Fear and its carry-over effects in a delay-induced predator-prey model with additional food to predator

Bijoy Kumar Das[a], Debgopal Sahoo[a], Guruprasad Samanta[a,*]

[a]*Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, India*

Abstract. Interaction between prey and predator is a natural phenomenon in ecology that significantly contributes to the structure of ecological variety. Recent studies indicate that the presence of predator can influence the physiological behaviour of prey species to such an extent that it can be more efficient than direct predation in decreasing the prey biomass. Moreover, such non-lethal effects can be carried over through seasons or generations. In this present article, we analyze the impact of predator-induced fear and its carry-over effect in a predator-prey model in which the predator species can access some alternative or additional food sources. Well-posedness of the system and some basic dynamical properties such as extinction criteria, stability analysis with global stability, uniform persistence etc. are discussed thoroughly. From the bifurcation analyzes, we can observe that fear and its carry-over effect can not switch the stability from one equilibrium state to other equilibrium state. However once the coexistence equilibrium state occurs in the system, a higher level of fear can stabilize it. On the other hand, higher level of carry-over effect promote the oscillatory dynamics around the coexistence state. Therefore, fear and its carry-over effects have two opposite roles in the stability of the coexistence equilibrium. Predator species may go extinct if the effective quantity of additional food is sufficiently low. Next we study the model system in presence of gestation delay and observe some interesting dynamics by taking the delay parameter as a bifurcation parameter. Our study demonstrates how non-lethal effects alter the dynamics of a prey-predator model and provides valuable biological insights, particularly into the dynamics of small food web.

1. Introduction

Predator-prey interactions have a long history in mathematical ecology. The relationship between prey and predator is one of the most important factors that contributes considerably to the formation of community establishment and the conservation of ecological diversity. Typical predator–prey interaction models usually focus on an organism's principal purpose within a food web. Due to its significant importance, the dynamical behaviour of predator–prey model has remained one of the most prevalent subjects in mathematical ecology.

In a predator–prey system, the access to additional food can have a major impact on the system's dynamics. Most predators' diets include additional food as a crucial key component. Despite the fact

that additional food receives less attention in the scientific literature, many predator species' life cycles are profoundly influenced by this type of food sources. Many scientists have shown interest in the effects of feeding additional food to predator and the resulting effects on predator-prey dynamics, as well as its application in mathematical biology. The primary goal of this strategy is to decrease predation pressure on prey species. In recent years, many scientists and ecologists have studied the effects of feeding additional food to predators in predator–prey systems. Field experiments have also revealed that diversionary/substitute feeding does not always serve the objective of biological conservation [4]. Mondal et al. [17] discussed a prey-predator model with the effect of fear and additional food and observed its dynamical behaviour. They demonstrated that feeding additional food to the predator biomass reduces the predator's attack rate on the prey biomass and promotes the growth of the prey population. Das and Samanta [18] studied similar type of prey-predator interactions in presence of additional food and prey refuge in the fluctuating environment. 'Quality' and 'quantity' are the critical factors that determine the impact of additional food on a model system. Huxel et al. [19, 20] and Srinivasu [21] observed in their research that a high-quality additional food source causes a high frequency of predation, but a low-quality source may help to save the target prey. Das and Samanta [22] have concluded that when the predator's effective food level of additional food is high, the predator dominates the prey population in a fluctuating environment.

Over a long period of time, it was thought that predators only had a long-term impact on prey populations through direct consumption. However, emerging research indicates that predator-prey interactions are influenced not just by direct predation but also by indirect impacts such as the cost of fear, which can affect preys' physiological, morphological, and life cycle responses [1–3, 38, 40]. Prey exhibits numerous anti-predator behaviours to avoid predation, such as habitat alterations, reduced foraging activities, vigilance, physiological changes, and many more [2]. These types of indirect effects are recognized as trait-mediated effects and are caused by the predator on prey characteristics rather than prey density. The risk of predation has a far greater influence on prey foraging behaviour than growth, survivorship, or fecundity [5]. Such non-lethal effects are also based on some field experiments: the interaction between larval dragonfly *Anax sp.* (predator species) and bullfrog tadpoles *Rana catesbeiana* (prey species) [6]. According to certain recent research works [7, 8], fear of predation may play a significant role in determining the long term behaviour of a predator-prey model. A field experiment on song sparrows (Melospiza melodia) in 2011 confirmed the impact of fear on prey populations [8]. In that experiment, song sparrows population were separated to an open atmosphere by eliminating direct predation and manipulating fake predation risk with predator sound playbacks and it was found that due to the fear of predation, the number of offspring generated each year is reduced by roughly 40%. Prey that perceives a high degree of predation risk may temporarily leave their area and returning only when the threat has passed and they are reasonably secure. Prey population are afraid to come to an open space because of predators, thus they don't have a free environment to complete their everyday tasks, such as coupling. As a result, the level of fear may reduce prey fertility and survival, and when compared to only direct predation, the overall drop in prey population may be significantly larger. One more field experiment was studied on snowshoe hares, which reveals that any physiological stress, not simply predatory stress, can lower reproductive output [23]. Das et al. [9] proposed a gestation delay-induced prey-predator model with the cost of predation fear into both the birth rate and death rate of prey species in presence of the Holling type II functional response. They found that the presence of a coexistence equilibrium point is enough to stop the extinction of predator population. The basic bifurcation analyzes reveal that a larger cost of fear or a larger intraspecific competition rate aids the population's survival in a coexistence stage. The suggested model may induce the bi-stable phenomena between two coexisting steady states with an appropriate choice of parametric variables.

Almost all biological and ecological processes involve time delay. Therefore introducing time delay into an ecological model makes it more realistic than the non-delayed model. Predator reproduction after prey consumption is not instantaneous, but is mediated by a time lag necessary for gestation. As a result, the time lag between prey capture and its associated contribution to predator's growth is an essential issue in model construction and biological elucidation [37, 39]. In 2018, Ma and Wang [25] studied how a delay-induced prey-predator system with habitat complexity respond dynamically. Dutta et al. [24] analyzed a delay-induced eco-epidemic model, where they found Bautin bifurcation and Gavrilov-Guckenheimer bifurcation for some preferable parametric values.

Understanding the components or conditions that influence the fitness of wild organisms is a primary objective of ecologists [14]. Repeated measurements of clinical tests gave rise to the phrase "carry-over effect". After that, it has recently been applied to the area of ecological and evolutionary studies and hence may be used to a wide range of research areas. O'Connor et al. [10] defined the term "carry-over" as follows:

> "In an ecological context, carry-over effects can occur in any situation in which an individual's previous history and experience explains their current performance in a given situation".

They recommended that ecological carry-over effect can also be happened in the stages of life-history, physiological states etc., and each of this will be associated with a discrete time-scale. Furthermore, several laboratory investigations have revealed that nonlethal carry-over effects have an impact on long-term population dynamics [11, 29, 41]. In 2022, Sasmal and Takeuchi [26] investigated the role of predation fear and its carry-over effects in a prey-predator model with Holling type I functional response. They identified that a phenomena known as the 'paradox of enrichment' may occur in this sort of model, which may be eliminated by selecting appropriate non-lethal parameters. Mondal and Samanta [27] studied similar type of model but with Holling type II functional response and prey refuge. They also compared this model to a stochastic variant that includes Gaussian white noise terms due to the effect of fluctuating environment.

In this present article, we propose a delay-induced predator-prey model that takes into account the influence of fear and its carry-over effect on prey reproduction caused by the predator. Further we also assume that growth of predator depends not only on prey species but also on some alternative/additional food sources. The main goal of this article is to figure out how the stability of the system dynamics is influenced by the fear and its carry-over effect in presence of additional food to predator. This study is organised as follows: In Section 2, we formulate a predator-prey model with (i) cost of fear and its carry-over effect in prey reproduction and (ii) additional food to predator. The well-posedness of our derived model system is verified in Section 3. In Section 4, we evaluate some conditions in terms of model parameters under which the prey or predator biomass goes extinct. In absence of prey, predator species can expand exponentially under some restriction due to the presence of additional food, as detailed more in this section. In Section 5, we discuss the existence of various steady states and explore the local stability analysis around these steady states. Global stability and uniform persistence are verified in Sections 6 and 7 respectively. Existence of various local bifurcations are discussed in Section 8. In Section 9, we introduce the gestation delay and investigate the impact of delay parameter on system dynamics. All of the theoretical results are illustrated by various numerical examples. Finally, we discuss and conclude our findings with biological significance in Section 10.

2. The Model

First, we model the growth of a prey species in absence of predator, which is governed by the logistic growth dynamics [32] and can be expressed by the following ODE:

$$\frac{dx}{dt} = rx - d_1 x - a_1 x^2, \tag{2.1}$$

where x is the prey biomass, r is the prey reproduction rate, d_1 is the natural mortality rate and a_1 is the density-dependent death rate. Next, we consider the prey biomass together with a constant predator biomass, y. Many experimental studies suggest that the presence of predator can indirectly affect the reproduction of prey species by generating the predator-induced fear and its carry-over effect in prey individuals [8, 29–31]. Ignoring the direct consumption of predator, we therefore, modify the above logistic growth model incorporating with fear and it's carry-over effect in prey reproduction, which is given by:

$$\frac{dx}{dt} = rx \left(\frac{1 + cx}{1 + cx + ky} \right) - d_1 x - a_1 x^2. \tag{2.2}$$

Here the parameter c defines the carry-over effect due to fear as quantified by the parameter k. The term $f(c, k, x, y) = \frac{1+cx}{1+cx+ky}$, which is related to the fear and its carry over effects, has the following properties:

(i) For $c = 0$, $f(0, k, x, y) = \frac{1}{1+ky}$ which modifies the model (2.2) to a single prey species population model incorporating with only fear effect [7, 9].

(ii) When there is no fear, i.e., $k = 0$, $f(c, 0, x, y) = 1$ and the model (2.2) becomes a simple logistic growth model as in (2.1).

(iii) For $y = 0$, $f(c, k, x, 0) = 1$, that means in absence of predator, prey species does not suffer any reproductive decay.

(iv) $\frac{\partial}{\partial k} f(c, k, x, y) < 0$ and $\lim_{k \to \infty} f(c, k, x, y) = 0$, i.e., increasing the level of fear (k) has a negative impact on the growth of prey biomass. Further, this detrimental impact may increase to such a stage where the reproduction of prey population suffers as well.

(v) Similarly, $\frac{\partial}{\partial y} f(c, k, x, y) < 0$ and $\lim_{y \to \infty} f(c, k, x, y) = 0$, i.e., increasing predator biomass gives adverse impact on prey reproduction.

(vi) $\frac{\partial}{\partial c} f(c, k, x, y) = \frac{kxy}{(1+cx+ky)^2} > 0$ and $\lim_{c \to \infty} f(c, k, x, y) = 1$, indicating that the carry-over effect (c) has positive impact on the growth of prey biomass (due to gain experience from past incidents) and furthermore the prey species can recover his natural reproduction rate when the carry-over effect is very high.

(vii) Similarly, $\frac{\partial}{\partial x} f(c, k, x, y) = \frac{cky}{(1+cx+ky)^2} > 0$ and $\lim_{x \to \infty} f(c, k, x, y) = 1$, that means increasing prey biomass has positive impact on their own growth and if the prey population is very large, no reduction in prey reproduction occurs due to anti-predator behaviour.

All of these properties ensure that the function $f(c, k, x, y) = \frac{1+cx}{1+cx+ky}$ can capture the main characteristics of predation-induced fear and its carry-over effect, and hence can be used in modelling framework. Pictorial representations of this fear function are shown in Fig.1, from which one can anticipate the behaviour of fear function for varying x and y, or c and k.

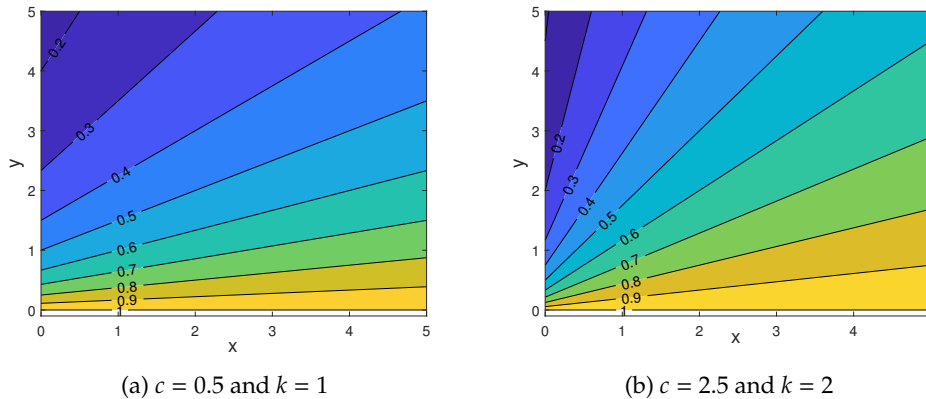(a) $c = 0.5$ and $k = 1$ (b) $c = 2.5$ and $k = 2$

Figure 1: Contour plots illustrating the nature of fear function $f(c, k, x, y)$ for variation of prey biomass (x) and predator biomass (y). Similar types of contour plots occur for varying c and k. Increasing x or c has positive impact and increasing y or k has negative impact on prey's reproduction.

Recently Mondal and Samanta [27] have provided a detailed analysis of single prey population model as in equation (2.2) from global perspective and investigated the impact of fear and its carry-over effect on the growth dynamics of prey species. It has been shown that the fear and its carry-over effect can be a cause of generating Allee dynamics. In other words, the single species system may represent three forms of growth dynamics, namely, strong Allee dynamics, weak Allee dynamics and logistic dynamics, depending

on the restrictions of model parameters. Moreover, the variation of reproduction rate (r) can produce two types of local bifurcations: (i) saddle-node bifurcation and (ii) transcritical bifurcation. Further they have discussed the model dynamics in the presence of predator population explicitly, with Holling type-I functional response and observed some substantial role of fear and its carry-over impact in the stability of coexistence equilibrium and the occurrence of 'paradox of enrichment'. Mondal and Samanta [27] have also investigated the similar type of predator-prey model but with Holling type II functional response and prey refuge.

