infty]$, is a state solution of the discontinuous system* (2.4) *on $[-\varsigma, b)$ if*

(1) *$x = (x_1, x_2, ..., x_n)^\top$ is continuous on $[-\varsigma, b)$ and absolutely continuous on any compact interval of $[0, b)$;*

(2) *there exists a measurable function $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^\top : [-\varsigma, b) \to \mathbb{R}^n$ such that $\gamma_j(t) \in \overline{co}[f_j((A_j^{-1}x_j)(t))]$ for a.e. $t \in [-\varsigma, b)$ and*

$$\begin{aligned} \frac{dx_i(t)}{dt} =& - d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n} h_{ij}(t)(A_i^{-1}x_i)(t - \delta_{ij}(t)) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) + \sum_{j=1}^{n} b_{ij}(t)\gamma_j(t - \tau_{ij}(t)) \\ &+ \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t} \gamma_j(s)ds + I_i(t), \ \text{for } a.e. \ t \in [0, b), \ i = 1, 2, ..., n. \end{aligned} \tag{2.5}$$

Any function $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^\top$ satisfying (2.5) *is called an output solution associated with the state $x = (x_1, x_2, ..., x_n)^\top$. With this definition it turns out that the state $x = (x_1, x_2, ..., x_n)^\top$ is a solution of* (2.4) *in the sense of Filippov since it satisfies*

$$\begin{aligned}\frac{dx_i(t)}{dt} \in & -d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n} h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n} a_{ij}(t)\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(t)\Big)\Big] \\ & + \sum_{j=1}^{n} b_{ij}(t)\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(t-\tau_{ij}(t))\Big)\Big] + \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t} \overline{co}\Big[f_j\Big((A_j^{-1}x_j)(s)\Big)\Big]ds + I_i(t), \\ & \text{for } a.e.\ t \in [0,b),\ i = 1,2,...,n. \end{aligned} \tag{2.6}$$

Definition 2.5. *(IVP). For any continuous function $\phi = (\phi_1, \phi_2, ..., \phi_n)^\top : [-\varsigma, 0] \to \mathbb{R}^n$ and any measurable selection $\psi = (\psi_1, \psi_2, ..., \psi_n)^\top : [-\varsigma, 0] \to \mathbb{R}^n$, such that $\psi_j(s) \in \overline{co}[g_j(\phi_j(s))](j = 1, 2, ..., n)$ for a.e. $s \in [-\varsigma, 0]$ by an initial value problem associated to* (2.4) *with initial condition $[\phi, \psi]$, we mean the following problem: find a couple of functions $[x, \gamma] : [-\varsigma, b) \to \mathbb{R}^n \times \mathbb{R}^n$, such that x is a solution of* (2.4) *on $[-\varsigma, b)$ for some $b > 0$, γ is an output solution associated to x, and*

$$\begin{cases} \frac{dx_i(t)}{dt} & = -d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n} h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) \\ & \quad + \sum_{j=1}^{n} b_{ij}(t)\gamma_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}\gamma_j(s)ds + I_i(t), \quad \text{for } a.e.\ t \in [0,b),\ i = 1,2,...,n, \\ \gamma_j(t) & \in \overline{co}[f_j((A_j^{-1}x_j)(t))], \quad \text{for } a.e.\ t \in [0,b), \\ x(s) & = \phi(s),\ \forall s \in [-\varsigma, 0], \\ \gamma(s) & = \psi(s), \quad \text{for } a.e.\ s \in [-\varsigma, 0]. \end{cases} \tag{2.7}$$

Definition 2.6. *Let $x^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))^\top$ be a solution of the given IVP of system* (2.4) *(or* (2.5)*), $x^*(t)$ is said to be globally exponentially stable, if for any solution $x(t) = (x_1(t), x_2(t), ..., x_n(t))^\top$ of* (2.4) *(or* (2.5)*), there exist constants $\alpha > 0$ and $M > 0$ such that*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \le Me^{-\alpha t}, \quad \text{for } t \ge t_0 \ge 0.$$

Definition 2.7. *(Clarke Regular[3]) $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be regular, if for each $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$,*

(1) there exists the usual right directional derivative

$$D^+V(x,v) = \lim_{h\to 0^+} \frac{V(x+hv) - V(x)}{h};$$

(2) the generalized directional derivative of V at x in the direction $v \in \mathbb{R}^n$ is defined as

$$\widetilde{D}V(x,v) = \lim_{h\to 0^+} \sup_{y\to x} \frac{V(y+hv) - V(y)}{h},$$

then $D^+V(x,v) = \widetilde{D}V(x,v)$.

Definition 2.8. *For a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, we can define Clarke's generalized gradient of V at point (x,t), as follows*

$$\partial V(x,t) = \overline{co}\Big[\lim_{k\to\infty} \nabla V(x_k, t_k) : (x_k, t_k) \to (x,t), (x_k, t_k) \notin N,\ (x_k, t_k) \notin \Omega\Big],$$

where $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is the set of points where V is not differentiable and $N \subset \mathbb{R}^n \times \mathbb{R}$ is an arbitrary set with measure zero.

Let $\partial_x V(x,t)$ denote the Clarke generalized gradient of $V(x,t)$ in the variable x and $\partial_t V(x,t)$ be the Clarke generalized gradient of $V(x,t)$ in the variable t. The next lemma gives a chain rule for computing the time derivative of a regular function $V(x,t)$ along the solution trajectories of differential system (2.1).

Lemma 2.9. *(Chain Rule, Guo and Huang [13]). Let $x(t)$ be a Filippov solution of system* (2.1) *on interval I containing t and $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a regular function. Then, $x(t)$ and $V(x(t),t)$ are differentiable for a.e. $t \in I$, and we have*

$$\frac{dV(x(t),t)}{dt}|_{(2.1)} = \eta + \zeta^\top \xi(t),$$

$\forall \eta \in \partial_t V(x,t)$ and $\zeta \in \partial_x V(x,t)$, where $\xi(t) \in F(t,x)$ is a measurable function and satisfies $\dot{x}(t) = \xi(t)$, for a.e. $t \in I$.

As for the concept of almost periodic function, we use the definition introduced by Fink [11] and He [15].

Definition 2.10. *A continuous function $x(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be almost periodic on $\mathbb{R}$ if for any $\epsilon > 0$, the set $T(x,\epsilon) = \{\omega : \|x(t+\omega) - x(t)\| < \epsilon, \forall t \in \mathbb{R}\}$ is relatively dense, that is, for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\omega(\epsilon)$ in this interval such that $\|x(t+\omega) - x(t)\| < \epsilon$, for all $t \in \mathbb{R}$.*

3. Existence of almost periodic solution

In this section, we investigate the existence of almost periodic solution to the system (2.4).

Theorem 3.1. *Suppose that the following assumption are satisfied:*

(H1) For $i = 1,2,...,n$, f_i is continuous expect on a countable set of isolate points ρ_k^i, where there exist finite right limits $\lim_{x_i \to (\rho_k^i)^+} f_i(x_i) := f_i^+(\rho_k^i)$ and left limits $\lim_{x_i \to (\rho_k^i)^-} f_i(x_i) := f_i^-(\rho_k^i)$, respectively. Moreover, f_i has a finite number of discontinuous points on any compact interval of $\mathbb{R}$.

(H2) For $i = 1,2,...,n$, $f_i = g_i + h_i$, where g_i is continuous on $\mathbb{R}$ and h_i is continuous expect on a countable set of isolate points ρ_k^i. For $\forall u, v \in \mathbb{R}$, there exists positive constants L_i such that

$$|g_i(u) - g_i(v)| \le L_i |u - v|, \ i = 1,2,...,n.$$

Moreover, $h_i (i = 1,2,...,n)$ is monotonically decreasing in $\mathbb{R}$.

(H3) For $i = 1,2,...,n$, and $s \in \mathbb{R}$, the delays $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ are nonnegative continuous almost periodic functions and $d_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$, $c_{ij}(t)$, $I_i(t)$ are continuous almost periodic functions, that is, for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that for any interval $[\alpha, \alpha + l]$, there is $\omega \in [\alpha, \alpha + l]$ such that

$$|d_i(t+\omega) - d_i(t)| < \epsilon, |I_i(t+\omega) - I_i(t)| < \epsilon, |a_{ij}(t+\omega) - a_{ij}(t)| < \epsilon,$$
$$|b_{ij}(t+\omega) - b_{ij}(t)| < \epsilon, |c_{ij}(t+\omega) - c_{ij}(t)| < \epsilon,$$
$$|h_{ij}(t+\omega) - h_{ij}(t)| < \epsilon, |\sigma_{ij}(t+\omega) - \sigma_{ij}(t)| < \epsilon,$$

hold for all i, $i = 1,2,...,n$ and $t \in \mathbb{R}$.

(H4) The delays $\tau_{ij}(t)$ and $\sigma_{ij}(t)$ are continuously differentiable function and satisfying $\tau'_{ij}(t) < 1$ for $i, j = 1,2,...,n$. Moreover, there exist positive constants $\xi_1, \xi_2, ..., \xi_n$ and $\delta > 0$ such that

$$\limsup_{t \to +\infty} \Gamma_i(t) < 0 \ \text{ and } \ \limsup_{t \to +\infty} \Upsilon_i(t) < 0, \ \ i = 1,2,...,n,$$

where

$$\begin{aligned}\Gamma_i(t) =&\xi_i\delta - \xi_i d_i(t) + \sum_{j=1}^{n}\xi_j\frac{|d_i(t)||h_{ij}(t)|}{1-\sum_{j=1}^{n}h_{ij}^{M}} + \sum_{j=1}^{n}\xi_j L_i|a_{ji}(t)| + \sum_{j=1}^{n}\xi_j\frac{L_i e^{\delta\tau_{ji}^{M}}|b_{ji}(\varphi_{ji}^{-1}(t))|}{1-\tau'_{ji}(\varphi_{ji}^{-1}(t))}\\ &+\sum_{j=1}^{n}\xi_j\sigma'_{ji}(t)L_i\int_{t-\sigma_{ji}(t)}^{t}|c_{ji}(u+\sigma_{ji}(t))|e^{\delta[u+\sigma_{ji}(t)-t]}du + \sum_{j=1}^{n}\xi_j L_i\int_{-\sigma_{ji}(t)}^{0}|c_{ji}(t-s)|e^{-\delta s}ds,\\ \Upsilon_i(t) =&-\xi_i a_{ii}(t) + \sum_{j=1,j\neq i}^{n}\xi_j|a_{ji}(t)| + \sum_{j=1}^{n}\xi_j\frac{|b_{ji}(\varphi_{ji}^{-1}(t))|}{1-\tau'_{ji}(\varphi_{ji}^{-1}(t))}e^{\delta\tau_{ji}^{M}}\\ &+\sum_{j=1}^{n}\xi_j\sigma'_{ji}(t)\int_{t-\sigma_{ji}(t)}^{t}|c_{ji}(u+\sigma_{ji}(t))|e^{\delta[u+\sigma_{ji}(t)-t]}du + \sum_{j=1}^{n}\xi_j\int_{-\sigma_{ji}(t)}^{0}|c_{ji}(t-s)|e^{-\delta s}ds,\end{aligned}$$

and φ_{ij}^{-1} is the inverse function of $\varphi_{ij}(t) = t - \tau_{ij}(t)$.

