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Existence and uniqueness of a mild solution for a class of the fractional evolution equation With nonlocal condition involving φ-Riemann Liouville fractional derivative

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Abstract. In this paper, by using the fractional power of operators and theory fixed point theorems, we discuss Existence and uniqueness of mild solution to initial value problems for fractional semilinear evolution equations with compact semigroup in Banach spaces with nonlocal conditions. In particular, we derive the form of fundamental solution in terms of semigroup induced by resolvent and φ-Riemann-Liouville fractional derivatives. These results generalize previous works where the classical Riemann-Liouville fractional derivative is considered. In the end, we give an example to illustrate the applications of the abstract results.

1. Introduction

Fractional differential equations have been an exciting field of applied mathematics, it s gives very important tools for describing and studying natural phenomena, on fractional calculus more authors are interesting by the theory of fractional evolution equations since they are abstract formulations for many problems in physics, engineering, chemistry, finance ... (see [1]–[12], [31], [32], [33]).
Our work is inspired by many studies on the existence and unique solutions of partial evolution equations based on semi-group and fixed point theory (see [13]–[17]).

Consider the following nonlocal Elliptic problem of fractional evolution equation with Riemann-Liouville fractional derivative:

$$\begin{cases} \mathcal{D}_0^{\alpha;\varphi}(u(t)) = Au(t) + f(t, u(t)), & \text{a.e. } t \in [0, a] = J \\ \mathcal{D}_0^{\alpha-1;\varphi}(u(0)) = u_0 - g(u), \end{cases} \tag{1}$$

where $\mathcal{D}_0^{\alpha;\varphi}$ is φ - Riemann–Liouville fractional derivative of order $\alpha, 0 < \alpha < 1$.
A is the infinitesimal generator of a C_0 - semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$ in Banach space $U.f : J \times U \to U$ is a given function, $g : C(J, U) \to L(J, U)$ is a given operator satisfying some assumptions

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and u_0 is an element of the Banach space U.

This study will be organized as follows. In Sect 2, we will briefly recall some definitions and preliminary concepts about fractional calculus and auxiliary results used in the following sections. We construct a mild solution by using semigroup for the problem (1) in Sect 3. We prove the existence and uniqueness of mild solutions of the problem (1) under compact analytic semigroup by Darbo-Sadovskii's fixed point theorem in Sect 4. Finally, we give some examples to illustrate the application of the results obtained in Sect 5.

2. Preliminaries

We give some indications about the semigroups of linear operators. [18], [19]. For a strongly continuous semigroup (i.e., C_0-semigroup) $\{S(t)\}_{t\geq 0}$, the infinitesimal generator of $\{S(t)\}_{t\geq 0}$ is defined by

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{u}, \quad u \in U.$$

We denote by $D(A)$ the domain of A, that is,

$$D(A) = \left\{ u \in U : \lim_{t\to 0^+} \frac{S(t)u - u}{u} \text{ exists} \right\}.$$

Lemma 2.1. . [18], [19] *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup, then there exist constants $C \geq 1$ and $a \geq 0$ such that $\|S(t)\| \leq Ce^{at}$ for all $t \geq 0$.*

Lemma 2.2. . *([18],[19]) A linear operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup if and only if:*

(i) A is closed and $\overline{D(A)} = U$
(ii) The resolvent set $\rho(A)$ of A contains $R^+$and, for every $\lambda > 0$, we have

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda},$$

where $R(\lambda, A) := (\lambda^\alpha I - A)^{-1} u = \int_0^\infty e^{-\lambda^\alpha t} S(t) u dt$
let A be the infinitesimal generator of a compact C_0-semigroup of uniformly bounded linear operators $\{T(t)\}_{t\geq 0}$ on U. Then there exists $\xi \geq 1$ such that $\xi = \sup_{t\in[0,\infty)} \|T(t)\|$.

Definition 2.3. *The gamma function $\Gamma(z)$ is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (\mathrm{Re}(z) > 0) \quad z \in \mathbb{C},$$

Definition 2.4. *([21]). Let $\alpha > 0$, f be an integrable function defined on $[a, b]$ and $\varphi \in C^1([a, b])$ be an increasing function with $\varphi'(t) \neq 0$ for all $t \in [a, b]$. The left φ-Riemann-Liouville fractional integral operator of order α of a function f is defined by:*

$$\mathcal{I}^{\alpha,\varphi}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\alpha-1} f(s) ds.$$

Definition 2.5. *([21]). Let $n - 1 < \alpha < n$, $f \in C^n([a, b])$ and $\varphi \in C^n([a, b])$ be an increasing function with $\varphi'(t) \neq 0$ for all $t \in [a, b]$. The left φ-Riemann-Liouville fractional derivative of order α of a function f is defined by:*

$$\begin{aligned} \mathcal{D}^{\alpha,\varphi}_{a^+} f(t) &= \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n I^{n-\alpha,\varphi}_{a^+} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{n-\alpha-1} f(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$.

Lemma 2.6. *([21],[22]). Let $\alpha > 0$ and $\beta > 0$, then*

(i) $I_{a^+}^{\alpha,\varphi}(\varphi(s) - \varphi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\varphi(t) - \varphi(a))^{\beta+\alpha-1}$,
(ii) $\mathcal{D}_{a^+}^{\alpha,\varphi}(\varphi(s) - \varphi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\varphi(t) - \varphi(a))^{\beta-\alpha-1}$.

Lemma 2.7. *([20]). Let $f \in C^n([a,b])$ and $n-1 < \alpha < n$. Then we have*
(1) $\mathcal{D}_{a^+}^{\alpha,\varphi} I_{a^+}^{\alpha,\varphi} f(t) = f(t)$;
(2) $I_{a^+}^{\alpha,\varphi}\mathcal{D}_{a^+}^{\alpha,\varphi} f(t) = f(t) - \sum_{k=1}^{n} \frac{f^{[k-1]}(a^+)}{\Gamma(k-\alpha)}(\varphi(t) - \varphi(a))^{k-a}$,

where $f^{[k]}(t) := \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^k f(t)$ *on* $[a,b]$. *In particular, given $\alpha \in (0,1)$, one has*

$$I_{a^+}^{\alpha,\varphi}\mathcal{D}_{a^+}^{\alpha,\varphi} f(t) = f(t) - c(t-a)^{\alpha-1},$$

where c is a constant.

Lemma 2.8. *([21]). Let $u, \varphi : [a,\infty) \to \mathbb{R}$ be real valued functions such that $\varphi(t)$ is continuous and $\varphi'(t) > 0$ on $[0,\infty)$. The generalized Laplace transform of f is denoted by*

$$\mathcal{L}_{\varphi}\{u(t)\}(s) = \int_a^{\infty} e^{-s(\varphi(t)-\varphi(a))} u(t)\varphi'(t)dt$$

for all s.

Lemma 2.9. *([21]). Let u and v be two functions which are piecewise continuous at each interval $[0,T]$ and of exponential order.*
We define the generalized convolution of u and v by

$$(u *_{\psi} v)(t) = \int_a^t u(\tau)v\left(\psi^{-1}(\psi(t) + \psi(a) - \psi(\tau))\right)\psi'(\tau)d\tau$$

Theorem 2.10. *(Gronwall's inequality [23], [24])*
Let u, v be two integrable functions and h be a continuous function on $[a,b]$. Let $\varphi \in C^1([a,b])$ be an increasing function such that $\varphi'(t) \neq 0$ for all $t \in [a,b]$. Assume that (1) u and v are nonnegative; (2) h is nonnegative and nondecreasing. If

$$u(t) \leq v(t) + h(t)\int_a^t (\varphi(t) - \varphi(s))^{\alpha-1} u(s)\varphi'(s)ds,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(n\alpha)}(\varphi(t) - \varphi(s))^{k\alpha-1} v(s)\varphi'(s)ds$$

for all $t \in [a,b]$.

