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Stability analysis of periodic solutions of the neutral-type neural networks with impulses and time-varying delays

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Abstract. This paper is concerned with a class of neutral-type neural networks with impulses and delays. By using continuation theorem due to Mawhin and constructing the appropriate Lyapunov-Krasovskii functional, several new sufficient conditions ensuring the existence and global exponential stability of the periodic solution are obtained. Moreover, a numerical example is provided to illustrate the main results. Our results can extend and improve some earlier publications.

1. Introduction

1.1. Previous works

In recent years, neutral-type neural network models have been extensively studied and successfully applied to various science and engineering fields such as mechanics, electrical engineering, automatic control, parallel computation and so on. As was pointed by Hale [9] that the main reason for considering the neutral equation with difference operators is that it will be included without imposing too many smoothness conditions on the initial data.

A source of instability for neural networks is time delay which inevitably exists in the implementation of artificial neural networks due to the finite switching speed of amplifiers or network congestion. Therefore, stability analysis for delayed neutral-type neural networks has become an important research topic and various criteria have been developed in the literature over the past decade, see [7], [12], [13], [21], [22], [25], [28] and the references therein. For example, Orman [21] derived the sufficient conditions for global stability of neutral-type neural networks with time delays, by using the new LMI conditions, Rakkiyappan and Balasubramaniam [22] considered the global asymptotic stability results for neutral-type neural networks with distributed time delays. Taken the discontinuous activations into account, Kong et al. in [12] and [13] studied the stability and synchronization of the discontinuous neutral-type neural networks with delays based on the Filippov solution theory and the Lyapunov-Krasovskii functionals.

On the other hand, some evolution processes are subject to sudden changes. The mathematical description of these processes leads to impulsive differential equations. This type of differential equations can use

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to describe population dynamics, biological phenomena or several physical situations. We refer the reader to [3], [11], [20], for some results and applications of the impulsive differential equations. Impulsive effects are also likely to exist in the neural network system. For example, in implementation of electronic networks in which state is subject to instantaneous perturbations and experiences abrupt change at certain moments, which may be caused by switching phenomenon, frequency change, or other sudden noise, that is, does exhibit impulsive effects [14]. So, it is worthwhile to study the neural networks with impulse. Recently, several kinds of neural networks with impulse have been investigated, see [1], [2], [4], [6], [10], [15], [19], [23], [24], [26] and the references therein. For example, Wang et al in [24] studied globally exponential stability of periodic solutions for impulsive neutral-type neural networks with delays, by establishing a singular impulsive delay differential inequality and employing contraction mapping principle, the authors established the new results of existence and global exponential stability of the periodic solution.

Although the neutral-type neural networks with impulses have been widely studied, there are still two problems needed to be considered further.

- First, through the research of neutral-type neural networks with impulses, we find that the neutral character in neural networks is often showed by the nonlinear term like $h_j(\dot{x}_j(t-\cdot)$ and rarely showed by the difference operator $Ax(t) = x(t) - c(t)x(t-\cdot)$. This may be due to the fact that the mechanism on which how the solution is influenced by the impulses and the difference operator A associated to neutral-type neural networks is far away from clear.

- Second, to the author's best knowledge, few papers applied the method of Mawhin's continuation theorem to study the generalized neutral-type neural networks with impulses and delays.

1.2. Model Formulation

Motivated by the above fact, in this paper, we consider the following neutral-type neural networks with impulses and delays:

$$\begin{cases} (A_ix_i)'(t) & = -a_i(t)x_i(t) + \sum_{j=1}^{n} \Big[b_{ij}(t)f_j(x_j(t)) \\ & \quad + d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\Big] + I_i(t), \ \ t > 0, \ \ t \neq t_k, \\ \Delta x_i(t_k) & = x_i(t_k^+) - x_i(t_k^-) = e_{ik}(x_i(t_k)), \\ & \quad i = 1,2,...,n, \ k = 1,2,..., \end{cases} \tag{1.1}$$

where A_i is the difference operator defined by

$$A_ix_i(t) = x_i(t) - \sum_{j=1}^{n} c_{ij}(t)x_i(t-\delta_{ij}(t)), \ \ i = 1,2,...,n, \tag{1.2}$$

$\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ denotes the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$; $x_i(t)$ and $I_i(t)$ represent the activation and external input of the *ith* neuron at time t, respectively; $a_i(t)$ represents the rate with which the *ith* unit will reset its potential to the resting state when disconnected from the network and external inputs at time t; $\delta_{ij}(t)$ and $\tau_{ij}(t)$ correspond to the finite speed of the axonal transmission of signal; $b_{ij}(t)$ denotes the strength of the *jth* unit on the *ith* unit at time t; $d_{ij}(t)$ denotes the strength of the *jth* unit on the *ith* unit at time $t - \tau_{ij}(t)$; f_j and g_j are signal transmission functions. Throughout this paper, we always assume that $a_i(t)$, $b_{ij}(t)$, $d_{ij}(t)$, $\tau_{ij}(t)$, $I_i(t)$, $\delta_{ij}(t)$ are continuously periodic functions defined on $t \in [0,\infty)$. Moreover, $a_i(\cdot)$, $b_{ij}(\cdot)$, $d_{ij}(\cdot)$, $\tau_{ij}(\cdot)$, $\delta_{ij}(\cdot)$ are positive everywhere, $f_j(x_j)$ and $g_j(x_j)$ are continuous. Let $\xi = \max\limits_{1\le i,j\le n}\{\sup_{t\in\mathbb{R}}|\tau_{ij}(t)|, \sup_{t\in\mathbb{R}}|\delta_{ij}(t)|\}$.

System (1.1) is supplemented with initial values given by

$$x_i(s) = \phi_i(s), \ \ s \in [-\xi, 0], \ \ i = 1,2,...,n,$$

where $\phi_i(\cdot)$ denote continuous T-periodic function defined on $[-\xi, 0]$.

1.3. Major contributions

In comparison to the existing results, the key contributions of this paper can be shown by the following four aspects:

- Unlike the previous neutral-type neural networks, the neutral-type neural networks considered in the paper shows the neutral character by the A_i operator, which is different from the corresponding results of other papers.
- When $c_{ij} \equiv 0$, $i, j = 1, 2, ..., n$, system (1.1) is changed into a non-neutral type neural networks with impulses which have been extensively studied.
- Since there is few paper concerning with the periodic solution and stability of the neutral-type neural networks with impulses and delays, this paper aims to investigate the new existence results of periodic solutions based on the new method of continuation theorem. Moreover, the periodic solutions are further proved to be global exponential asymptotic stable.

The remainder of this paper is organized as follows: In Section 2, we present some preliminary results. In Section 3, under suitable hypotheses, we show that system (1.1) possesses at least one T-periodic solution. In Section 4, by constructing the appropriate Lyapunov function, we derive several sufficient conditions ensuring that the periodic solutions of (1.1) are global exponential stable. In Section 5, with the help of an example, we demonstrate the applicability of our main results.