Then for any IVP associated to (2.7)*, there exists a solution $[x, \gamma]$ of the neural network system* (2.4) *on* $[0, +\infty)$*, i.e., the solution x of* (2.4) *is defined for $t \in [0, +\infty)$ and γ is defined for $t \in [0, +\infty)$ up to a set with measure zero. Moreover, there exists constant $M > 0$ such that $\|x\| < M$ for $t \in [-\varsigma, +\infty)$ and $\|\gamma\| < M$ for a.e. $t \in [-\varsigma, +\infty)$.*

Proof. Define the set-valued map

$$\begin{aligned}x_i(t) \hookrightarrow& -d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n}h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n}a_{ij}(t)\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(t)\Big)\Big]\\ &+\sum_{j=1}^{n}b_{ij}(t)\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(t-\tau_{ij}(t))\Big)\Big] + \sum_{j=1}^{n}c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(s)\Big)\Big]ds + I_i(t),\ i = 1, 2, ..., n.\end{aligned}$$

By (H2), one can easily see that the above set-valued map is upper semi-continuous with nonempty compact convex values and the local existence of a solution $x(t)$ of (2.6) is obviously([10], [16]). That meas, the IVP of system (2.5) has at least one solution $x(t) = (x_1(t), x_2(t), ..., x_n(t))^\top$ on $[0, b)$ for some $b \in [0, +\infty)$ and the derivative of $x_i(t)$ is a measurable selection from

$$\begin{aligned}&-d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n}h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n}a_{ij}(t)\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(t)\Big)\Big]\\ &+\sum_{j=1}^{n}b_{ij}(t)\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(t-\tau_{ij}(t))\Big)\Big] + \sum_{j=1}^{n}c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}\overline{co}\Big[f_j\Big((A_j^{-1}x_j)(s)\Big)\Big]ds + I_i(t),\\ &\text{for } a.e.\ t \in [0, b),\ i = 1, 2, ..., n.\end{aligned}$$

It follows from the Continuation Theorem [[1], Theorem 2, P78] that either $b = +\infty$, or $b < +\infty$ and $\lim_{t\to b^-}\|x(t)\| = +\infty$, where $\|x(t)\| = \sum_{i=1}^{n}|x_i(t)|$ is defined as above. Next, we will show that $\lim_{t\to b^-}\|x(t)\| < +\infty$ if $b < +\infty$, which means that the maximal existing interval of $x(t)$ can be extended to $+\infty$. From Definition 2.4 , there exists $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^\top : [-\varsigma, b) \to \mathbb{R}^n$ such that $\gamma_j(t) \in \overline{co}[f_j(A_j^{-1}x_j)(t)]$ for *a.e.* $t \in [-\varsigma, b)$ and

$$\begin{aligned}\frac{dx_i(t)}{dt} =& -d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n}h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n}a_{ij}(t)\gamma_j(t)\\ &+\sum_{j=1}^{n}b_{ij}(t)\gamma_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n}c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}\gamma_j(s)ds + I_i(t),\ \text{for } a.e.\ t \in [0, b),\ i = 1, 2, ..., n.\end{aligned}$$

By (H2), we can see that there exists a vector variable $\eta = (\eta_1, \eta_2, ..., \eta_n)^\top : [-\varsigma, b) \to \mathbb{R}^n$ and $\eta_i(t) \in \overline{co}[h_i((A_i^{-1}x_i)(t))]$ such that

$$\gamma_i(t) = g_i(x(t)) + \eta_i(t), \ i = 1, 2, ..., n.$$

Consider the following candidate Lyapunov function:

$$\begin{aligned} V(t) = &\sum_{i=1}^{n} \xi_i e^{\delta t}|x_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{t-\tau_{ij}(t)}^{t} \frac{|b_{ij}(\varphi_{ij}^{-1}(u))|}{1-\tau'_{ij}(\varphi_{ij}^{-1}(u))}\Big[|g_j(x_j(u))| + |\eta_j(u)|\Big]e^{\delta(u+\tau_{ij}^M)}du \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0}\int_{t+s}^{t} |c_{ij}(u-s)|\Big[|g_j(x_j(u))| + |\eta_j(u)|\Big]e^{\delta(u-s)}duds. \end{aligned} \tag{3.1}$$

Obviously, $V(t)$ is regular. Meanwhile, the solution $x(t)$ of the system (2.4) are all absolutely continuous. Then, $V(t)$ is differential for *a.e.* $t \geq 0$ and the time derivative can be evaluated by Lemma 2.9 .

Define $\nu_i(t) = sign\{x_i(t)\}$ if $x_i(t) \neq 0$; while $\nu_i(t)$ can be arbitrarily choosen in $[-1, 1]$ if $x_i(t) = 0$. In particular, we can choose $\nu_i(t)$ as follows

$$\nu_i(t) = \begin{cases} 0, & x_i(t) = \gamma_i(t) = 0, \\ -sign\{\eta_i(t)\}, & x_i(t) = 0 \text{ and } \gamma_i(t) \neq 0, \\ sign\{x_i(t)\}, & x_i(t) \neq 0. \end{cases}$$

Then, we have

$$\nu_i(t)\{x_i(t)\} = |x_i(t)|, \ \nu_i(t)\{\eta_i(t)\} = -|\eta_i(t)|, \ i = 1, 2, ..., n.$$

Now, by applying the chain rule in Lemma 2.9 , calculate the time derivative of $V(t)$ along the solution trajectories of the system (2.4) in the sense of (2.5), then we can get for *a.e.* $t \geq 0$ that

$$\begin{aligned} \frac{dV(t)}{dt} = &\sum_{i=1}^{n} \xi_i \delta e^{\delta t}|x_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta t}\nu_i(t)\frac{dx_i(t)}{dt} \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1-\tau'_{ij}(\varphi_{ij}^{-1}(t))}\Big[|g_j(x_j(t))| + |\eta_j(t)|\Big]e^{\delta(t+\tau_{ij}^M)} \\ &- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i |b_{ij}(t)|\Big[|g_j(x_j(t-\tau_{ij}(t)))| + |\eta_j(t-\tau_{ij}(t))|\Big]e^{\delta(t-\tau_{ij}(t)+\tau_{ij}^M)} \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u+\sigma_{ij}(t))|\Big[|g_j(x_j(u))| + |\eta_j(u)|\Big]e^{\delta(u+\sigma_{ij}(t))}du \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t-s)|\Big[|g_j(x_j(t))| + |\eta_j(t)|\Big]e^{\delta(t-s)}ds \\ &- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t)|\Big[|g_j(x_j(t+s))| + |\eta_j(t+s)|\Big]e^{\delta(t)}ds, \end{aligned}$$

then, we have

$$
\begin{aligned}
\frac{dV(t)}{dt} &= \sum_{i=1}^{n} \xi_i \delta e^{\delta(t)} |x_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta(t)} \nu_i(t) \frac{dx_i(t)}{dt} \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))} \Big[|g_j(x_j(t))| + |\eta_j(t)| \Big] e^{\delta(t+\tau_{ij}^M)} \\
&- \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i |b_{ij}(t)| \Big[|g_j(x_j(t - \tau_{ij}(t)))| + |\eta_j(t - \tau_{ij}(t))| \Big] e^{\delta(t - \tau_{ij}(t) + \tau_{ij}^M)} \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u + \sigma_{ij}(t))| \Big[|g_j(x_j(u))| + |\eta_j(u)| \Big] e^{\delta(u + \sigma_{ij}(t))} du \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t - s)| \Big[|g_j(x_j(t))| + |\eta_j(t)| \Big] e^{\delta(t-s)} ds \\
&- \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(t)| \Big[|g_j(x_j(s))| + |\eta_j(s)| \Big] e^{\delta(t)} ds,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\frac{dV(t)}{dt} &\leq \sum_{i=1}^{n} \xi_i \delta e^{\delta t} |x_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta(t)} \nu_i(t) \Big[- d_i(t) x_i(t) - d_i(t) \sum_{j=1}^{n} h_{ij}(t) (A_i^{-1} x_i)(t - \delta_{ij}(t)) \\
&+ \sum_{j=1}^{n} a_{ij}(t) [g_j(x_j(t)) + \eta_j(t)] + \sum_{j=1}^{n} b_{ij}(t) [g_j(x_j(t - \tau_{ij}(t))) + \eta_j(t - \tau_{ij}(t))] \\
&+ \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} [g_j(x_j(s)) + \eta_j(s)] ds + I_i \Big] \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))} \Big[|g_j(x_j(t))| + |\eta_j(t)| \Big] e^{\delta(t+\tau_{ij}^M)} \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u + \sigma_{ij}(t))| \Big[|g_j(x_j(u))| + |\eta_j(u)| \Big] e^{\delta(u + \sigma_{ij}(t))} du \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t - s)| \Big[|g_j(x_j(t))| + |\eta_j(t)| \Big] e^{\delta(t-s)} ds,
\end{aligned}
$$

then, we have

$$
\begin{aligned}
\frac{dV(t)}{dt} &\leq \sum_{i=1}^{n} \xi_i \delta e^{\delta t} |x_i(t)| - \sum_{i=1}^{n} \xi_i e^{\delta t} d_i(t) |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i e^{\delta t} |d_i(t)| |h_{ij}(t)| |(A_i^{-1} x_i)(t - \delta_{ij}(t))| \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i e^{\delta t} |a_{ij}(t)| |g_j(x_j(t))| - \sum_{i=1}^{n} \xi_i e^{\delta t} a_{ii}(t) |\eta_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \xi_i e^{\delta t} |a_{ij}(t)| |\eta_j(t)| \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))} \Big[|g_j(x_j(t))| + |\eta_j(t)| \Big] e^{\delta(t+\tau_{ij}^M)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\sigma'_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}|c_{ij}(u+\sigma_{ij}(t))|\Big[|g_j(x_j(u))|+|\eta_j(u)|\Big]e^{\delta(u+\sigma_{ij}(t))}du\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{-\sigma_{ij}(t)}^{0}|c_{ij}(t-s)|\Big[|g_j(x_j(t))|+|\eta_j(t)|\Big]e^{\delta(t-s)}ds+\sum_{i=1}^{n}\xi_ie^{\delta t}I_i(t),
\end{aligned}
\tag{3.2}
$$