Definition 2.11. *([25],[26]) The Wright function ϕ_α is defined by*

$$\begin{aligned}\phi_\alpha(z) &:= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^n}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha), \quad z \in \mathbb{C}\end{aligned}$$

with $0 < \alpha < 1$

Proposition 2.12. *([25],[26]) For $-1 < r < \infty, \lambda > 0$, the following results hold.*

(i) $\phi_\alpha(\theta) \geq 0$ *for* $\theta \geq 0$ *and* $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$;
(ii) $\phi_\alpha(t) \geq 0, t > 0$
(iii) $\int_0^\infty \frac{\alpha}{t^{\alpha+1}}\phi_\alpha\left(\frac{1}{t^\alpha}\right)e^{-\lambda t}dt = e^{-\lambda^\alpha}$
(iv) $\int_0^\infty \phi_\alpha(t)t^r dt = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$
(v) $\int_0^\infty \phi_\alpha(t)e^{-zt}dt = E_\alpha(-z), z \in \mathbb{C}$
(iv) $\int_0^\infty \alpha t\phi_\alpha(t)e^{-zt}dt = e_\alpha(-z), z \in \mathbb{C}$.

When E_α and e_α are a Mittag-Leffler functions

We introduce the definition for Kuratowski measure of noncompactness, which will be used in the proofs of our results.

Definition 2.13. *([27])Let U be a Banach space and $\mathcal{B}(U)$ be the bounded subset of U. The Kuratowski measure of noncompactness is the map $\mu : \mathcal{B}(U) \to [0, \infty)$ define by*

$$\mu(B) = \inf\left\{\varepsilon > 0 : B \subset \bigcup_{j=1}^{\infty} B_j, \operatorname{diam}\left(B_j\right) < \varepsilon \text{ for } i = 1, 2, \ldots, n\right\},$$

where $\operatorname{diam}\left(B_j\right) = \sup\left\{|x - y| : x, y \in B_j\right\}$.

Lemma 2.14. *([27],[28]) Let U be Banach spaces and $W, V \subset U$ be bounded. Then the noncompactness measure has the following properties:*
(i) $\mu(W) = 0$ *if and only if $\bar{W}$ is compact, where $\bar{W}$ means the closure hull of W;*
(ii) $\mu(\lambda W) = |\lambda|\mu(W)$, *where* $\lambda \in \mathbb{R}$;
(iii) $\mu(W) = \mu(\bar{W}) = \mu(\operatorname{conv} W)$, *where conv W means the convex hull of W;*
(iv) $\mu(W \cup V) = \max\{\mu(W), \mu(V)\}$;
(v) $\mu(W) \leq \mu(V)$ *if* $W \subset V$;
(vi) $\mu(W + V) \leq \mu(W) + \mu(V)$; *where* $W + V = \{u \mid u = w + v, w \in W, v \in V\}$;
(vii) $\mu(W + u) = \mu(W)$, *for any* $u \in U$;
(viii) If the map $Q : \operatorname{dom}(Q) \subset U \to X$ is Lipschitz continuous with constant k, then $\mu(Q(S)) \leq k\mu(S)$ for any bounded subset $S \subset \operatorname{dom}(Q)$, where X is another Banach space.

Lemma 2.15. *([29])Let U be a Banach space, and let $D \subset U$ be bounded. Then there exists a countable set $D_0 \subset D$ such that $\mu(D) \leq 2\mu(D_0)$.*

Theorem 2.16. *(Darbo-Sadovskii's fixed point theorem [28]). If B is a bounded, closed and convex subset of a Banach space U, and the continuous map $\mathscr{T} : B \to B$ is an α-contraction, then the map $\mathscr{T}$ has at least one fixed point in B.*

3. Mild Solutions

Lemma 3.1. *The nonlocal Elliptic problem 1 is equivalent to the integral equation,*

$$\begin{aligned} u(t) = {} & \frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\Gamma(\alpha)}[u_0 - g(u)] \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi'(s)(\varphi(t) - \varphi(s))^{\alpha-1}[Au(s) + f(s, u(s))]ds. \end{aligned} \tag{2}$$

for $t \in (0, a]$,

Proof. Suppose 2 is true, then:

$$\mathcal{D}_0^{\alpha-1;\varphi}u(t)=\mathcal{D}_0^{\alpha-1;\varphi}\left(\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\Gamma(\alpha)}[u_0-g(u)]\right)$$
$$+\mathcal{D}_0^{\alpha-1;\varphi}\left(\frac{1}{\Gamma(\alpha)}\int_0^t\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}[Au(s)+f(s,u(s))]ds\right)$$

We use de lemma 2.6 so:

$$\mathcal{D}_0^{\alpha-1;\varphi}u(t)=[u_0-g(u)]+\int_0^t Au(s)+f(s,u(s))ds$$

When t=0 we conclue the intial codition, $\mathcal{D}_0^{\alpha-1;\varphi}$ is absolutely continuous on $[0,a]$ then:

$$\mathcal{D}_0^{\alpha;\varphi}(u(t))=\frac{d}{dt}\mathcal{D}_0^{\alpha-1;\varphi}(u(t))\quad=[Au(s)+f(s,u(s))]$$

for the other side:
If u satisfies the problem 2, then applying to both sides of 1, we have

$$\mathcal{I}_0^{\alpha;\varphi}\mathcal{D}_0^{\alpha;\varphi}(u(t))=\mathcal{I}_0^{\alpha;\varphi}[Au(s)+f(s,u(s))],\ \text{almost all } t\in[0,a]$$

By Lemma 2.7, we obtain

$$u(t)=c(\varphi(t)-\varphi(0))^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}[Au(s)+f(s,u(s))]ds. \tag{3}$$

for $t\in(0,a]$,
And

$$u(t)=c(\varphi(t)-\varphi(0))^{\alpha-1}+\mathcal{I}_0^{\alpha;\varphi}[Au(s)+f(s,u(s))]. \tag{4}$$

for $t\in(0,a]$,
Then, we applique $\mathcal{D}_0^{\alpha-1;\varphi}$:

$$\mathcal{D}_0^{\alpha-1;\varphi}u(t)=c\mathcal{D}_0^{\alpha-1;\varphi}(\varphi(t)-\varphi(0))^{\alpha-1}+\mathcal{D}_0^{\alpha-1;\varphi}\mathcal{I}_0^{\alpha;\varphi}[Au(s)+f(s,u(s))]. \tag{5}$$

Also

$$\mathcal{D}_0^{\alpha-1;\varphi}u(t)=c+\int_0^t Au(s)+f(s,u(s))ds. \tag{6}$$

of the following condition:

$$\mathcal{D}_0^{\alpha-1;\varphi}(u(0))=u_0-g(u)$$

Then:

$$c=u_0-g(u)$$

The proof is complete.