2. Preliminary

In this section, we make some necessary preparations. Firstly, we introduce the following notations

$$\begin{aligned}
&\bar{a}_i = \frac{1}{T}\int_0^T a_i(t)dt,\ \bar{b}_{ij} = \frac{1}{T}\int_0^T b_{ij}(t)dt,\ \bar{d}_{ij} = \frac{1}{T}\int_0^T d_{ij}(t)dt,\\
&\bar{I}_i = \frac{1}{T}\int_0^T I_i(t)dt, a_i^+ = \max_{t\in[0,T]} |a_i(t)|,\ b_{ij}^+ = \max_{t\in[0,T]} |b_{ij}(t)|,\\
&d_{ij}^+ = \max_{t\in[0,T]} |d_{ij}(t)|, I_i^+ = \max_{t\in[0,T]} |I_i(t)|, a_i^- = \min_{t\in[0,T]} |a_i(t)|,\\
&b_{ij}^- = \min_{t\in[0,T]} |b_{ij}(t)|,\ d_{ij}^- = \min_{t\in[0,T]} |d_{ij}(t)|,\ I_i^- = \min_{t\in[0,T]} |I_i(t)|,\\
&\tau = \max_{1\le i\le n, 1\le j\le n} \{\tau_{ij}^+\},\ \tau_{ij}^+ = \max_{t\in[0,T]} \tau_{ij}(t),\ i, j = 1, 2, ..., n.
\end{aligned}$$

Define

$$\begin{aligned}
\|x\|_2 =& \Big(\int_0^T |x(t)|^2 dt\Big)^{\frac{1}{2}},\ \text{for } x \in C(\mathbb{R}, \mathbb{R}),\\
B_i(t) =& -a_i(t)x_i(t) + \sum_{j=1}^{n} \Big[b_{ij}(t)f_j(x_j(t)) + d_{ij}(t)g_j(x_j(t - \tau_{ij}(t)))\Big]\\
&+ I_i(t),\ i = 1, 2, ..., n,\\
C_T =& \Big\{\phi \in C(\mathbb{R}, \mathbb{R}), \phi(t + T) \equiv \phi(t)\Big\}.
\end{aligned}$$

From Hale's terminology [9], a solution of the system (1.1) is $x_i \in C(\mathbb{R}, \mathbb{R})$ such that $A_i x_i \in C^1(\mathbb{R}, \mathbb{R})$ and system (1.1) is satisfied on $\mathbb{R}$. In general, x_i is not from $C^1(\mathbb{R}, \mathbb{R})$. Nevertheless, it is easy to see that $(A_i x_i)' = A_i x_i'$. Thus, a T-periodic solution x_i of the (1.1) must be from $C^1(\mathbb{R}, \mathbb{R})$.

For any solution $x(t) = (x_1, x_2, ..., x_n)^\top$ and periodic solution of system (1.1), $x^*(t) = (x_1^*, x_2^*, ..., x_n^*)^\top$, let

$$\|\varphi - x^*\| = \sum_{i=1}^{n} \max_{t\in[-\tau,0]} |\varphi_i(t) - x^*(t)|.$$

Consider the impulsive system

$$\begin{cases} x'(t) = f(t, x_t),\ t \neq t_k,\ k = 1, 2, ..., \\ \Delta x(t_k) = e_k(x(t_k)), \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(t + T, x) = f(t, x)$, $e_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous, $x_t(s) = x(t + s)$, $-\tau \le s \le 0$, and there exists a positive integer q such that $t_{k+q} = t_k + T$, $e_{k+q}(x) = e_k(x)$ with $t_k \in \mathbb{R}$, $t_{k+1} > t_k$, $\lim_{k\to\infty} t_k \to \infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$. For $t_k \neq 0 (k = 1, 2, ...)$. $[0, T] \cap \{t_k\} = \{t_1, t_2, ..., t_m\}$, where t_k are called the set of jump points.

Let us recall some definitions. For the Cauchy problem

$$\begin{cases} x'(t) = f(t, x_t),\ t \neq t_k,\ t \in [0, T],\ k = 1, 2, ..., \\ \Delta x(t_k) = e_k(x(t_k^-)),\ x(0) = x_0. \end{cases} \tag{2.2}$$

Definition 2.1. A map $x : [0, T] \to \mathbb{R}^n$ is said to be a solution of (2.2), if it satisfies the following conditions:

(1) $x(t)$ is a piecewise continuous map with first-class discontinuity points in $[0, T] \cap \{t_k\}$, and at each discontinuity point it is continuous on the left;

(2) $x(t)$ satisfies (2.2).

Definition 2.2. (See [3]) A map $x : [0, T] \to \mathbb{R}^n$ is said to be a T-periodic solution of (2.2), if

(1) $x(t)$ satisfies (1) and (2) of Definition 2.1 in the interval $[0, T]$;

(2) $x(t)$ satisfies $x(t + T^-) = x(t^-)$, $t \in \mathbb{R}$.

Obviously, if $x(t)$ is a solution of (2.2) defined on $[0, T]$ such that $x(0) = x(T)$, then by the periodicity of (2.2) in t, the function $x^*(t)$ defined by

$$x^*(t) = \begin{cases} x(t - hT), t \in [hT, (h + 1)T] \setminus \{t_k\}, \\ x^*(t) \text{ is left continuous at } t = t_k \end{cases}$$

is a T-periodic solution of (2.1).

Definition 2.3. (See [3]) The periodic solution x^* of system (1.1) is said to be globally exponentially stable, if there exist constants $\lambda > 0$ and $M > 1$ such that

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \le Me^{-\lambda t}\|\varphi - x^*\|.$$

Lemma 1. *(See [8, 18])* Let X and Y be two real Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \bar{\Omega} \subset X \to Y$ be L-compact on $\bar{\Omega}$. Suppose that all of the following conditions hold:

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L),\ \forall \lambda \in (0, 1)$;

(2) $QNx \neq 0, \forall x \in \partial\Omega \cap \ker L$;

(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where $J : \mathrm{Im}Q \to \ker L$ is an homeomorphism map.

Then the equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

Lemma 2. *(See [16])* Suppose $\delta_{ij} \in C^1(\mathbb{R}, \mathbb{R})$ with $\delta_{ij}(t+T) \equiv \delta_{ij}(t)$ and $0 < \delta'_{ij}(t) < 1$ for all $t \in [0, T]$. Then the function $t - \delta_{ij}(t)$ has an inverse $\mu_{ij}(t)$ satisfying $\mu \in C(\mathbb{R}, \mathbb{R})$ with $\mu_{ij}(t+T) = \mu_{ij}(t) + T$ for all $t \in [0, T]$.

Throughout this paper, besides $\delta_{ij}(i, j = 1, 2, ..., n)$ being a periodic function with period T, we assume in addition that $\delta_{ij} \in C^1(\mathbb{R}, \mathbb{R})$ with $0 < \delta'_{ij}(t) < 1$ for all $t \in [0, T]$.

Remark 1. From the above assumption, one can see from Lemma 3 that the function $t - \delta_{ij}(t)$ has an inverse. Denote the inverse of the function $t - \delta_{ij}(t)$ by the function $\mu_{ij}(t)$.

Define

$$\sigma_0 = -\min_{t\in[0,T]} \delta'_{ij}(t),\ \sigma_1 = \max_{t\in[0,T]} \delta'_{ij}(t),\ \delta^+_{ij} = \max_{t\in[0,T]} |\delta_{ij}(t)|.$$

Clearly, $\sigma_0 \geq 0$ and $0 \leq \sigma_1 < 1$.

Lemma 3. Let

$$W_i : C_T \to C_T,$$

$$(W_i x_i)(t) = \sum_{j=1}^{n} c_{ij}(t) x_i(t - \delta_{ij}(t)),\ i = 1, 2, ..., n,$$

if $\sum_{j=1}^n |c^+_{ij}| < 1$, then W_i satisfies the following conditions:

(1) $\|W_i\| \leq \sum_{j=1}^n |c^+_{ij}| < 1,\ i = 1, 2, ..., n;$

(2) $\int_0^T \left|(W_i x_i)(t)\right| dt \leq \frac{\sum_{j=1}^n |c^+_{ij}|}{1-\sigma_1} \int_0^T |x_i(t)| dt,\ i = 1, 2, ..., n;$

(3) $\int_0^T \left|(W_i x_i)(t)\right|^2 dt \leq \frac{\sum_{j=1}^n |c^+_{ij}|^2}{1-\sigma_1} \int_0^T |x_i(t)|^2 dt,\ i = 1, 2, ..., n,$
where $\sigma_1 = \max_{t\in[0,T]} \delta'_{ij}(t),\ i, j = 1, 2, ..., n$, and $0 \leq \sigma_1 < 1$.