which together with Lemma 2.3 and (H2) yields

$$
\begin{aligned}
\frac{dV(t)}{dt}\le&\sum_{i=1}^{n}\xi_i\delta e^{\delta t}|x_i(t)|-\sum_{i=1}^{n}\xi_ie^{\delta t}d_i(t)|x_i(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_ie^{\delta t}\frac{|d_i(t)||h_{ij}(t)|}{1-\sum_{j=1}^{n}h_{ij}^M}|x_i(t-\delta_{ij}(t))|\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_ie^{\delta t}L_j|a_{ij}(t)||x_j(t)|-\sum_{i=1}^{n}\xi_ie^{\delta t}a_{ii}(t)|\eta_i(t)|+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\xi_ie^{\delta t}|a_{ij}(t)||\eta_j(t)|\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1-\tau'_{ij}(\varphi_{ij}^{-1}(t))}\Big[L_j|x_j(t)|+|\eta_j(t)|\Big]e^{\delta(t+\tau_{ij}^M)}\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\sigma'_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}|c_{ij}(u+\sigma_{ij}(t))|\Big[L_j|x_j(u)|+|\eta_j(u)|\Big]e^{\delta(u+\sigma_{ij}(t))}du\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{-\sigma_{ij}(t)}^{0}|c_{ij}(t-s)|\Big[L_j|x_j(t)|+|\eta_j(t)|\Big]e^{\delta(t-s)}ds+\sum_{i=1}^{n}\xi_ie^{\delta t}I_i(t)\\
\le&\sum_{i=1}^{n}e^{\delta t}|x_i(t)|\Bigg[\xi_i\delta-\xi_id_i(t)+\sum_{j=1}^{n}\xi_j\frac{|d_i(t)||h_{ij}(t)|}{1-\sum_{j=1}^{n}h_{ij}^M}+\sum_{j=1}^{n}\xi_jL_i|a_{ji}(t)|+\sum_{j=1}^{n}\xi_j\frac{L_ie^{\delta\tau_{ji}^M}|b_{ji}(\varphi_{ji}^{-1}(t))|}{1-\tau'_{ji}(\varphi_{ji}^{-1}(t))}\\
&+\sum_{j=1}^{n}\xi_j\sigma'_{ji}(t)L_i\int_{t-\sigma_{ji}(t)}^{t}|c_{ji}(u+\sigma_{ji}(t))|e^{\delta[u+\sigma_{ji}(t)-t]}du+\sum_{j=1}^{n}\xi_jL_i\int_{-\sigma_{ji}(t)}^{0}|c_{ji}(t-s)|e^{-\delta s}ds\Bigg]\\
&+\sum_{i=1}^{n}e^{\delta t}|\eta_i(t)|\Bigg[-\xi_ia_{ii}(t)+\sum_{j=1,j\neq i}^{n}\xi_j|a_{ji}(t)|+\sum_{j=1}^{n}\xi_j\frac{|b_{ji}(\varphi_{ji}^{-1}(t))|}{1-\tau'_{ji}(\varphi_{ji}^{-1}(t))}e^{\delta\tau_{ji}^M}\\
&+\sum_{j=1}^{n}\xi_j\sigma'_{ji}(t)\int_{t-\sigma_{ji}(t)}^{t}|c_{ji}(u+\sigma_{ji}(t))|e^{\delta[u+\sigma_{ji}(t)-t]}du+\sum_{j=1}^{n}\xi_j\int_{-\sigma_{ji}(t)}^{0}|c_{ji}(t-s)|e^{-\delta s}ds\Bigg]\\
&+\sum_{i=1}^{n}\xi_ie^{\delta t}I_i(t).
\end{aligned}
\tag{3.3}
$$

From the assumption (H4), we can see that there exist positive constants ϑ_i, $\upsilon_i(i=1,2,...,n)$ and $t_0\ge 0$ such that if $t\ge t_0$

$$\Gamma_i(t)\le-\vartheta_i<0,\quad \Upsilon_i(t)\le-\upsilon_i<0,\quad i=1,2,...,n,$$

which together with (3.3) gives

$$\frac{dV(t)}{dt}\le\sum_{i=1}^{n}\xi_ie^{\delta t}I_i^M,\ \ a.e.\ \ t\ge 0.$$

Moreover, since

$$V(t)=V(0)+\int_0^t V'(s)ds\le V(0)+\int_0^t\sum_{i=1}^{n}\xi_ie^{\delta s}I_i^Mds=V(0)+\frac{1}{\delta}\sum_{i=1}^{n}\xi_ie^{\delta t}I_i^M,\ \ t\ge 0. \tag{3.4}$$

From (3.1), we can also have

$$V(t) \geq e^{\delta t}\|x(t)\|. \tag{3.5}$$

Combining with (3.4) and (3.5), we obtain

$$\|x(t)\| \leq e^{-\delta t}V(t) \leq V(0) + \frac{1}{\delta}\sum_{i=1}^{n}\xi_i e^{\delta t} I_i^M, \quad t \in [0, b), \tag{3.6}$$

which implies that $x(t)$ is bounded on the interval $[-\tau, b)$, where $\varsigma = \max_{1\leq i,j\leq n}\{\tau_{ij}^M, \sigma_{ij}^M\}$. Then, we can see that $\lim_{t\to b^-}\|x(t)\| < +\infty$. Thus, we can conclude that $b = +\infty$. Therefore, in view of (3.6), we have

$$\|x(t)\| \leq V(0) + \frac{1}{\delta}\sum_{i=1}^{n}\xi_i e^{\delta t} I_i^M + \|\phi\| := M_0, \quad t \in [-\varsigma, +\infty). \tag{3.7}$$

Note that, f_i has a finite number of discontinuous points on any compact interval of $\mathbb{R}$. In particular, f_i has a finite number of discontinuous points on compact interval $[-M_0, M_0]$. Without loss of generality, let f_i discontinuous at points $\{\rho_k^i : k = 1, 2, ..., l_i\}$ on the interval $[-M_0, M_0]$ and assume that $-M_0 < \rho_1^i < \rho_2^i < \cdots < \rho_{l_i}^i < M_0$. Let us consider a series of continuous functions:

$$f_i^0(x) = \begin{cases} f_i(x), & x \in [-M_0, \rho_1^i), \\ f_i(\rho_1^i - 0), & x = \rho_1^i; \end{cases} \qquad f_i^{l_i}(x) = \begin{cases} f_i(\rho_{l_i}^i + 0), & x = \rho_{l_i}^i, \\ f_i(x), & x \in (\rho_{l_i}^i, M_o]; \end{cases}$$

and

$$f_i^k(x) = \begin{cases} f_i(\rho_k^i - 0), & x = \rho_k^i, \\ f_i(x), & x \in (\rho_k^i, \rho_{k+1}^i), \\ f_i(\rho_{k+1}^i + 0), & x = \rho_{k+1}^i, \end{cases} \quad k = 1, 2, ..., l_i - 1.$$

Let

$$M_i = \max\Big\{\max_{x\in[-M_0,\rho_1^i]}\{f_i^0(x)\}, \max_{1\leq k\leq l_i-1}\Big\{\max_{x\in[\rho_k^i,\rho_{k+1}^i]}\{f_i^k(x)\}\Big\}, \max_{x\in[\rho_{l_i}^i,M_0]}\{f_i^{l_i}(x)\}\Big\},$$

$$m_i = \max\Big\{\min_{x\in[-M_0,\rho_1^i]}\{f_i^0(x)\}, \min_{1\leq k\leq l_i-1}\Big\{\min_{x\in[\rho_k^i,\rho_{k+1}^i]}\{f_i^k(x)\}\Big\}, \min_{x\in[\rho_{l_i}^i,M_0]}\{f_i^{l_i}(x)\}.\Big\}$$

It is clear that

$$|\overline{co}[f_i((A_i^{-1}x_i)(t))]| \leq \max\{|M_i|, |m_i|\}, \quad i = 1, 2, ..., n.$$

Since, $\gamma_i(t) \in \overline{co}[f_i((A_i^{-1}x_i)(t))]$ for a.e. $t \in [-\varsigma, +\infty)$ and $i = 1, 2, ..., n$, we have

$$|\gamma_i(t)| \leq \max\{|M_i|, |m_i|\}, \text{ for a.e. } t \in [-\varsigma, +\infty), \ i = 1, 2, ..., n.$$

Thus,

$$\|\gamma(t)\| \leq \max\Big\{\sum_{i=1}^{n}|M_i|, \sum_{i=1}^{n}|m_i|\Big\}, \text{ for a.e. } t \in [-\varsigma, +\infty). \tag{3.8}$$

Let

$$M = \max\Big\{M_0, \sum_{i=1}^{n}|M_i|, \sum_{i=1}^{n}|m_i|\Big\}.$$

Then, from (3.7) and (3.8), it follows that

$$\|x(t)\| \leq M, \ \|\gamma(t)\| \leq M, \ t \in [-\varsigma, +\infty). \tag{3.9}$$

Therefore, the proof is complete.

Theorem 3.2. *suppose that the assumptions (H1), (H3), (H4) and the following assumption holds:*

(H2) For $i = 1, 2, ..., n$, $f_i = g_i + h_i$, where g_i is continuous on $\mathbb{R}$ and h_i is continuous expect on a countable set of isolate points ρ^i_k. For $\forall u, v \in \mathbb{R}$, there exists positive constants L_i such that*

$$|g_i(u) - g_i(v)| \leq L_i|u - v|, \ i = 1, 2, ..., n.$$

Moreover, $h_i(i = 1, 2, ..., n)$ is monotonically nondecreasing in $\mathbb{R}$.

Then for any IVP associated to (2.7), *there exists a solution $[x, \gamma]$ of the neural network system* (2.4) *on $[0, +\infty)$, i.e., the solution x of* (2.4) *is defined for $t \in [0, +\infty)$ and γ is defined for $t \in [0, +\infty)$ up to a set with measure zero. Moreover, there exists constant $M > 0$ such that $\|x\| < M$ for $t \in [-\varsigma, +\infty)$ and $\|\gamma\| < M$ for a.e. $t \in [-\varsigma, +\infty)$.*

Proof. The proof is similar to the proof of Theorem 3.1, we omit it here.

The following lemma points out that any solution of system (2.4) is asymptotically almost periodic.