Lemma 3.2. *If (2) holds, then we have*

$$u(t)=\alpha\int_0^\infty(\varphi(t)-\varphi(0))^{\alpha-1}\phi_\alpha(\theta)S\left((\varphi(t)-\varphi(0)^\alpha\theta)\right)(u_0-g(u))d\theta$$
$$+\alpha\int_0^t\int_0^\infty(\varphi(t)-\varphi(s))^{\alpha-1}\theta\phi_\alpha(\theta)S\left((\varphi(t)-\varphi(s)^\alpha\theta)\right)$$
$$\times f\left(s,u\left(s\right)\right)d\theta\varphi'(t)ds$$

Proof. Let $\lambda > 0$. Applying the generalized Laplace transforms to (2) , we have

$$U(\lambda) = \frac{u_0 - g(u)}{\lambda^\alpha} + \frac{1}{\lambda^\alpha}(AU(\lambda) + F(\lambda)),$$

where

$$U(\lambda) = \int_0^\infty e^{-\lambda(\varphi(\tau)-\varphi(0))} u(\tau)\varphi'(\tau)d\tau, \quad F(\lambda) = \int_0^\infty e^{-\lambda(\varphi(\tau)-\varphi(0))} f(\tau, u(\tau))\varphi'(\tau)d\tau.$$

Then we have:

$$\begin{aligned}
U(\lambda) &= (\lambda^\alpha I - \mathscr{A})^{-1}(u_0 - g(u)) + (\lambda^\alpha I - \mathscr{A})^{-1} F(\lambda) \\
&= \int_0^\infty e^{-\lambda^\alpha s} S(s)(u_0 - g(u))ds + \int_0^\infty e^{-\lambda^\alpha s} S(s)F(\lambda)ds \\
&= \alpha \int_0^\infty \hat{t}^{\alpha-1} e^{-(\lambda\hat{t})^\alpha} S\left(\hat{t}^\alpha\right)(u_0 - g(u))dt + \alpha \int_0^\infty \hat{t}^{\alpha-1} e^{-(\lambda t)^\alpha} S\left(\hat{t}^\alpha\right) F(\lambda)dt \\
&= e_1 + e_2.
\end{aligned}$$

Now putting: $\hat{t} = \varphi(t) - \varphi(0)$, next we consider the following one-sided stable probability density in [30]

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=1}^\infty (-1)^{k-1}\theta^{-\alpha k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha), \quad \theta \in (0, \infty)$$

whose integration is given by

$$\int_0^\infty e^{-\lambda\theta}\rho_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \quad \text{where } \alpha \in (0, 1).$$

$$\begin{aligned}
e_1 &= \alpha \int_0^\infty (\varphi(t) - \varphi(0))^{\alpha-1} e^{-(\lambda(\varphi(t)-\varphi(0)))^\alpha} S\left((\varphi(t) - \varphi(0))^\alpha\right)(u_0 - g(u))\varphi'(t)dt \\
&= \alpha \int_0^\infty \int_0^\infty (\varphi(t) - \varphi(0))^{\alpha-1} e^{-\lambda(\varphi(t)-\varphi(0))\theta} \rho_\alpha(\theta) S\left((\varphi(t) - \varphi(0))^\alpha\right)(u_0 - g(u))\varphi'(t)d\theta dt \\
&= \alpha \int_0^\infty \int_0^\infty e^{-\lambda(\varphi(t)-\varphi(0))} \frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\theta^\alpha}\rho_\alpha(\theta) S\left(\frac{(\varphi(t) - \varphi(0))^\alpha}{\theta^\alpha}\right)(u_0 - g(u))\varphi'(t)d\theta dt \\
&= \int_0^\infty e^{-\lambda(\varphi(t)-\varphi(0))} \int_0^\infty \alpha\frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\theta^\alpha}\rho_\alpha(\theta) S\left(\frac{(\varphi(t) - \varphi(0))^\alpha}{\theta^\alpha}\right)(u_0 - g(u))\varphi'(t)d\theta dt
\end{aligned}$$

Similar procedure:

$$\begin{aligned}
e_2 &= \alpha \int_0^\infty \int_0^\infty (\varphi(t) - \varphi(0))^{\alpha-1} e^{-(\lambda(\varphi(t)-\varphi(0)))^\alpha} S\left((\varphi(t) - \varphi(0))^\alpha\right) \\
&\quad \times e^{-(\lambda(\varphi(s)-\varphi(0)))^\alpha} f(s, u(s))\varphi'(s)\varphi'(t)dsdt \\
&= \int_0^\infty e^{-\lambda(\varphi(t)-\varphi(0))} \int_0^\infty \int_0^\infty \alpha\frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\theta^\alpha}\rho_\alpha(\theta) S\left(\frac{(\varphi(t) - \varphi(0))^\alpha}{\theta^\alpha}\right) \\
&\quad \times e^{-(\lambda(\varphi(s)-\varphi(0)))} f(s, u(s))\varphi'(s)d\theta ds\varphi'(t)dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda(\varphi(t)+\varphi(s)-2\varphi(0))} \alpha\frac{(\varphi(t) - \varphi(0))^{\alpha-1}}{\theta^\alpha}\rho_\alpha(\theta) S\left(\frac{(\varphi(t) - \varphi(0))^\alpha}{\theta^\alpha}\right) \\
&\quad \times f(s, u(s))\varphi'(s)d\theta ds\varphi'(t)dt
\end{aligned}$$

$$= \int_0^\infty \int_t^\infty \int_0^\infty e^{-\lambda(\varphi(\tau)-\varphi(0))} \alpha \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(t)-\varphi(0))^\alpha}{\theta^\alpha}\right)$$
$$\times f\left(\varphi^{-1}(\varphi(\tau)-\varphi(t)+\varphi(0)), u\left(\varphi^{-1}(\varphi(\tau)-\varphi(t)+\varphi(0))\right)\right) \varphi'(\tau) d\theta d\tau \varphi'(t) dt$$
$$= \int_0^\infty \int_0^\tau \int_0^\infty e^{-\lambda(\varphi(\tau)-\psi(0))} \alpha \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(t)-\varphi(0))^\alpha}{\theta^\alpha}\right)$$
$$\times f\left(\varphi^{-1}(\varphi(\tau)-\varphi(t)+\varphi(0)), u\left(\varphi^{-1}(\varphi(\tau)-\varphi(t)+\varphi(0))\right)\right) \varphi'(\tau) d\theta \varphi'(t) dt d\tau$$
$$= \int_0^\infty e^{-\lambda(\varphi(\tau)-\varphi(0))} \int_0^\tau \int_0^\infty \alpha \frac{(\varphi(\tau)-\varphi(s))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(\tau)-\varphi(s))^\alpha}{\theta^\alpha}\right)$$
$$\times f(s, u(s)) \varphi'(\tau) d\theta \varphi'(t) ds d\tau$$

Then we get:

$$U(\lambda) = \int_0^\infty e^{-\lambda(\varphi(t)-\varphi(0))} \int_0^\infty \alpha \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(t)-\varphi(0))^\alpha}{\theta^\alpha}\right)$$
$$\times (u_0 - g(u)) \varphi'(t) d\theta dt$$
$$+ \int_0^\infty e^{-\lambda(\varphi(\tau)-\varphi(0))} \int_0^\tau \int_0^\infty \alpha \frac{(\varphi(\tau)-\varphi(s))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(\tau)-\varphi(s))^\alpha}{\theta^\alpha}\right)$$
$$\times f(s, u(s)) \varphi'(\tau) d\theta \varphi'(t) ds d\tau$$

Now we invert the generalized Laplace transforms:

$$u(t) = \alpha \int_0^\infty \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(t)-\varphi(0))^\alpha}{\theta^\alpha}\right) (u_0 - g(u)) d\theta$$
$$+ \alpha \int_0^t \int_0^\infty \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta) S\left(\frac{(\varphi(t)-\varphi(s))^\alpha}{\theta^\alpha}\right)$$
$$\times f(s, u(s)) d\theta \varphi'(s) ds$$
$$= \alpha \int_0^\infty (\varphi(t)-\varphi(0))^{\alpha-1} \theta \phi_\alpha(\theta) S\left((\varphi(t)-\varphi(0)^\alpha \theta)\right) (u_0 - g(u)) d\theta$$
$$+ \alpha \int_0^t \int_0^\infty (\varphi(t)-\varphi(s))^{\alpha-1} \theta \phi_\alpha(\theta) S\left((\varphi(t)-\varphi(s)^\alpha \theta)\right)$$
$$\times f(s, u(s)) d\theta \varphi'(s) ds$$

where $\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \rho_\alpha\left(\theta^{-\frac{1}{\alpha}}\right)$ is the probability density function defined on $(0, \infty)$.