Predator species considered in the preceding articles are specialist predators whose growth is exclusively dependent on the indicated prey species, i.e., predator species goes extinct in the absence of their specific prey. However, this type of predator-prey relation is very rare in ecology. In absence of their primary food source, most predator become dependent on the alternative/additional food sources in order to avoid their extinction crisis. Therefore, provision of additional food to predator has great impact on regulating ecological diversity. It is also one of the commonly accepted strategies in the field of biological control. Hence it is quite reasonable to consider the additional food of constant biomass A, which is provided to the predator species and allocated equally through out the habitat. We now modify the standard Holling type II functional response in the presence of some additional food to the predator.

Let us presume that T represents the total amount of time needed by the predator to obtain food from prey. This total time includes the time T_s : spent by predator for seeking prey and/or additional food, the time T_x : spent for handling prey, the time T_A : spent for handling additional food.

$$\therefore T = T_s + T_x + T_A.$$

Let, $V(x)$ is the total amount of predated prey per predator individual which is proportional to the available prey biomass (x) as well as the time budget (T_s) for seeking prey. Then $V(x) = p_1 x T_s$, where p_1 is the proportionality constant. Similarly if $V(A)$ be the amount of predated additional food, then $V(A) = p_2 A T_s$, where p_2 is the proportionality constant. Biologically the constants p_1 and p_2 are defined as attack rates or rates of predation. Further, assume h_1 and h_2 are the average handling times for each hunted prey and additional food respectively, then $T_x = h_1 V(x)$ and $T_A = h_2 V(A)$. Therefore the total time, T can be written as:

$$T = T_s + h_1 V(x) + h_2 V(A) = (1 + h_1 p_1 x + h_2 p_2 A) T_s.$$

Now, the size of hunted prey and hunted additional food per predator individual per unit time are given by

$$\frac{V(x)}{T} = \frac{p_1 x}{(1 + h_1 p_1 x + h_2 p_2 A)} \quad \text{and} \quad \frac{V(A)}{T} = \frac{p_2 A}{(1 + h_1 p_1 x + h_2 p_2 A)}$$

respectively. After some simplifications, we rewrite the expressions as follows:

$$\frac{V(x)}{T} = \frac{a_2 x}{b + \alpha \mu A + x} \quad \text{and} \quad \frac{V(A)}{T} = \frac{a_2 \mu A}{b + \alpha \mu A + x}$$

which represent the functional responses of predator towards prey and additional food respectively. Here $a_2 = \frac{1}{h_1}$ = maximum predation rate, $b = \frac{1}{h_1 p_1}$ = half saturation constant of predator biomass, $\alpha = \frac{h_2}{h_1}$ = quality of additional food and $\mu = \frac{p_2}{p_1}$ = coefficient of effective quantity of additional food. So the growth of the prey population in presence of direct predation, follows the following ODE:

$$\frac{dx}{dt} = rx\left(\frac{1 + cx}{1 + cx + ky}\right) - d_1 x - a_1 x^2 - \frac{a_2 xy}{b + \alpha \mu A + x}, \tag{2.3}$$

and the growth of predator biomass together with the derived functional responses on prey and additional food together with natural mortality rate of predator, can be written as:

$$\frac{dy}{dt} = \frac{c_1 a_2 (x + \mu A) y}{b + \alpha \mu A + x} - d_2 y. \tag{2.4}$$

Here, c_1 $(0 < c_1 < 1)$ is the conversion factor considered as the energetic efficiency in converting consumption of prey into reproduction (of predator) for each captured prey and d_2 is the mortality rate of predator. So, together with the equations (2.3) and (2.4), our final model looks like as:

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(\frac{1 + cx}{1 + cx + ky}\right) - d_1 x - a_1 x^2 - \frac{a_2 xy}{b + \alpha \mu A + x} \equiv F_1(x, y), \\
\frac{dy}{dt} &= \frac{c_1 a_2 (x + \mu A) y}{b + \alpha \mu A + x} - d_2 y \equiv F_2(x, y),
\end{aligned} \tag{2.5}$$

with the following initial conditions

$$x(0) > 0, \quad y(0) > 0. \tag{2.6}$$

Description and range of model parameters are summarized in Table 1.

Table 1: Description and range of system parameters

Parameter	Description	Range
r	Reproduction rate of prey	$0.2 - 6$
d_1	Natural death rate of prey	$0.01 - 1$
a_1	Density dependent death rate of prey	$0.001 - 1$
k	Level of fear	$0.05 - 10$
c	Carry-over effect rate due to fear	$0.01 - 10$
a_2	Coefficient of consumption rate of predator	$0.1 - 3$
b	Half saturation constant	$0.1 - 5$
α	Quality of additional food	$0.01 - 2$
μ	Coefficient of effective quantity of additional food	$0.01 - 2$
A	Additional food	$0.1 - 10$
c_1	Conversion coefficient	$0.01 - 1$
d_2	Death rate of predator	$0.01 - 1$

We first verify the well-posedness of our proposed dynamical system before investigating its stability.

3. Preliminaries

The well-posedness of a system refers that the solution of that system exists uniquely and is positive for any time $t \geq 0$ and also the solution does not grow abruptly for long time interval. Biologically, these assert that the interacting species in the system are ecologically well behaved and abundance of each species is restricted due to limited resources.

Theorem 3.1. *Solution of system (2.5), with initial conditions (2.6), exists uniquely and is positive for any time $t \geq 0$.*

Proof. It can be easily verified that the right hand side of system (2.5) are continuous and locally lipschitzian in $\mathbb{R}^2_+$. Hence the solution $(x(t), y(t))$ of the system exists uniquely on $[0, \xi)$ where $0 < \xi \leq \infty$ [33]. Now from system (2.5) with initial conditions $x(0) > 0$ and $y(0) > 0$, we have

$$\begin{aligned}
x(t) &= x(0) \exp\left[\int_0^t \left\{\frac{r(1 + cx(s))}{1 + cx(s) + ky(s)} - d_1 - a_1 x(s) - \frac{a_2 y(s)}{b + \alpha \mu A + x(s)}\right\} ds\right] > 0, \\
y(t) &= y(0) \exp\left[\int_0^t \left\{\frac{c_1 a_2 (x(s) + \mu A)}{b + \alpha \mu A + x(s)} - d_2\right\} ds\right] > 0.
\end{aligned} \tag{3.1}$$

This proves that the solution of the proposed system remains positive for any time $t \geq 0$. $\square$

Next, we show the uniform boundedness of the system's solution under certain parametric constraint.

Theorem 3.2. *Solutions of system (2.5), initiating in $\mathbb{R}_+^2$, are uniformly bounded, provided $\mu A < \frac{bd_2}{c_1 a_2}$.*

Proof. From the first equation of system (2.5), we have

$$\frac{dx}{dt} \leq rx\left(\frac{1+cx}{1+cx+ky}\right) - d_1 x - a_1 x^2$$

$$\leq (r-d_1)x - a_1 x^2 \quad \left[\because \frac{1+cx}{1+cx+ky} \leq 1\right]$$

$$= (r-d_1)x\left(1 - \frac{x}{\frac{r-d_1}{a_1}}\right).$$

$$\therefore \lim_{t\to\infty} \sup x(t) \leq \frac{r-d_1}{a_1}.$$

Next, let us assume $W = x + \frac{y}{c_1}$. Then,

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{c_1}\frac{dy}{dt}$$

$$= rx\left(\frac{1+cx}{1+cx+ky}\right) - d_1 x - a_1 x^2 + \frac{a_2\mu A y}{b+\alpha\mu A + x} - \frac{d_2 y}{c_1}$$

$$\leq rx - d_1 x - a_1 x^2 + \frac{a_2\mu A y}{b} - \frac{d_2 y}{c_1} \quad \left[\because \frac{1+cx}{1+cx+ky} \leq 1 \text{ and } \frac{a_2\mu A y}{b+\alpha\mu A + x} \leq \frac{a_2\mu A y}{b}\right]$$

$$= rx - a_1 x^2 - d_1 x - \left(d_2 - \frac{c_1 a_2\mu A}{b}\right)\frac{y}{c_1}.$$

Taking $\mu A < \frac{bd_2}{c_1 a_2}$ and $\zeta = \min\left\{d_1, d_2 - \frac{c_1 a_2\mu A}{b}\right\}$, we have

$$\frac{dW}{dt} + \zeta W \leq rx - a_1 x^2 - (d_1 - \zeta)x - \left(d_2 - \frac{c_1 a_2\mu A}{b} - \zeta\right)\frac{y}{c_1}$$

$$\leq rx - a_1 x^2$$

$$= -a_1\left(x - \frac{r}{2a_1}\right)^2 + \frac{r^2}{4a_1}$$

$$\leq \frac{r^2}{4a_1}.$$

Then by applying the theorem of differential inequality [15] for $W(t)$, we obtain

$$0 \leq W\left(x(t), y(t)\right) \leq \frac{r^2}{4a_1\zeta}(1 - e^{-\zeta t}) + W\left(x(0), y(0)\right)e^{-\zeta t}$$

$$\implies 0 \leq W\left(x(t), y(t)\right) \leq \frac{r^2}{4a_1\zeta} + \epsilon, \text{ for any } \epsilon > 0, \text{ as } t \to \infty.$$

Therefore, every positive solution in $\mathbb{R}_+^2$ confined in the region

$$\Omega = \left\{(x,y) \in \mathbb{R}_+^2 : 0 < x(t) \leq \frac{r-d_1}{a_1}, \ 0 \leq W\left(x(t), y(t)\right) \leq \frac{r^2}{4a_1\zeta} + \epsilon, \text{ for any } \epsilon > 0\right\}. \tag{3.2}$$

This demonstrates that any solution of system (2.5) is uniformly bounded provided $\mu A < \frac{bd_2}{c_1 a_2}$. $\square$

4. Extinction Criterion

In this section, we derive certain parametric restrictions under which one or both of the prey and predator go extinct.

Theorem 4.1. *If $r < d_1$, then $\lim\limits_{t \to \infty} x(t) = 0$.*

Proof. From the first equation of system (2.5), we have

$$\frac{dx}{dt} \leq rx \left(\frac{1 + cx}{1 + cx + ky} \right) - d_1 x$$
$$\leq rx - d_1 x.$$

Then clearly $\lim\limits_{t \to \infty} x(t) = 0$ as $r < d_1$. Therefore higher mortality rate always drives the prey population to extinction. This is ecologically realistic and intuitive. $\square$

Next, we show that in absence of prey, predator species either goes extinct or grows in larger biomass.

Theorem 4.2. *For $r < d_1$:*

$$\lim\limits_{t \to \infty} y(t) = 0 \text{ if } \mu A(c_1 a_2 - \alpha d_2) < bd_2, \text{ and } y(t) \to \infty \text{ as } t \to \infty \text{ if } \mu A(c_1 a_2 - \alpha d_2) > bd_2.$$

Proof. For large time t, the second equation of system (2.5) gives

$$\frac{dy}{dt} = \left(\frac{c_1 a_2 \mu A}{b + \alpha \mu A} - d_2 \right) y.$$

Then it is easy to conclude that $\lim\limits_{t \to \infty} y(t) = 0$ if $\mu A (c_1 a_2 - \alpha d_2) < bd_2$ and $y(t) \to \infty$ as $t \to \infty$ if $\mu A(c_1 a_2 - \alpha d_2) > bd_2$. Biologically, it states that in absence of prey species, predator growth is regulated by the effective quantity of additional food. The provision of sufficient additional food increases the size of predator biomass; otherwise, predator species goes extinct due to a lack of food source. $\square$

Now we show that under some certain parametric constraints, the prey species goes extinct even its reproduction rate is higher than the natural mortality rate.

Theorem 4.3. *If $r > d_1$, $\mu A(c_1 a_2 - \alpha d_2) > bd_2 + d_2 \left(\frac{r - d_1}{a_1} \right)$, then $y(t) \to \infty$ as $t \to \infty$ and hence $\lim\limits_{t \to \infty} x(t) = 0$.*

Proof. From the second equation of system (2.5), we have (for large t):

$$\frac{dy}{dt} \geq \frac{c_1 a_2 \mu A y}{b + \alpha \mu A + x} - d_2 y$$
$$\geq \left[\frac{c_1 a_2 \mu A}{b + \alpha \mu A + \frac{r - d_1}{a_1}} - d_2 \right] y \quad \left(\because \lim\limits_{t \to \infty} \sup x(t) \leq \frac{r - d_1}{a_1} \right)$$

Therefore, $y(t) \to \infty$ as $t \to \infty$ for $r > d_1$ and $\mu A(c_1 a_2 - \alpha d_2) > bd_2 + d_2 \left(\frac{r - d_1}{a_1} \right)$.
Now from the first equation of system (2.5):

$$\frac{dx}{dt} \leq rx \left(\frac{1 + cx}{1 + cx + ky} \right) - d_1 x$$
$$\leq rx \left(\frac{1 + cx}{1 + ky} \right) - d_1 x$$
$$\leq \left[\frac{r \left\{ 1 + c \left(\frac{r - d_1}{a_1} \right) \right\}}{1 + ky} - d_1 \right] x.$$

So, for large t, $\frac{dx}{dt} < 0$, as $y(t) \to \infty$ as $t \to \infty$ and this implies $\lim\limits_{t \to \infty} x(t) = 0$. Hence the theorem is proved. $\square$

Remark 4.4. *The provision of sufficient additional food increases the predator population to large extent, where it causes a massive predation burden on prey species and drives them to extinction.*

Next on the other side, for the extinction of predator population, we have the following theorem:

Theorem 4.5. *If $r > d_1$ and $0 < \mu A < \frac{bd_2}{c_1 a_2} - \left(\frac{r-d_1}{a_1}\right)$, then $\lim_{t \to \infty} y(t) = 0$.*

Proof. We have from the second equation of (2.5), for large time t:

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{c_1 a_2 (x + \mu A) y}{b + \alpha \mu A + x} - d_2 y \\
&\leq \left[\frac{c_1 a_2 (x + \mu A)}{b} - d_2\right] y \\
&\leq \left[\frac{c_1 a_2}{b}\left(\frac{r - d_1}{a_1} + \mu A\right) - d_2\right] y.
\end{aligned}
$$

Therefore, $\lim_{t \to \infty} y(t) = 0$ for $r > d_1$ and $0 < \mu A < \frac{bd_2}{c_1 a_2} - \left(\frac{r-d_1}{a_1}\right)$. This proves the theorem. $\square$

Different types of extinction scenarios are presented in Fig.2. In Fig.2a, (i) $r < d_1$ and (ii) $\mu A(c_1 a_2 - \alpha d_2) < bd_2$; hence both the prey and predator population go extinct. But for Fig.2b, (i) $r < d_1$ and (ii) $\mu A(c_1 a_2 - \alpha d_2) > bd_2$ which imply that prey species goes extinct and predator population survives with the consumption of additional food sources. However, even if $r > d_1$, prey species may experience extinction possibility depending upon the effective quantity of additional food to predator. If $\mu A(c_1 a_2 - \alpha d_2) > bd_2 + d_2\left(\frac{r-d_1}{a_1}\right)$, predator biomass enlarges in size and prey species goes extinct due to massive predation pressure as illustrated in Fig.2c. But for Fig.2d, $0 < \mu A < \frac{bd_2}{c_1 a_2} - \left(\frac{r-d_1}{a_1}\right)$; hence predator population goes extinct and prey population grows towards the environmental carrying capacity.