Proof. (2) By Lemma 2 and Remark 1, we have

$$\begin{aligned}
\int_0^T \left|(W_i x_i)(t)\right| dt &\leq \sum_{j=1}^n \int_0^T |c_{ij}(t)||x_i(t - \delta_{ij}(t))| dt \\
&\leq \sum_{j=1}^n |c^+_{ij}| \int_0^T |x_i(t - \delta_{ij}(t))| dt \\
&= \sum_{j=1}^n |c^+_{ij}| \int_{-\delta_{ij}(0)}^{T-\delta_{ij}(T)} \frac{1}{1 - \delta'_{ij}(\mu_{ij}(s))} |x_i(s)| ds \\
&= \sum_{j=1}^n |c^+_{ij}| \int_0^T \frac{1}{1 - \delta'_{ij}(\mu_{ij}(s))} |x_i(s)| ds \\
&\leq \frac{\sum_{j=1}^n |c^+_{ij}|}{1 - \sigma_1} \int_0^T |x_i(t)| dt.
\end{aligned}$$

(3) By Lemma 2 and Remark 1, we have

$$\int_0^T \left|(W_i x_i)(t)\right|^2 dt \leq \sum_{j=1}^n \int_0^T |c_{ij}(t)|^2 |x_i(t - \delta_{ij}(t))|^2 dt$$

$$\begin{aligned}
&\le \sum_{j=1}^{n} |c_{ij}^{+}|^2 \int_0^T |x_i(t-\delta_{ij}(t))|^2 dt \\
&= \sum_{j=1}^{n} |c_{ij}^{+}|^2 \int_{-\delta_{ij}(0)}^{T-\delta_{ij}(T)} \frac{1}{1-\delta'_{ij}(\mu_{ij}(s))} |x_i(s)|^2 ds \\
&= \sum_{j=1}^{n} |c_{ij}^{+}|^2 \int_0^T \frac{1}{1-\delta'_{ij}(\mu_{ij}(s))} |x_i(s)|^2 ds \\
&\le \frac{\sum_{j=1}^{n} |c_{ij}^{+}|^2}{1-\sigma_1} \int_0^T |x_i(t)|^2 dt.
\end{aligned}$$

Thus, the proof is completed. □

Lemma 4. If $\sum_{j=1}^{n} |c_{ij}^{+}| < 1$ and $\frac{\sum_{j=1}^{n} |c_{ij}^{+}|^2}{1-\sigma_1} < 1$, then the inverse of difference operator A_i, denoted by A_i^{-1}, exists and

(1) $\|A_i^{-1}\| \le \frac{1}{1-\sum_{j=1}^{n} |c_{ij}^{+}|}$, $i = 1, 2, ..., n$;

(2) $\int_0^T |(A_i^{-1}x_i)(t)| dt \le \frac{1}{1-\frac{\sum_{j=1}^{n} |c_{ij}^{+}|}{1-\sigma_1}} \cdot \int_0^T |x_i(t)| dt$, $i = 1, 2, ..., n$;

(3) $\int_0^T |(A_i^{-1}x_i)(t)|^2 dt \le \frac{1}{1-\frac{\sum_{j=1}^{n} |c_{ij}^{+}|^2}{1-\sigma_1}} \cdot \int_0^T |x_i(t)|^2 dt$, $i = 1, 2, ..., n$.

Proof. (1) From the first part of Lemma 3, we can know that $\|W_i\| \le \sum_{j=1}^{n} |c_{ij}^{+}| < 1$, by $A_i^{-1} = (I_i - W_i)^{-1}$, we can have $\|A_i^{-1}\| = \|(I_i - W_i)^{-1}\| \le \frac{1}{1-\sum_{j=1}^{n} |c_{ij}^{+}|}$.

(2) By $A_i^{-1} = (I_i - W_i)^{-1}$ and $\forall k \in \mathbb{Z}$, we have

$$\begin{aligned}
&\int_0^T |(A_i^{-1}x_i)(t)| dt \le \int_0^T \Big|[(I_i - W_i)^{-1}x_i](t)\Big| dt \\
&\le \int_0^T \Big|\sum_{k\ge 0} (W_i^k x_i)(t)\Big| dt \le \sum_{k\ge 0} \int_0^T \Big|(W_i^k x_i)(t)\Big| dt \\
&= \int_0^T |x_i(t)| dt + \sum_{k\ge 1} \int_0^T \Big|(W_i^k x_i)(t)\Big| dt, \ i = 1, 2, ..., n.
\end{aligned}$$

By using the second part of Lemma 3, we find if $k \ge 1$ and $k \in \mathbb{Z}$, then

$$\begin{aligned}
&\int_0^T \Big|(W_i^k x_i)(t)\Big| dt \le \int_0^T \Big|(W_i W_i^{k-1} x_i)(t)\Big| dt \\
&\le \frac{\sum_{j=1}^{n} |c_{ij}^{+}|}{1-\sigma_1} \int_0^T \Big|(W_i^{k-1} x_i)(t)\Big| dt \\
&\le \Big(\frac{\sum_{j=1}^{n} |c_{ij}^{+}|}{1-\sigma_1}\Big)^k \int_0^T \Big|x_i(t)\Big| dt, \ i = 1, 2, ..., n.
\end{aligned}$$

Thus, we can obtain

$$\int_0^T |(A_i^{-1}x_i)(t)| dt \le \int_0^T |x_i(t)| dt + \sum_{k\ge 1} \int_0^T \Big|(W_i^k x_i)(t)\Big|$$

$$\leq \sum_{k\geq 0}\Big(\frac{\sum_{j=1}^{n}|c_{ij}^{+}|}{1-\sigma_1}\Big)^k \int_0^T \Big|x_i(t)\Big|dt$$
$$= \frac{1}{1-\frac{\sum_{j=1}^{n}|c_{ij}^{+}|}{1-\sigma_1}}\cdot \int_0^T \Big|x_i(t)\Big|dt,\ i=1,2,...,n.$$

(3) Similar to the proof of (2). By $A_i^{-1}=(I_i-W_i)^{-1}$ and $\forall k\in\mathbb{Z}$, we have

$$\int_0^T |(A_i^{-1}x_i)(t)|^2dt \leq \int_0^T \Big|[(I_i-W_i)^{-1}x_i](t)\Big|^2 dt$$
$$\leq \int_0^T \Big|\sum_{k\geq 0}(W_i^k x_i)(t)\Big|^2 dt \leq \sum_{k\geq 0}\int_0^T \Big|(W_i^k x_i)(t)\Big|^2 dt$$
$$= \int_0^T |x_i(t)|^2 dt + \sum_{k\geq 1}\int_0^T \Big|(W_i^k x_i)(t)\Big|^2 dt,\ i=1,2,...,n.$$

By using the third part of Lemma 3, we find if $k\geq 1$ and $k\in\mathbb{Z}$, then

$$\int_0^T \Big|(W_i^k x_i)(t)\Big|^2 dt \leq \int_0^T \Big|(W_iW_i^{k-1}x_i)(t)\Big|^2 dt$$
$$\leq \frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1}\int_0^T \Big|(W_i^{k-1}x_i)(t)\Big|^2 dt$$
$$\leq \Big(\frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1}\Big)^k \int_0^T \Big|x_i(t)\Big|^2 dt,\ i=1,2,...,n.$$

Thus, we can get

$$\int_0^T |(A_i^{-1}x_i)(t)|^2dt \leq \int_0^T |x_i(t)|^2 dt + \sum_{k\geq 1}\int_0^T \Big|(W_i^k x_i)(t)\Big|^2 dt$$
$$\leq \sum_{k\geq 0}\Big(\frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1}\Big)^k \int_0^T \Big|x_i(t)\Big|^2 dt$$
$$= \frac{1}{1-\frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1}}\cdot \int_0^T \Big|x_i(t)\Big|^2 dt,\ i=1,2,...,n.$$

Therefore, the proof is completed. □

For any nonnegative integer q, let $t_q < T < t_{q+1} = T + t_1$ and

$$C[0,T;t_1,t_2,...,t_q] = \Big\{x|x:[0,T]\to\mathbb{R}^n, x(t) \text{ exists everywhere except } t_k, x(t_k^+), x(t_k^-) \text{ exist and } x(t_k)=x(t_k^-), k=1,2,...,q\Big\}.$$

Take

$$X = \Big\{x|x\in C[0,T;t_1,t_2,...,t_q], x(0)=x(T), x(t_k^+), x(t_k^-) \text{ exists}, x(t_k)=x(t_k^-), k=1,2,...q\Big\},$$

$Y = X \times \mathbb{R}^{n\times(q+1)}$ and $\|x\| = \sum_{i=1}^{n} \max_{t\in[0,T]} |x_i(t)|$, then X and Y are all Banach spaces.