Proof is completed.

We define the operator $\Upsilon_\varphi^\alpha(t)$:

$$\Upsilon_\varphi^\alpha(t) u = \alpha \int_0^\infty \theta \phi_\alpha(\theta) S(t^\alpha \theta) u d\theta$$

for $u \in E$, $0 \leq s \leq t \leq T$.

Definition 3.3. *A function $u \in C([0,T], E)$ is called a mild solution of 1 if it satisfies:*

$$u(t) = (\varphi(t)-\varphi(0))^{\alpha-1} \Upsilon_\varphi^\alpha(\varphi(t)-\varphi(0))(u_0 - g(u))$$
$$+ \int_0^t (\varphi(t)-\varphi(s))^{\alpha-1} \Upsilon_\varphi^\alpha(\varphi(t)-\varphi(s)) f(s, u(s)) \varphi'(s) ds \quad t \in [0,T].$$

Lemma 3.4. *The operator $\Upsilon_{\varphi}^{\alpha}$ have the following properties:*

(i) For any fixed $t \geq 0, \Upsilon_{\varphi}^{\alpha}(t)$ are bounded linear operator with

$$\left\|\Upsilon_{\varphi}^{\alpha}(t)(u)\right\| \leq \frac{\alpha\xi}{\Gamma(1+\alpha)}\|u\| = \frac{\xi}{\Gamma(\alpha)}\|u\|$$

for all $u \in E$.

(ii) The operator $\Upsilon_{\varphi}^{\alpha}(t)$ are strongly continuous for all $t \geq 0$, that is, for every $u \in E$ and $0 \leq t_1 < t_2 \leq T$ we have:

$$\left\|\Upsilon_{\varphi}^{\alpha}(t_2)\,u - \Upsilon_{\varphi}^{\alpha}(t_1)\,u\right\| \to 0$$

as $t_1 \longrightarrow t_2$.

(iii) If $S(t)$ is compact operator for every $t > 0$, then $\Upsilon_{\varphi}^{\alpha}(t)$ are compact for all $t > 0$.

(iv) If $\Upsilon_{\varphi}^{\alpha}(t)$ is compact strongly continuous semigroup of bounded linear operator for $t > 0$, then $\Upsilon_{\varphi}^{\alpha}(t)$ are continuous in the uniform operator topology.

(v)Assume that $S(t)_{t>0}$ is compact operator. Then $S(t)_{t>0}$ is equicontinuous.

Proof. See the argument of [17].

4. Main results

U is a Banach space with the norm $|.|$. Let $a \in \mathbb{R}^+$, $J' = (0,a]$. Denote $C(J,U)$ as the Banach space of continuous functions from J into U with the norm:

$$\|u\| = \sup_{t\in[0,a]} |u(t)|,$$

where $u \in C(J,U)$, and $B(U)$ be the space of all bounded linear operators from U to U with the norm $\|S\|_{B(U)} = \sup\{|S(u)|; |u| = 1\}$, where $S \in B(U)$ and $u \in U$.

We put

$$U_{\varphi}^{(\alpha)}(J') = \left\{u \in C(J',U) : \lim_{t\to 0+}(\varphi(t)-\varphi(0))^{1-\alpha}u(t) \text{ exists and is finite}\right\}.$$

For any $x \in U_{\varphi}^{(\alpha)}(J')$, let the norm $\|\cdot\|_{\alpha,\varphi}$ defined by:

$$\|u\|_{\alpha,\varphi} = \sup_{t\in(0,a]}\left\{(\varphi(t)-\varphi(0))^{1-\alpha}|u(t)|\right\}.$$

is not difficult to verify $\|\cdot\|_{\alpha,\varphi}$ is a norm, the norm is covenable with $U_{\varphi}^{(\alpha)}(J')$, Then $\left(U_{\varphi}^{(\alpha)}(J'), \|\cdot\|_{\alpha,\varphi}\right)$ is a Banach space .

For $r > 0$, define a closed subset $B_r^{(\alpha)}(J') \subset U_{\varphi}^{(\alpha)}(J')$ as follows

$$B_r^{(\alpha)}(J') = \left\{u \in U_{\varphi}^{(\alpha)}(J') : \|u\|_{\alpha,\varphi} \leq r\right\}.$$

Thus, $B_r^{(\alpha)}(J')$ is a bounded closed and convex subset of $U_\varphi^{(\alpha)}(J')$. Let $B(J)$ be the closed ball of the space $C(J,U)$ with radius r and center at 0 , that is

$$B(J) = \{u \in C(J,U) : \|u\| \leq r\}.$$

Thus $B(J)$ is a bounded closed and convex subset of $C(J,U)$.

We introduce the following hypotheses :

(H0) $S(t)(t > 0)$ is equicontinuous, i.e., $S(t)$ is continuous in the uniform operator topology for $t > 0$;
(H1) for each $t \in J'$, the function $f(t,\cdot) : U \to U$ is continuous and for each $u \in U$, the function $f(\cdot,u) : J' \to U$ is strongly measurable;
(H2) there exists a function $\hbar \in L(J',\mathbb{R}^+)$such that

$$\mathcal{I}_0^{\alpha;\varphi}\hbar \in C(J',\mathbb{R}^+), \quad \lim_{t\to 0+}(\varphi(t)-\varphi(0))^{1-\alpha}\mathcal{I}_0^{\alpha;\varphi}\hbar(t) = 0,$$

and

$$|f(t,u)| \leq \hbar(t) \text{ for all } u \in B_r^{(\alpha)}(J') \text{ and almost all } t \in [0,a];$$

(H3) there exists a constant $L \in \left(0, \frac{\Gamma(\alpha)}{M}\right)$ such that the operator $g : C(J',U) \to L(J',U)$ satisfies:

$$|g(u_1) - g(u_2)| \leq L\|u_1 - u_2\|_{\alpha,\varphi}, \quad \text{for } u_1, u_2 \in B_r^{(\alpha)}(J')$$

(H4) there exists a constant $r > 0$ such that

$$\frac{M}{\Gamma(\alpha) - ML}\left(|u_0| + |g(0)| + \sup_{t\in(0,a]}\left\{(\varphi(t)-\varphi(0))^{1-\alpha}\mathcal{I}_0^{\alpha;\varphi}\hbar(t)\right\}\right) \leq r$$

(H5) For any $r > 0$, there exists $k(t) \in L^\infty([0,a],U)$ such that

$$|f(t,u_1(t)) - f(t,u_2(t))| \leq k(t)|u_1 - u_2|, \quad \text{for } u_1, u_2 \in B_r^{(\alpha)}(J')$$

For any $u \in B_r^{(\alpha)}(J')$, define an operator $\mathfrak{R}$ as follows

$$(\mathfrak{R}u)(t) = (\mathfrak{R}_1 u)(t) + (\mathfrak{R}_2 u)(t),$$

where

$$\begin{aligned} (\mathfrak{R}_1 u)(t) &= (\varphi(t)-\varphi(0))^{\alpha-1}\Upsilon_\varphi^\alpha(t,0)(u_0 - g(u)), && \text{for } t \in (0,a], \\ (\mathfrak{R}_2 u)(t) &= \int_0^t (\varphi(t)-\varphi(s))^{\alpha-1}\Upsilon_\varphi^\alpha(t,s)f(s,u(s))\varphi'(s)ds, && \text{for } t \in (0,a] \end{aligned}$$

