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On suitable sets for countable rectifiable spaces

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Abstract. In this note, we give a definition of suitable sets for rectifiable spaces. We show that every T_0 countable rectifiable space has a suitable set.

1. Introduction

Recall that a *paratopological group* is a group with a topology such that the multiplication on the group is jointly continuous. A *topological group* is a paratopological group such that the inverse mapping of G into itself associating x^{-1} with $x \in G$ is continuous [3]. In [6], M.M. Choban introduced the notion of rectifiable spaces. A topological space X is said to be a *rectifiable* space provided that there are a surjective homeomorphism $\varphi : X \times X \to X \times X$ and an element $e \in X$ such that $\pi_1 \circ \varphi = \pi_1$ and $\varphi(x, x) = (x, e)$ for each $x \in X$, where $\pi_1 : X \times X \to X$ is the projection to the first coordinate ([6] and [10]). We call the mapping φ a *rectification* of X, the element e is a *right unit element* [10]. It is clear that every topological group G is rectifiable by means of the mapping $\varphi(x, y) = (x, xy^{-1})$. Thus rectifiable spaces are generalizations of topological groups. V.V. Uspenskii pointed out that there exists a rectifiable space which is not a topological group [22].

Every T_0 first-countable topological group is metrizable ([3], Theorem 3.3.12). In 1996, A.S. Gul'ko proved that every T_0 first-countable rectifiable space is metrizable ([10], Theorem 3.2). In 2008, A.V. Arhangel'skii proved that for any Hausdorff topological group G, any remainder $bG\backslash G$ of G in a Hausdorff compactification bG of G is either pseudocompact or Lindelöf ([1], Theorem 2.4). In 2010, A.V. Arhangel'skii and M.M. Choban proved that for any Hausdorff compactification bG of an arbitrary Tychonoff rectifiable space G, the remainder $bG\backslash G$ is either pseudocompact or Lindelöf ([2], Theorem 3.1). In 2011, F.C Lin and R.X. Shen discussed cardinal invariants, and generalized metric properties on paratopological groups and rectifiable spaces [15]. In 2012, F.C. Lin, C. Liu and S. Lin proved that a locally compact rectifiable space with the Souslin property is σ-compact ([14], Theorem 4.3). In 2012, L.-X. Peng and S.-J. Guo proved that every rectifiable p-space with a countable Souslin number is Lindelöf [18]. In 2015, F.C. Lin, J. Zhang and K.X. Zhang proved that each locally compact Hausdorff rectifiable space is paracompact [17]. In 2015, L.-X. Peng and D.-Z. Kong proved that the family of (topological) cofinalities of elements of a rectifiable GO-space has at most one infinite element [19].

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Hoffmann and Morris introduced the notion of a suitable set for topological groups and proved that every locally compact Hausdorff topological group has a suitable set [12]. Recall that a subset S of a topological group G is said to be a *suitable set* if (a) it has the discrete topology, (b) it is a closed subset of $G\backslash\{1\}$ and (c) the subgroup generated by S is dense in G ([8] and [12]), where 1 is the identity of G. In [8], it was proved that every metrizable topological group and every countable Hausdorff topological group has a suitable set. In [11], Guran studied suitable sets for paratopological groups. Later, F.C. Lin, A. Ravsky and T.T. Shi discussed when paratopological groups of different classes have suitable sets [13].

The notion of a gyrogroup was introduced by A.A. Ungar [20] in 2002 as a generalization of a group. In 2017, W. Atiponrat [4] introduced the concept of topological gyrogroups, which is a generalization of a topological group. Namely, a *topological gyrogroup G* is a gyrogroup $(G, \oplus)$ endowed with a topology such that the multiplication map $\oplus : G \times G \to G$ is jointly continuous and the inverse map $\ominus : G \to G$ is continuous. Z.Y. Cai, S. Lin and W. He proved that every topological gyrogroup is a rectifiable space ([5], in the proof of Theorem 2.3). Thus every topological group is a topological gyrogroup and every topological gyrogroup is a rectifiable space. In 2020, F.C. Lin, T.T. Shi and M. Bao proved that each countable Hausdorff topological gyrogroup has a suitable set ([16], Theorem 3.3).

In this note, we give a definition of suitable sets for rectifiable spaces (see Definition 2.9) and prove that every T_0 countable rectifiable space has a suitable set.

The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [9]. Every regular space satisfies T_1 and T_3.