Let

$$\begin{aligned} &L : D(L) \subset C^1[0,T;t_1,t_2,...,t_q] \cap X \to Y, \\ &Lx(t) = \Big(x'(t), \Delta x(t_1), \Delta x(t_2), ..., \Delta x(t_q)\Big), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &D(L) = \Big\{x|x \in C^1([0,T;t_1,t_2,...,t_q],\mathbb{R}) : x(t+T) = x(t)\Big\}, \\ &\Delta(x(t_k)) = x(t_k^+) - x(t_k^-), \ \ k = 1,2,...,q. \end{aligned}$$

Let

$$\begin{aligned} &N : X \to Y, \\ &Nx(t) = \left(\begin{pmatrix} B_1(t), \\ B_2(t) \\ \cdots \\ B_n(t) \end{pmatrix}, \begin{pmatrix} \Delta x_1(t_1), \\ \Delta x_2(t_1) \\ \cdots \\ \Delta x_n(t_1) \end{pmatrix}, \cdots, \begin{pmatrix} \Delta x_1(t_q), \\ \Delta x_2(t_1) \\ \cdots \\ \Delta x_n(t_q) \end{pmatrix}\right). \end{aligned} \tag{2.4}$$

Obviously, $\ker L = \Big\{x|x = c \in \mathbb{R}^n, t \in [0,T]\Big\}$, and

$$\begin{aligned} \operatorname{Im} L =& \Big\{y|y = (f, c_1, c_2, ..., c_q, d) \in Y, \int_0^T f(t)dt + \sum_{i=1}^{q} c_i + d = 0\Big\} \\ =& X \times \mathbb{R}^{n\times q} \times \{0\}. \end{aligned}$$

Then $\operatorname{Im} L$ is closed in Y and $\dim\ker L = \operatorname{codim}\operatorname{Im} L = n$. Hence, $\operatorname{Im} L$ is closed in Y and L is a Fredholm mapping of index zero.

Define $P : X \to X,\ \ Px(t) = \frac{1}{T}\int_0^T x(t)dt$, and $Q : Y \to Y$,

$$Q\Big(f(t), c_1, c_2, ..., c_q, d\Big) = \Big(\frac{1}{T}\Big[\int_0^T f(t)dt + \sum_{i=1}^{q} c_i + d\Big],\ 0, 0, ..., 0\Big).$$

It is easy to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

The inverse $K_p : \operatorname{Im} L \to \ker P \cap D(L)$ of L_p has the form

$$\begin{aligned} &K_p\Big(f(t), a_1, a_2, ..., a_q\Big) = \int_0^t f(s)ds + \sum_{t>t_k} c_k \\ &- \frac{1}{T}\int_0^T \int_0^t f(s)dsdt - \sum_{k=1}^{q} c_k, \end{aligned} \tag{2.5}$$

then

$$QNx(t) = \left(\begin{pmatrix} \frac{1}{T}\int_0^T B_1(t) - \frac{1}{T}\sum_{k=1}^{q} e_{1k}(x_1(t_k)) \\ \cdots \\ \frac{1}{T}\int_0^T B_n(t) - \frac{1}{T}\sum_{k=1}^{q} e_{nk}(x_n(t_k)) \end{pmatrix}, 0, \cdots, 0\right)_{n\times(q+1)},$$

and

$$K_p(I-Q)Nx(t)=\begin{pmatrix}\int_0^t B_1(s)ds-\sum_{t>t_k}e_{1k}(x_1(t_k))\\ \cdots\\ \int_0^t B_n(s)ds-\sum_{t>t_k}e_{nk}(x_n(t_k))\end{pmatrix}_{n\times 1}$$
$$-\begin{pmatrix}\frac{1}{T}\int_0^T\int_0^t B_1(s)dsdt-\left(\frac{t}{T}-\frac{1}{2}\right)\int_0^T B_1(t)dt\\ \cdots\\ \frac{1}{T}\int_0^T\int_0^t B_n(s)dsdt-\left(\frac{t}{T}-\frac{1}{2}\right)\int_0^T B_n(t)dt\end{pmatrix}_{n\times 1}$$
$$-\begin{pmatrix}\sum_{k=1}^q e_{1k}(x_1(t_k))\\ \cdots\\ \sum_{k=1}^q e_{nk}(x_n(t_k))\end{pmatrix}_{n\times 1}.$$

Clearly, QN and $K_p(I-Q)N$ are continuous. For any bounded open subset $\Omega\subset X$, $QN(\bar{\Omega})$ is bounded, moreover, applying the Arzela-Ascoli theorem, it is not difficult to show that $K_p(I-Q)N(\bar{\Omega})$ are relatively compact. Therefore, N is L-compact on X for any open bounded set Ω.

Throughout the rest of this paper, we always assume that:

- [H1] There exist constants $p_j\geq 0$ and $q_j\geq 0$ such that
$$|f_j(x_j)|\leq p_j,\quad |g_j(x_j)|\leq q_j,\quad j=1,2,...,n.$$
- [H2] Functions $f_j(u)$ and $g_j(u)(j=1,2,...,n)$ satisfy the Lipschitz condition, *i.e.*, there are constants $L_{1j}>0$ and $L_{2j}>0$ such that, for all u_1, $u_2\in\mathbb{R}$,
$$|f_j(u_1)-f_j(u_2)|\leq L_{1j}|u_1-u_2|,$$
$$|g_j(u_1)-g_j(u_2)|\leq L_{2j}|u_1-u_2|.$$
- [H3] $\sum_{j=1}^n|c_{ij}^+|<1$, $\frac{\sum_{j=1}^n|c_{ij}^+|^2}{1-\sigma_1}<1$ and $a_i^->\frac{a_i^+\sum_{j=1}^n c_{ij}^+}{\sqrt{(1-\sigma_1)-\sum_{j=1}^n|c_{ij}^+|^2}}$, $i=1,2,...,n$.
- [H4] There exists a positive integer q such that
$$t_{k+q}=t_k+T,\quad e_{k+q}(x)=e_k(x),\quad k=1,2,....$$

3. Existence of periodic solution

In this section, we study the existence of periodic solution of (1.1).

Theorem 3.1. Suppose that conditions [H1]-[H4] hold, then there exist positive constants K_i, which are independent of λ such that
$$x_i(t)\leq K_i,\ i=1,2,...,n,\ t\in\mathbb{R},$$
where $x=(x_1,x_2,...,x_n)^\top$ is any solution to the equation $Lx=\lambda Nx$, $\lambda\in(0,1]$.

Proof. First of all, consider the following operator equation
$$Lx=\lambda Nx,\ \lambda\in(0,1),$$

where L and N are defined by (2.3) and (2.4), respectively, then we have

$$\begin{cases} (A_i x_i)'(t) &= \lambda\Big[-a_i(t)x_i(t)+\sum_{j=1}^{n}\Big(b_{ij}(t)f_j(x_j(t)) \\ &\quad +d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\Big)\Big] \\ &\quad +\lambda I_i(t),\ t\in[0,T],\ \ t\neq t_k, \\ \Delta x_i(t_k) &= x_i(t_k^+)-x_i(t_k^-)=\lambda e_{ik}(x_i(t_k)), \\ &\quad x_i(0)=x_i(T), i=1,2,...,n,\ k=1,2,.... \end{cases} \tag{3.1}$$

Suppose that $(x_1(t),x_2(t),...,x_n(t))^\top\in X$ is a solution of system (3.1) for a certain $\lambda\in(0,1)$. Integrating (3.1) over the interval $[0,T]$, we obtain

$$\int_0^T a_i(t)x_i(t)dt=\int_0^T\Big[\sum_{j=1}^{n}\Big(b_{ij}(t)f_j(x_j(t))+d_{ij}(t)\cdot g_j(x_j(t-\tau_{ij}(t)))\Big)+I_i(t)\Big]dt+\sum_{k=1}^{q}e_{ik}(x_i(t_k)). \tag{3.2}$$