It is easy to see that $\lim_{t\to 0+}(\varphi(t)-\varphi(0))^{1-\alpha}(\mathfrak{R}u)(t) = \frac{u_0 - g(u)}{\Gamma(\alpha)}$. For any $v \in B(J)$, set

$$u(t) = (\varphi(t)-\varphi(0))^{\alpha-1}v(t), \text{ for } t \in (0,a].$$

Then, $u \in B_r^{(\alpha)}(J')$. Define Ψ as follows

$$(\Psi v)(t) = (\Psi_1 v)(t) + (\Psi_2 v)(t),$$

where

$$(\Psi_1 v)(t) = \begin{cases} (\varphi(t)-\varphi(0))^{1-\alpha}(\mathfrak{R}_1 u)(t), & \text{for } t \in (0,a] \\ \frac{u_0 - g(u)}{\Gamma(\alpha)}, & \text{for } t = 0 \end{cases}$$

$$(\Psi_2 v)(t) = \begin{cases} (\varphi(t)-\varphi(0))^{1-\alpha}(\mathfrak{R}_2 u)(t), & \text{for } t \in (0,a] \\ 0, & \text{for } t = 0 \end{cases}$$

Obviously, u is a mild solution of (1) in $B_r^{(\alpha)}(J')$ if and only if the operator equation $u = \Re u$ has a solution $u \in B_r^{(\alpha)}(J')$.

Before giving the main results, we firstly prove the following lemmas.

Lemma 4.1. *Assume that* $(H_0) - (H_4)$ *hold, then* $\{\Psi v : v \in B(J)\}$ *is equicontinuous.*

Proof. **I.** $\{\Psi_1 v : v \in B(J)\}$ is equicontinuous.
For any $v \in B(J)$, let $u(t) = (\varphi(t) - \varphi(0))^{\alpha-1}v(t), t \in (0,a]$. Then $u \in B_r^{(q)}(J')$. For $t_1 = 0, 0 < t_2 \leq a$, we get

$$\begin{aligned}
|(\Psi_1 v)(t_2) - (\Psi_1 v)(0)| &\leq |\Upsilon_\varphi^\alpha(t_2)) - \frac{u_0 - g(u)}{\Gamma(\alpha)}| \\
&\leq |\left(\Upsilon_\varphi^\alpha(t_2) - \frac{1}{\Gamma(\alpha)}\right)(u_0 - g(u))| \\
&\leq |\left(\Upsilon_\varphi^\alpha(t_2) - \frac{1}{\Gamma(\alpha)}\right)|\left(|u_0| + L\|u\|_{\alpha,\varphi} + |g(0)|\right) \\
&\leq |\left(\Upsilon_\varphi^\alpha(t_2) - \frac{1}{\Gamma(\alpha)}\right)|(|u_0| + Lr + |g(0)|) \\
&\to 0, \text{ as } t_2 \to 0
\end{aligned}$$

For $0 < t_1 < t_2 \leq a$, we get

$$\begin{aligned}
|(\Psi_1 v)(t_2) - (\Psi_1 v)(t_1)| &\leq |\Upsilon_\varphi^\alpha(t_2)(u_0 - g(u)) - \Upsilon_\varphi^\alpha(t_1)(u_0 - g(u))| \\
&\leq |\left(\Upsilon_\varphi^\alpha(t_2) - \Upsilon_\varphi^\alpha(t_1)\right)(u_0 - g(u))| \\
&\leq |\left(\Upsilon_\varphi^\alpha(t_2) - \Upsilon_\varphi^\alpha(t_1)\right)|\left(|u_0| + L\|u\|_q + |g(0)|\right) \\
&\leq |\left(\Upsilon_\varphi^\alpha(t_2) - \Upsilon_\varphi^\alpha(t_1)\right)|(|u_0| + Lr + |g(0)|) \\
&\to 0, \text{ as } t_2 \to t_1.
\end{aligned}$$

Hence, $\{\Psi_1 v : v \in B(J)\}$ is equicontinuous.
II. $\{\Psi_2 v : v \in B(J)\}$ is equicontinuous.
For any $v \in B(J)$, let $u(t) = (\varphi(t) - \varphi(0))^{\alpha-1}v(t), t \in (0,a]$. Then $u \in B_r^{(q)}(J')$. For $t_1 = 0, 0 < t_2 \leq a$, we get

$$\begin{aligned}
|(\Psi_2 v)(t_2) - (\Psi_2 v)(0)| &= |(\varphi(t_2) - \varphi(0))^{1-\alpha} \times \\
&\int_0^{t_2} ((\varphi(t_2) - \varphi(s)))^{\alpha-1} \Upsilon_\varphi^\alpha((\varphi(t_2) - \varphi(s))) f(s, u(s)) ds| \\
&\leq \frac{M}{\Gamma(q)}(\varphi(t_2) - \varphi(s))^{1-\alpha} \int_0^{t_2} (\varphi(t_2) - \varphi(s))^{\alpha-1} \hbar(s) ds \\
&\to 0, \text{ as } t_2 \to 0.
\end{aligned}$$

For $0 < t_1 < t_2 \leq a$, we have

$$\begin{aligned}
&|(\Psi_2 v)(t_2) - (\Psi_2 v)(t_1)| \\
&\leq |\int_{t_1}^{t_2} (\varphi(t_2) - \varphi(0))^{1-\alpha}(\varphi(t_2) - \varphi(s))^{\alpha-1}\Upsilon_\varphi^\alpha((\varphi(t_2) - \varphi(s))) f(s, u(s)) ds| \\
&+ |(\varphi(t_2) - \varphi(0))^{1-\alpha}(\varphi(t_2) - \varphi(s))^{\alpha-1}\Upsilon_\varphi^\alpha((\varphi(t_2) - \varphi(s))) f(s, u(s)) ds \\
&- \int_0^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha}(\varphi(t_1) - \varphi(s))^{\alpha-1}\Upsilon_\varphi^\alpha((\varphi(t_2) - \varphi(s))) f(s, u(s)) ds|
\end{aligned}$$

$$
\begin{aligned}
&+ \Big| \int_0^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \Upsilon_\varphi^\alpha \left((\varphi(t_2) - \varphi(s))\right) f(s, u(s)) ds \\
&- \int_0^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \Upsilon_\varphi^\alpha \left((\varphi(t_1) - \varphi(s))\right) f(s, u(s)) ds \Big| \\
\le& \frac{M}{\Gamma(q)} \Big| \int_{t_1}^{t_2} (\varphi(t_2) - \varphi(0))^{1-\alpha} (\varphi(t_2) - \varphi(s))^{\alpha-1} \hbar(s) ds \Big| \\
&+ \frac{M}{\Gamma(q)} \int_0^{t_1} \left((\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} - (\varphi(t_2) - \varphi(0))^{1-\alpha} (\varphi(t_2) - \varphi(s))^{\alpha-1} \right) \\
&\times \hbar(s) ds \\
&+ \Big| \int_0^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \\
&\times \left(\Upsilon_\varphi^\alpha \left((\varphi(t_2) - \varphi(s))\right) f(s, u(s)) - \Upsilon_\varphi^\alpha \left((\varphi(t_2) - \varphi(s))\right) f(s, u(s)) \right) ds \Big| \\
\le& E_1 + E_2 + E_3,
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= \frac{M}{\Gamma(q)} \Big| \int_0^{t_2} (\varphi(t_2) - \varphi(0))^{1-\alpha} (\varphi(t_2) \\
&- \varphi(s))^{\alpha-1} \hbar(s) ds - \int_0^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \hbar(s) ds \Big| \\
E_2 &= \frac{2M}{\Gamma(q)} \int_0^{t_1} ((\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \\
&- (\varphi(t_2) - \varphi(0))^{1-\alpha} (\varphi(t_2) - \varphi(s))^{\alpha-1}) \hbar(s) ds \\
E_3 &= \Big| \int_0^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \\
&\times \left(\Upsilon_\varphi^\alpha \left((\varphi(t_2) - \varphi(s)\right) - \Upsilon_\varphi^\alpha \left((\varphi(t_1) - \varphi(s))\right) \right) f(s, x(s)) ds \Big|
\end{aligned}
$$