5. Local stability

Here we derive some parametric conditions under which various types of steady state solutions of system (2.5) are emerged, and further we also explore the stability conditions around these steady states. Since the condition $\mu A < \frac{bd_2}{c_1 a_2}$ is sufficient for boundedness of the system solutions, we, therefore restrict our analysis under this parametric constraint.

5.1. Equilibrium Points

System (2.5) always possesses two following boundary equilibrium points:
(i) Trivial equilibrium point: $E_0(0, 0)$ and
(ii) Axial equilibrium point: $E_a\left(\frac{r-d_1}{a_1}, 0\right)$, provided $r > d_1$.
But it is biologically intuitive that higher mortality rate is always harmful for any kind of species and drives that species to extinction, so throughout this manuscript, we assume that $r > d_1$. The interior equilibrium point $E_c(x^*, y^*)$ satisfies the two non-trivial prey and predator nullclines together in the interior of the first quadrant:
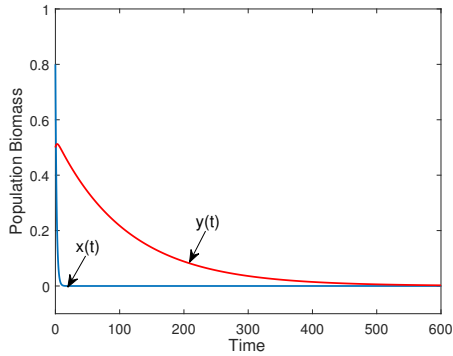
$$
\Phi(x, y) \equiv r\left(\frac{1 + cx}{1 + cx + ky}\right) - d_1 - a_1 x - \frac{a_2 y}{b + \alpha \mu A + x} = 0, \tag{5.1}
$$

$$
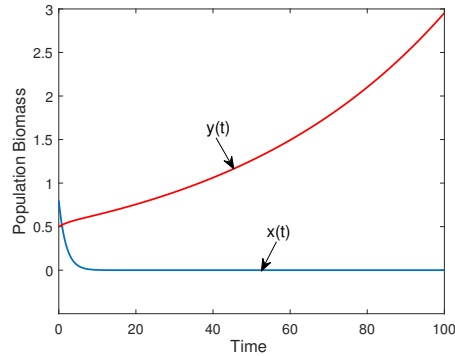\Psi(x, y) \equiv \frac{c_1 a_2 (x + \mu A)}{b + \alpha \mu A + x} - d_2 = 0. \tag{5.2}
$$

From equation (5.2), we get the explicit expression of x^* which is given by

$$
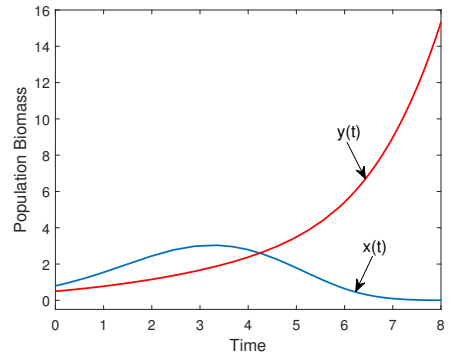x^* = \frac{(d_2 \alpha - c_1 a_2)\mu A + bd_2}{c_1 a_2 - d_2},
$$

provided $c_1 a_2 - d_2 \neq 0$. For positive x^*, any of the following two sets of conditions must be satisfied:
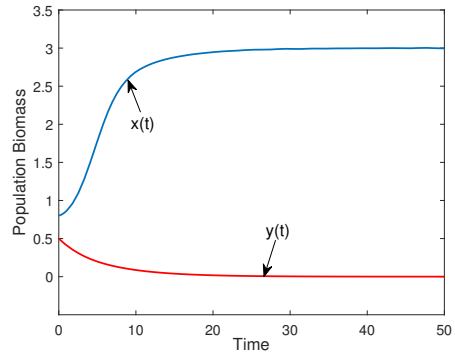
(a) { $r = 0.5$, $d_1 = 0.6$, $a_1 = 0.2$, $k = 3$, $c = 2$, $a_2 = 0.7$, $b = 2.5$, $\alpha = 0.2$, $\mu = 0.4$, $A = 3.5$, $c_1 = 0.4$, $d_2 = 0.15$}

(b) { $r = 0.5$, $d_1 = 0.6$, $a_1 = 0.2$, $k = 3$, $c = 2$, $a_2 = 0.7$, $b = 2.5$, $\alpha = 0.1$, $\mu = 0.4$, $A = 3.5$, $c_1 = 0.45$, $d_2 = 0.15$}

(c) { $r = 2.5$, $d_1 = 0.6$, $a_1 = 0.2$, $k = 3$, $c = 2$, $a_2 = 0.7$, $b = 2.5$, $\alpha = 0.1$, $\mu = 1$, $A = 7$, $c_1 = 0.45$, $d_2 = 0.15$}

(d) { $r = 1.5$, $d_1 = 0.6$, $a_1 = 0.3$, $k = 3$, $c = 2$, $a_2 = 0.5$, $b = 2.5$, $\alpha = 0.1$, $\mu = 0.3$, $A = 1.5$, $c_1 = 0.3$, $d_2 = 0.25$}

Figure 2: Extinction scenarios for different parametric set. **(a):** Both prey and predator go extinct. **(b):** Prey population goes extinct and predator biomass increases in size. **(c):** Predator biomass increases in size and prey population goes extinct due to massive predation pressure. **(d):** Predator species goes extinct and prey biomass increases to the environmental carrying capacity.

1. $\frac{c_1 a_2 \mu A}{b + \alpha \mu A} < d_2 < c_1 a_2$,

2. $c_1 a_2 < d_2 < \frac{c_1 a_2 \mu A}{b + \alpha \mu A}$.

But the feasibility conditions mentioned in the second set violate the boundedness criterion stated in Theorem 3.2. Hence in the rest of the article, we consider only the first set of feasibility condition. Now by using the value of x^* in the first equation of system (2.5), we derive the following quadratic equation in y, say $f(y) = 0$, where

$$f(y) \equiv Py^2 + Qy + R. \tag{5.3}$$

Here,

$P = a_2 k$,
$Q = a_2(1 + cx^*) + k(b + \alpha \mu A + x^*)(d_1 + a_1 x^*)$ and
$R = -(1 + cx^*)(b + \alpha \mu A + x^*)(r - d_1 - a_1 x^*)$.

Clearly $P > 0$, $Q > 0$ and hence the existence of positive root of equation (5.3), say y^*, depends upon the sign of R. If $R > 0$, no positive root occurs but for $R < 0$, exactly one positive root exists. Therefore, the
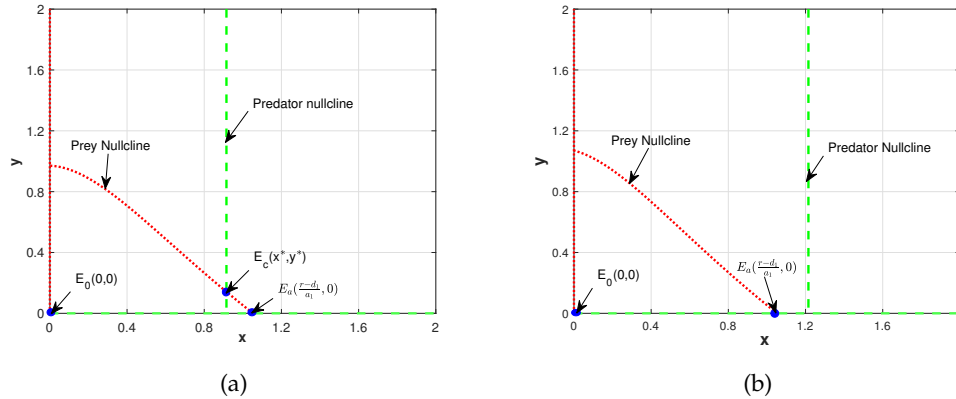
Figure 3: Prey and Predator nullclines and the existence of equilibrium points. Trivial and axial equilibrium points always exist, whereas the existence of interior equilibrium point depends upon the condition: $(d_2\alpha - c_1a_2)\mu A + bd_2 < (c_1a_2 - d_2)\left(\frac{r-d_1}{a_1}\right)$. **(a):** $(b = 0.4)$ Interior equilibrium point exists. **(b):** $(b = 0.5)$ Interior equilibrium point fails to exist. Other parametric values are given as $r = 0.5$, $c = 0.5$, $k = 1$, $d_1 = 0.1$, $a_1 = 0.38$, $a_2 = 0.08$, $\alpha = 0.7$, $\mu = 0.3$, $A = 0.5$, $c_1 = 0.5$, $d_2 = 0.03$.

interior equilibrium $E_c(x^*, y^*)$, where y^* is a positive root of the equation (5.3) exists provided $R < 0$, that means when $x^* < \frac{r-d_1}{a_1}$ i.e., $(d_2\alpha - c_1a_2)\mu A + bd_2 < (c_1a_2 - d_2)\left(\frac{r-d_1}{a_1}\right)$. One can anticipate these parametric restrictions by analyzing the predator-prey nullclines, as plotted in Fig.3. Clearly the non-trivial predator nullcline is a line given by $x = \frac{(d_2\alpha - c_1a_2)\mu A + bd_2}{c_1a_2 - d_2}$ which is parallel to the y-axis and the non-trivial prey nullcline is a curve that cuts the x-axis at $\left(\frac{r-d_1}{a_1}, 0\right)$. Then the intersection between these two non-trivial nullclines will be possible if $(d_2\alpha - c_1a_2)\mu A + bd_2 < (c_1a_2 - d_2)\left(\frac{r-d_1}{a_1}\right)$.

Now we investigate the stability behaviour of the system around these equilibrium states by examining the sign of the eigenvalues of the corresponding Jacobian matrix. The Jacobian matrix $J(x, y)$ of system (2.5) is given by

$$J(x, y) = \begin{pmatrix} \frac{r(1+ky)(1+2cx)+rc^2x^2}{(1+cx+ky)^2} - d_1 - 2a_1x - \frac{(b+\alpha\mu A)a_2y}{(b+\alpha\mu A+x)^2} & -\frac{krx(1+cx)}{(1+cx+ky)^2} - \frac{a_2x}{b+\alpha\mu A+x} \\ \frac{c_1a_2(b+(\alpha-1)\mu A)y}{(b+\alpha\mu A+x)^2} & \frac{c_1a_2(x+\mu A)}{b+\alpha\mu A+x} - d_2 \end{pmatrix}. \tag{5.4}$$

At the trivial equilibrium point $E_0(0, 0)$, the Jacobian matrix reduces to

$$J_0 = \begin{pmatrix} r - d_1 & 0 \\ 0 & \frac{c_1a_2\mu A}{b+\alpha\mu A} - d_2 \end{pmatrix}.$$

Since one of the eigenvalues is $r - d_1 > 0$, $E_0(0, 0)$ is unstable.

Theorem 5.1. *The trivial equilibrium point E_0 is always unstable.*

Now the Jacobian matrix at the predator free equilibrium point $E_a\left(\frac{r-d_1}{a_1}, 0\right)$ is given by

$$J_a = \begin{pmatrix} -(r - d_1) & -(r - d_1)\left[\frac{kr}{a_1+c(r-d_1)} + \frac{a_2}{a_1b+a_1\alpha\mu A+(r-d_1)}\right] \\ 0 & \frac{c_1a_2(r-d_1+a_1\mu A)}{a_1b+a_1\alpha\mu A+r-d_1} - d_2 \end{pmatrix}.$$

Then the eigenvalues are $-(r - d_1)$ and $\frac{c_1 a_2 (r - d_1 + a_1 \mu A)}{a_1 b + a_1 \alpha \mu A + r - d_1} - d_2$. Now as $(r - d_1) > 0$, the first eigenvalue is negative in sign, so E_a is stable if $\frac{c_1 a_2 (r - d_1 + a_1 \mu A)}{a_1 b + a_1 \alpha \mu A + r - d_1} - d_2 < 0$, i.e., $(\alpha d_2 - c_1 a_2) \mu A + b d_2 > (c_1 a_2 - d_2) \left(\frac{r - d_1}{a_1} \right)$. Otherwise it will be unstable.