2. Main results

Lemma 2.1. ([7], [10], [21]) *A topological space G is rectifiable if and only if there are two continuous mappings $p : G^2 \to G, q : G^2 \to G$ such that for any $x \in G, y \in G$ and some $e \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y, q(x, x) = e.$$

Lemma 2.2. *A topological space G is rectifiable if and only if there are two continuous open mappings $p : G^2 \to G, q : G^2 \to G$ such that for any $x \in G, y \in G$ and some $e \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y, q(x, x) = e.$$

Proof. The sufficiency follows from Lemma 2.1. To assist the reader, we give an explication for the necessity. Let $\varphi : G^2 \to G^2$ be a rectification. Let $p = \pi_2 \circ \varphi^{-1}, q = \pi_2 \circ \varphi$, where $\pi_2 : G^2 \to G$ is the projection to the second coordinate. Since the mappings $\varphi^{-1}, \varphi, \pi_2$ are open and continuous, the mappings p and q are open and continuous. □

In what follows, in discussing a rectifiable space, we let p and q denote the two continuous open mappings appearing in Lemma 2.2. If G is a rectifiable space and $A, B \subset G$, then we denote $p(A \times B)$ and $q(A \times B)$ by $p(A, B)$ and $q(A, B)$, respectively.

Notation 2.3. If S is a nonempty subset of a rectifiable space G, then let $S_0 = S, S_1 = S \cup \{p(a, b), q(a, b) : a, b \in S_0\}$, $S_{n+1} = \{p(a, b), q(a, b) : a, b \in S_n\}$ for every $n \geq 1$. Denote $\langle S\rangle = \bigcup\{S_n : n \in \omega\}$. If $S = \{a_1, ..., a_m\}$ for some $m \in \mathbb{N}$, then denote S_n by $\{a_1, ..., a_m\}_n$.

Proposition 2.4. *If S is a nonempty subset of a rectifiable space G, then $S_n \subset S_{n+1}$ for every $n \in \omega$ and $p(\langle S\rangle, \langle S\rangle) \subset \langle S\rangle, q(\langle S\rangle, \langle S\rangle) \subset \langle S\rangle, e \in S_1 \subset \langle S\rangle$.*

Proof. It is obvious $S_0 \subset S_1$. Since $q(x, x) = e$ for every $x \in S_0$, we have $e \in S_1$. Let $n \in \mathbb{N}$. Since $q(x, x) = e$ for every $x \in S_{n-1}$, the point $e = q(x, x) \in S_n$. Assume $S_i \subset S_{i+1}$ for each $i < n$. Since $e \in S_n$ and $p(x, e) = x$ for every $x \in S_n$, we have $x = p(x, e) \in S_{n+1}$ for every $x \in S_n$. Then $S_n \subset S_{n+1}$. If $a, b \in \langle S\rangle$, then there exist $n, m \in \omega$, such that $a \in S_n, b \in S_m$. We assume that $n \leq m$. Then $a, b \in S_m$. Thus $p(a, b) \in S_{m+1} \subset \langle S\rangle$ and $q(a, b) \in S_{m+1} \subset \langle S\rangle$. Thus $p(\langle S\rangle, \langle S\rangle) \subset \langle S\rangle$ and $q(\langle S\rangle, \langle S\rangle) \subset \langle S\rangle$. □

Lemma 2.5. ([18], Lemma 2.6) *Let G be a rectifiable space. If $A \subset G$ and V is an open neighborhood of the right neutral element e of G, then $\overline{A} \subset p(A, V)$.*

Lemma 2.6. *Let G be a rectifiable space. If S is an open subspace of G, then $\langle S\rangle$ is clopen in G.*

Proof. Since S is open, it follows from Lemma 2.2 that $p(S, S)$ and $q(S, S)$ are open subspaces of G. Thus $S_1 = S \cup \{p(a, b), q(a, b) : a, b \in S\} = S \cup p(S, S) \cup q(S, S)$ is open in G. Let $n \in \mathbb{N}$. Assume that S_i is open for each $i \le n$. By Proposition 2.4, $e \in S_1 \subset S_n \subset \langle S\rangle$, where e is the right neutral element of G. By Lemma 2.2, the mappings p and q are open. Then $S_{n+1} = \{p(a, b), q(a, b) : a, b \in S_n\} = p(S_n, S_n) \cup q(S_n, S_n)$ is open in G. Thus $\langle S\rangle$ is open and $e \in \langle S\rangle$. By Lemma 2.5, $\overline{\langle S\rangle} \subset p(\langle S\rangle, \langle S\rangle)$. By Proposition 2.4, $p(\langle S\rangle, \langle S\rangle) \subset \langle S\rangle$. Thus $\overline{\langle S\rangle} = \langle S\rangle$. Then $\langle S\rangle$ is clopen in G. □