One can reduce that $\lim_{t_2 \to t_1} E_1 = 0$, since $\mathcal{I}_0^{\alpha;\varphi} \hbar \in C(J', \mathbb{R}^+)$. We know

$$
\begin{aligned}
&\left(\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} - \varphi(t_2) - \varphi(0))^{1-\alpha} (\varphi(t_2) - \varphi(s))^{\alpha-1} \right) \hbar(s) \\
&\le \varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \hbar(s)
\end{aligned}
$$

and $\int_0^{t_1} \varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \hbar(s) ds$ exists ($s \in [0, t_1]$), then by Lebesgue dominated convergence theorem, we have

$$
\int_0^{t_1} \left(\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} - \varphi(t_2) - \varphi(0))^{1-\alpha} (\varphi(t_2) - \varphi(s))^{\alpha-1} \right) \hbar(s) ds \to 0,
$$

as $t_2 \to t_1$

then one can deduce that $\lim_{t_2 \to t_1} E_2 = 0$.
For $\varepsilon > 0$ be enough small, we have

$$
\begin{aligned}
E_3 \le& \int_0^{t_1-\varepsilon} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \\
&\times \left\| \Upsilon_\varphi^\alpha \left((\varphi(t_2) - \varphi(s)\right) - \Upsilon_\varphi^\alpha \left((\varphi(t_1) - \varphi(s))\right) \right\|_{B(U)} |f(s, u(s))| ds \\
&+ \int_{t_1-\varepsilon}^{t_1} (\varphi(t_1) - \varphi(0))^{1-\alpha} (\varphi(t_1) - \varphi(s))^{\alpha-1} \\
&\times \left\| \Upsilon_\varphi^\alpha \left((\varphi(t_2) - \varphi(s)\right) - \Upsilon_\varphi^\alpha \left((\varphi(t_1) - \varphi(s))\right) \right\|_{B(U)} |f(s, u(s))| ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\varphi(t_1)-\varphi(0))^{1-\alpha}\int_0^{t_1}(\varphi(t_1)-\varphi(s))^{\alpha-1}\hbar(s)ds\\
&\quad\times \sup_{s\in[0,t_1-\varepsilon]}\left\|\Upsilon_\varphi^\alpha\left((\varphi(t_2)-\varphi(s)\right)-\Upsilon_\varphi^\alpha\left((\varphi(t_1)-\varphi(s)\right)\right\|_{B(U)}\\
&\quad+\frac{2M}{\Gamma(q)}\int_{t_1-\varepsilon}^{t_1}(\varphi(t_1)-\varphi(0))^{1-\alpha}\int_0^{t_1}(\varphi(t_1)-\varphi(s))^{\alpha-1}\hbar(s)ds\\
&\leq E_{3,1}+E_{3,2}+E_{3,3},
\end{aligned}
$$

where

$$
\begin{aligned}
E_{3,1}&=\frac{r\Gamma(\alpha)}{M}\sup_{s\in[0,t_1-\varepsilon]}\left\|\Upsilon_\varphi^\alpha\left((\varphi(t_2)-\varphi(s)\right)-\Upsilon_\varphi^\alpha\left((\varphi(t_1)-\varphi(s)\right)\right\|_{B(U)},\\
E_{3,2}&=\frac{2M}{\Gamma(\alpha)}|\int_0^{t_1}(\varphi(t_1)-\varphi(0))^{1-\alpha}(\varphi(t_1)-\varphi(s))^{\alpha-1}\hbar(s)ds\\
&-\int_0^{t_1-\varepsilon}(\varphi(t_1-\varepsilon)-\varphi(0))^{1-\alpha}(\varphi(t_1-\varepsilon)-\varphi(s))^{\alpha-1}\hbar(s)ds|,\\
E_{3,3}&=\frac{2M}{\Gamma(\alpha)}\int_0^{t_1-\varepsilon}((\varphi(t_1-\varepsilon)-\varphi(0))^{1-\alpha}(\varphi(t_1-\varepsilon)-\varphi(s))^{\alpha-1}\\
&-(\varphi(t_1)-\varphi(0))^{1-\alpha}(\varphi(t_1)-\varphi(s))^{\alpha-1})\hbar(s)ds.
\end{aligned}
$$

By H_0, it is easy to see that $E_{3,1}\to 0$ as $t_2\to t_1$. Similar to the proof that E_1, E_2 tend to zero, we get $E_{3,2}\to 0$ and $E_{3,3}\to 0$ as $\varepsilon\to 0$. Thus, E_3 tends to zero independently of $v\in B(J)$ as $t_2\to t_1, \varepsilon\to 0$. Therefore, $|(\Psi_2 v)(t_2)-(\Psi_2 v)(t_1)|$ tends to zero independently of $v\in B(J)$ as $t_2\to t_1$, which means that $\{\Psi_2 v : v\in B(J)\}$ is equicontinuous. Therefore, $\{\Psi v : v\in B(J)\}$ is equicontinuous.

Lemma 4.2. *Assume that $(H_1)-(H_4)$ hold. Then Ψ maps $B(J)$ into $B(J)$, and Ψ is continuous in $B(J)$.*

Proof. . **I.** Ψ maps $B(J)$ into $B(J)$. For any $v\in B(J)$, let $u(t)=(\varphi(t_1)-\varphi(0))^{\alpha-1}v(t)$. Then $u\in B_r^{(q)}(J')$. For $t\in[0,a]$, by $(H_1)-(H_4)$, we have

$$
\begin{aligned}
|(\Psi y)(t)| &\leq |\Upsilon_\varphi^\alpha(t)(u_0-g(u))|\\
&\quad+(\varphi(t)-\varphi(0))^{1-\alpha}|\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\Upsilon_\varphi^\alpha(\varphi(t)-\varphi(s))f\left(s,u\left(s\right)\right)\varphi'(s)ds|\\
&\leq \frac{M}{\Gamma(\alpha)}\left(|u_0|+L\|u\|_{\alpha,\varphi}+|g(0)|\right)\\
&\quad+\frac{M(\varphi(t)-\varphi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}|f(s,u(s))|\varphi'(s)ds\\
&\leq \frac{M}{\Gamma(q)}\left(|u_0|+Lr+|g(0)|+\sup_{t\in[0,a]}\left\{(\varphi(t)-\varphi(0))^{1-\alpha}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\hbar(s)\varphi'(s)ds\right\}\right)\\
&\leq r
\end{aligned}
$$

Hence, $\|\Psi v\|\leq r$, for any $v\in B(J)$.