Theorem 5.2. *The axial equilibrium point $E_a \left(\frac{r - d_1}{a_1}, 0 \right)$ is stable if $(\alpha d_2 - c_1 a_2) \mu A + b d_2 > (c_1 a_2 - d_2) \left(\frac{r - d_1}{a_1} \right)$ and unstable if $(\alpha d_2 - c_1 a_2) \mu A + b d_2 < (c_1 a_2 - d_2) \left(\frac{r - d_1}{a_1} \right)$.*

Remark 5.3. *Existence of interior equilibrium point E_c violates the stability condition of axial equilibrium point E_a. Therefore, the existence of E_c is enough to stop the predator species from extinction.*

Now the Jacobian matrix at the coexistence equilibrium point $E_c(x^*, y^*)$ is given by

$$J_c = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = \left[\frac{a_2 x^* y^*}{(b + \alpha \mu A + x^*)^2} + \frac{rck x^* y^*}{(1 + cx^* + ky^*)^2} - a_1 x^* \right],$$

$$a_{12} = -x^* \left[\frac{kr(1 + cx^*)}{(1 + cx^* + ky^*)^2} + \frac{a_2}{b + \alpha \mu A + x^*} \right],$$

$$a_{21} = \frac{a_2 c_1 (b - \mu A (1 - \alpha)) y^*}{(b + \alpha \mu A + x^*)^2} \text{ and}$$

$$a_{22} = 0.$$

Then the characteristic equation of the above matrix is

$$\lambda^2 - a_{11} \lambda - a_{21} a_{12} = 0.$$

According to the Routh–Hurwitz criteria, $a_{11} < 0$ and $-a_{12} a_{21} > 0$ ensure that the two eigenvalues of J_c are negative or having negative real parts. But as a_{12} is always negative, a_{21} should be positive for stability, i.e., $b > (1 - \alpha) \mu A$.

Theorem 5.4. *The coexistence equilibrium point E_c is locally asymptotically stable if (i) $r < \frac{1}{ck y^*} \left(a_1 - \frac{a_2 y^*}{(b + \alpha \mu A + x^*)^2} \right) (1 + cx^* + ky^*)^2$ and (ii) $b > (1 - \alpha) \mu A$.*

6. Global Stability

In this section, we will look at the global stability analysis of all possible equilibrium points arising from system (2.5).

Theorem 6.1. *The trivial equilibrium E_0 is globally asymptotically stable if $r < d_1$ and $\mu A (c_1 a_2 - \alpha d_2) < b d_2$.*

Proof. First we choose a Lyapunov function as:

$$V_0 = x + \frac{1}{c_1} y.$$

Then

$$\frac{dV_0}{dt} = \frac{dx}{dt} + \frac{1}{c_1} \frac{dy}{dt}.$$

Now putting the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from system (2.5), we have

$$\frac{dV_0}{dt} = rx\left(\frac{1+cx}{1+cx+ky}\right) - d_1x - a_1x^2 + \frac{a_2\mu Ay}{b+\alpha\mu A + x} - \frac{d_2}{c_1}y$$

$$\leq (r-d_1)x - a_1x^2 + \left(\frac{a_2\mu A}{b+\alpha\mu A} - \frac{d_2}{c_1}\right)y \quad \left[\text{as } rx\left(\frac{1+cx}{1+cx+ky}\right) \leq rx \text{ and } \frac{a_2\mu Ay}{b+\alpha\mu A + x} \leq \frac{a_2\mu Ay}{b+\alpha\mu A}\right]$$

So, $\frac{dV_0}{dt} < 0$, provided $r < d_1$ and $\mu A(c_1a_2 - \alpha d_2) < bd_2$. Also, $\left.\frac{dV_0}{dt}\right|_{E_0} = 0$.

Therefore, by Lyapunov theorem [12], E_0 is globally asymptotically stable for $r < d_1$ and $\mu A(c_1a_2 - \alpha d_2) < bd_2$. $\square$

Theorem 6.2. *The axial equilibrium E_a of system (2.5) is globally asymptotically stable if $\frac{(\alpha d_2 - c_1a_2)\mu A + bd_2}{c_1a_2} > \frac{r-d_1}{a_1}$.*

Proof. From the first equation of system (2.5),

$$\frac{dx}{dt} \leq (r-d_1)x - a_1x^2 = (r-d_1)x\left[1 - \frac{x}{\left(\frac{r-d_1}{a_1}\right)}\right].$$

$$\therefore \limsup_{t\to\infty} x(t) \leq \frac{r-d_1}{a_1}.$$

Now for large time t,

$$\frac{dy}{dt} = \left(\frac{c_1a_2(x+\mu A)}{b+\alpha\mu A + x} - d_2\right)y$$

$$\leq \left(\frac{c_1a_2(x+\mu A)}{b+\alpha\mu A} - d_2\right)y$$

$$\leq -\frac{1}{(b+\alpha\mu A)}\left[(b+\alpha\mu A)d_2 - c_1a_2\left(\frac{r-d_1}{a_1} + \mu A\right)\right].$$

Therefore, $\lim_{t\to\infty} y(t) = 0$ if $\frac{(\alpha d_2 - c_1a_2)\mu A + bd_2}{c_1a_2} > \frac{r-d_1}{a_1}$. Hence the theorem. $\square$

Theorem 6.3. *Interior equilibrium point E_c is globally asymptotically stable if $r - d_1 > a_2c_1 - d_2 > 0$, $(b+\alpha\mu A)d_2 > a_2c_1\mu A$ and $\min\left\{\frac{2a_1 - rc - (r-d_1)c}{c(c_1a_2 - d_2)}, \frac{(a_1 - rc)(b+\alpha\mu A) - (r-d_1)}{c_1a_2 - d_2 + a_2cc_1\mu A}\right\} > 0$.*

Proof. Let $F_1(x,y)$ and $F_2(x,y)$ represent the R.H.S of system (2.5). Next we assume the Dulac function $G(x,y)$ as:

$$G(x,y) = x^{-1}y^\beta (1+cx+ky)(b+\alpha\mu A + x),$$

where β is to be calculated later. Now we have:

$$D(x,y) = \frac{\partial}{\partial x}\{G(x,y)F_1(x,y)\} + \frac{\partial}{\partial y}\{G(x,y)F_2(x,y)\}$$

$$\leq x^{-1}y^\beta\{g_1(x,\beta)ky + g_2(x,\beta)\},$$

(6.1)

where

$$g_1(x,\beta) = \left[\{(\beta+2)(a_2c_1 - d_2) - (b+\alpha\mu A)a_1\}x - 2a_1x^2 + (\beta+2)a_2c_1\mu A - (b+\alpha\mu A)(\beta+2)d_2\right]$$

and

$$g_2(x,\beta) = [\{(r-d_1)c - 2a_1 + (\beta+1)c(c_1a_2 - d_2) + rc\}x^2 + \{(r-d_1) - a_1(b+\alpha\mu A)$$
$$+ (\beta+1)(c_1a_2 - d_2) + (1+\beta)a_2cc_1\mu A + rc(b+\alpha\mu A)\}x + a_2c_1\mu A(1+\beta)$$
$$- (1+\beta)(b+\alpha\mu A)d_2].$$

So,

$$g_2(x,\beta) - g_1(x,\beta) = [(d_2 - a_2c_1) + (r-d_1) + (1+\beta)a_2cc_1\mu A]x + rcx(b+\alpha\mu A + x) - a_2c_1\mu A$$
$$+ (b+\alpha\mu A)d_2 + \{(1+\beta)(a_2c_1 - d_2) + (r-d_1)\}cx^2.$$

$\therefore g_2 - g_1 > 0$ if $(d_2 - a_2c_1) + (r-d_1) > 0$ i.e., $r - d_1 > a_2c_1 - d_2 > 0$ and
$-a_2c_1\mu A + (b+\alpha\mu A)d_2 > 0$, i.e., $(b+\alpha\mu A)d_2 > a_2c_1\mu A$.

$$\therefore g_2(x,\beta) > g_1(x,\beta).$$

So, $D(x,y) < x^{-1}y^\beta(1+ky)g_2(x,\beta)$. Thus, $D(x,y) < 0$ for $(x,y) \in \mathbb{R}_+^2$, if

$$g_2(x,\beta) < 0, \quad for \ x \in [0,\infty). \tag{6.2}$$

Therefore, it is sufficient to find a β such that the equation (6.2) holds. Now (6.2) is satisfied if

$$\{(r-d_1)c - 2a_1 + (\beta+1)c(c_1a_2 - d_2) + rc\} < 0,$$
$$\{(r-d_1) - a_1(b+\alpha\mu A) + (\beta+1)(c_1a_2 - d_2) + (1+\beta)a_2cc_1\mu A + rc(b+\alpha\mu A)\} < 0 \quad and$$
$$(1+\beta)\{a_2c_1\mu A - (b+\alpha\mu A)d_2\} < 0.$$

From the first and second conditions, we obtain $\beta + 1 < \frac{2a_1 - rc - (r-d_1)c}{c(c_1a_2 - d_2)}$ and $\beta + 1 < \frac{(a_1 - rc)(b+\alpha\mu A) - (r-d_1)}{c_1a_2 - d_2 + a_2cc_1\mu A}$ respectively, and from the third one, we get $\beta + 1 > 0$ as $\{a_2c_1\mu A - (b+\alpha\mu A)d_2\} < 0$. Therefore, one can find a β if the following parametric restriction is satisfied:

$$0 < \min\left\{\frac{2a_1 - rc - (r-d_1)c}{c(c_1a_2 - d_2)}, \frac{(a_1 - rc)(b+\alpha\mu A) - (r-d_1)}{c_1a_2 - d_2 + a_2cc_1\mu A}\right\}.$$

Therefore, there exists β such that $D(x,y) < 0$ for $(x,y) \in \mathbb{R}_+^2$. Hence by the Bendixson-Dulac theorem, system (2.5) has no periodic orbits in the positive quadrant. $\square$

7. Uniform Persistent

In evolutionary biology, persistence refers to the long-term survival of all interacting species which exist initially. Here, we prove the permanence of system (2.5) by utilizing average Lyapunov function [28, 34].

Theorem 7.1. *System (2.5) is uniformly persistent if* $(c_1a_2 - d_2)\left(\frac{r-d_1}{a_1}\right) > (\alpha d_2 - c_1a_2)\mu A + bd_2$.

Proof. For $(x,y) \in \mathbb{R}_+^2$, we consider the following average Lyapunov function:

$$\mathcal{L}(x,y) = N^{\sigma_1}P^{\sigma_2},$$

where σ_1 and σ_2 are some positive constants. Now, differentiating with respect to t, we obtain:

$$\frac{\dot{\mathcal{L}}}{\mathcal{L}} = \sigma_1\frac{\dot{x}}{x} + \sigma_2\frac{\dot{y}}{y}$$
$$= \sigma_1\left\{\frac{r(1+cx)}{1+cx+ky} - d_1 - a_1x - \frac{a_2y}{b+\alpha\mu A + x}\right\} + \sigma_2\left\{\frac{c_1a_2(x+\mu A)}{b+\alpha\mu A + x} - d_2\right\}.$$
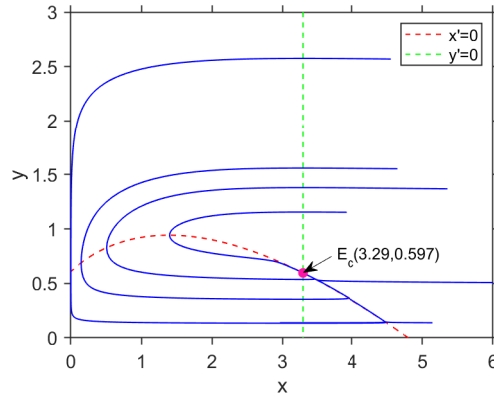
Figure 4: Phase portrait for the stable coexistence equilibrium point $E_c(3.29, 0.597)$ with different initial population biomass. Parameter values are: $r = 1.6, c = 2, k = 3, d_1 = 0.4, a_1 = 0.25, a_2 = 0.7, b = 2.15, \alpha = 0.4, \mu = 0.25, A = 4, c_1 = 0.35, d_2 = 0.18$.

Let us denote the right hand side expression by $\Delta(x, y)$. The system will be uniformly persistent if $\Delta(x, y) > 0$ at the boundary equilibrium points $(0, 0)$ and $\left(\frac{r-d_1}{a_1}, 0\right)$ for some $\sigma_1 > 0$ and $\sigma_2 > 0$. Computing Δ at the prescribed boundary equilibrium points, we obtain

$$\Delta(0, 0) = \sigma_1(r - d_1) + \sigma_2\left(\frac{c_1 a_2 \mu A}{b + \alpha \mu A} - d_2\right) \quad \text{and}$$

$$\Delta\left(\frac{r - d_1}{a_1}, 0\right) = \sigma_2\left\{\frac{c_1 a_2\left(\frac{r-d_1}{a_1} + \mu A\right)}{b + \alpha \mu A + \frac{r-d_1}{a_1}} - d_2\right\}.$$

Since $r > d_1$, $\Delta(0, 0) > 0$ for some suitable choices of $\sigma_i > 0$, $i = 1, 2$. But $\Delta\left(\frac{r-d_1}{a_1}, 0\right) > 0$, provided $\frac{c_1 a_2\left(\frac{r-d_1}{a_1} + \mu A\right)}{b + \alpha \mu A + \frac{r-d_1}{a_1}} - d_2 > 0$, i.e., $(c_1 a_2 - d_2)\left(\frac{r-d_1}{a_1}\right) > (\alpha d_2 - c_1 a_2)\mu A + b d_2$.

Thus $\Delta > 0$ at the boundary equilibrium points under the above stated conditions for some $\sigma_i > 0$, $i = 1, 2$. Hence system (2.5) is uniformly persistent. $\square$

Remark 7.2. *Persistency implies instability of the boundary equilibria.*

8. Local Bifurcations

In this section, we investigate whether the system has any local bifurcations that may result in various sorts of dynamical implications such as stability exchange from one state to another, emergence of new steady state, appearance of limit cycle, and so on. Such local bifurcations can also be analyzed from the positions of nontrivial nullclines.

8.1. Transcritical bifurcation

Theorem 8.1. *System (2.5) undergoes a transcritical bifurcation at the bifurcation threshold*

$$A^{[TC]} = \frac{1}{\mu}\left[\frac{b d_2}{c_1 a_2 - \alpha d_2} + \left(\frac{d_2 - c_1 a_2}{c_1 a_2 - \alpha d_2}\right)\left(\frac{r - d_1}{a_1}\right)\right]$$

provided $(c_1 a_2 - \alpha d_2) \neq 0$.