Theorem 3.3. *Suppose that the assumptions (H1), (H2), (H3) and (H4) are satisfied, then any solution $x(t)$ of the system* (2.4) *associated with an output $\gamma(t)$ is asymptotically almost periodic, i.e., for any $\epsilon > 0$, there exist $T > 0$, $l = l(\epsilon)$ and $\omega = \omega(\epsilon)$ in any interval with the length of $l(\epsilon)$, such that*

$$\|x(t + \omega) - x(t)\| \leq \epsilon, \ \text{for all } t \geq T.$$

Proof. From (H3), it follows that, for any $\epsilon > 0$, there exists $l = l(\epsilon)$ such that for any $\alpha \in \mathbb{R}$ there exists $\omega \in [\omega, \omega + l]$ satisfying the following inequalities:

$$\begin{aligned}
&|d_i(t + \omega) - d_i(t)| \leq \frac{\xi^m \delta\epsilon}{25nM\xi^M}, && |I_i(t + \omega) - I_i(t)| \leq \frac{\xi^m \delta\epsilon}{25n\xi^M},\\
&|a_{ij}(t + \omega) - a_{ij}(t)| \leq \frac{\xi^m \delta\epsilon}{25n^2M\xi^M}, && |b_{ij}(t + \omega) - b_{ij}(t)| \leq \frac{\xi^m \delta\epsilon}{25n^2M\xi^M},\\
&|h_{ij}(t + \omega) - h_{ij}(t)| \leq \frac{\xi^m \delta\epsilon}{25n^2M\xi^M}, && |c_{ij}(t + \omega) - c_{ij}(t)| \leq \frac{\xi^m \delta\epsilon}{25n^2\varsigma M\xi^M},\\
&|\sigma_{ij}(t + \omega) - \sigma_{ij}(t)| \leq \frac{\xi^m \delta\epsilon}{25n^2c^M_{ij}M\xi^M}.
\end{aligned} \tag{3.10}$$

where $\xi^m := \min_{1\leq i\leq n}\{\xi_i\} \leq \max_{1\leq i\leq n}\{\xi_i\} := \xi^M$. Furthermore, in view of (H1) and $\gamma_j(t) \in \overline{co}[f_j((A^{-1}_i x_i)(t))]$, $j = 1, 2, ..., n$, for a.e. $t \in [-\varsigma, +\infty)$, then we can see that

$$\Big|\gamma_j(t + \omega - \tau_{ij}(t + \omega)) - \gamma_j(t + \omega - \tau_{ij}(t))\Big| < \frac{\xi^m \delta\epsilon}{25nb^M_{ij}\xi^M}, \ \text{for a.e. } t \in [-\varsigma, +\infty), \tag{3.11}$$

and

$$\Big| \int_{t+\omega-\sigma_{ij}(t+\omega)}^{t+\omega} \gamma_j(s)ds - \int_{t+\omega-\sigma_{ij}(t)}^{t+\omega} \gamma_j(s)ds\Big| < \frac{\xi^m \delta\epsilon}{25nc^M_{ij}\xi^M}, \ \text{for a.e. } t \in [-\varsigma, +\infty). \tag{3.12}$$

Let

$$\begin{aligned}\Phi_i(t,\omega) = &-[d_i(t+\omega)-d_i(t)]x_i(t+\omega)\\&-[d_i(t+\omega)-d_i(t)]\sum_{j=1}^{n}h_{ij}(t+\omega)(A_i^{-1}x_i)(t+\omega-\delta_{ij}(t+\omega))\\&-d_i(t)\sum_{j=1}^{n}[h_{ij}(t+\omega)-h_{ij}(t)](A_i^{-1}x_i)(t+\omega-\delta_{ij}(t+\omega))\\&+\sum_{j=1}^{n}[a_{ij}(t+\omega)-a_{ij}(t)]\gamma_j(t+\omega)\\&+\sum_{j=1}^{n}[b_{ij}(t+\omega)-b_{ij}(t)]\gamma_j(t+\omega-\tau_{ij}(t+\omega))\\&+\sum_{j=1}^{n}[c_{ij}(t+\omega)-c_{ij}(t)]\int_{t+\omega-\sigma_{ij}(t+\omega)}^{t+\omega}\gamma_j(s)ds+[I_i(t+\omega)-I_i(t)],\end{aligned}$$

and

$$\begin{aligned}\Psi_i(t,\omega) = &\sum_{j=1}^{n}b_{ij}(t)[\gamma_j(t+\omega-\tau_{ij}(t+\omega))-\gamma_j(t+\omega-\tau_{ij}(t))]\\&+\sum_{j=1}^{n}c_{ij}(t)\Big[\int_{t+\omega-\sigma_{ij}(t+\omega)}^{t+\omega}\gamma_j(s)ds-\int_{t+\omega-\sigma_{ij}(t)}^{t+\omega}\gamma_j(s)ds\Big].\end{aligned}$$

Then, by (3.10), (3.11) and (3.12), we have

$$\begin{aligned}|\Phi_i(t,\omega)| &\le \frac{\xi^m\delta\epsilon}{5n\xi^M}+\frac{\xi^m\delta\epsilon}{25n\xi^M}\frac{h_{ij}^M+d_i^M}{1-\sum_{j=1}^{n}h_{ij}^M}, \quad \text{for a.e. } t\in[-\varsigma,+\infty),\\|\Psi_i(t,\omega)| &\le \frac{\xi^m\delta\epsilon}{25\xi^M}+\frac{\xi^m\delta\epsilon}{25\xi^M}\\&\le \frac{\xi^m\delta\epsilon}{5n\xi^M}+\frac{\xi^m\delta\epsilon}{25n\xi^M}\frac{h_{ij}^M+d_i^M}{1-\sum_{j=1}^{n}h_{ij}^M}, \quad \text{for a.e. } t\in[-\varsigma,+\infty).\end{aligned}\tag{3.13}$$

Now, we consider the following candidate Lyapunov function:

$$\begin{aligned}W(t) = &\sum_{i=1}^{n}\xi_i e^{\delta t}|x_i(t+\omega)-x_i(t)|\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{t-\tau_{ij}(t)}^{t}\frac{|b_{ij}(\varphi_{ij}^{-1}(u))|}{1-\tau_{ij}'(\varphi_{ij}^{-1}(u))}|g_j(x_j(u+\omega))-g_j(x_j(u))|e^{\delta(u+\tau_{ij}^M)}du\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{t-\tau_{ij}(t)}^{t}\frac{|b_{ij}(\varphi_{ij}^{-1}(u))|}{1-\tau_{ij}'(\varphi_{ij}^{-1}(u))}|\eta_j(u+\omega)-\eta_j(u)|e^{\delta(u+\tau_{ij}^M)}du\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{-\sigma_{ij}(t)}^{0}\int_{t+s}^{t}|c_{ij}(u-s)||g_j(x_j(u+\omega))-g_j(x_j(u))|e^{\delta(u-s)}duds\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{-\sigma_{ij}(t)}^{0}\int_{t+s}^{t}|c_{ij}(u-s)||\eta_j(u+\omega)-\eta_j(u)|e^{\delta(u-s)}duds.\end{aligned}\tag{3.14}$$

Obviously, $W(t)$ is regular. Meanwhile, the solutions $x(t+\omega)$, $x(t)$ of the neural network system (2.4) are all absolutely continuous. Then, $W(t)$ is differential for *a.e.* $t \geq 0$ and the time derivative can be evaluated by Lemma 2.9 .

Define $\nu_i(t) = sign\{x_i(t+\omega) - x_i(t)\}$ if $x_i(t+\omega) \neq x_i(t)$; while $\nu_i(t)$ can be arbitrarily choosen in $[-1, 1]$ if $x_i(t+\omega) = x_i(t)$. In particular, we can choose $\nu_i(t)$ as follows

$$\nu_i(t) = \begin{cases} 0, & x_i(t+\omega) - x_i(t) = \gamma_i(t+\omega) - \gamma_i(t) = 0, \\ -sign\{\eta_i(t+\omega) - \eta_i(t)\}, & x_i(t+\omega) = x_i(t) \text{ and } \gamma_i(t+\omega) \neq \gamma_i(t), \\ sign\{x_i(t+\omega) - x_i(t)\}, & x_i(t+\omega) \neq x_i(t). \end{cases}$$

Then, we have

$$\nu_i(t)\{x_i(t+\omega) - x_i(t)\} = |x_i(t+\omega) - x_i(t)|, \ i = 1, 2, ..., n,$$
$$\nu_i(t)\{\eta_i(t+\omega) - \eta_i(t)\} = -|\eta_i(t+\omega) - \eta_i(t)|, \ i = 1, 2, ..., n.$$

Now, by applying the chain rule in Lemma 2.9 , calculate the time derivative of $V(t)$ along the solution trajectories of the system (2.4) in the sense of (2.5), then we can get for *a.e.* $t \geq 0$ that

$$\begin{aligned}
\frac{dW(t)}{dt} =& \sum_{i=1}^{n} \xi_i \delta e^{\delta t} |x_i(t+\omega) - x_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta t} \nu_i(t) \frac{d[x_i(t+\omega) - x_i(t)]}{dt} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))} |g_j(x_j(t+\omega)) - g_j(x_j(t))| e^{\delta(t+\tau_{ij}^M)} \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i (1 - \tau'_{ij}(t)) |b_{ij}(t)| |g_j(x_j(t - \tau_{ij}(t) + \omega)) - g_j(x_j(t - \tau_{ij}(t)))| e^{\delta[t - \tau_{ij}(t) + \tau_{ij}^M]} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))} |\eta_j(t+\omega) - \eta_j(t)| e^{\delta(t+\tau_{ij}^M)} \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i (1 - \tau'_{ij}(t)) |b_{ij}(t)| |\eta_j(t - \tau_{ij}(t) + \omega) - \eta_j(t - \tau_{ij}(t))| e^{\delta[t - \tau_{ij}(t) + \tau_{ij}^M]} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u + \sigma_{ij}(t))| |g_j(x_j(u+\omega)) - g_j(x_j(u))| e^{\delta(u+\sigma_{ij}(t))} du \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t-s)| |g_j(x_j(t+\omega)) - g_j(x_j(t))| e^{\delta(t-s)} ds \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t)| |g_j(x_j(t+s+\omega)) - g_j(x_j(t+s))| e^{\delta t} ds \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u + \sigma_{ij}(t))| |\eta_j(u+\omega) - \eta_j(u)| e^{\delta(u+\sigma_{ij}(t))} du \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t-s)| |\eta_j(t+\omega) - \eta_j(t)| e^{\delta(t-s)} ds \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t)| |\eta_j(t+s+\omega) - \eta_j(t+s)| e^{\delta t} ds.
\end{aligned} \tag{3.15}$$