Lemma 2.7. *Let G be a rectifiable space and $S \subset G$. If $n \in \mathbb{N}$ and $x \in S_n$, then there exist an open continuous mapping $l_x : G^{2^n} \to G$ and $a_i \in S$ for each $i \le 2^n$ such that $l_x(a_1, ..., a_{2^n}) = x$ with the following property: If O_x is an open neighborhood of x, then there exists an open neighborhood W_i of a_i for each $i \le 2^n$ such that $l_x(\prod_{1\le i\le 2^n} W_i) \subset O_x \cap \langle\bigcup\{W_i : i \le 2^n\}\rangle$.*

Proof. We prove it by induction. If $x \in S_1$, then $x \in S$ or there exist $a, b \in S$ such that $x = p(a, b)$ or $x = q(a, b)$.

Case 1. $x \in S$. Then denote $l_x : G \times G \to G$ be the usual projection to the first coordinate. Thus l_x is an open continuous mapping. If O_x is any open neighborhood of the point x, then we let $W_i = O_x$ for $i = 1, 2$. Then $l_x(W_1 \times W_2) = W_1$. Since by Proposition 2.4 $W_1 \subset \langle W_1\rangle$, we have $l_x(W_1 \times W_2) \subset O_x \cap \langle W_1 \cup W_2\rangle$.

Case 2. Now we assume $x = p(a, b)$ or $x = q(a, b)$ for some points $a, b \in S$. We just prove the case of $x = p(a, b)$, the proof of the other case is similar. Since $x \in O_x$ and O_x is open, there exist open sets O_a, O_b of G such that $a \in O_a$, $b \in O_b$ and $x \in p(O_a \times O_b) = p(O_a, O_b) \subset O_x$. If $l_x = p$, then the mapping l_x is open and continuous such that $x \in l_x(O_a \times O_b) \subset O_x \cap \langle O_a \cup O_b\rangle$.

Let $n \in \mathbb{N}$. Assume that the result holds for each $i \le n$. Now let $x \in S_{n+1}$. By the definition of S_{n+1}, there exist $b, d \in S_n$ such that $x = p(b, d)$ or $x = q(b, d)$. Without loss of generality, we assume that $x = p(b, d)$. Since $b, d \in S_n$, by assumption there exist open continuous mappings $l_b : G^{2^n} \to G, l_d : G^{2^n} \to G$ and points $b_1, ..., b_{2^n} \in S, d_1, ..., d_{2^n} \in S$ such that $b = l_b(b_1, ..., b_{2^n}), d = l_d(d_1, ..., d_{2^n})$ and the property of this result holds.

For each $i \le 2^{n+1}$,

let $a_i = \begin{cases} b_i, & i \le 2^n; \\ d_{i-2^n}, & 2^n < i \le 2^{n+1}. \end{cases}$

Then $x = p(b, d) = p(l_b(b_1, ..., b_{2^n}), l_d(d_1, ..., d_{2^n}))$. Let $l_x : G^{2^{n+1}} \to G$ be a mapping from $G^{2^{n+1}}$ to G such that $l_x(y_1, ..., y_{2^{n+1}})= p(l_b(y_1, ..., y_{2^n}), l_d(y_{2^n+1}, ..., y_{2^{n+1}}))$ for each $(y_1, ..., y_{2^{n+1}}) \in G^{2^{n+1}}$. Since the mappings p, l_b and l_d are open and continuous, the mapping $l_x : G^{2^{n+1}} \to G$ is open and continuous such that $l_x(a_1, ..., a_{2^{n+1}}) = x$ and $\{a_1, ..., a_{2^{n+1}}\} \subset S$.

Let O_x be any open neighborhood of x. Since the mapping p is continuous, there exist open neighborhoods O_b and O_d of b and d, respectively, such that $p(O_b, O_d) \subset O_x$. Since $l_b(b_1, ..., b_{2^n}) = b$ and $b \in O_b$, there exists an open neighborhood W_i of b_i for each $1 \le i \le 2^n$ such that $l_b(\prod_{1\le i\le 2^n} W_i) \subset O_b \cap \langle\bigcup\{W_i : 1 \le i \le 2^n\}\rangle$. Since $l_d(d_1, ..., d_{2^n}) = d$ and $d \in O_d$, there exists an open neighborhood W_{i+2^n} of d_i for each $1 \le i \le 2^n$ such that $l_d(\prod_{1+2^n\le i\le 2^{n+1}} W_i) \subset O_d \cap \langle\bigcup\{W_{i+2^n} : 1 \le i \le 2^n\}\rangle$.