Proof. We apply Sotomayor's theorem [12] to prove this theorem. For $A = A^{[TC]}$, the Jacobian matrix at E_a is evaluated as:

$$J\left(E_a; A = A^{[TC]}\right) = \begin{pmatrix} -(r - d_1) & -(r - d_1)\left[\frac{kr}{a_1 + c(r - d_1)} + \frac{a_2}{a_1 b + a_1 \alpha \mu A^{[TC]} + (r - d_1)}\right] \\ 0 & 0 \end{pmatrix}. \tag{8.1}$$

Clearly, zero is an eigen value of this Jacobian matrix. Now corresponding to the zero eigenvalue, the eigenvector of $J\left(E_a; A = A^{[TC]}\right)$ and $\left[J\left(E_a; A = A^{[TC]}\right)\right]^T$ are obtained as $V = \begin{pmatrix} v_1 \\ 1 \end{pmatrix}$ and $W = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, where $v_1 = -\left[\frac{kr}{a_1 + c(r - d_1)} + \frac{a_2}{a_1 b + a_1 \alpha \mu A^{[TC]} + (r - d_1)}\right]$. Utilizing the identical formulae from [12], we evaluate the transversality criteria for transcritical bifurcation given as:

$$\Delta_1 = W^T\left[F_A\left(E_a; A = A^{[TC]}\right)\right] = 0,$$

$$\Delta_2 = W^T\left[DF_A\left(E_a; A = A^{[TC]}\right)V\right] = \frac{c_1 a_2 \mu \left\{b + (1 - \alpha)\left(\frac{r - d_1}{a_1}\right)\right\}}{\left(b + \alpha \mu A^{[TC]} + \frac{r - d_1}{a_1}\right)^2} \neq 0,$$

$$\Delta_3 = W^T\left[D^2F\left(E_a; A = A^{[TC]}\right)(V, V)\right] = \frac{c_1 a_2 \left\{b + (\alpha - 1)\mu A^{[TC]}\right\}}{\left(b + \alpha \mu A^{[TC]} + \frac{r - d_1}{a_1}\right)^2} v_1 \neq 0.$$

Here $F \equiv \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, where F_1 and F_2 are defined in (2.5). Since these satisfy all the requirements of Sotomayor's theorem, a transcritical bifurcation occurs at $A = A^{[TC]}$ through which the system exchanges its stability from E_a to E_c or vice versa. $\square$

Theorem 8.2. *For each of the bifurcation parameters r, μ and c_1, system (2.5) undergoes a transcritical bifurcation respectively at $r^{[TC]} = d_1 + \frac{1}{c_1 a_2 - d_2}\{d_2(a_1 b + \alpha a_1 \mu A) - c_1 a_1 a_2 \mu A\}$, provided $(c_1 a_2 - d_2) \neq 0$; $\mu^{[TC]} = \frac{1}{A}\left[\frac{bd_2}{c_1 a_2 - \alpha d_2} + \left(\frac{d_2 - c_1 a_2}{c_1 a_2 - \alpha d_2}\right)\left(\frac{r - d_1}{a_1}\right)\right]$, provided $(c_1 a_2 - \alpha d_2) \neq 0$ and $c_1^{[TC]} = \frac{d_2(a_1 b + a_1 \alpha \mu A + r - d_1)}{a_2(r - d_1 + a_1 \mu A)}$.*

Proof. The proofs are same as above and so omitted. $\square$

8.2. Hopf-bifurcation

To explore the instability of the interior equilibrium point $E_c(x^*, y^*)$, we consider the birth rate, r as a free parameter. The characteristic equation of the Jacobian matrix J_c can be written in terms of r as

$$\lambda^2 - T(r)\lambda + D(r) = 0, \tag{8.2}$$

where $T(r)$ and $D(r)$ are smooth functions of r, generated by trace and determinant respectively of the said matrix given as:

$$T = \left[\frac{a_2 x^* y^*}{(b + \alpha \mu A + x^*)^2} + \frac{rckx^* y^*}{(1 + cx^* + ky^*)^2} - a_1 x^*\right]$$

and

$$D = a_2 c_1 x^* y^* \left[\frac{kr(1 + cx^*)}{(1 + cx^* + ky^*)^2} + \frac{a_2}{b + \alpha \mu A + x^*}\right]\left[\frac{b - (1 - \alpha)\mu A}{(b + \alpha \mu A + x^*)^2}\right].$$

If equation (8.2) has a pair of imaginary roots, the stability of E_c depends upon the sign of their real parts. For negative real parts, E_c is locally asymptotically stable and for positive real part of at least one root, E_c is unstable. Therefore, stability changes when equation (8.2) creates purely imaginary roots. In relation to this type of stability change, we have the following theorem.

Theorem 8.3. *(Hopf-Bifurcation Theorem [13]) If $T(r)$ and $D(r)$ are differentiable functions of r in a neighbourhood $N_\epsilon\left(r^{[H]}\right)$ ($\epsilon > 0$) such that equation (8.2) has a pair of imaginary roots, say $\lambda = q_1(r) + iq_2(r)$ with $q_1(r), q_2(r) \in \mathbb{R}$ which become purely imaginary at $r = r^{[H]}$ and $\left[\frac{dq_1(r)}{dr}\right]_{r=r^{[H]}} \neq 0$, then a Hopf-bifurcation occurs around E_c at $r = r^{[H]}$.*

Next we will show that system (2.5) exhibits Hopf-bifurcation at $r = r^{[H]}$.

Theorem 8.4. *System (2.5) undergoes a Hopf-bifurcation around the coexistence equilibrium point E_c when the bifurcating parameter r passes through its threshold value $r = r^{[H]}$ provided $T(r^{[H]}) = 0$, $D(r^{[H]}) > 0$ and $\left[\frac{dT}{dr}\right]_{r=r^{[H]}} \neq 0$. The threshold value of r can be obtained by solving the equation:*

$$\frac{a_2 y^*}{(b + \alpha\mu A + x^*)^2} + \frac{rcky^*}{(1 + cx^* + ky^*)^2} - a_1 = 0,$$

where $x^* = \frac{(d_2\alpha - c_1 a_2)\mu A + bd_2}{c_1 a_2 - d_2}$ and y^* satisfies (5.3).

Proof. At $r = r^{[H]}$, it is given that $T(r^{[H]}) = 0$, $D(r^{[H]}) > 0$. Then the equation (8.2) has two purely imaginary roots $\lambda_1 = i\sqrt{D(r^{[H]})}$ and $\lambda_2 = -i\sqrt{D(r^{[H]})}$. Now if $r \in N_\epsilon\left(r^{[H]}\right)$ ($\epsilon > 0$) the roots have of the form $\lambda_1 = q_1(r) + iq_2(r)$ and $\lambda_2 = q_1(r) - iq_2(r)$, where $q_1(r)$ and $q_2(r)$ are real valued functions of r. Therefore, by the help of the theorem of Hopf-bifurcation [13], we can easily say that system (2.5) shifts its stability via Hopf-bifurcation at $r = r^{[H]}$ provided the transversality conditions

$$\left[\frac{d}{dr}(Re\lambda_i(r))\right]_{r=r^{[H]}} = \left[\frac{dq_1(r)}{dr}\right]_{r=r^{[H]}} \neq 0, \quad i = 1, 2,$$

are satisfied. Putting $\lambda_1(r) = q_1(r) + iq_2(r)$ in (8.2) and differentiating with respect to r, we have

$$2(q_1(r) + iq_2(r))(\dot{q}_1(r) + i\dot{q}_2(r)) - T(r)(\dot{q}_1(r) + i\dot{q}_2(r)) - \dot{T}(k)(q_1(r) + iq_2(r)) + \dot{D}(r) = 0.$$

Comparing the real and imaginary parts, we obtain:

$$\dot{q}_1(2q_1 - T) + \dot{q}_2(-2q_2) - \dot{T}q_1 + \dot{D} = 0 \implies \dot{q}_1 X_1 - \dot{q}_2 X_2 + X_3 = 0, \tag{8.3}$$

$$\dot{q}_1(2q_2) + \dot{q}_2(2q_1 - T) - \dot{T}q_2 = 0 \implies \dot{q}_1 X_2 + \dot{q}_2 X_1 + X_4 = 0, \tag{8.4}$$

where $X_1 = (2q_1 - T)$, $X_2 = 2q_2$, $X_3 = -\dot{T}q_1 + \dot{D}$ and $X_4 = -\dot{T}q_2$.
Solving the equations (8.3) and (8.4), we get

$$\dot{q}_1 = -\frac{(X_1 X_3 + X_2 X_4)}{X_1^2 + X_2^2}. \tag{8.5}$$

Now at $r = r^{[H]}$, two cases arise:
Case 1: When $q_1 = 0$, $q_2 = \sqrt{D}$. Then $X_1 = 0$, $X_2 = 2\sqrt{D}$, $X_3 = \dot{D}$ and $X_4 = -\dot{T}\sqrt{D}$. Hence from (8.5), we get

$$[\dot{q}_1]_{r=r^{[H]}} = \left[\frac{dq_1(r)}{dr}\right]_{r=r^{[H]}} = \frac{1}{2}\left[\frac{dT(r)}{dr}\right]_{r=r^{[H]}} \neq 0,$$

since $T(r^{[H]}) = 0$, $D(r^{[H]}) > 0$ and $\left[\frac{dT}{dr}\right]_{r=r^{[H]}} \neq 0$.
Case 2: When $q_1 = 0$, $q_2 = -\sqrt{D}$. Then $X_1 = 0$, $X_2 = -2\sqrt{D}$, $X_3 = \dot{D}$ and $X_4 = \dot{T}\sqrt{D}$. Hence from (8.5), we get

$$[\dot{q}_1]_{r=r^{[H]}} = \left[\frac{dq_1(r)}{dr}\right]_{r=r^{[H]}} = \frac{1}{2}\left[\frac{dT(r)}{dr}\right]_{r=r^{[H]}} \neq 0,$$

since $T(r^{[H]}) = 0$, $D(r^{[H]}) > 0$ and $\left[\frac{dT}{dr}\right]_{r=r^{[H]}} \neq 0$. Hence the theorem. $\square$

Remark 8.5. *Similarly, system (2.5) may experience Hopf-bifurcations with respect to k (fear effect) and c (carry-over effect).*

Now we conduct some numerical simulations to illustrate these local bifurcations consistent with the analytical findings. We, therefore consider the following parametric set which is although hypothetical but biologically feasible.

Parametric Values: Set 1											
r	c	k	d_1	a_1	a_2	b	α	μ	A	c_1	d_2
1.6	2	3	0.4	0.25	0.7	2.15	0.4	0.25	4	0.35	0.18

Table 2: Parameter values used for numerical simulation of system (2.5)

.