Let $\xi(\xi\neq t_k)\in[0,T]$, $k=1,2,...,q$, such that $x_i(\xi)=\inf\limits_{t\in[0,T]}x_i(t)$, $i=1,2,...,n$. Then, it follows from (3.2) and [H1] that

$$\begin{aligned} T\overline{a}_i x_i(\xi) &\leq \int_0^T\Big[\sum_{j=1}^{n}\Big|b_{ij}(t)f_j(x_j(t))+d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))+I_i(t)\Big|\Big]dt+\sum_{k=1}^{q}|e_{ik}(x_i(t_k))| \\ &\leq \int_0^T\Big[\sum_{j=1}^{n}\Big|b_{ij}(t)\Big|\Big|f_j(x_j(t))\Big|+\Big|d_{ij}(t)\Big|g_j(x_j(t-\tau_{ij}(t)))\Big|+\Big|I_i(t)\Big|\Big]dt+\sum_{k=1}^{q}|e_{ik}(x_i(t_k))| \\ &\leq T\sum_{j=1}^{n}\Big(b_{ij}^+p_j+d_{ij}^+q_j\Big)+TI_i^++\sum_{k=1}^{q}\overline{e}_{ik}. \end{aligned}$$

Thus, we have

$$\begin{aligned} x_i(\xi) \leq & \frac{1}{\overline{a}_i}\Big[\sum_{j=1}^{n}\Big(b_{ij}^+p_j+d_{ij}^+q_j\Big)+I_i^++\frac{1}{T}\sum_{k=1}^{q}\overline{e}_{ik}\Big] \\ :=& M_i,\ i=1,2,...,n. \end{aligned} \tag{3.3}$$

Let $t_0=t_0^+=0$ and $t_{q+1}=T$. From (3.1) and (3.2), and by using the Hölder inequality, we obtain

$$\begin{aligned} \int_0^T\Big|(A_ix_i)'(t)\Big|dt &= \sum_{k=1}^{q}\int_{t_k^+-1}^{t_k}\Big|(A_ix_i)'(t)\Big|dt+\sum_{k=1}^{q}\Big|(A_ix_i)(t_k^+)-(A_ix_i)(t_k)\Big| \\ &\leq \int_0^T|a_i(t)||x_i(t)|dt+\int_0^T\sum_{j=1}^{n}\Big[|b_{ij}(t)||f_j(x_j(t))| \\ &\quad +|d_{ij}(t)||g_j(x_j(t-\tau_{ij}(t)))|\Big]dt+\int_0^T|I_i(t)|dt+\sum_{k=1}^{q}|e_{ik}((A_ix_i)(t_k))| \\ &\leq a_i^+\sqrt{T}\Big(\int_0^T|x_i(t)|^2dt\Big)^{1/2}+T\sum_{j=1}^{n}\Big(b_{ij}^+p_j+d_{ij}^+q_j\Big)+TI_i^++\sum_{k=1}^{q}\overline{e}_{ik}, \end{aligned}$$

which together with the third part of Lemma 4 yields

$$\begin{aligned}
&\int_0^T \left|(A_ix_i)'(t)\right|dt \le a_i^+\sqrt{T}\Big(\int_0^T |x_i(t)|^2dt\Big)^{1/2} + T\sum_{j=1}^n \left(b_{ij}^+p_j + d_{ij}^+q_j\right) + TI_i^+ + \sum_{k=1}^q \bar{e}_{ik} \\
&\le a_i^+\sqrt{T}\Big(\frac{1}{1-\frac{\sum_{j=1}^n |c_{ij}^+|^2}{1-\sigma_1}}\cdot\int_0^T |(A_ix_i)(t)|^2dt\Big)^{1/2} + T\sum_{j=1}^n \left(b_{ij}^+p_j + d_{ij}^+q_j\right) + TI_i^+ + \sum_{k=1}^q \bar{e}_{ik} \\
&= \frac{a_i^+\sqrt{T(1-\sigma_1)}}{\sqrt{(1-\sigma_1)-\sum_{j=1}^n |c_{ij}^+|^2}}\cdot\|A_ix_i\|_2 + T\sum_{j=1}^n \left(b_{ij}^+p_j + d_{ij}^+q_j\right) + TI_i^+ + \sum_{k=1}^q \bar{e}_{ik}.
\end{aligned} \tag{3.4}$$

Furthermore, multiplying both sides of system (3.1) by $(A_ix_i)(t)$ and integrating over $[0, T]$, since

$$\begin{aligned}
&\int_0^T (A_ix_i)'(t)(A_ix_i)(t)dt = \frac{\lambda}{2}\cdot\Big\{(A_ix_i)^2(t_1) - (A_ix_i)^2(0) \\
&+ \sum_{l=2}^q \left[(A_ix_i)^2(t_l) - (A_ix_i)^2(t_{l-1}^+)\right] + (A_ix_i)^2(T) - (A_ix_i)^2(t_q^+)\Big\} \\
&= \frac{\lambda}{2}\cdot\sum_{l=1}^q \left[(A_ix_i)^2(t_l) - (A_ix_i)^2(t_l^+)\right] \\
&= -\lambda\cdot\sum_{k=1}^q \left[(A_ix_i)(t_k) + \frac{1}{2}e_{ik}((A_ix_i)(t_k))\right]e_{ik}((A_ix_i)(t_k)),
\end{aligned} \tag{3.5}$$

we obtain

$$\begin{aligned}
&0 = \int_0^T (A_ix_i)'(t)(A_ix_i)(t)dt = -\lambda\int_0^T a_i(t)x_i(t)(A_ix_i)(t)dt \\
&+ \lambda\int_0^T \sum_{j=1}^n \left[b_{ij}(t)f_j(x_j(t)) + d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\right](A_ix_i)(t)dt \\
&+ \lambda\int_0^T I_i(t)(A_ix_i)(t)dt + \lambda\sum_{k=1}^q \left[(A_ix_i)(t_k) + \frac{1}{2}e_{ik}((A_ix_i)(t_k))\right]\cdot e_{ik}((A_ix_i)(t_k)).
\end{aligned} \tag{3.6}$$

Moreover,

$$\begin{aligned}
&\int_0^T a_i(t)x_i(t)(A_ix_i)(t)dt = \int_0^T a_i(t)(A_ix_i)(t)\Big[x_i(t) \\
&- \sum_{j=1}^n c_{ij}(t)x_i(t-\delta_{ij}(t)) + \sum_{j=1}^n c_{ij}(t)x_i(t-\delta_{ij}(t))\Big]dt \\
&= \int_0^T a_i(t)(A_ix_i)^2(t)dt + \int_0^T a_i(t)(A_ix_i)(t)\sum_{j=1}^n c_{ij}(t)x_i(t-\delta_{ij}(t))dt.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we can have

$$
\begin{aligned}
0 = &-\Big[\int_0^T a_i(t)(A_ix_i)^2(t)dt + \int_0^T a_i(t)(A_ix_i)(t)\cdot\sum_{j=1}^{n} c_{ij}(t)x_i(t-\delta_{ij}(t))dt\Big]\\
&+\int_0^T\sum_{j=1}^{n}\Big[b_{ij}(t)f_j(x_j(t)) + d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\Big](A_ix_i)(t)dt\\
&+\int_0^T I_i(t)(A_ix_i)(t)dt + \sum_{k=1}^{q}\Big[(A_ix_i)(t_k) + \frac{1}{2}e_{ik}((A_ix_i)(t_k))\Big]\cdot e_{ik}((A_ix_i)(t_k)).
\end{aligned}
\tag{3.8}
$$

It follows from (3.8) and [H1] that

$$
\begin{aligned}
&a_i^-\|A_ix_i\|_2^2\\
&\le a_i^+\sum_{j=1}^{n}c_{ij}^+\Big(\int_0^T|(A_ix_i)(t)|^2dt\Big)^{1/2}\Big(\int_0^T|x_i(t-\delta_{ij}(t))|^2dt\Big)^{1/2}\\
&+\Big(\sum_{j=1}^{n}\Big[b_{ij}^+p_j + d_{ij}^+q_j\Big] + I_i^+\Big)\sqrt{T}\Big(\int_0^T|(A_ix_i)(t)|^2dt\Big)^{1/2}\\
&+\sum_{k=1}^{q}\Big[(A_ix_i)(t_k) + \frac{1}{2}e_{ik}((A_ix_i)(t_k))\Big]e_{ik}((A_ix_i)(t_k)).
\end{aligned}
\tag{3.9}
$$