Furthermore, by (2.5), we have

$$\begin{aligned}\frac{d[x_i(t+\omega)-x_i(t)]}{dt} &= -d_i(t)[x_i(t+\omega)-x_i(t)] \\ &- d_i(t)\sum_{j=1}^{n} h_{ij}(t)[(A_i^{-1}x_i)(t+\omega-\delta_{ij}(t+\omega))-(A_i^{-1}x_i)(t-\delta_{ij}(t))] \\ &+ \sum_{j=1}^{n} a_{ij}(t)[\gamma_j(t+\omega)-\gamma_j(t)] + \sum_{j=1}^{n} b_{ij}(t)[\gamma_j(t+\omega-\tau_{ij}(t))-\gamma_j(t-\tau_{ij}(t))] \\ &+ \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}[\gamma_j(s+\omega)-\gamma_j(s)]ds + \Phi_i(t,\omega)+\Psi_i(t,\omega).\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15), in view of (H2) and Lemma 2.3, we have

$$\begin{aligned}\frac{dW(t)}{dt} &\le \sum_{i=1}^{n}\xi_i\delta e^{\delta t}|x_i(t+\omega)-x_i(t)| + \sum_{i=1}^{n}\xi_i e^{\delta t}\Big[|\Phi_i(t,\omega)|+|\Psi_i(t,\omega)|\Big] \\ &- \sum_{i=1}^{n}\xi_i e^{\delta t}d_i(t)|x_i(t+\omega)-x_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i e^{\delta t}\frac{|d_i(t)||h_{ij}(t)|}{1-\sum_{j=1}^{n}h_{ij}^{M}}|x_i(t+\omega-\delta_{ij}(t+\omega))-x_i(t-\delta_{ij}(t))| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i e^{\delta t}|a_{ij}(t)||g_j(x_j(t+\omega))-g_j(x_j(t))| \\ &- \sum_{i=1}^{n}\xi_i e^{\delta t}a_{ii}(t)|\eta_i(t+\omega)-\eta_i(t)| + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\xi_i e^{\delta t}|a_{ij}(t)||\eta_i(t+\omega)-\eta_i(t)| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1-\tau_{ij}'(\varphi_{ij}^{-1}(t))}\Big[|g_j(x_j(t+\omega))-g_j(x_j(t))|+|\eta_j(t+\omega)-\eta_j(t)|\Big]e^{\delta(t+\tau_{ij}^{M})} \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\sigma_{ij}'(t)\int_{t-\sigma_{ij}(t)}^{t}|c_{ij}(u+\sigma_{ij}(t))|\Big[|g_j(x_j(u+\omega))-g_j(x_j(u))|+|\eta_j(u+\omega)-\eta_j(u)|\Big]e^{\delta(u+\sigma_{ij}(t))}du \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{-\sigma_{ij}(t)}^{0}|c_{ij}(t-s)|\Big[|g_j(x_j(t+\omega))-g_j(x_j(t))|+|\eta_j(t+\omega)-\eta_j(t)|\Big]e^{\delta(t-s)}ds \\ &\le \sum_{i=1}^{n}\xi_i\delta e^{\delta t}|x_i(t+\omega)-x_i(t)| + \sum_{i=1}^{n}\xi_i e^{\delta t}[|\Phi_i(t,\omega)|+|\Psi_i(t,\omega)|] \\ &- \sum_{i=1}^{n}\xi_i e^{\delta t}d_i(t)|x_i(t+\omega)-x_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i e^{\delta t}\frac{|d_i(t)||h_{ij}(t)|}{1-\sum_{j=1}^{n}h_{ij}^{M}}|x_i(t+\omega-\delta_{ij}(t+\omega))-x_i(t-\delta_{ij}(t))| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i e^{\delta t}|a_{ij}(t)|L_j|x_j(t+\omega)-x_j(t)| \\ &- \sum_{i=1}^{n}\xi_i e^{\delta t}a_{ii}(t)|\eta_i(t+\omega)-\eta_i(t)| + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\xi_i e^{\delta t}|a_{ij}(t)||\eta_i(t+\omega)-\eta_i(t)| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1-\tau_{ij}'(\varphi_{ij}^{-1}(t))}\Big[L_j|x_j(t+\omega)-x_j(t)|+|\eta_j(t+\omega)-\eta_j(t)|\Big]e^{\delta(t+\tau_{ij}^{M})}\end{aligned}$$

$$\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\sigma'_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}|c_{ij}(u+\sigma_{ij}(t))|\Big[L_j|x_j(u+\omega)-x_j(u)|+|\eta_j(u+\omega)-\eta_j(u)|\Big]e^{\delta(u+\sigma_{ij}(t))}du\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\int_{-\sigma_{ij}(t)}^{0}|c_{ij}(t-s)|\Big[L_j|x_j(t+\omega)-x_j(t)|+|\eta_j(t+\omega)-\eta_j(t)|\Big]e^{\delta(t-s)}ds\\
&\le\sum_{i=1}^{n}\xi_i e^{\delta t}\Big[|\Phi_i(t,\omega)|+|\Psi_i(t,\omega)|\Big]+\sum_{i=1}^{n}e^{\delta t}\Gamma_i(t)|x_i(t+\omega)-x_i(t)|\\
&+\sum_{i=1}^{n}e^{\delta t}\Upsilon_i(t)|\eta_i(t+\omega)-\eta_i(t)|.
\end{aligned}\tag{3.17}$$

It follows from the assumption (H4) that, there exists positive constants ϑ_i, $\upsilon_i (i=1,2,...,n)$ such that for $t\ge 0$, we have

$$\Gamma_i(t)\le-\vartheta_i<0,\quad \Upsilon_i(t)\le-\upsilon_i<0,\ \ i=1,2,...,n,$$

which together with (3.17) gives

$$\frac{dW(t)}{dt}\le\sum_{i=1}^{n}\xi_i e^{\delta t}\Big[|\Phi_i(t,\omega)|+|\Psi_i(t,\omega)|\Big],\ \ a.e.\ \ t\ge 0,$$

which together with (3.13) yields

$$\begin{aligned}
\frac{dW(t)}{dt}&\le\sum_{i=1}^{n}\xi_i e^{\delta t}\Big[|\Phi_i(t,\omega)|+|\Psi_i(t,\omega)|\Big]\\
&\le\sum_{i=1}^{n}\xi_i e^{\delta t}\Big(\frac{\xi^m\delta}{5n\xi^M}+\frac{\xi^m\delta}{25n\xi^M}\frac{h_{ij}^M+d_i^M}{1-\sum_{j=1}^n h_{ij}^M}\Big)\epsilon,\ \ a.e.\ \ t\ge 0,
\end{aligned}$$

Thus, by (3.14), we have

$$\begin{aligned}
&\xi^m e^{\delta t}\|x(t+\omega)-x(t)\|\le W(t)\\
&\le W(0)+\sum_{i=1}^{n}e^{\delta t}\Big(\frac{\xi^m\delta}{5n}+\frac{\xi^m\delta}{25n}\frac{h_{ij}^M+d_i^M}{1-\sum_{j=1}^n h_{ij}^M}\Big)\epsilon,
\end{aligned}$$

which leads to

$$\begin{aligned}
\|x(t+\omega)-x(t)\|&\le\frac{1}{\xi^m}e^{-\delta t}W(0)+\frac{1}{\xi^m}e^{-\delta t}\sum_{i=1}^{n}e^{\delta t}\Big(\frac{\xi^m\delta}{5n}+\frac{\xi^m\delta}{25n}\frac{h_{ij}^M+d_i^M}{1-\sum_{j=1}^n h_{ij}^M}\Big)\epsilon\\
&\le\frac{1}{\xi^m}e^{-\delta t}W(0)+\Big(\frac{\delta}{5}+\frac{\delta}{25}\frac{h_{ij}^M+d_i^M}{1-\sum_{j=1}^n h_{ij}^M}\Big)\epsilon.
\end{aligned}\tag{3.18}$$

Moreover, from (3.14), it follows that $W(0)$ is a constant. Then, we can choose a sufficiently large $T>0$ such that

$$\frac{1}{\xi^m}e^{-\delta t}W(0)\le\epsilon,\ \text{for}\ t\ge T,$$

which together with (3.18) and the arbitrariness of ϵ gives

$$\|x(t+\omega)-x(t)\| \le \epsilon, \text{ for } t \ge T.$$

Therefore, the proof is complete.

Now we prove that system (2.4) possesses at least one almost periodic solution.

Theorem 3.4. *Suppose that the assumptions (H1), (H2), (H3) and (H4) are satisfied, then there exists at least one almost periodic solution of the neural network system* (2.4).

Proof. Let $x(t)$ be any solution of the neural network system (2.4), that is, there exists $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^\top$ such that $\gamma_j(t) \in \overline{co}[f_j((A_j^{-1}x_j)(t))]$ for a.e. $t \in [-\varsigma, b)$ and

$$\begin{aligned}\frac{dx_i(t)}{dt} = &-d_i(t)x_i(t) - d_i(t)\sum_{j=1}^n h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^n a_{ij}(t)\gamma_j(t)\\ &+\sum_{j=1}^n b_{ij}(t)\gamma_j(t-\tau_{ij}(t)) + \sum_{j=1}^n c_{ij}(t)\int_{t-\sigma_{ij}(t)}^t \gamma_j(s)ds + I_i(t),\ i=1,2,...,n.\end{aligned}$$

It follows from (3.13) that, we can select a sequence t_k satisfying $\lim\limits_{k\to+\infty} t_k = +\infty$ and

$$|\Phi_i(t,t_k)| \le \frac{1}{k},\ \ |\Psi_i(t,t_k)| \le \frac{1}{k},\ \text{ for a.e. } t \in [-\varsigma, +\infty). \tag{3.19}$$

where

$$\begin{aligned}\Phi_i(t,t_k) = &-[d_i(t+t_k)-d_i(t)]x_i(t+t_k)\\ &-[d_i(t+t_k)-d_i(t)]\sum_{j=1}^n h_{ij}(t+t_k)(A_i^{-1}x_i)(t+t_k-\delta_{ij}(t+t_k))\\ &-d_i(t)\sum_{j=1}^n [h_{ij}(t+t_k)-h_{ij}(t)](A_i^{-1}x_i)(t+t_k-\delta_{ij}(t+t_k))\\ &+\sum_{j=1}^n [a_{ij}(t+t_k)-a_{ij}(t)]\gamma_j(t+t_k) + \sum_{j=1}^n [b_{ij}(t+t_k)-b_{ij}(t)]\gamma_j(t+t_k-\tau_{ij}(t+t_k))\\ &+\sum_{j=1}^n [c_{ij}(t+t_k)-c_{ij}(t)]\int_{t+t_k-\sigma_{ij}(t+t_k)}^{t+t_k}\gamma_j(s)ds + [I_i(t+t_k)-I_i(t)],\end{aligned}$$

and

$$\begin{aligned}\Psi_i(t,t_k) = &\sum_{j=1}^n b_{ij}(t)[\gamma_j(t+t_k-\tau_{ij}(t+t_k)) - \gamma_j(t+t_k-\tau_{ij}(t))]\\ &+\sum_{j=1}^n c_{ij}(t)\Big[\int_{t+t_k-\sigma_{ij}(t+t_k)}^{t+t_k}\gamma_j(s)ds - \int_{t+t_k-\sigma_{ij}(t)}^{t+t_k}\gamma_j(s)ds\Big].\end{aligned}$$