II. Ψ is continuous in $B(J)$. For any $v_n, v\in B(J), n=1,2,\ldots$, with $\lim_{n\to\infty}v_n=v$, we have

$$\lim_{n\to\infty}v_n(t)=v(t) \text{ and } \lim_{n\to\infty}t^{q-1}v_n(t)=(\varphi(t)-\varphi(0))^{\alpha-1}v(t), \text{ for } t\in(0,a].$$

Then by (H_1), we have

$$
\begin{aligned}
&f\left(t,u_n(t)\right)=f\left(t,(\varphi(t)-\varphi(0))^{\alpha-1}v_n(t)\right)\to f\left(t,(\varphi(t)-\varphi(0))^{\alpha-1}v(t)\right)=f(t,u(t)),\\
&\quad\text{as } m\to\infty,
\end{aligned}
$$

where $u_n(t) = (\varphi(t) - \varphi(0))^{\alpha-1}v_n(t)$ and $u(t) = (\varphi(t) - \varphi(0))^{\alpha-1}v(t)$. On the one hand, using (H_2), we get for each $t \in J'$,

$$(\varphi(t) - \varphi(s))^{\alpha-1}|f(s, u_n(s)) - f(s, u(s))| \leq (\varphi(t) - \varphi(s))^{\alpha-1}2\hbar(s) \quad , \text{a.e. in } [0, t].$$

On the other hand, the function $s \to (\varphi(t) - \varphi(s))^{\alpha-1}2\hbar(s)$ is integrable for $s \in [0, t]$ and $t \in J$. By Lebesgue dominated convergence theorem, we get

$$\int_0^t (\varphi(t) - \varphi(s))^{\alpha-1}|f(s, u_n(s)) - f(s, u(s))|ds \to 0, \text{ as } n \to 0.$$

For $t \in [0, a]$

$$\begin{aligned}
|(\Psi v_n)(t) - (\Psi v)(t)| &= |(\varphi(t) - \varphi(0))^{1-\alpha}(\mathfrak{R}u_n(t) - \mathfrak{R}u(t))| \leq |\Upsilon_\varphi^\alpha(t)(g(u_n) - g(u))| \\
&+ (\varphi(t) - \varphi(0))^{1-\alpha}|\int_0^t (\varphi(t) - \varphi(s))^{\alpha-1}\Upsilon_\varphi^\alpha(t)(\varphi(t) - \varphi(s))(f(s, u_n(s)) - f(s, u(s)))ds| \\
&\leq \frac{ML}{\Gamma(q)}\|u_n - u\|_{\alpha,\varphi} + \frac{M(\varphi(t) - \varphi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (\varphi(t) - \varphi(s))^{\alpha-1}|f(s, u_n(s)) - f(s, u(s))|ds \\
&\leq \frac{ML}{\Gamma(q)}\|v_n - v\| + \frac{M(\varphi(t) - \varphi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (\varphi(t) - \varphi(s))^{\alpha-1}|f(s, u_n(s)) - f(s, u(s))|ds
\end{aligned}$$

Therefore, $\Psi v_n \to \Psi v$ pointwise on J as $m \to \infty$, by which Lemma 4.1 implies that $\Psi v_n \to \Psi v$ uniformly on J as $n \to \infty$ and so Ψ is continuous.

4.1. Existence result

In the following, we suppose that the operator A generates a compact C_0-semigroup $\{S(t)\}_{t\geq 0}$ on U, that is, for any $t > 0$, the operator $S(t)$ is compact.

Theorem 4.3. *Assume that $(H_1) - (H_4)$ hold. Then nonlocal problem (1) has at least one mild solution in $B_r^{(\alpha)}(J')$.*

Proof. . Obviously, u is a mild solution of (1) in $B_r^{(q)}(J')$ if and only if v is a fixed point of $v = \Psi v$ in $B(J)$, where $u(t) = (\varphi(t) - \varphi(0))^{1-\alpha}v(t)$. So, it is enough to prove that $v = \Psi v$ has a fixed point in $B(J)$. For any $v_1, v_2 \in B(J)$, according to (H_3), we have

$$\begin{aligned}
|\Psi_1 v_1(t) - \Psi_1 v_2(t)| &= (\varphi(t) - \varphi(0))^{1-\alpha}|(\mathfrak{R}_1 u_1)(t) - (\mathfrak{R}_1 u_2)(t)| \\
&\leq \frac{M}{\Gamma(\alpha)}|g(u_1) - g(u_2)| \\
&\leq \frac{ML}{\Gamma(\alpha)}\|u_1 - u_2\|_{\alpha,\varphi} \\
&= \frac{ML}{\Gamma(\alpha)}\|v_1 - v_2\|
\end{aligned}$$

which implies that $\|\Psi_1 v_1 - \Psi_1 v_2\| \leq \frac{ML}{\Gamma(\alpha)}\|v_1 - v_2\|$. Thus, we obtain that

$$\alpha(\Psi_1(B(J))) \leq \frac{ML}{\Gamma(\alpha)}\alpha(B(J)). \tag{7}$$

Next, we show that for any $t \in [0, a], W(t) = \{(\Psi_2 v)(t), v \in B(J)\}$ is relatively compact in U. Obviously, $W(0)$ is relatively compact in U. Let $t \in (0, a]$ be fixed. For every $\varepsilon \in (0, t)$ and $\delta > 0$, define an operator $\Psi_{\varepsilon,\delta}$

on $B(J)$ by the formula

$$\begin{aligned}(\Psi_{\varepsilon,\delta}v)(t) &= \alpha(\varphi(t)-\varphi(0))^{\alpha-1}\int_0^{t-\varepsilon}\int_\delta^\infty \theta(\varphi(t)-\varphi(s))^{1-\alpha}\phi_\alpha(\theta)S\left((\varphi(t)-\varphi(s))^\alpha\theta\right)\\ &\times f(s,u(s))d\theta ds\\ &= \alpha(\varphi(t)-\varphi(0))^{\alpha-1}S(\varphi(\varepsilon^\alpha\delta))\int_0^{t-\varepsilon}\int_\delta^\infty \theta(\varphi(t)-\varphi(s))^{1-\alpha}\phi_\alpha(\theta)S\left((\varphi(t)-\varphi(s))^\alpha\theta-\varphi(\varepsilon^\alpha\delta)\right)\\ &\times f(s,u(s))d\theta ds,\end{aligned}$$

where $u \in B_r^{(q)}(J')$. Then from the compactness of $S(\varphi(\varepsilon^\alpha\delta))\,(\varepsilon^q\delta > 0)$, we obtain that the set $W_{\varepsilon,\delta}(t) = \{(\Psi_{\varepsilon,\delta})(t), v \in B(J)\}$ is relatively compact in U for every $\varepsilon \in (0,t)$ and $\delta > 0$. Moreover, for every $v \in B(J)$, we have

$$\begin{aligned}|(\Psi_2 v)(t)-(\Psi_{\varepsilon,\delta}v)(t)| &\le |\alpha(\varphi(t)-\varphi(0))^{\alpha-1}\int_0^t\int_0^\delta \theta(\varphi(t)-\varphi(s))^{1-\alpha}\phi_\alpha(\theta)\\ &\times S\left((\varphi(t)-\varphi(s))^\alpha\theta\right)f(s,u(s))d\theta ds|\\ &+|\alpha(\varphi(t)-\varphi(0))^{\alpha-1}\int_{t-\varepsilon}^t\int_\delta^\infty \theta(\varphi(t)-\varphi(s))^{1-\alpha}\phi_\alpha(\theta)\\ &\times S\left((\varphi(t)-\varphi(s))^\alpha\theta\right)f(s,u(s))d\theta ds|\\ &\le \alpha(\varphi(t)-\varphi(0))^{1-\alpha}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\hbar(s)ds\int_0^\delta\theta\phi_\alpha(\theta)d\theta\\ &+\alpha(\varphi(t)-\varphi(0))^{1-\alpha}\int_{t-\varepsilon}^t(\varphi(t)-\varphi(s))^{\alpha-1}\hbar(s)ds\int_0^\infty\theta\phi_\alpha(\theta)d\theta\\ &\le \alpha M(\varphi(t)-\varphi(0))^{1-\alpha}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\hbar(s)ds\int_0^\delta\theta\phi_\alpha(\theta)d\theta\\ &+\frac{M}{\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{1-\alpha}\int_{t-\varepsilon}^t(\varphi(t)-\varphi(s))^{\alpha-1}\hbar(s)ds\\ &\to 0 \quad \text{as } \varepsilon\to 0, \delta\to 0.\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $W(t), t > 0$. Hence the set $W(t), t > 0$ is also relatively compact in U. Therefore, $\{(\Psi_2 v)(t), v \in B(J)\}$ is relatively compact by Arzela-Ascoli theorem. Indeed