Furthermore,

$$
\int_0^T|x_i(t-\delta_{ij}(t))|^2dt = \int_{-\delta_{ij}(0)}^{T-\delta_{ij}(T)}\frac{1}{1-\delta'_{ij}(\mu_{ij}(s))}|x_i(s)|^2ds.
$$

It follows from Lemma 2 that

$$
\int_{-\delta_{ij}(0)}^{T-\delta_{ij}(T)}\frac{1}{1-\delta'_{ij}(\mu_{ij}(s))}|x_i(s)|^2ds = \int_0^T\frac{1}{1-\delta'_{ij}(\mu_{ij}(s))}|x_i(s)|^2ds.
$$

By Remark 1, we have

$$
\begin{aligned}
\frac{1}{1+\sigma_0}\int_0^T|x_i(s)|^2ds &\le \int_0^T\frac{1}{1-\delta'_{ij}(\mu_{ij}(s))}|x_i(s)|^2ds\\
&\le \frac{1}{1-\sigma_1}\int_0^T|x_i(s)|^2ds.
\end{aligned}
\tag{3.10}
$$

Substituting (3.10) into (3.9), we get

$$
\begin{aligned}
a_i^-\|A_ix_i\|^2 \le &\frac{a_i^+\sum_{j=1}^{n}c_{ij}^+}{\sqrt{1-\sigma_1}}\cdot\Big(\int_0^T|(A_ix_i)(t)|^2dt\Big)^{1/2}\cdot\Big(\int_0^T|x_i(t)|^2dt\Big)^{1/2}\\
&+\sqrt{T}\Big(\sum_{j=1}^{n}\Big[b_{ij}^+p_j + d_{ij}^+q_j\Big] + I_i^+\Big)\cdot\Big(\int_0^T|(A_ix_i)(t)|^2dt\Big)^{1/2}\\
&+\sum_{k=1}^{q}\Big[(A_ix_i)(t_k) + \frac{1}{2}e_{ik}((A_ix_i)(t_k))\Big]e_{ik}((A_ix_i)(t_k)).
\end{aligned}
\tag{3.11}
$$

Since $\frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1} < 1$, it follows from the third part of Lemma 4 that

$$\begin{aligned}\Big(\int_0^T |x_i(t)|^2 dt\Big)^{1/2} &= \Big(\int_0^T |(A_i^{-1}A_i)(x_i(t))|^2 dt\Big)^{1/2} \\ &\le \Big(\frac{1}{1-\frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1}} \cdot \int_0^T |(A_i x_i)(t)|^2 dt\Big)^{1/2} \\ &= \Big(\frac{1}{1-\frac{\sum_{j=1}^{n}|c_{ij}^{+}|^2}{1-\sigma_1}}\Big)^{1/2}\Big(\int_0^T |(A_i x_i)(t)|^2 dt\Big)^{1/2}.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain

$$\begin{aligned}a_i^{-}\|A_i x_i\|_2^2 \le &\frac{a_i^{+}\sum_{j=1}^{n} c_{ij}^{+}}{\sqrt{(1-\sigma_1)-\sum_{j=1}^{n}|c_{ij}^{+}|^2}} \cdot \|A_i x_i\|_2^2 + \sqrt{T}\Big(\sum_{j=1}^{n}\Big[b_{ij}^{+}p_j + d_{ij}^{+}q_j\Big] + I_i^{+}\Big) \cdot \|A_i x_i\|_2 \\ &+ \sum_{k=1}^{q}\Big[(A_i x_i)(t_k) + \frac{1}{2}e_{ik}((A_i x_i)(t_k))\Big]e_{ik}((A_i x_i)(t_k)),\end{aligned}$$

which together with [H3] implies that there exists a positive constant N_i, $i = 1, 2, ..., n$, such that

$$\|A_i x_i\|_2 \le N_i, \quad i = 1, 2, ..., n. \tag{3.13}$$

Clearly, $N_i (i = 1, 2, ..., n)$ is independent with λ.

Substituting (3.13) into (3.4), we can have

$$\begin{aligned}\int_0^T |(A_i x_i)'(t)| dt \le &\frac{a_i^{+}\sqrt{T(1-\sigma_1)}}{\sqrt{(1-\sigma_1)-\sum_{j=1}^{n}|c_{ij}^{+}|^2}} \cdot \|A_i x_i\|_2 \\ &+ T\sum_{j=1}^{n}\Big(b_{ij}^{+}p_j + d_{ij}^{+}q_j\Big) + TI_i^{+} + \sum_{k=1}^{q}\bar{e}_{ik} \\ \le &\frac{a_i^{+}N_i\sqrt{T(1-\sigma_1)}}{\sqrt{(1-\sigma_1)-\sum_{j=1}^{n}|c_{ij}^{+}|^2}} + T\sum_{j=1}^{n}\Big(b_{ij}^{+}p_j + d_{ij}^{+}q_j\Big) + TI_i^{+} + \sum_{k=1}^{q}\bar{e}_{ik}.\end{aligned} \tag{3.14}$$

By using the second part of Lemma 4, we can have

$$\begin{aligned}\int_0^T |x_i'(t)| dt &= \int_0^T |(A_i^{-1}A_i)x_i'(t)| dt \\ &\le \frac{1}{1-\frac{\sum_{j=1}^{n}|c_{ij}^{+}|}{1-\sigma_1}}\int_0^T |(A_i x_i)'(t)| dt \\ &\le \frac{1}{1-\frac{\sum_{j=1}^{n}|c_{ij}^{+}|}{1-\sigma_1}} \cdot \Big[\frac{a_i^{+}N_i\sqrt{T(1-\sigma_1)}}{\sqrt{(1-\sigma_1)-\sum_{j=1}^{n}|c_{ij}^{+}|^2}} \\ &+ T\sum_{j=1}^{n}\Big(b_{ij}^{+}p_j + d_{ij}^{+}q_j\Big) + TI_i^{+} + \sum_{k=1}^{q}\bar{e}_{ik}\Big].\end{aligned} \tag{3.15}$$

Since

$$|x_i(t)| \leq |x_i(\xi)| + \int_0^T |x_i'(t)|dt, \ \forall t \in [0,T], \ i = 1,2,...,n,$$

then it follows from (3.3) and (3.15) that there exists constants $K_i(i = 1,2,...,n)$ such that

$$|x_i(t)| \leq K_i, \ t \in [0,T], \ i = 1,2,...,n, \tag{3.16}$$

and clearly, $K_i(1,2,...,n)$ is independent of λ. Therefore, the proof is completed. □

Theorem 3.2. *Assume that all the conditions in Theorem 3.1 hold, then system* (1.1) *has at least one T-periodic solution.*

Proof. Denote

$$H^* = \sum_{i=1}^n K_i + C,$$

where $C > 0$ is taken enough large so that

$$\min_{1\leq i\leq n} \bar{a}_i H^* > \min_{1\leq i\leq n} \Big[\sum_{j=1}^n \Big(|\bar{b}_{ij}|p_j + |\bar{d}_{ij}|q_j \Big) + \bar{I}_i - \frac{1}{T}\sum_{k=1}^q \bar{e}_{ik} \Big]. \tag{3.17}$$

Set

$$\Omega = \Big\{ x | x = (x_1(t),...,x_n(t))^\top \in X, \|x\| \leq H^* \Big\}.$$

Then, we can see that the condition (1) of Lemma 1 is satisfied.