Since the mapping l_x satisfies $l_x(y_1, ..., y_{2^{n+1}}) = p(l_b(y_1, ..., y_{2^n}), l_d(y_{2^n+1}, ..., y_{2^{n+1}}))$ for each $(y_1, ..., y_{2^{n+1}}) \in G^{2^{n+1}}$, we have $l_x(\prod_{1\le i\le 2^{n+1}} W_i) = p(l_b(\prod_{1\le i\le 2^n} W_i), l_d(\prod_{1+2^n\le i\le 2^{n+1}} W_i))$. Since $l_b(\prod_{1\le i\le 2^n} W_i) \subset O_b \cap \langle\bigcup\{W_i : 1 \le i \le 2^n\}\rangle$ and $l_d(\prod_{1+2^n\le i\le 2^{n+1}} W_i) \subset O_d \cap \langle\bigcup\{W_{i+2^n} : 1 \le i \le 2^n\}\rangle$ and $p(O_b, O_d) \subset O_x$, we have $l_x(\prod_{1\le i\le 2^{n+1}} W_i) = p(l_b(\prod_{1\le i\le 2^n} W_i), l_d(\prod_{1+2^n\le i\le 2^{n+1}} W_i)) \subset O_x \cap \langle\bigcup\{W_i : 1 \le i \le 2^{n+1}\}\rangle$. □

Lemma 2.8. *Let G be a rectifiable space and $S \subset G$. If $n \in \mathbb{N}$ and $x \in S_n$, then there exist $a_1, ..., a_{2^n} \in S$ such that $x \in \{a_1, ..., a_{2^n}\}_n$.*

Proof. Case 1. $n = 1$. If $x \in S$, then $x \in \{x, x\}_1 = \{x\} \cup \{p(x, x), e\}$. Now we assume that $x \in S_1 \setminus S$. Then there exist $a, b \in S$ such that $x = p(a, b)$ or $x = q(a, b)$. Then $x \in \{a, b\}_1$.

Case 2. Let $m \in \mathbb{N}$. Suppose that for every $n \leq m$ and every $x \in S_n$, there exists $\{a_1, ..., a_{2^n}\} \subset S$ such that $x \in \{a_1, ..., a_{2^n}\}_n$.

Case 3. Now we assume that $x \in S_{m+1}$. Then there exist $b, d \in S_m$ such that $x = p(b, d)$ or $x = q(b, d)$. Without loss of generality, we assume $x = p(b, d)$. By induction, there exist $\{b_1, ..., b_{2^m}\} \subset S$ and $\{d_1, ..., d_{2^m}\} \subset S$ such that $b \in \{b_1, ..., b_{2^m}\}_m$ and $d \in \{d_1, ..., d_{2^m}\}_m$. Since $x = p(b, d)$, the point $x \in \{b, d\}_1$. Hence $x \in \{b_1, ..., b_{2^m}, d_1, ..., d_{2^m}\}_{m+1}$ and $\{b_1, ..., b_{2^m}, d_1, ..., d_{2^m}\} \subset S$. □

Definition 2.9. A subset S of a rectifiable space G is said to be a *suitable set* for G if (a) it has the discrete topology, (b) it is a closed subset of $G \setminus \{e\}$ and (c) the set $\langle S \rangle$ is dense in G, where e is the right neutral element of G.

Proposition 2.10. *If G is a rectifiable space and has a suitable set S, then G is a T_1-space or $G = \overline{\langle S \rangle}$ is a two-point set.*