For any $\{u_n\} \subset B_r^{(\alpha)}(J')$, set

$$v_n(t) = \begin{cases}(\varphi(t)-\varphi(0))^{1-\alpha}u_n(t), & \text{if } t \in (0,a],\\ v_n(0_+), & \text{if } t = 0.\end{cases}$$

Then $\{v_n\} \subset B(J)$. We can find at least one sequence $\{\Psi_2 v_{n_m}\}_{m=1}^\infty$ which is convergent. Hence,

$$\lim_{m\to\infty}(\varphi(t)-\varphi(0))^{1-\alpha}\left(\mathfrak{R}_2 x_{n_m}\right)(t) = \lim_{m\to\infty}\left(\Psi_2 v_{n_m}\right)(t), \quad \text{for } t \in (0,a].$$

This means that $\{\mathfrak{R}_2 u_{n_m}\}_{m=1}^\infty$ is convergent in $B_r^{(q)}(J')$. Therefore, $\left\{(\mathfrak{R}_2 u)(t), u \in B_r^{(\alpha)}(J')\right\}$ is relatively compact. Thus, we have $\alpha\left(\Psi_2\left(B_r^{(\alpha)}(J')\right)\right) = 0$. By (7), we have

$$\begin{aligned}\alpha(\Psi(B(J))) &\le \alpha(\Psi_1(B(J))) + \alpha(\Psi_2(B(J)))\\ &\le \frac{ML}{\Gamma(\alpha)}\alpha(B(J))\end{aligned}$$

Thus, the operator Ψ is an α-contraction in $B(J)$. By Lemma 4.2, we know that Ψ is continuous. Hence, Darbo-Sadovskii's fixed point theorem 2.16 shows that Ψ has a fixed point $v^* \in B(J)$. Let $u^*(t) = (\varphi(t) - \varphi(0))^{\alpha-1}v^*(t)$. Then u^* is a mild solution of (1).

4.2. uniqueness result

Theorem 4.4. *Assume that* $(H_1)-(H_5)$ *hold. Then problem (1) has a unique mild solution in* $B_r^{(\alpha)}(J')$*. Provided that*

$$\frac{\xi L}{\Gamma(\alpha)}(\varphi(a)-\varphi(0))^{\alpha-1}<1$$

Proof. Let u_1 and u_2 be the solutions of the problem (1) in $B_r^{\alpha}((0,a])$. Then, for each $i\in\{1,2\}$, the solution u_i satisfies

$$\begin{aligned}u_i(t)&=(\varphi(t)-\varphi(0))^{\alpha-1}\Upsilon_{\varphi}^{\alpha}(\varphi(t)-\varphi(0))(u_0-g(u_i))\\&+\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\Upsilon_{\varphi}^{\alpha}(\varphi(t)-\varphi(s))f(s,u_i(s))\,\varphi'(s)ds\quad t\in[0,T].\end{aligned}$$

Then, for any $t\in[0,a]$,

$$\begin{aligned}\|u_1-u_2\|&\leq(\varphi(t)-\varphi(0))^{\alpha-1}\|\Upsilon_{\varphi}^{\alpha}(\varphi(t)-\varphi(0))(g(u_1)-g(u_2))\|\\&+\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\|\Upsilon_{\varphi}^{\alpha}(\varphi(t)-\varphi(s))f(s,u_1(s))-f(s,u_2(s))\|\varphi'(s)ds\\&\leq\frac{\xi}{\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{\alpha-1}\|(g(u_1)-g(u_2))\|\\&+\frac{\xi}{\Gamma(\alpha)}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\|f(s,u_1(s))-f(s,u_2(s))\|\varphi'(s)ds\\&\leq\frac{\xi L}{\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{\alpha-1}\|u_1-u_2\|+\frac{\xi\nu}{\Gamma(\alpha)}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\|u_1-u_2\|\varphi'(s)ds\end{aligned}$$

Then

$$\left(1-\frac{\xi L}{\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{\alpha-1}\right)\|u_1-u_2\|\leq\frac{\xi\nu}{\Gamma(\alpha)}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\|u_1-u_2\|\varphi'(s)ds$$

under the previous condition

$$\|u_1-u_2\|\leq\frac{\xi\nu}{\Gamma(\alpha)-\xi L(\varphi(t)-\varphi(0))^{\alpha-1}}\int_0^t(\varphi(t)-\varphi(s))^{\alpha-1}\|u_1-u_2\|\varphi'(s)ds$$

where $\nu=\sup_{0\leq t\leq T}|\hbar(t)|$. By Gronwall inequality (2.10), we obtain

$$\|u_1(t)-u_2(t)\|=0\quad\text{for all}\quad t\in[0,T]$$

implies that $u_1\equiv u_2$, then the result.

5. Illustrative application

Let $U=L^2([0,\pi],\mathbb{R})$ equipped with the norm, for all $u,v\in L^2([0,\pi])$ by:

$$\|u\|=\left(\int_0^{\pi}|u(x)|^2dx\right)^{\frac{1}{2}}.$$

Consider the following problem of time-fractional parabolic partial differential equation:

$$\begin{cases}\mathcal{D}_0^{\frac{3}{4};\ln(1+t)}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=t^{-\frac{1}{4}}\cos(u(x,t)),\quad t\in(0,a],x\in[0,\pi]\\u(0,t)=u(\pi,t)=0,\quad t\in(0,a]\\\mathcal{D}_0^{-\frac{1}{4};\ln(1+t)}u(x,0)+\sum_{i=0}^{n}\int_0^{\pi}e^{-(y+x)}u(t_i,y)\,dy=u_0(x),\quad x\in[0,\pi]\end{cases}$$

where $\mathcal{D}_0^{\frac{3}{4};\ln(1+t)}$ is $\ln(1+t)$ -Riemann-Liouville fractional derivative of order $\frac{3}{4}$, $a > 0$, n is a positive integer, $0 < t_0 < t_1 < \ldots < t_n \leq a$, $u_0(z) \in U = L^2([0,\pi],\mathbb{R})$.

We define an operator A by $Au = \frac{\partial^2}{\partial x^2}u$ with the domain

$$D(A) = \{v(\cdot) \in X : v, v' \text{ absolutely continuous}, \quad v'' \in X, v(0) = v(\pi) = 0\}.$$

Then A generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ which is compact, analytic and selfadjoint.

$$f(t,u(x,t)) = t^{-\frac{1}{4}}\cos(u(x,t)),$$

and the operator $g : C(J',U) \to L(J',U)$ is given by

$$g(u) = \sum_{i=0}^{n}\int_0^{\pi} e^{-(y+x)}u(t_i,y)\,dy$$

for $v \in U = L^2([0,\pi],\mathbb{R}), x \in [0,\pi]$, and choose:

$$\hbar(t) = t^{-1/4}, \quad L = \frac{n+1}{2}\left(1 - e^{-2\pi}\right)$$

and

$$r = \frac{1}{\Gamma\left(\frac{3}{4}\right) - L}\left(|u_0| + 2\sqrt{a^{\frac{1}{2}}\ln(1+a)}\right), \quad \text{provided that} \quad \frac{L}{\Gamma(3/4)} < 1.$$

Then, (H1)-(H4) are satisfied. According to Theorem 4.3, system (1) has a mild solution in $B_r^{(3/4)}((0,a])$. By using the theorem (4.4) provided that we had the uniquensses of solution:

$$\frac{(n+1)\left(1-e^{-2\pi}\right)}{2\Gamma(\frac{3}{4})\ln(1+t)^{\frac{1}{4}}} < 1$$

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