The feasibility criteria $\frac{c_1 a_2 \mu A}{b + \alpha \mu A} < d_2 < c_1 a_2$ and $(d_2 \alpha - c_1 a_2)\mu A + bd_2 < (c_1 a_2 - d_2)\left(\frac{r - d_1}{a_1}\right)$ are satisfied for the existence of E_c which is obtained as $E_c(3.29, 0.5965)$ from the above mentioned parametric set (Table 2). Also the coexistence state $E_c(3.29, 0.5965)$ is globally asymptotically stable which is depicted in Fig.4.
Now, if we gradually increase the level of additional food from $A = 4$ to $A = 6.905368$, the system exhibits oscillatory behaviour around the coexistence equilibrium point, E_c. Again, if we decrease A to $A = 1.734104$, the predator species can not survive in the ecosystem but the prey species can survive properly in absence of predation and approach to its carrying capacity, i.e., the system undergoes a transcritical bifurcation at $A^{[TC]} = 1.734104$. Also, it should be noted that increasing value of A have a positive impact on predator biomass, which is biologically intuitive. However, the increased predator biomass generates additional predation burden on prey species, resulting in a decrease in prey biomass. All of these scenarios are depicted in Figs.5(a),(b). Similar types of qualitative behaviour are observed for the bifurcating parameter μ (see Figs.5(c),(d)). A transcritical bifurcation occurs at $\mu^{[TC]} = 0.1084$ and a Hopf-bifurcation appears at $\mu^{[H]} = 0.4316$.
In every prey-predator model, the reproduction rate r plays a crucial role in system dynamics. So, it is quite reasonable to study the impact of this parameter on the proposed model. For this purpose, we vary the parameter r while keeping the rest of the parameters fixed as in Table 2. If $r < d_1 = 0.4$, both species go extinct from the environment and the system becomes stable at E_0. If r exceeds the value $r^{[TC_1]} = 0.4$, prey species can survive in the environment, but predator species cannot. Therefore, the system experiences a transcritical bifurcation at $r^{[TC_1]} = 0.4$. Another transcritical bifurcation occurs at $r^{[TC_2]} = 1.223$, through which the system exchanges its stability with the interior equilibrium point E_c. Higher value of r drives the system oscillatory around E_c. More precisely the system undergoes a Hopf-bifurcation at $r^{[H]} = 3.018$ (see Figs.6(a),(b)). We also analyze the change of system behaviour by varying the conversion coefficient c_1. In this case, a transcritical bifurcation occurs at $c_1^{[TC]} = 0.3259$ and a Hopf-bifurcation appears at $c_1^{[H]} = 0.424$ (see Figs.6(c),(d)).
Next, we investigate the qualitative change of system behaviour in $r - A$ parametric plane as depicted in Fig.7. A Hopf-bifurcation curve and two transcritical bifurcation curves are shown in this diagram. The whole $r - A$ parametric plane is divided by these three bifurcation curves into four sub-regions that are referred to as R_1, R_2, R_3 and R_4. In R_1, none of the species can survive in ecosystem. Moreover, in this parametric region, provision of additional food upto $(c_1 a_2 - \alpha d_2)A < bd_2$ have no impact on predator growth. But for $(c_1 a_2 - \alpha d_2)A > bd_2$, predator species follows the exponential growth model. Analytical explanations of these scenarios are described in Theorem 4.2. We can now only enter the region R_2 by raising the value of r. In this region, predator-free equilibrium point E_a is globally asymptotically stable. Next if we raise the value of r or A and enter into the region R_3, the system will go through another transcritical bifurcation, which will result in a stability switch between E_a and E_c. Stability of E_c will be lost if the values of r and A are taken from R_4. In this case, oscillatory behaviour around E_c is observed. Therefore, any of the higher value of r and A can produce a oscillatory behaviour around the coexistence state. Now we investigate the impact of fear (k) and its carry-over effect (c) on the system dynamics. We have already observed that if the parametric condition $(d_2 \alpha - c_1 a_2)\mu A + bd_2 = (c_1 a_2 - d_2)\left(\frac{r - d_1}{a_1}\right)$ is satisfied, a transcritical
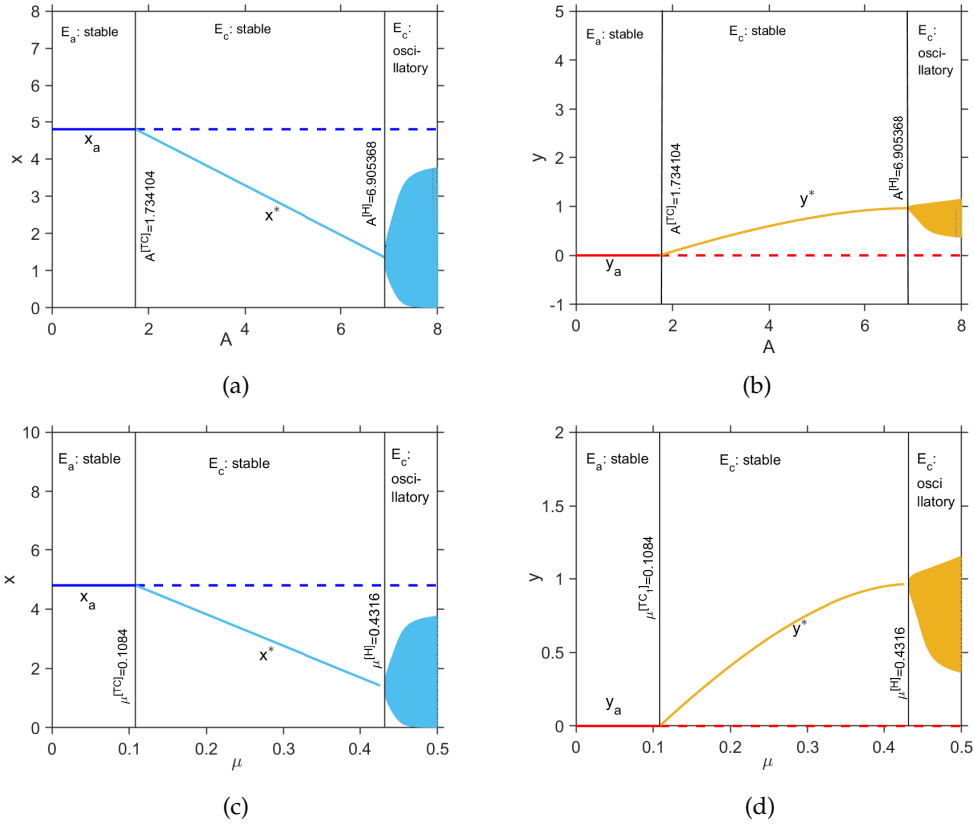
Figure 5: Bifurcation diagrams of system (2.5) with respect to the bifurcation parameters A and μ. Dotted line describes the unstable behaviour and solid line represents the stable behaviour of the system. First column depicts the change of prey biomass and the second column describes the change of predator biomass. **(a),(b):** A transcritical and a Hopf-bifurcation occurs for the bifurcating parameter A. **(c),(d):** Similarly the system undergoes a transcritical and a Hopf-bifurcation for the bifurcating parameter μ. The parametric values are taken from Table 2.

bifurcation occurs through which the system trades its stability between E_a and E_c. It can be noticed that this parametric condition is independent of k and c. Therefore, no transcritical bifurcation occurs for the varying parameters k and c. On the other hand, the parametric criteria that cause the system to experience a Hopf-bifurcation around E_c, explicitly depend on k and c. Hence the only bifurcation that the system faces for varying the parameters k and c is the Hopf-bifurcation. For numerical demonstration, we consider the following set of parametric values.

Parametric Values: Set 2											
r	c	k	d_1	a_1	a_2	b	α	μ	A	c_1	d_2
4	0.1	5	0.4	0.25	0.7	2.15	0.4	0.25	4	0.35	0.18

Table 3: Parameter values used for numerical simulation of system (2.5)

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For this particular set of parametric values, the coexistence equilibrium point E_c is globally asymptotically stable. Now if the level of fear (k) is decreased to $k = 0.1204$, a Hopf-bifurcation occurs through which the system exhibits oscillatory behaviour around E_c as depicted in Figs.8(a),(b). Except for a decrease in
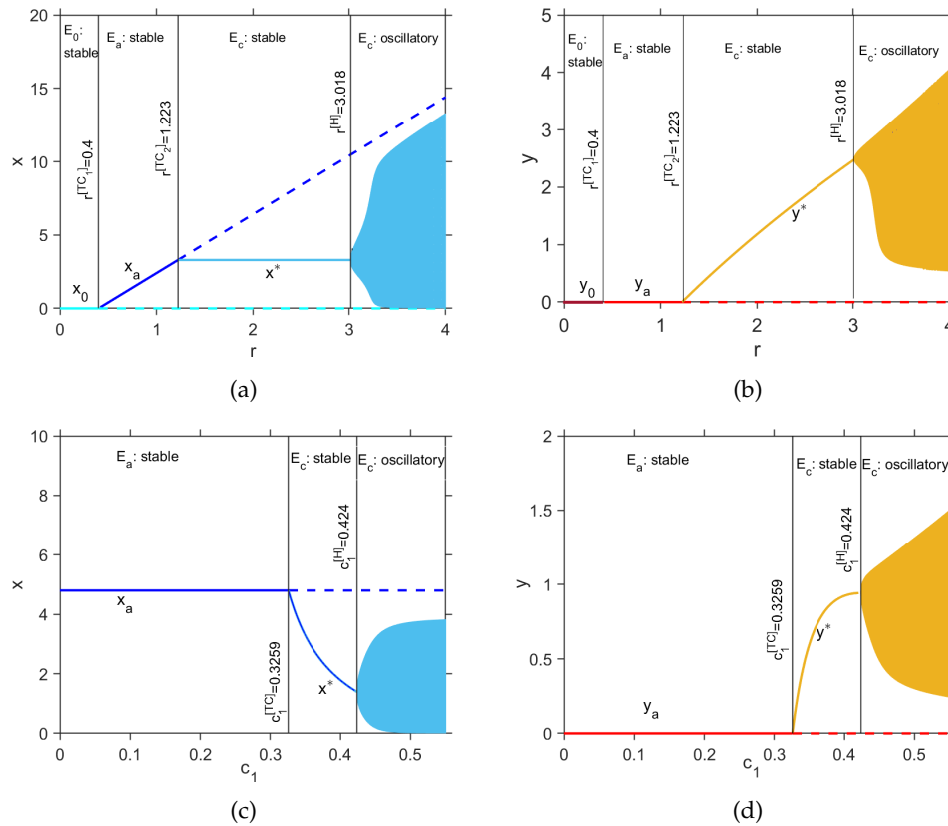
Figure 6: Bifurcation diagrams of system (2.5) with respect to the bifurcation parameters r and c_1. Dotted line describes the unstable behaviour and solid line represents the stable behaviour of the system. First column depicts the change of prey biomass and the second column describes the change of predator biomass. **(a),(b):** Two transcritical and a Hopf-bifurcation occurs for the bifurcating parameter r. **(c),(d):** Similarly a transcritical and a Hopf-bifurcation occurs for the bifurcating parameter c_1. The parametric values are taken from Table 2.

predator biomass, a higher level of fear has no effect on the system's qualitative behaviour. Therefore, higher level of fear has stabilizing effect on the system dynamics. On the other hand, if we gradually increase c to $c = 1.065$, oscillatory behaviour around E_c is emerged, but for lower value of c, no bifurcation occurs (see Figs.8(c),(d)). Hence higher carry-over has destabilizing effect on the system dynamics.

To investigate the combine effect of k and c, we analyze the system behaviour in $k - c$ parametric plane (see Fig.9(a)) and it is observed that a Hopf-bifurcation curve divides the whole parametric plane into two sub-regions which are labeled as R_1 and R_2. In R_1, the coexistence equilibrium point E_c is globally asymptotically stable and in R_2, the coexistence becomes oscillatory through the Hopf-bifurcation. It should be emphasised that higher level of fear or lower level of carry-over effect decreases the possibility of occurrence of Hopf-bifurcation. From Fig.9(b), one can investigate how the different combinations of A and k affect the system behaviour. This figure contains a transcritical bifurcation curve and a Hopf-bifurcation curve that divides the whole $A - k$ parametric plane into three sub-regions: R_1, R_2 and R_3. In R_1, E_a is globally asymptotically stable; In R_2, the coexistence equilibrium E_c is globally asymptotically stable and in R_3, the coexistence is oscillatory. Similar types of dynamical changes are observed in $A - c$ parametric plane as depicted in Fig.9(c). It should also be observed that there is a tiny range of A for which the system exhibits two Hopf-bifurcations with respect to the bifurcation parameter k, and the coexistence follows the stable-oscillatory-stable transition (see Fig.9(b) and Fig.10).
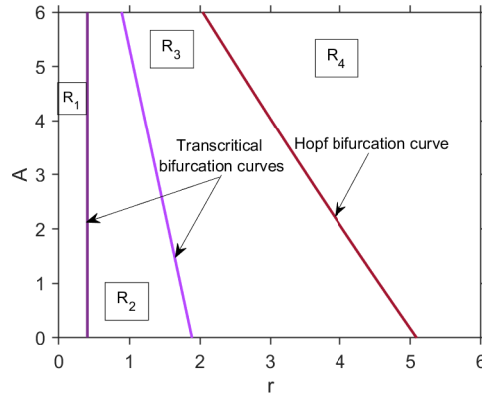
Figure 7: Two-parametric bifurcation diagram and stability zone of various equilibrium points of system (2.5) in the $r - A$ plane. In R_1, E_0 is stable; In R_2, the axial equilibrium point E_a is stable; In R_3, the coexistence equilibrium point E_c is stable and in R_4, the system exhibits oscillatory behaviour around E_c. R_1, R_2 and R_3 are separated by the transcritical bifurcation curves, whereas R_3 and R_4 are separated by the Hopf-bifurcation curve. The parametric values are taken from Table 2.

9. Effect of discrete time delay

Almost all biological processes exhibit time delay. When it comes to mathematical modelling, incorporation of time delay in any ecological system, makes the system more genuine than the corresponding non-delayed system [9, 35, 36]. During the time of prey-predator interaction, the energy conversion process that takes place during predation does not happen instantly. The entire conversion process takes time, which is referred to as gestation delay. So, in presence of gestation delay (τ), system (2.5) can be modified as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(\frac{1+cx}{1+cx+ky}\right) - d_1 x - a_1 x^2 - \frac{a_2 xy}{b + \alpha\mu A + x}, \\ \frac{dy}{dt} &= \frac{c_1 a_2 (x(t-\tau) + \mu A)y(t-\tau)}{b + \alpha\mu A + x(t-\tau)} - d_2 y. \end{aligned} \tag{9.1}$$

The initial history function is assumed as:

$$x(\theta) = \phi_1(\theta), \; y(\theta) = \phi_2(\theta), \tag{9.2}$$

where $\theta \in [-\tau, 0]$ and $\Phi = (\phi_1, \phi_2)$ belongs to the Banach space of continuous functions $\Phi : [-\tau, 0] \to \mathbb{R}_+^2$ with

$$\|\Phi\| = \sup_{-\tau \le \theta \le 0} \left(|\phi_1(\theta)|, |\phi_2(\theta)|\right).$$

For biological feasibility, we further consider $x(0) > 0$, $y(0) > 0$.
In the system, τ represents the time lag for predator's digestion and gestation. System (9.1) has the same steady state as the non-delayed system discussed previously. To preserve biological biodiversity, we primarily focus on determining the parametric conditions under which this delay-induced system is stable around the interior equilibrium point E_c. Using the transformation $X(t) = x(t) - x^*$, $Y(t) = y(t) - y^*$, system (9.1) can be transformed into a linearized system as follows:

$$\frac{d}{dt}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = C\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + D\begin{pmatrix} X(t-\tau) \\ Y(t-\tau) \end{pmatrix}. \tag{9.3}$$
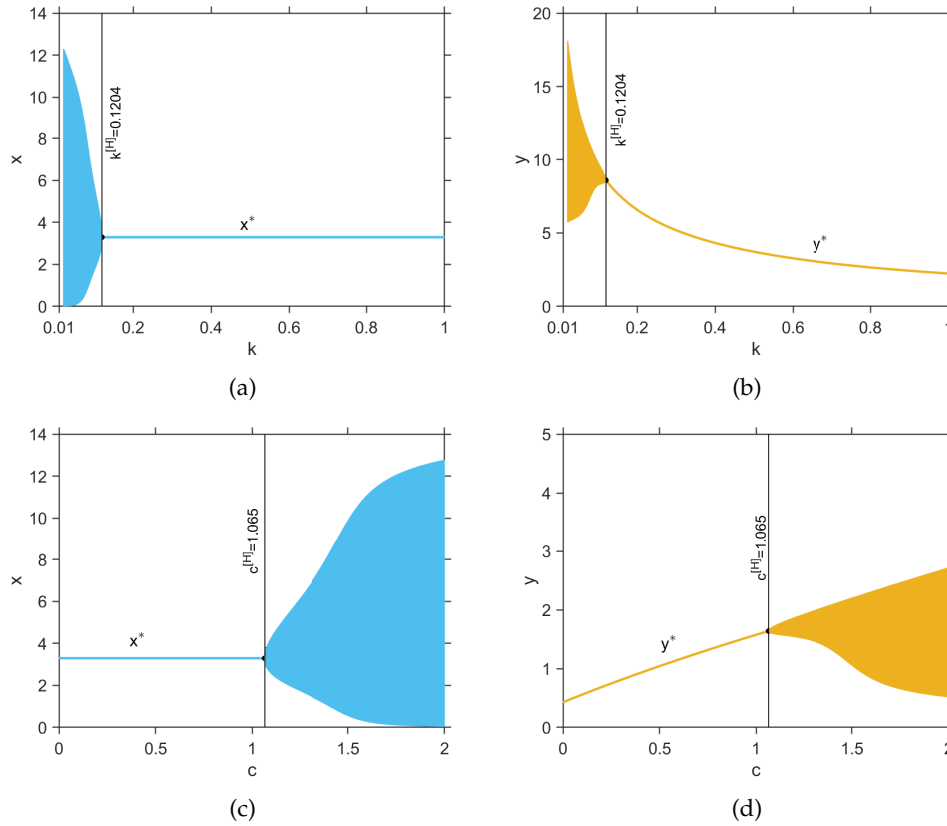
Figure 8: Bifurcation diagrams of system (2.5) with respect to the bifurcation parameters k and c. Dotted line describes the unstable behaviour and solid line represents the stable behaviour of the system. First column depicts the change of prey biomass and the second column describes the change of predator biomass. **(a),(b):** A Hopf-bifurcation occurs for the bifurcating parameter k. **(c),(d):** Similarly the system undergoes a Hopf-bifurcation for the bifurcating parameter c. The parametric values are taken from Table 3.