Proof. Since S is a suitable set for X, it follows that (a) S has the discrete topology, (b) S is a closed subset of $G \setminus \{e\}$ and (c) the set $\langle S \rangle$ is dense in G, where e is the right neutral element of G. Since S is a discrete subspace of G and $S \cup \{e\}$ is closed in G, $\overline{\{x\}} \subset \{x, e\}$ for every $x \in S$. If there exists some $x \in S$ such that $\overline{\{x\}} = \{x\}$, then G is a T_1-space following from that every rectifiable space is homogeneous. Now we assume that $\overline{\{x\}} \neq \{x\}$ for every $x \in S$ and $\overline{\{e\}} \neq \{e\}$. Then $\overline{\{e\}} = \{x, e\}$ for every $x \in S$. Let $x_0 \in S$. Then $\overline{\{x_0\}} = \{x_0, e\}$ and $\overline{\{e\}} = \{x, e\}$ for every $x \in S$. Thus $x = x_0$ or $x = e$ for every $x \in S$. Since $\overline{\{e\}} = \{x_0, e\}$ and the mapping p is continuous, $p(e, x_0) \in p(\{e\} \times \{x_0\}) \subset p(\{e\} \times \overline{\{e\}}) \subset \overline{\{p(e, e)\}} = \overline{\{e\}}$. Similarly, $q(e, x_0) \in \overline{\{e\}}$. Since the mapping p is continuous and $e, x_0 \in \overline{\{e\}}$, we have $p(x_0, e) \in \overline{\{e\}}$ and $q(x_0, e) \in \overline{\{e\}}$. We also know that $p(x_0, e) = x_0$. Then $\langle S \rangle \subset \{x_0, e\} = \overline{\{e\}}$. Thus $G = \overline{\langle S \rangle}$ is a two-point set. □

Theorem 2.11. *If G is a non-T_1 rectifiable space with at least three elements, then G does not have a suitable set.*

Proof. Suppose G has a suitable set S. By Proposition 2.10, G is a T_1-space or $G = \overline{\langle S \rangle}$ is a two-point set. Since $|G| \geq 3$, it follows from Proposition 2.10 that the space G is a T_1-space. A contradiction. □

Corollary 2.12. *If G is rectifiable space such that $|G| \geq 3$ and has a suitable set, then G is a regular space.*

Proof. By Corollary 2.2 in [10], every rectifiable space is a T_3-space. Since $|G| \geq 3$ and the rectifiable space G has a suitable set, it follows from Theorem 2.11 that G is a T_1-space. Thus G is a regular space. □

Recall that a topological space is said to be 0-*dimensional* if it has a basis of clopen subsets.

Lemma 2.13. *Let G be a non-discrete rectifiable T_1-space and let U be a non-empty open subset of G such that $G = \langle U \rangle$. Then for every point $x \in U$ there exists an open neighborhood V_x of x such that $x \in V_x \subset U$ and $\langle U \setminus \overline{V_x} \rangle = G$. Further, if G is 0-dimensional, then V_x can be chosen to be clopen in G.*

Proof. Let x be any point of U. Since G is a T_1-space, the set $S = U \setminus \{x\}$ is open in G. If $x = e$, where e is the right neutral element of G, then $q(y, y) = e$ for any $y \in S$. Thus $\langle S \rangle = \langle U \rangle = G$.

Now we assume that $x \neq e$. Since G is non-discrete, the point x is not an isolated point of G. Then $x \in \overline{S}$. Since $S \subset \langle S \rangle$ and by Lemma 2.6 $\langle S \rangle$ is clopen, we have $x \in \overline{S} \subset \langle S \rangle$. Thus $\langle S \rangle = \langle U \rangle = G$. Then there exists $n \in \mathbb{N}$ such that $x \in S_n$ (see Notation 2.3). Since S is open and the mappings p and q are open, the set S_n is open. Since $x \in S_n$, it follows from Lemma 2.7 that there exist an open continuous mapping $l_x : G^{2^n} \to G$ and $a_i \in S$ for each $i \leq 2^n$ such that $l_x(a_1, ..., a_{2^n}) = x$. Since G is a T_1-space and $x \notin \{a_i : i \leq 2^n\}$, there exists an open set O_x such that $x \in O_x$ and $O_x \cap \{a_i : i \leq 2^n\} = \emptyset$. By Corollary 2.2 in [10], G is a regular space. Then there exists an open neighborhood V_x^* of x such that $x \in V_x^* \subset \overline{V_x^*} \subset U \cap O_x \cap S_n$. By Lemma 2.7, for each $i \leq 2^n$ there exists an open neighborhood W_i of a_i such that $W_i \subset U$, $W_i \cap \overline{V_x^*} = \emptyset$

and $x \in l_x(\prod_{1\le i\le 2^n} W_i) \subset V_x^* \cap \langle \bigcup\{W_i : 1 \le i \le 2^n\}\rangle$. Since the mapping l_x is open, the set $l_x(\prod_{1\le i\le 2^n} W_i)$ is open. Then there exists an open neighborhood V_x of x such that $x \in V_x \subset \overline{V_x} \subset l_x(\prod_{1\le i\le 2^n} W_i)$. Then $\overline{V_x} \subset V_x^* \cap \langle \bigcup\{W_i : 1 \le i \le 2^n\}\rangle$. Since $\bigcup\{W_i : 1 \le i \le 2^n\} \subset G\backslash\overline{V_x^*} \subset G\backslash\overline{V_x}$, we have $\langle U\backslash\overline{V_x}\rangle = \langle U\rangle = G$. The last statement of the lemma is obvious. □