When $x = (x_1(t),...,x_n(t))^\top \in \partial\Omega \cap \mathbb{R}^n$, $x = (x_1,...,x_n)^\top$ is a constant in $\mathbb{R}^n$ with

$$|x_1| + |x_2| + \cdots + |x_n| = H^*.$$

Then,

$$\begin{aligned} QNx = QN(x_1,...,x_n)^\top = \Big(& -\bar{a}_i x_i + \sum_{j=1}^n \Big[\bar{b}_{ij} f_j(x_j(t)) \\ & + \bar{d}_{ij} g_j(x_j(t-\tau_{ij}(t))) \Big] + \bar{I}_i - \frac{1}{T}\sum_{k=1}^q e_{ik}(x_i(t_k)) \Big)_{n\times 1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\|QNx\| = \|QN(x_1,...,x_n)^\top\| \\ &= \sum_{i=1}^n \Big| \bar{a}_i x_i + \frac{1}{T}\sum_{k=1}^q e_{ik}(x_i(t_k)) - \sum_{j=1}^n \Big[\bar{b}_{ij} f_j(x_j(t)) \\ &\quad + \bar{d}_{ij} g_j(x_j(t-\tau_{ij}(t))) \Big] - \bar{I}_i \Big| \\ &\geq \sum_{i=1}^n \bar{a}_i |x_i| + \frac{1}{T}\sum_{i=1}^n \sum_{k=1}^q \bar{e}_{ik} - \sum_{i=1}^n \sum_{j=1}^n \Big[|\bar{b}_{ij}|p_j + |\bar{d}_{ij}|q_j \Big] - \sum_{i=1}^n |\bar{I}_i| \\ &\geq \sum_{i=1}^n \Big(\bar{a}_i |x_i| + \frac{1}{T}\sum_{k=1}^q \bar{e}_{ik} \Big) - \sum_{i=1}^n \sum_{j=1}^n \Big(\Big[|\bar{b}_{ij}|p_j + |\bar{d}_{ij}|q_j \Big] + |\bar{I}_i| \Big) \end{aligned}$$

$$\geq \min_{1\leq i\leq n}\sum_{i=1}^{n}\left(\bar{a}_i|x_i| + \frac{1}{T}\sum_{k=1}^{q}\bar{e}_{ik}\right) - \max_{1\leq i\leq n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left[|\bar{b}_{ij}|p_j + |\bar{d}_{ij}|q_j\right]\right.$$
$$\left. + |\bar{I}_i|\right),$$

which together with (3.17) gives

$$\|QNx\| = \|QN(x_1, ..., x_n)^\top\| > 0.$$

This leads to a contradiction with $x = (x_1(t), ..., x_n(t))^\top \in \partial\Omega \cap \mathbb{R}^n$. Thus, the condition (2) of Lemma 1 is satisfied.

Finally, we show that the condition (3) of Lemma 1 is also satisfied.

Define

$$\varphi : \ker L \times [0, 1] \to X,$$

by

$$\varphi(x_1, ..., x_n, \mu) = -\mu(x_1, ..., x_n)^\top + (1 - \mu)QN(x_1, ..., x_n)^\top.$$

If $(x_1, ..., x_n)^\top \in \partial\Omega \cap \ker L$, $(x_1, ..., x_n)^\top$ is a constant in $\mathbb{R}^n$ with $\sum_{i=1}^{n}|x_i| = H^*$, then we can obtain $\varphi(x_1, ..., x_n, \mu) \neq (0, 0, ..., 0)^\top$. Thus,

$$\begin{aligned}&\deg\left(QN(x_1, ..., x_n)^\top, \Omega \cap \ker L, (0, 0..., 0)^\top\right)\\&= \deg\left((-x_1, ..., -x_n)^\top, \Omega \cap \ker L, (0, 0..., 0)^\top\right)\\&\neq 0,\end{aligned}$$

which implies that the condition (3) of Lemma 1 is also satisfied. Therefore, by Lemma 1, we can conclude that system (1.1) has at least one T-periodic solution. □

4. Global Exponential Stability

In this section, we will prove that the periodic solution of (1.1) is global exponential asymptotic stable.

Theorem 4.1. Assume that [H1]-[H4] hold, Furthermore, assume that

- [H5] The following inequalities hold:

$$a_i^- > \sum_{j=1}^{n} 2b_{ij}^+ L_{1j} + \sum_{j=1}^{n} d_{ij}^+ L_{2j}\tau,\ i = 1, 2, ..., n,$$

  where L_{1j} and L_{2j} are defined in [H2], $\tau_{ij}^+ = \max_{t\in[0,T]} \tau_{ij}(t)$, $\tau = \max_{1\leq i,j\leq n}\{\tau_{ij}^+\}$, $i, j = 1, 2, ..., n$.

- [H6] $e_{ik}(x_i(t_k)) = -\gamma_{ik}x_i(t_k)$, $0 < \gamma_{ik} < 2$, $i = 1, 2, ..., n$, $k \in \mathbb{Z}$.

Then there exists a positive constant α such that the periodic solutions of system (1.1) satisfy

$$\sum_{i=1}^{n}|x_i(t) - x_i^*(t)| \leq e^{-\alpha t}|x_i(0) - x_i^*(0)|,\ t > 0,$$

where $x^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))^\top$.

Proof. From Theorem 3.1-3.2, we can see that system (1.1) possesses a T-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))^\top$.

Suppose that $x(t) = (x_1(t), x_2(t), ..., x_n(t))^\top$ is an arbitrary solution of system (1.1). Then, it follows from system (1.1) that

$$\begin{aligned}
&\frac{d}{dt}\Big(x_i(t) - x_i^*(t)\Big) = -a_i\Big(x_i(t) - x_i^*(t)\Big) \\
&+ \sum_{j=1}^{n} \Big[b_{ij}(t)\Big(f_j(x_j(t)) - f_j(x_j^*(t))\Big) \\
&+ d_{ij}(t)\Big(g_j(x_j(t-\tau_{ij}(t))) - g_j(x_j^*(t-\tau_{ij}(t)))\Big)\Big],
\end{aligned}$$

for $i = \{1, 2, ..., n\}$, $t > 0$, $t \neq t_k$, $k \in \mathbb{Z}$. Then by [H2], we can have

$$\begin{aligned}
&\frac{d^+}{dt}\Big|x_i(t) - x_i^*(t)\Big| \le -a_i^-\Big|x_i(t) - x_i^*(t)\Big| \\
&+ \sum_{j=1}^{n} \Big[b_{ij}^+ L_{1j}\Big|x_j(t) - x_j^*(t)\Big| \\
&+ d_{ij}^+ L_{2j}\Big|x_j(t-\tau_{ij}(t)) - x_j^*(t-\tau_{ij}(t))\Big|\Big],
\end{aligned} \tag{4.1}$$

for $i = \{1, 2, ..., n\}$, $t > 0$, $t \neq t_k$, $k \in \mathbb{Z}$ and d^+/dt denotes the upper-right derivative. Moreover,

$$\begin{aligned}
x_i(t_k+0) - x_i^*(t_k+0) =& x_i(t_k) + e_i(x_i(t_k)) - \Big[x_i^*(t_k) + e_i(x_i^*(t_k))\Big] \\
=& (1-\gamma_{ik})\Big(x_i(t_k) - x_i^*(t_k)\Big),
\end{aligned}$$

which together with [H6] yields

$$\begin{aligned}
\Big|x_i(t_k+0) - x_i^*(t_k+0)\Big| \le& \Big|1-\gamma_{ik}\Big|\Big|x_i(t_k) - x_i^*(t_k)\Big| \\
\le& \Big|x_i(t_k) - x_i^*(t_k)\Big|,
\end{aligned}$$

for $i = \{1, 2, ..., n\}$, $k \in \mathbb{Z}$.