Here,

$$C = \begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix},$$

where

$$c_{11} = \frac{a_2 x^* y^*}{(b + \alpha \mu A + x^*)^2} + \frac{rckx^* y^*}{(1 + cx^* + ky^*)^2} - a_1 x^*,$$

$$c_{12} = -x^* \left[\frac{kr(1 + cx^*)}{(1 + cx^* + ky^*)^2} + \frac{a_2}{b + \alpha \mu A + x^*} \right],$$

$$c_{22} = -d_2,$$

$$d_{21} = \frac{c_1 a_2 y^* (b + (\alpha - 1)\mu A)}{(b + \alpha \mu A + x^*)^2},$$

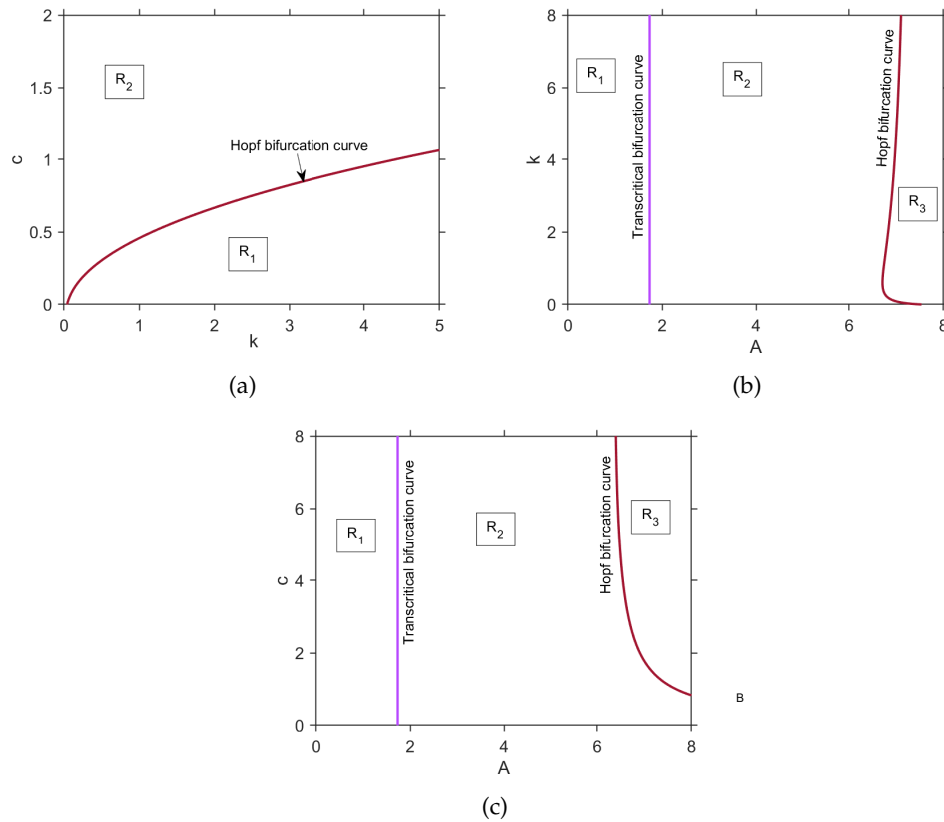$$d_{22} = \frac{c_1 a_2 (x + \mu A)}{(b + \alpha \mu A + x)}.$$

Figure 9: Two-parametric bifurcation diagrams and stability zone of various equilibrium points of system (2.5) in the $k-c$, $A-k$ and $A-c$ parametric planes. **(a):** In R_1, E_c is stable; In R_2, oscillatory behaviour around E_c appears; **(b):** In R_1, E_a is stable; In R_2, E_c is stable and in R_3 the system exhibits oscillatory behaviour around E_c. **(c):** In R_1, E_a is stable; In R_2, E_c is stable and in R_3, E_c is oscillatory. For Fig.(a), the parametric values are taken from Table 3 and for Fig.(b) and Fig.(c), the parametric values are taken from Table 2.

The characteristic equation of the linearized system (9.3) is

$$\lambda^2 + M_1\lambda + M_2 + (N_1\lambda + N_2)e^{-\lambda\tau} = 0, \tag{9.4}$$

where

$$M_1 = -c_{11} - c_{22},$$
$$M_2 = c_{11}c_{22},$$
$$N_1 = -d_{22},$$
$$N_2 = c_{11}d_{22} - c_{12}d_{21}.$$

For $\tau = 0$, the above characteristic equation reduces to

$$\lambda^2 + (M_1 + N_1)\lambda + M_2 + N_2 = 0. \tag{9.5}$$

Therefore, the coexistence equilibrium point will be stable if the following two conditions are satisfied:

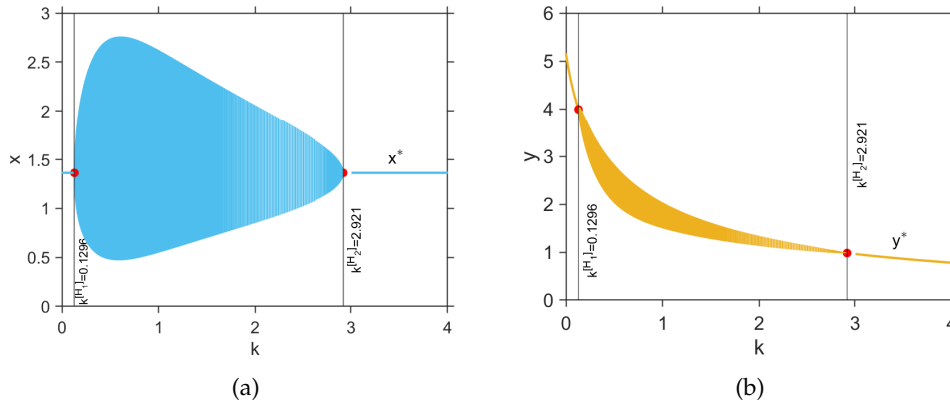$$M_1 + N_1 > 0, \quad \text{and} \quad M_2 + N_2 > 0. \tag{9.6}$$

(a) (b)

Figure 10: Bifurcation diagrams of system (2.5) with respect to the bifurcation parameter k. Two Hopf-bifurcations occur at $k^{[H_1]} = 0.1296$ and at $k^{[H_2]} = 2.921$. In this case, the level of additional food is taken as $A = 6.9$. The parametric values are taken from Table 2.

Now, when $\tau \neq 0$, the sign of the real part of the roots of the associated characteristic equation (9.4), determines the stability behaviour of the delay-induced system around the coexistence steady state, $E_c(x^*, y^*)$. E_c is locally asymptotically stable if all the real parts are negative; otherwise, it is unstable. First of all, it is assumed that E_c is asymptotically stable for $\tau = 0$. Then in presence of delay, we evaluate the parametric conditions for which E_c is stable for $\tau > 0$. Continuity property of τ ensures that the characteristic equation (9.4) has roots with positive real parts if and only if it has purely imaginary roots. This property helps to deduce the parametric conditions for which the equation (9.4) has roots with negative real parts.
Now substituting $\lambda = \xi + i\omega$ in equation (9.4), we obtain

$$(\xi + i\omega)^2 + M_1(\xi + i\omega) + M_2 + (N_1(\xi + i\omega) + N_2)e^{-\xi\tau}(\cos(\omega\tau) - i\sin(\omega\tau)) = 0. \tag{9.7}$$

Comparing the real and complex parts from both sides, we obtain:

$$\xi^2 - \omega^2 + M_1\xi + M_2 + \xi N_1 e^{-\xi\tau}\cos(\omega\tau) + N_2 e^{-\xi\tau}\cos(\omega\tau) + N_1\omega e^{-\xi\tau}\sin(\omega\tau) = 0, \tag{9.8}$$

$$2\xi\omega + M_1\omega + N_1\omega e^{-\xi\tau}\cos(\omega\tau) - N_1\xi e^{-\xi\tau}\sin(\omega\tau) - N_2 e^{-\xi\tau}\sin(\omega\tau) = 0. \tag{9.9}$$

Now for purely imaginary roots, let us take $\xi = 0$ and then the above two equations are reduced to

$$N_1\omega\sin(\omega\tau) + N_2\cos(\omega\tau) = \omega^2 - M_2, \tag{9.10}$$
$$N_1\omega\cos(\omega\tau) - N_2\sin(\omega\tau) = -M_1\omega. \tag{9.11}$$

By squaring and adding, we obtain

$$\omega^4 + (M_2^2 - 2M_2 - N_1^2)\omega^2 + M_2^2 - N_2^2 = 0. \tag{9.12}$$

Now by considering $\omega^2 = \sigma$, we have

$$Z(\sigma) \equiv \sigma^2 + (M_2^2 - 2M_2 - N_1^2)\sigma + M_2^2 - N_2^2 = 0. \tag{9.13}$$

Clearly, $Z(\infty) > 0$ and so the equation (9.13) has at least one positive real root if $Z(0) < 0$, i.e., $M_2^2 < N_2^2$. Let us assume that the positive root is σ_+, then $\omega = \pm\sqrt{\sigma_+}$.
Now we state a lemma proved by Ruan and Wang [16].

Lemma 9.1. *Consider the exponential polynomial*

$$
\begin{aligned}
P(\lambda, e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, ..., e^{-\lambda \tau_m}) = {} & \lambda^n + p_1^{(0)} \lambda^{n-1} + ... + p_{n-1}^{(0)} \lambda + p_n^0 \\
& + \left[p_1^{(1)} \lambda^{n-1} + ... + p_{n-1}^{(1)} \lambda + p_n^{(1)} \right] e^{-\lambda \tau_1} \\
& + ... \\
& + \left[p_1^{(m)} \lambda^{n-1} + ... + p_{n-1}^{(m)} \lambda + p_n^{(m)} \right] e^{-\lambda \tau_m},
\end{aligned}
$$

where $\tau_i \geq 0$ ($i = 1, 2, 3, ...m$) and $p_j^{(i)}$ ($i = 1, 2, 3, ...m; j = 1, 2, 3, ...n$) are constants. As $(\tau_1, \tau_2, ..., \tau_m)$ vary, the sum of the orders of zero of $P(\lambda, e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, ...e^{-\lambda \tau_m})$ in the open half plane can change only if a zero appears on or crosses the imaginary axis.

Now by considering τ as a bifurcation parameter, we investigate the existence of Hopf-bifurcation around E_c which is given below as a theorem:

Theorem 9.2. *Suppose the coexistence steady state E_c exists and is locally asymptotically stable under the parametric restrictions stated in Theorem 5.4 for $\tau = 0$. If $M_2^2 < N_2^2$, one can obtain a threshold value of τ, say τ^*, so that E_c is locally asymptotically stable for $0 \leq \tau < \tau^*$ and unstable for $\tau > \tau^*$, where*

$$
\tau^* = \min_n \tau_+^{(n)} = \min_n \left[\frac{1}{\sqrt{\sigma_+}} \cos^{-1} \left(\frac{(N_2 - M_1 N_1)\sigma_+ - M_2 N_2}{N_2^2 + N_1^2 \sigma_+} \right) + \frac{2\pi n}{\sqrt{\sigma_+}} \right] \quad (n = 0, 1, 2, 3, ...)
$$

In other words, system (9.1) exhibits a Hopf-bifurcation around E_c at $\tau = \tau^*$ provided $KL > LN$ where K, L, M and N are defined within the proof.

Proof. It is shown that the equation (9.13) has one positive root, say σ_+ under the parametric condition $M_2^2 < N_2^2$. Then from equations (9.10) and (9.11), we can evaluate $\tau_+^{(n)}$ ($n = 0, 1, 2, 3...$) in terms of σ_+ which is given as follows:

$$
\tau_+^{(n)} = \frac{1}{\sqrt{\sigma_+}} \cos^{-1} \left(\frac{(N_2 - M_1 N_1)\sigma_+ - M_2 N_2}{N_2^2 + N_1^2 \sigma_+} \right) + \frac{2\pi n}{\sqrt{\sigma_+}} \quad (n = 0, 1, 2, 3, ...)
$$

Now we assume that $\tau^* = \min_n \tau_+^{(n)}$. Therefore, using Butler's Lemma, we can conclude that the coexistence equilibrium point E_c is stable for $\tau < \tau^*$, provided E_c is stable for $\tau = 0$.

Next we check the transversality condition: $\left[\frac{d}{d\tau}(Re\lambda(\tau)) \right]_{\tau=\tau^*} = \left[\frac{d\xi}{d\tau} \right]_{\tau=\tau^*} > 0$.