Lemma 2.14. ([9], Theorem 6.2.6 and Corollary 6.2.8) *Every countable regular space is* 0*-dimensional.*

Theorem 2.15. *Every countable rectifiable T_0-space G has a closed discrete subset S such that $\langle S\rangle = G$. In particular, S is a suitable set for G.*

Proof. By Corollary 2.2 in [10], every rectifiable space satisfies T_3 separation axiom. Since every T_0-space satisfying T_3 separation axiom is regular, the rectifiable T_0-space G is regular.

If G is discrete or there exists a finite subset F of G such that $G = \langle F\rangle$ (in this case, G is called finitely generated), then the claim is trivial. Now we assume that G is neither discrete nor finitely generated.

Let $G = \{g_n : n < \omega\}$. It suffices to find a subset S of G such that $\langle S\rangle = G$ and, for each $n < \omega$, an open neighborhood U_n of g_n such that $U_n \cap S$ is finite.

For this it will suffice to find for each $n < \omega$ a clopen set V_n in G and a finite set $A_n \subset G$ such that the following conditions hold:

(1) $g_n \in V_0 \cup V_1 \cup ... \cup V_n$;
(2) $G = \langle G\backslash(V_0 \cup V_1 \cup ... \cup V_n)\rangle$;
(3) for $n > 0$, $V_n \subset G\backslash(V_0 \cup V_1 \cup ... \cup V_{n-1})$;
(4) $V_i \cap A_n = \emptyset$, for $i < n$;
(5) $g_n \in \langle A_0 \cup A_1 \cup ... \cup A_n\rangle$.

That the above suffices is clear by putting $U_n = V_0 \cup V_1 \cup ... \cup V_n$ and $S = \bigcup_{n<\omega} A_n$. We shall define the sets A_n and V_n inductively.

Put $A_0 = \{g_0\}$. Since G is a countable regular space, it follows Lemma 2.14 that G is 0-dimensional. By Lemma 2.13, there exists a clopen neighborhood V_0 of g_0 such that $G = \langle G\backslash V_0\rangle$.

Now assume that $k \in \omega$ and there exist finite sets $A_0, A_1, ..., A_k$ and clopen sets $V_0, V_1, ..., V_k$ which have the above properties (1)-(5) for each $n \le k$. If $g_{k+1} \in \langle A_0 \cup A_1 \cup ... \cup A_k\rangle$, put $A_{k+1} = \emptyset$. Now we assume $g_{k+1} \notin \langle A_0 \cup A_1 \cup ... \cup A_k\rangle$. By (2), $G = \langle G\backslash(V_0 \cup V_1 \cup ... \cup V_k)\rangle$. If $S = G\backslash(V_0 \cup V_1 \cup ... \cup V_k)$, then there exists $n \in \mathbb{N}$ such that $g_{k+1} \in S_n$ (see Notation 2.3). By Lemma 2.8, there exist $y_1, y_2, ..., y_{2^n} \in S$ such that $g_{k+1} \in \{y_1, y_2, ..., y_{2^n}\}_n$. Put $A_{k+1} = \{y_1, y_2, ..., y_{2^n}\}$. Then $V_i \cap A_{k+1} = \emptyset$ for $i \le k$ and $g_{k+1} \in \langle A_0 \cup A_1 \cup ... \cup A_k\rangle$.

Now if $g_{k+1} \in V_0 \cup ... \cup V_k$, put $V_{k+1} = \emptyset$. If $g_{k+1} \notin V_0 \cup ... \cup V_k$, then by Lemma 2.13 there exists a clopen neighborhood V_{k+1} of g_{k+1} such that $V_{k+1} \subset G\backslash(V_0 \cup ... \cup V_k)$ and $G = \langle G\backslash(V_0 \cup V_1 \cup ... \cup V_{k+1})\rangle$. Then conditions (1)-(3) are satisfied in both cases.

By induction, the sets A_n and V_n can be defined for all n with the required properties, which complete the proof. □

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