Furthermore, by (3.9), it is easy to see that there exists $M' > 0$ such that $\|x_i'\| \le M'$ for a.e. $t \in [-\varsigma, +\infty)$. Thus, the sequence $\{x(t+t+k)\}_{k\in\mathbb{N}}$ is equi-continuous and uniformly bounded. By using the Arzela-Ascoli theorem and diagonal selection principle, we can select a sub-sequence of $\{t_k\}$(still denoted by $\{t_k\}$), such that $x(t+t_k)$ uniformly converges to a continuous function $x*(t)$ on any compact set of $(-\infty, +\infty)$. In addition, from (3.9), for any $t \in [-\varsigma, +\infty)$, we can see that $\{\gamma(t+t_k)\}_{k\in\mathbb{N}}$ is uniformly bounded. Therefore, for any

$t \in [-\varsigma, +\infty)$, we can select a sub-sequence of $\{t_k\}$(still denoted by $\{t_k\}$), such that $\gamma(t+t_k)$ converges weakly to a measurable function $\gamma*(t)$.

In order to prove that x^* is a solution of system (2.4) on $t \in [-\varsigma, +\infty)$, we firstly claim that $\gamma_j^*(t) \in \overline{co}[f_j((A_j^{-1}x_j)(t))]$. Indeed, based on the following facts: (1) $\overline{co}[f_j(\cdot)]$ is upper semi-continuous set-valued map; (2) for any $t \in [-\varsigma, +\infty)$, $\lim_{k\to+\infty} x(t+t_k) = x^*(t)$, we know that, for any $\epsilon > 0$, there exists $N > 0$ such that

$$\overline{co}[f_i((A_i^{-1}x_i)(t+t_k))] \subseteq \overline{co}[f_i((A_i^{-1}x_i)(t))] + \epsilon\Lambda, \text{ for } k > N, \ t \in [-\varsigma, +\infty),$$

where $\Lambda = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Thus, for any $k > N$, we have

$$\gamma_i(t+t_k) \in \overline{co}[f_i((A_i^{-1}x_i)(t))] + \epsilon\Lambda.$$

Then, by the compactness of $\overline{co}[f_i((A_i^{-1}x_i)(t))] + \epsilon\Lambda$, we have

$$\gamma^*(t) = \lim_{k\to+\infty} \gamma(t+t_k) \in \overline{co}[f_i((A_i^{-1}x_i)(t))] + \epsilon\Lambda,$$

which leads to $\gamma_i^*(t) \in \overline{co}[f_i((A_i^{-1}x_i)(t))]$ for a.e. $t \in [-\varsigma, +\infty)$ by the arbitrariness of ϵ.

Next, we prove that $x^*(t)$ is a solution of system (2.4). In fact, for any $t \in [-\varsigma, +\infty)$ and $\Delta t \in \mathbb{R}$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&x_i^*(t+\Delta t) - x_i^*(t) = \lim_{k\to+\infty} \int_t^{t+\Delta t} \Big[\Phi_i(\theta, t_k) + \Psi_i(\theta, t_k) - d_i(\theta)x_i(\theta+t_k) \\
&- d_i(\theta)\sum_{j=1}^n h_{ij}(\theta+t_k)(A_i^{-1}x_i)(\theta+t_k-\delta_{ij}(\theta+t_k)) \\
&+ d_i(\theta)\sum_{j=1}^n [h_{ij}(\theta+t_k) - h_{ij}(\theta)](A_i^{-1}x_i)(\theta+t_k-\delta_{ij}(\theta+t_k)) + \sum_{j=1}^n a_{ij}(\theta)\gamma_j(\theta+t_k) \\
&+ \sum_{j=1}^n b_{ij}(\theta)\gamma_j(\theta+t_k-\tau_{ij}(\theta)) + \sum_{j=1}^n c_{ij}(\theta)\int_{\theta-\sigma_{ij}(\theta)}^{\theta} \gamma_j(s+t_k)ds + I_i(\theta)\Big]d\theta \\
&= \int_t^{t+\Delta t} \Big[- d_i(\theta)x_i^*(\theta) - d_i(\theta)\sum_{j=1}^n h_{ij}(\theta)(A_i^{-1}x_i^*)(\theta-\delta_{ij}(\theta)) + \sum_{j=1}^n a_{ij}(\theta)\gamma_j^*(\theta) \\
&+ \sum_{j=1}^n b_{ij}(\theta)\gamma_j^*(\theta-\tau_{ij}(\theta)) + \sum_{j=1}^n c_{ij}(\theta)\int_{\theta-\sigma_{ij}(\theta)}^{\theta} \gamma_j^*(s)ds + I_i(\theta)\Big]d\theta \\
&+ \lim_{k\to+\infty}\int_t^{t+\Delta t} \Big[\Phi_i(\theta, t_k) + \Psi_i(\theta, t_k)\Big]d\theta,
\end{aligned} \tag{3.20}$$

which together with (3.19) yields

$$\begin{aligned}
x_i^*(t+\Delta t) - x_i^*(t) = \int_t^{t+\Delta t} \Big[&- d_i(\theta)x_i^*(\theta) - d_i(\theta)\sum_{j=1}^n h_{ij}(\theta)(A_i^{-1}x_i^*)(\theta-\delta_{ij}(\theta)) \\
&+ \sum_{j=1}^n a_{ij}(\theta)\gamma_j^*(\theta) + \sum_{j=1}^n b_{ij}(\theta)\gamma_j^*(\theta-\tau_{ij}(\theta)) \\
&+ \sum_{j=1}^n c_{ij}(\theta)\int_{\theta-\sigma_{ij}(\theta)}^{\theta} \gamma_j^*(s)ds + I_i(\theta)\Big]d\theta,
\end{aligned}$$

which shows that $x^*(t)$ is a solution of system (2.4).

At last, we show that $x^*(t)$ is the almost periodic solution of the system (2.4). By Theorem 3.3, we know that for any $\epsilon > 0$, there exist $T > 0$, $l = l(\epsilon)$ and $\omega = \omega(\epsilon)$ in any interval with the length of $l(\epsilon)$, such that

$$\|x(t+\omega) - x(t)\| < \epsilon, \text{ for all } t \geq T.$$

Then, there exists sufficiently large constant $K > 0$ such that

$$\|x(t+t_k+\omega) - x(t+t_k)\| < \epsilon, \text{ for all } t \geq [-\varsigma, +\infty), \ k > K,$$

let $k \to +\infty$, we can have

$$\|x^*(t+\omega) - x^*(t)\| < \epsilon, \text{ for all } t \geq [-\varsigma, +\infty),$$

which implies that $x^*(t)$ is the almost periodic solution of the system (2.4). Therefore, the proof is complete.

4. Uniqueness and global exponential stability

In this section, we shall study the uniqueness and global exponential stability of the almost periodic solution for the neural network system (2.4).

Theorem 4.1. *Suppose that the assumptions (H1), (H2), (H3) and (H4) are satisfied, then the unique almost periodic solution of neural network system* (2.4) *is globally exponentially stable.*

Proof. Let $x(t)$ and $\widetilde{x}(t)$ be any two solutions of the neural network system (2.4) associated with outputs $\gamma(t)$ and $\widetilde{\gamma}(t)$, $[\phi, \varphi]$ and $[\widetilde{\phi}, \widetilde{\varphi}]$ are the corresponding initial values, respectively. From (H2), it follows that $f_i = g_i + h_i$. There exist two vectors variable $\eta(t) = (\eta_1(t), \eta_2(t), ..., \eta_n(t))^\top$ and $\widetilde{\eta}(t) = (\widetilde{\eta}_1(t), \widetilde{\eta}_2(t), ..., \widetilde{\eta}_n(t))^\top$, such that $\eta_i(t) + g_i(x_i(t)) = \gamma_i(t)$ and $\widetilde{\eta}_i(t) + g_i(\widetilde{x}_i(t)) = \widetilde{\gamma}_i(t)$ where $\eta_i(t) \in \overline{co}[f_i((A_i^{-1}x_i)(t))]$ and $\widetilde{\eta}_i(t) \in \overline{co}[f_i((A_i^{-1}\widetilde{x}_i)(t))]$ for a.e. $t \in [-\varsigma, +\infty)$, respectively.

Consider the following candidate Lyapunov function:

$$\begin{aligned} U(t) = &\sum_{i=1}^{n} \xi_i e^{\delta t} |x_i(t) - \widetilde{x}_i(t)| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{t-\tau_{ij}(t)}^{t} \frac{|b_{ij}(\varphi_{ij}^{-1}(u))|}{1-\tau'_{ij}(\varphi_{ij}^{-1}(u))} |g_j(x_j(u)) - g_j(\widetilde{x}_j(u))| e^{\delta(u+\tau_{ij}^M)} du \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{t-\tau_{ij}(t)}^{t} \frac{|b_{ij}(\varphi_{ij}^{-1}(u))|}{1-\tau'_{ij}(\varphi_{ij}^{-1}(u))} |\eta_j(u) - \widetilde{\eta}_j(u)| e^{\delta(u+\tau_{ij}^M)} du \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0}\int_{t+s}^{t} |c_{ij}(u-s)||g_j(x_j(u)) - g_j(\widetilde{x}_j(u))| e^{\delta(u-s)} du ds \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0}\int_{t+s}^{t} |c_{ij}(u-s)||\eta_j(u) - \widetilde{\eta}_j(u)| e^{\delta(u-s)} du ds. \end{aligned} \tag{4.1}$$

Obviously, $U(t)$ is regular. Meanwhile, the solutions $x(t+\omega)$, $x(t)$ of the neural network system (2.4) are all absolutely continuous. Then, $U(t)$ is differential for *a.e.* $t \geq 0$ and the time derivative can be evaluated by Lemma 2.9 .