Differentiating (9.8) and (9.9) with respect to τ and setting $\xi = 0$ and $\tau = \tau^*$, we obtain

$$
K \left[\frac{d\xi}{d\tau} \right]_{\tau=\tau^*} + L \left[\frac{d\omega}{d\tau} \right]_{\tau=\tau^*} = M, \tag{9.14}
$$

$$
-L \left[\frac{d\xi}{d\tau} \right]_{\tau=\tau^*} + K \left[\frac{d\omega}{d\tau} \right]_{\tau=\tau^*} = N, \tag{9.15}
$$

where

$$
\begin{aligned}
K &= M_1 + N_1 \cos(\sqrt{\sigma_+}\tau^*) - N_2 \tau^* \cos(\sqrt{\sigma_+}\tau^*) - N_1 \sqrt{\sigma_+}\tau^* \sin(\sqrt{\sigma_+}\tau^*), \\
L &= -2\sqrt{\sigma_+} - N_2 \tau^* \sin(\sqrt{\sigma_+}\tau^*) + N_1 \sin(\sqrt{\sigma_+}\tau^*) + N_1 \sqrt{\sigma_+}\tau^* \cos(\sqrt{\sigma_+}\tau^*), \\
M &= N_2 \sqrt{\sigma_+} \sin(\sqrt{\sigma_+}\tau^*) - N_1 \sigma_+^2 \cos(\sqrt{\sigma_+}\tau^*), \\
N &= N_1 \sigma_+ \sin(\sqrt{\sigma_+}\tau^*) + N_2 \sqrt{\sigma_+} \cos(\sqrt{\sigma_+}\tau^*).
\end{aligned}
$$

Solving (9.14) and (9.15), we obtain

$$\left[\frac{d\xi}{d\tau}\right]_{\tau=\tau^*} = \left[\frac{d}{d\tau}(Re\lambda(\tau))\right]_{\tau=\tau^*} = \left[\frac{KM - LN}{K^2 + L^2}\right].$$

Then clearly $\left[\frac{d}{d\tau}(Re\lambda(\tau))\right]_{\tau=\tau^*} > 0$, provided $KM > LN$. Hence the system undergoes a Hopf-bifurcation when τ crosses its threshold value τ^*. $\square$

Next, we study the delayed model numerically and investigate the influence of gestation delay on model dynamics. For this purpose, we consider the set of parametric values stated in Table 4. One can easily

Parametric Values: Set 3											
r	c	k	d_1	a_1	a_2	b	α	μ	A	c_1	d_2
0.5	0.5	1	0.1	0.38	0.08	0.4	0.7	0.3	0.5	0.9	0.03

Table 4: Parameter values used for numerical simulation of system (9.1)
.

check that for this particular set of parametric values the non-delayed system is asymptotically stable around $E_c(0.1036, 0.9442)$. We can now see how the system's dynamics vary when the delay parameter τ is gradually increased from 0. Fig.11 shows that the stability of E_c does not change initially, but as soon as τ crosses some threshold value τ^* (in this example approximate value of τ^* is 8.1), the system exhibits oscillatory behaviour around E_c. Therefore the system experiences a Hopf-bifurcation around the coexistence steady state at $\tau = \tau^*$. Fig.12(a) depicts the stable behaviour of the system around E_c for a typical value $\tau = 5 < \tau^*$. In Fig.12(b), oscillatory behaviour around E_c is observed for $\tau = 12 > \tau^*$.

Now we investigate numerically how the critical value of τ varies as the amount of additional food A fluctuates. In Fig.13(a), we have plotted the bifurcation diagram for three different values of A and it is observed that higher levels of A enhance the possibility of oscillatory dynamics. In particular, for $A = 0.4$, the critical value of τ is obtained as $\tau^* = 14$ but for $A = 0.6$, that critical value reduces to $\tau^* = 4.8$. Similarly from Fig.13(b), we can observe that higher level of fear promotes the stable coexistence in nature. In this case, the critical value of τ increases to around 18.5 for $k = 1.5$. Fig.13(c) depicts the behaviour of τ^* for three different levels of carry-over effect (c) and it is noticed that higher level of c reduces the critical value of τ, implying that higher level of c promotes the oscillatory dynamics.

10. Discussion

The ecological research focuses on determining the environmental components that control the dynamical complexity of the ecosystem. Numerous mathematical models have been developed in recent years to investigate the diverse ecological processes under various environmental restrictions. The predator-prey interaction model is one such type of ecological models. The majority of studies has concentrated only on the predator's lethal effects in prey-predator interactions. However, recent study has discovered that, in addition to lethal effects, there are certain non-lethal consequences that are equally as significant as the preceding one. Predation-induced fear is one of such non-lethal effects that plays an important role in population fitness. Non-lethal effects are not just restricted to a single generation but can also be carried over through generations. So, it is biologically relevant to consider the predation-induced fear, together with its carry-over effect in our predator-prey model. Further we modify the standard Holling type II functional response in the presence of some additional food to the predator. In addition, to obtain more realistic dynamics, we include the gestation delay in the predation term.

The well-posedness of our proposed model and extinction criteria for both species are determined under various parametric constraints. Existence of various equilibrium states and their local stability are discussed with standard linear stability analysis. We have also developed certain parametric criteria under which all of the system's interacting species can persist in nature. Various local bifurcations are also analyzed by
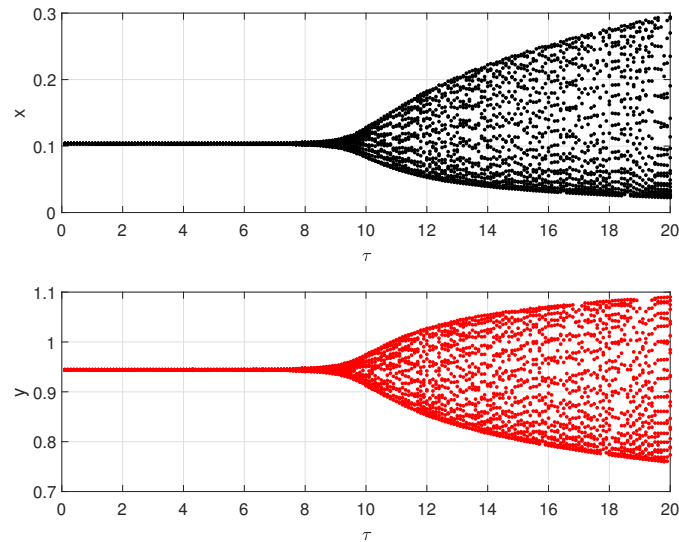
Figure 11: Bifurcation diagram of the delay-induced system (9.1) for the bifurcation parameter τ. A Hopf-bifurcation occurs when τ crosses a critical value $\tau^* = 8.1$. The parametric values are taken from Table 4.

taking r, k, c, A and μ as bifurcating parameters. It is also established that the gestation delay (τ) can cause the system to oscillate around the coexistence state. According to our analytical and numerical outcomes, fear and its carry-over effects with additional food to predator can jointly impact the proposed system in a variety of ways, as explained below:

1. The provision of additional food (A) to predators on a wide scale promotes the expansion of predator biomass, resulting in a significant predation burden on prey species and driving them to extinction. However if the effective quantity of additional food is very minimal, the predator species may go extinct. This scenario arises only when the predator species can not properly access the prey species either for their (prey) lower reproduction rate or for predator's lower hunting efficiency. The stable coexistence is possible only for the intermediate values of A. For some critical value of A, the presence of oscillatory dynamics around the coexistence state is also demonstrated.

2. Predation-induced fear (k) and its carry-over effect (c) are not capable to switch the system's stability from one equilibrium state to other equilibrium state, i.e., if the predator-free equilibrium becomes stable for some set of parametric values, we can not change its stability only by changing the values of k and c. However once the coexistence equilibrium state occurs in the system, a higher level of fear can stabilize it. For the lower level of fear, the system may experience a Hopf-bifurcation around the interior steady state E_c. Therefore, higher level of fear has stabilizing effect on the system dynamics. It should also be mentioned that there is a limited range of A for which the system exhibits two Hopf-bifurcations with respect to the bifurcation parameter k, indicating that the coexistence follows the stable-oscillatory-stable transition.

3. For higher level of carry-over effect, in particular if $c > c^{[H]}$, the system exhibits oscillatory dynamics around E_c. Hence higher carry-over has destabilizing effect on the system dynamics. To make the system stable in a coexistence state, the critical value ($c^{[H]}$) of c should be increased. This may be accomplished in two ways: by reducing the amount of additional food or by enlarging the costs of fear.

4. Lastly we analyse the delay-induced model by changing the delay parameter τ and obtain a critical value of τ, say τ^*, where a Hopf-bifurcation takes place, i.e., the system is stable for $0 < \tau < \tau^*$ and

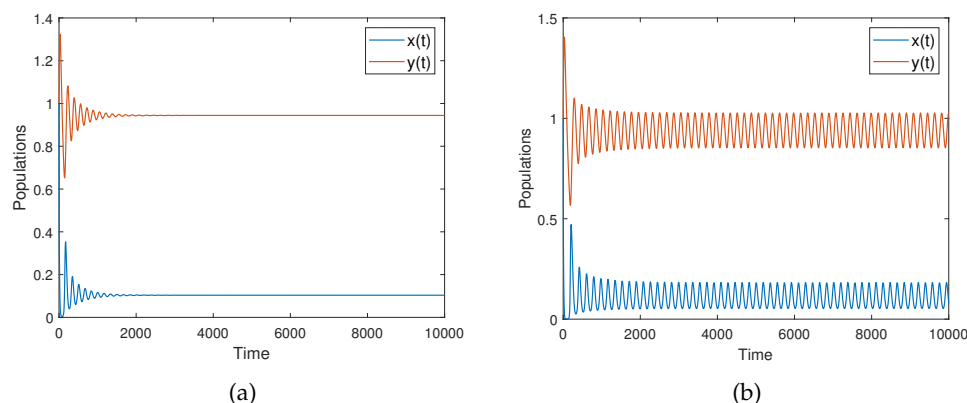Figure 12: Time series for prey species, x and predator species, y. **(a):** Stable behaviour around E_c is observed for $\tau = 5 < \tau^*$. **(b):** Oscillatory behaviour around E_c occurs for $\tau = 12 > \tau^*$. The parametric values are taken from Table 4.

unstable for $\tau > \tau^*$. Our primary goal is to raise the critical value τ^* so that the system becomes stable across a wider range of τ. Based on our numerical simulations, we can conclude that the value of τ^* may be increased in one of three ways: (i) by reducing the quantity of additional food, (ii) by raising the fear level, or (iii) by decreasing the degree of carry-over impact.

Thus the current work explores many conceivable dynamical behaviours of the predator-prey system, which will help in understanding the interplay between predator and prey with greater ecological implications. The proposed model may be enhanced for a two prey and one predator species, which may be important in forming community establishment and sustaining biological variety. Furthermore, a realistic mathematical model may be developed to examine the impacts of spatial diffusion in pattern formation via diffusion-driven instability. In addition, stochastic noise may be introduced in this model via environment-dependent parameters.
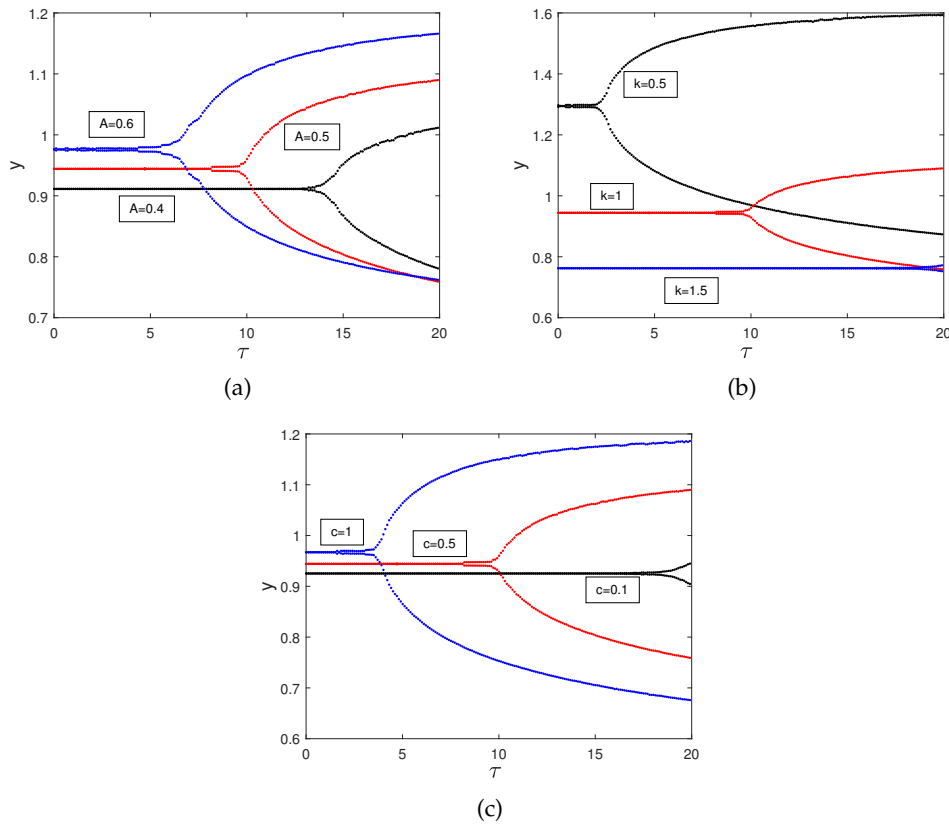
(a)

(b)

(c)

Figure 13: Variations of τ^* (critical value of τ) for three different values of A, k and c. **(a):** Higher level of A reduces the value of τ^*. **(b):** Higher level of k increases the value of τ^*. **(c):** Higher level of c reduces the value of τ^*. The parametric values are taken from Table 4.

Acknowledgement

Data Availability Statement

The data used to support the findings of the study are available within the article.

Conflict of Interest

With reference to the study, the authors claim that they do not have any conflicts of interest.

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