Choose the Lyapunov functional in the following form:

$$\begin{aligned}
V(t) = \sum_{i=1}^{n} \Big(&\Big|x_i(t) - x_i^*(t)\Big| + \int_0^t b_{ij}^+\Big|x_j(s) - x_j^*(s)\Big|ds \\
&+ \int_{t-\tau_{ij}}^t \frac{d_{ij}^+}{1-\tau_{ij}'(\mu(s))}\Big|x_j(s) - x_j^*(s)\Big|ds\Big), \ t > 0.
\end{aligned}$$

Then, combining with (4.1), we can get

$$\begin{aligned}
&\frac{d^+V(t)}{dt} \le \sum_{i=1}^{n}\Big\{-a_i^-\Big|x_i(t) - x_i^*(t)\Big| + \sum_{j=1}^{n}\Big[b_{ij}^+ L_{1j}|x_j(t) - x_j^*(t)| \\
&+ d_{ij}^+ L_{2j}|x_j(t-\tau_{ij}(t)) - x_j^*(t-\tau_{ij}(t))|\Big] \\
&+ \sum_{j=1}^{n} b_{ij}^+ L_{1j}|x_j(t) - x_j^*(t)| + \sum_{j=1}^{n} \frac{d_{ij}^+ L_{2j}}{1-\tau_{ij}'(\mu(t))}|x_j(t) - x_j^*(t)|
\end{aligned}$$

$$- \sum_{j=1}^{n} \frac{d_{ij}^{+} L_{2j}}{1 - \tau'_{ij}(\mu(t - \tau_{ij}(t)))} |x_j(t - \tau_{ij}(t)) - x_j^*(t - \tau_{ij}(t))|$$
$$\cdot (1 - \tau'_{ij}(t)) \Big\}$$
$$= - \sum_{i=1}^{n} \Big[a_i^- - \sum_{j=1}^{n} 2b_{ij}^{+} L_{1j} - \sum_{j=1}^{n} d_{ij}^{+} L_{2j} \tau \Big] |x_i(t) - x_i^*(t)|.$$

From [H5], we can see that there exists a positive constant $\alpha > 0$ such that

$$a_i^- \geq \sum_{j=1}^{n} 2b_{ij}^{+} L_{1j} + \sum_{j=1}^{n} d_{ij}^{+} L_{2j} \tau + \alpha, \ i = 1, 2, ..., n,$$

then we can have

$$\frac{d^+ V(t)}{dt} \leq -\alpha V(t), \ t > 0, \ t \neq t_k. \tag{4.2}$$

Moreover,

$$\begin{aligned} V(t_k + 0) &= \sum_{i=1}^{n} \left| x_i(t_k + 0) - x_i^*(t_k + 0) \right| \\ &\leq \sum_{i=1}^{n} \left| x_i(t_k) - x_i^*(t_k) \right| = V(t_k), \ k \in \mathbb{Z}. \end{aligned} \tag{4.3}$$

By using the exponential stability theorem [14], (4.2) and (4.3), we have

$$\frac{d^+ V(t)}{dt} \leq -\alpha V(t) \to V(t) \leq e^{-\alpha t} V(0), \ \forall t > 0,$$

Thus, we can obtain

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \leq e^{-\alpha t} |x_i(0) - x_i^*(0)|, \ \forall t > 0.$$

By Definition 2.3, we can conclude that the periodic solution of system (1.1) is globally exponentially stable. The proof is now completed. □

5. Numerical example

In this section, we present an example to demonstrate the results obtained in previous sections.

Example 5.1. *Consider the following neutral-type neural networks with impulses and delays:*

$$\begin{cases} (A_i x_i)'(t) & = -a_i(t) x_i(t) + \sum_{j=1}^{2} \Big[b_{ij}(t) f_j(x_j(t)) \\ & + d_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) \Big] + I_i(t), \\ \Delta x_i(t_k) & = x_i(t_k^+) - x_i(t_k^-) = e_{ik}(x_i(t_k)), \ t = t_k = kT, \\ & i = 1, 2, \ k = 1, 2, ..., \end{cases} \tag{5.1}$$

where

$$A_i x_i(t) = x_i(t) - \sum_{j=1}^{2} c_{ij}(t) x_i(t - \delta_{ij}(t)), \;\; i = 1, 2,$$
$$I_1(t) = 1 + \sin(\pi t), \; I_2(t) = 1 + \cos(\pi t),$$
$$a_i(t) = 1, \; b_{ij}(t) = d_{ij}(t) = 0.01, \;\; f_j(u) = g_j(u) = 0.2 \sin(u),$$
$$\tau_{ij}(t) = \frac{1}{8\pi} \sin(\pi t), \; \delta_{ij}(t) = 1 - \frac{1}{8\pi} \cos(\pi t),$$
$$c_{ij}(t) = 0.01 + 0.01 \sin(\pi T),$$
$$e_{1k} x_1(t_k) = -0.5 x_1(t_k), \; e_{2k} x_2(t_k) = -0.4 x_2(t_k).$$

For $i, j = 1, 2$, *we have,*

$$a_i^+ = a_i^- = 1, \; c_{ij}^+ = 0.02, \; L_{1j} = L_{2j} = 0.2,$$
$$\delta'_{ij}(t) = \frac{1}{8} \sin(\pi t), \tau = \max_{1 \le i \le n, 1 \le j \le n} \{\tau_{ij}^+\} = \frac{1}{8\pi}, \; b_{ij}^+ = d_{ij}^+ = 0.01,$$

then, we can see that assumptions [H1], [H2] and [H4] hold.

Moreover, by a simple calculation, we have

$$\sum_{j=1}^{n} |c_{ij}^+| < 1, \; \frac{\sum_{j=1}^{n} |c_{ij}^+|^2}{1 - \sigma_1} < 1, \; \sigma_1 = \max_{t \in [0,T]} \delta'_{ij}(t) = \frac{1}{8},$$
$$a_i^- - \frac{a_i^+ \sum_{j=1}^{n} |c_{ij}^+|}{\sqrt{(1 - \sigma_1) - \sum_{j=1}^{n} |c_{ij}^+|^2}} \approx 0.95722 > 0,$$

and

$$a_i^- - \sum_{j=1}^{n} 2 b_{ij}^+ L_{1j} - \sum_{j=1}^{n} d_{ij}^+ L_{2j} \tau \approx 0.99043 > 0, \; i = 1, 2.$$

thus, we can see that assumptions [H3], [H5] and [H6] hold. Therefore, by Theorem 3.1-3.2 and 4.1, we can obtain system (5.1) *has a global exponential stable* 2*-periodic solution. This fact can be presented in the following Figure 1 and Figure 2 .*

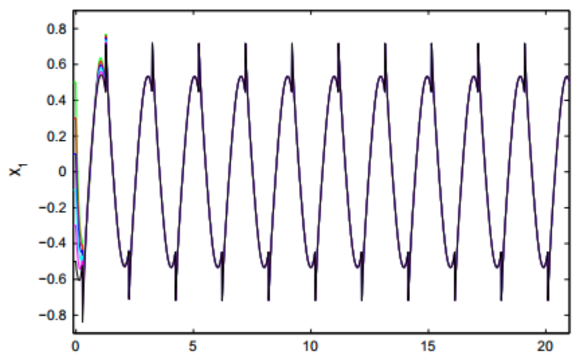

Figure 1: Time-domain behavior of the state variable x_1 with impulsive effects.

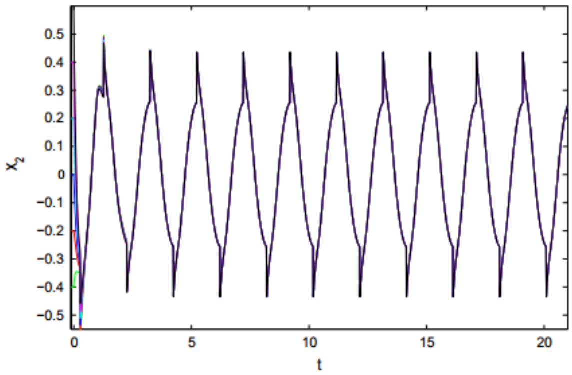

Figure 2: Time-domain behavior of the state variable x_2 with impulsive effects.

6. Conclusion

In the real world, impulsive differential equations are suitable for the mathematical simulation of evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change in their values. Processes of this type are often investigated in various fields of science and technology. In this paper, we investigated a generalized neutral-type neural networks with impulses and delays and the neural network model with impulses shows the neutral character by the $A_i(i = 1, 2, ..., n)$ operator, which is different from the corresponding ones known in the literature. The existence and global exponential stability of T-periodic solution have been completely established by means of the Mawhin's continuation theorem and by constructing the appropriate Lyapunov functional. These results extend previous works. Some interesting questions deserve further investigation.

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