Define $\nu_i(t) = sign\{x_i(t) - \widetilde{x}_i(t)\}$ if $x_i(t) \neq \widetilde{x}_i(t)$; while $\nu_i(t)$ can be arbitrarily choosen in $[-1, 1]$ if $x_i(t) = \widetilde{x}_i(t)$. In particular, we can choose $\nu_i(t)$ as follows

$$\nu_i(t) = \begin{cases} 0, & x_i(t) - \widetilde{x}_i(t) = \gamma_i(t) - \widetilde{\gamma}_i(t) = 0, \\ -sign\{\eta_i(t) - \widetilde{\eta}_i(t)\}, & x_i(t) = \widetilde{x}_i(t) \text{ and } \gamma_i(t) \neq \widetilde{\gamma}_i(t), \\ sign\{x_i(t) - \widetilde{x}_i(t)\}, & x_i(t) \neq \widetilde{x}_i(t). \end{cases}$$

Then, we have

$$\begin{aligned}
&\nu_i(t)\{x_i(t)-\widetilde{x}_i(t)\} = |x_i(t)-\widetilde{x}_i(t)|,\ i=1,2,...,n,\\
&\nu_i(t)\{\eta_i(t)-\widetilde{\eta}_i(t)\} = -|\eta_i(t)-\widetilde{\eta}_i(t)|,\ i=1,2,...,n.
\end{aligned} \tag{4.2}$$

Now, by applying the chain rule in Lemma 2.9 , calculate the time derivative of $U(t)$ along the solution trajectories of the system (2.4) in the sense of (2.5), then we can get for *a.e.* $t \geq 0$ that

$$\begin{aligned}
\frac{dU(t)}{dt} &= \sum_{i=1}^{n} \xi_i \delta e^{\delta t}|x_i(t)-\widetilde{x}_i(t)| + \sum_{i=1}^{n} \xi_i e^{\delta t}\nu_i(t)\frac{d[x_i(t)-\widetilde{x}_i(t)]}{dt}\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1-\tau'_{ij}(\varphi_{ij}^{-1}(t))}\Big[|g_j(x_j(t))-g_j(\widetilde{x}_j(t))| + |\eta_j(t)-\widetilde{\eta}_j(t)|\Big]e^{\delta(t+\tau_{ij}^M)}\\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i |b_{ij}(t)|\Big[|g_j(x_j(t-\tau_{ij}(t)))-g_j(\widetilde{x}_j(t-\tau_{ij}(t)))| + |\eta_j(t-\tau_{ij}(t))-\widetilde{\eta}_j(t-\tau_{ij}(t))|\Big]e^{\delta(t-\tau_{ij}(t)+\tau_{ij}^M)}\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \sigma'_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u+\sigma_{ij}(t))|\Big[|g_j(x_j(u))-g_j(\widetilde{x}_j(u))| + |\eta_j(u)-\widetilde{\eta}_j(u)|\Big]e^{\delta(u+\sigma_{ij}(t))}du\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t-s)|\Big[|g_j(x_j(t))-g_j(\widetilde{x}_j(t))| + |\eta_j(t)-\widetilde{\eta}_j(t)|\Big]e^{\delta(t-s)}ds\\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t)|\Big[|g_j(x_j(t+s))-g_j(\widetilde{x}_j(t+s))| + |\eta_j(t+s)-\widetilde{\eta}_j(t+s)|\Big]e^{\delta t}ds.
\end{aligned} \tag{4.3}$$

Mote that, by (2.5), we have

$$\begin{aligned}
\frac{d[x_i(t)-\widetilde{x}_i(t)]}{dt} &= -d_i(t)x_i(t) - d_i(t)\sum_{j=1}^{n} h_{ij}(t)(A_i^{-1}x_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t)\\
&+ \sum_{j=1}^{n} b_{ij}(t)\gamma_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}\gamma_j(s)ds + I_i(t)\\
&- \Big[-d_i(t)\widetilde{x}_i(t) - d_i(t)\sum_{j=1}^{n} h_{ij}(t)(A_i^{-1}\widetilde{x}_i)(t-\delta_{ij}(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{\gamma}_j(t)\\
&+ \sum_{j=1}^{n} b_{ij}(t)\widetilde{\gamma}_j(t-\tau_{ij}(t)) + \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}\widetilde{\gamma}_j(s)ds + I_i(t)\Big]\\
&= -d_i(t)[x_i(t)-\widetilde{x}_i(t)] - d_i(t)h_{ij}(t)[(A_i^{-1}x_i)(t-\delta_{ij}(t)) - (A_i^{-1}\widetilde{x}_i)(t-\delta_{ij}(t))]\\
&+ \sum_{j=1}^{n} a_{ij}(t)[\gamma_j(t)-\widetilde{\gamma}_j(t)] + \sum_{j=1}^{n} b_{ij}(t)[\gamma_j(t-\tau_{ij}(t))-\widetilde{\gamma}_j(t-\tau_{ij}(t))]\\
&+ \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma_{ij}(t)}^{t}[\gamma_j(s)-\widetilde{\gamma}_j(s)]ds.
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3), in view of (4.2) and (H4), we have

$$\frac{dU(t)}{dt} \le \sum_{i=1}^{n} \xi_i \delta e^{\delta t}|x_i(t) - \widetilde{x}_i(t)| - d_i(t) \sum_{i=1}^{n} \xi_i e^{\delta t}|x_i(t) - \widetilde{x}_i(t)|$$
$$+ \sum_{i=1}^{n} \xi_i e^{\delta t}|d_i(t)||h_{ij}(t)||(A_i^{-1}x_i)(t - \delta_{ij}(t)) - (A_i^{-1}\widetilde{x}_i)(t - \delta_{ij}(t))|$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i e^{\delta t}|a_{ij}(t)||g_j(x_j(t)) - g_j(\widetilde{x}_j(t))|$$

$$- \sum_{i=1}^{n} \xi_i e^{\delta t} a_{ii}(t)|\eta_i(t) - \widetilde{\eta}_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \xi_i e^{\delta t}|a_{ij}(t)||\eta_j(t) - \widetilde{\eta}_j(t)|$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))}\Big[|g_j(x_j(t)) - g_j(\widetilde{x}_j(t))| + |\eta_j(t) - \widetilde{\eta}_j(t)|\Big] e^{\delta(t+\tau_{ij}^M)}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u + \sigma_{ij}(t))|\Big[|g_j(x_j(u)) - g_j(\widetilde{x}_j(u))| + |\eta_j(u) - \widetilde{\eta}_j(u)|\Big] e^{\delta(u+\sigma_{ij}(t))} du$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t - s)|\Big[|g_j(x_j(t)) - g_j(\widetilde{x}_j(t))| + |\eta_j(t) - \widetilde{\eta}_j(t)|\Big] e^{\delta(t-s)} ds,$$

which together with Lemma 2.3 and (H2) further gives

$$\frac{dU(t)}{dt} \le \sum_{i=1}^{n} \xi_i \delta e^{\delta t}|x_i(t) - \widetilde{x}_i(t)| - d_i(t) \sum_{i=1}^{n} \xi_i e^{\delta t}|x_i(t) - \widetilde{x}_i(t)|$$
$$+ \sum_{i=1}^{n} \xi_i e^{\delta t} \frac{|d_i(t)||h_{ij}(t)|}{1 - \sum_{j=1}^{n} h_{ij}^M}|x_i(t - \delta_{ij}(t)) - \widetilde{x}_i(t - \delta_{ij}(t))| + \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i e^{\delta t}|a_{ij}(t)|L_j|x_j(t) - \widetilde{x}_j(t)|$$
$$- \sum_{i=1}^{n} \xi_i e^{\delta t} a_{ii}(t)|\eta_i(t) - \widetilde{\eta}_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \xi_i e^{\delta t}|a_{ij}(t)||\eta_j(t) - \widetilde{\eta}_j(t)|$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \frac{|b_{ij}(\varphi_{ij}^{-1}(t))|}{1 - \tau'_{ij}(\varphi_{ij}^{-1}(t))}\Big[L_j|x_j(t) - \widetilde{x}_j(t)| + |\eta_j(t) - \widetilde{\eta}_j(t)|\Big] e^{\delta(t+\tau_{ij}^M)}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \sigma'_{ij}(t) \int_{t-\sigma_{ij}(t)}^{t} |c_{ij}(u + \sigma_{ij}(t))|\Big[L_j|x_j(u) - \widetilde{x}_j(u)| + |\eta_j(u) - \widetilde{\eta}_j(u)|\Big] e^{\delta(u+\sigma_{ij}(t))} du$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \int_{-\sigma_{ij}(t)}^{0} |c_{ij}(t - s)|\Big[L_j|x_j(t) - \widetilde{x}_j(t)| + |\eta_j(t) - \widetilde{\eta}_j(t)|\Big] e^{\delta(t-s)} ds$$
$$\le \sum_{i=1}^{n} e^{\delta t}\Gamma_i(t)|x_i(t) - \widetilde{x}_i(t)| + \sum_{i=1}^{n} e^{\delta t}\Upsilon_i(t)|\eta_i(t) - \widetilde{\eta}_i(t)| < 0, \text{ for a.e. } t \ge 0.$$

Furthermore, from (4.1), it follows that

$$U(t) \ge \sum_{i=1}^{n} \xi_i e^{\delta t}|x_i(t) - \widetilde{x}_i(t)|,$$

thus,

$$\|x(t)-\widetilde{x}(t)\| = \sum_{i=1}^{n} |x_i(t)-\widetilde{x}_i(t)| \leq \frac{e^{-\delta t}}{\xi^m} U(t) \leq \frac{e^{-\delta t}}{\xi^m} U(0),$$

where $\xi^m := \min_{1\leq i\leq n}\{\xi_i\} > 0$. Moreover, since $U(0)$ is a constant. Thus, by Definition 2.6, we can conclude that the unique almost periodic solution of neural network system (2.4) is globally exponentially stable. Therefore, the proof is complete.

5. Numerical Examples

In this section, we present two topical examples to demonstrate the results obtained in previous sections.

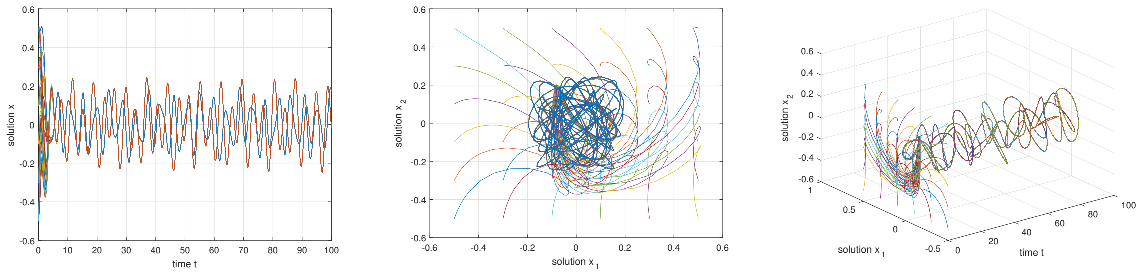
Example 5.1. *Consider the following neutral-type neural networks with discontinuous activations and mixed delays:*

$$\begin{cases} (A_1x_1)'(t) & = -x_1(t) + 0.3f_1(x_1(t)) - 0.01f_2(x_2(t)) + 0.01f_1(x_1(t-\tau_{11}(t))) \\ & \quad -0.01f_2(x_2(t-\tau_{12}(t))) + c_{11}(t)\int_{t-0.1\cos t}^{t} f_1(x_1(s))ds \\ & \quad +c_{12}(t)\int_{t-0.1\cos t}^{t} f_2(x_2(s))ds + 0.2\sin\sqrt{2}t + 0.1\sin\sqrt{5}t, \\ (A_2x_2)'(t) & = -x_2(t) + 0.01f_1(x_1(t)) + 0.3f_2(x_2(t)) + 0.01f_1(x_1(t-\tau_{21}(t))) \\ & \quad -0.01f_2(x_2(t-\tau_{22}(t))) + c_{21}(t)\int_{t-0.1\cos t}^{t} f_1(x_1(s))ds \\ & \quad +c_{22}(t)\int_{t-0.1\cos t}^{t} f_2(x_2(s))ds + 0.3\cos\sqrt{3}t - 0.1\sin t, \end{cases} \tag{5.1}$$

where $A_ix_i(t) = x_i(t) - \sum_{j=1}^{2} h_{ij}(t)x_i(t-\delta_{ij}(t))$, $i = 1,2$, *and* $d_i(t) = 1$, $a_{11}(t) = 0.3$, $a_{12}(t) = -0.01$, $a_{21}(t) = 0.01$, $a_{22}(t) = 0.3$, $b_{11}(t) = 0.01$, $b_{12}(t) = -0.01$, $b_{21}(t) = 0.01$, $b_{22}(t) = -0.01$, $c_{11}(t) = 0.03$, $c_{12}(t) = -0.02$, $c_{21}(t) = 0.04$, $c_{22}(t) = -0.03$, $h_{ij}(t) = 0.1 + 0.1\sin t$, $\delta_{ij}(t) = 0.2 + 0.1\cos t$, $\tau_{ij}(t) = 0.3 + 0.2\cos t$, $\sigma_{ij}(t) = 0.1\cos t$, $I_1(t) = 0.2\sin\sqrt{2}t + 0.1\sin\sqrt{5}t$, $I_2(t) = 0.3\cos\sqrt{3}t - 0.1\sin t$, *and*

$$f_1(x) = \begin{cases} \sin x + x^2, & x < -1, \\ \sin x, & |x| \leq 1, \\ \sin x - x^2, & x > 1. \end{cases} \qquad f_2(x) = \begin{cases} e^{-x} + \arctan x, & x < -1, \\ \arctan x, & |x| \leq 1, \\ -e^{-x} + \arctan x, & x > 1. \end{cases}$$

It is easy to see that the activation functions $f_1(x)$ *and* $f_2(x)$ *are discontinuous, unbounded, non-monotonic, and super linear growth condition (In fact,* $f_2(x)$ *satisfies the exponential growth condition). Meanwhile, for the activation functions* $f_1(x)$ *and* $f_2(x)$*, the unilateral Lipschitz-like condition does not be satisfied. Moreover, the activation functions* $f_1(x)$ *and* $f_2(x)$ *are discontinuous at* $x = \pm 1$*. Consider the IVP of the system* (5.1) *with random initial conditions. It is easy to verify that the system* (5.1) *satisfies all the assumptions in Theorem 3.1. Therefore, by Theorem 3.1, Theorem 3.3 and Theorem 4.1, we can see that the neutral-type neural network system* (5.1) *has a unique almost periodic solution which is globally exponentially stable. This fact can be presented in the following Figure 1.*



(a) (b) (c)

Figure 1: (a) Time-domain behavior of the state variables x_1 and x_2 for system (5.1) with random initial conditions for $t \in [-0.5, 0]$; (b) Phase plane behavior of the state variables x_1 and x_2 for system (5.1); (c) Three-dimensional trajectory of state variables x_1 and x_2 for system (5.1).

Example 5.2. *Let $\sigma_{ij}(t) \equiv 0$ and further consider the following neutral-type neural networks with discontinuous activations and discrete time-varying delays:*

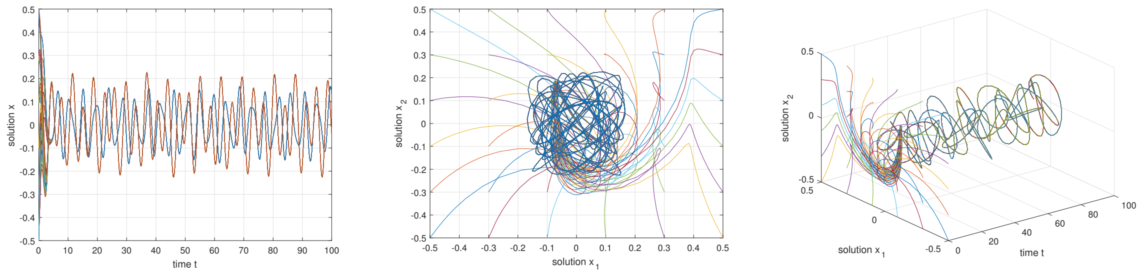
$$\begin{cases} (A_1x_1)'(t) &= -x_1(t) + 0.3f_1(x_1(t)) - 0.01f_2(x_2(t)) + 0.01f_1(x_1(t-\tau_{11}(t))) \\ &\quad -0.01f_2(x_2(t-\tau_{12}(t))) + 0.2\sin\sqrt{2}t + 0.1\sin\sqrt{5}t, \\ (A_2x_2)'(t) &= -x_2(t) + 0.01f_1(x_1(t)) + 0.3f_2(x_2(t)) + 0.01f_1(x_1(t-\tau_{21}(t))) \\ &\quad -0.01f_2(x_2(t-\tau_{22}(t))) + 0.3\cos\sqrt{3}t - 0.1\sin t, \end{cases} \tag{5.2}$$

where $A_ix_i(t) = x_i(t) - \sum_{j=1}^{2} h_{ij}(t)x_i(t-\delta_{ij}(t))$, $i = 1,2$, $d_i(t) = 1$, $a_{11}(t) = 0.3$, $a_{12}(t) = -0.01$, $a_{21}(t) = 0.01$, $a_{22}(t) = 0.3$,

$$\begin{aligned} &b_{11}(t) = 0.01, b_{12}(t) = -0.01, b_{21}(t) = 0.01, b_{22}(t) = -0.01, \\ &h_{ij}(t) = 0.1 + 0.1\sin t, \delta_{ij}(t) = 0.2 + 0.1\cos t, \tau_{ij}(t) = 0.3 + 0.2\cos t, \\ &I_1(t) = 0.2\sin\sqrt{2}t + 0.1\sin\sqrt{5}t, I_2(t) = 0.3\cos\sqrt{3}t - 0.1\sin t, \\ &f_1(x) = f_2(x) = \begin{cases} \frac{0.2x}{x^2+1}, & x < 1; \\ \frac{0.2x-1}{x^2+1}, & x > 1. \end{cases} \end{aligned}$$

It is easy to see that the activation function $f(x) = (f_1(x_1), f_2(x_2))^\top$ is discontinuous, bounded, monotonically nondecreasing. Meanwhile, 1 is a discontinuous point of the activation function $f_i(s)$, $f_i(1^+) < f_i(1^-)$ and $\overline{co}[f_i(1)] = [f_i(1^+), f_i(1^-)] = [-0.4, 0.1]$, $i = 1,2$.

Consider the IVP of the system (5.2) *with random initial conditions. It is easy to verify that the system* (5.2) *satisfies all the assumptions in Theorem 3.1. Therefore, by Theorem 3.1, Theorem 3.3 and Theorem 4.1, we can see that the neutral-type neural network system* (5.2) *has a unique almost periodic solution which is globally exponentially stable. This fact can be presented in the following Figure 2 .*



(a) (b) (c)

Figure 2: (a) Time-domain behavior of the state variables x_1 and x_2 for system (5.1) with random initial conditions for $t \in [-0.5, 0]$; (b) Phase plane behavior of the state variables x_1 and x_2 for system (5.2); (c) Three-dimensional trajectory of state variables x_1 and x_2 for system (5.2).

Remark 5.3. *In the Example 5.2, the distributed time-varying delay $\sigma_{ij}(t) \equiv 0$ for all $i, j = 1, 2$, and a class of neutral-type neural networks with discontinuous activations and discrete time-varying delays is considered which is different from the neural model studied in Example 5.1. By using the results established in the paper, we can see that the neutral-type neural network system* (5.2) *has a unique almost periodic solution which is globally exponentially stable. Figure 2 can show the fact.*

Remark 5.4. *For all we know, there is no research on the existence, uniqueness and global exponential stability of almost periodic solutions for the neutral-type neural networks with mixed delays. We also mention that all results in the [7], [8], [9], [20], [21], [22], [23], [39], [49], [52], [54] and the related references therein cannot be directly applied to imply the existence, uniqueness and global exponential stability of almost periodic solutions of* (5.1) *and* (5.2)*. This implies that the results of this paper are essentially new.*

6. Conclusion

In this paper, we investigate a class of neutral-type neural networks with discontinuous activations and mixed delays. The considered neural networks shows the neutral character by the neutral operator $A_i(i = 1, 2, ..., n)$ and the activation functions with time-varying delays of the considered neural networks are discontinuous continuous, which are different from the corresponding ones known in the literature. Different from the existing approaches to study the neutral-type neural networks with continuous activations, in this paper, under the concept of Filippov solution, by means of differential inclusions theory, the non-smooth analysis theory with Lyapunov-like approach, we employ a novel argument and the easily verifiable sufficient conditions have been provided to determine the existence, uniqueness, global exponential stability of the almost periodic solutions for the considered neural networks. It should be pointed out that it is the first time to investigate almost periodic dynamic behavior of the neutral-type neural networks with discontinuous activations and mixed delays. Finally, two numerical examples and the corresponding simulations have been presented at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion. Consequently, our results can enrich and extend the corresponding ones known in the literature.

Acknowledgements

The authors thank the anonymous reviewers for their insightful suggestions which improved this work significantly. This work was jointly supported by the National Natural Science Foundation of China (12001011), Natural Science Fund Project of the University in Anhui Province (2022AH030023).

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