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Deferred Cesàro means of fuzzy number-valued sequences with applications to Tauberian theorems

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Abstract. In this paper, the deferred Cesàro means of fuzzy number-valued sequences are studied and their summability by the deferred Cesàro method with respect to the supremum metric is introduced. Also, Tauberian conditions to retrieve the convergence of a fuzzy number-valued sequence from its deferred Cesàro summability are investigated.

1. Introduction

The concept of fuzzy sets has been initiated by L. A. Zadeh [49] as an alternative way of dealing with uncertainties. Following the seminal work of Zadeh, many researchers from different fields of science have made significant contributions to the development of fuzzy set theory and its applications.

The commonly accepted theory of fuzzy numbers was presented by Dubois and Prade [12]. Fuzzy numbers extend "classical" real numbers and more precisely a fuzzy number is regarded as a fuzzy subset of the real line, which fulfills some supplementary properties. The value of a fuzzy number is not as precise as that of a classical number. Hereby, fuzzy numbers describe the real world more accurately than classical numbers in various aspects.

The study on sequences and series of fuzzy numbers has been started with the works of Matloka [26] and Nanda [30] presenting the classes of fuzzy number-valued convergent, bounded, and Cauchy sequences and their several properties regarding the supremum metric. Later, various summation techniques such as Cesàro summability (Çanak [8], Talo and Çakan [43]), Abel summability (Yavuz and Talo [47]), logarithmic summability (Sezer [38]), weighted mean summability (Braha and Et [6], Tripathy and Baruah [45]), power series summability (Sezer and Çanak [37]), Borel summability (Yavuz and Coşkun [46]), statistical convergence (Savaş [36], Talo [44]) and statistical summability (Altin et al. [2], Yavuz [48]) have been introduced to deal with fuzzy number-valued sequences that are not convergent in the fuzzy number space and Tauberian type theorems are investigated to retrieve the convergence. Moreover, using several summability techniques (see, e.g., Das et al. [10], Gogoi and Dutta [19], Kadak et al. [22], Mohiuddine et al. [27]) approximation theorems for sequences of fuzzy positive linear operators have been obtained.

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The deferred Cesàro means of any sequence (u_n) are given by

$$D_n^{p,q}u = \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} u_k, \quad n \in \mathbb{N}, \tag{1}$$

where $p = (p_n)$ and $q = (q_n)$ are sequences of non-negative integers satisfying

$$p_n < q_n, \quad n \in \mathbb{N}, \tag{2}$$

and

$$\lim_{n\to\infty} q_n = +\infty. \tag{3}$$

The notion of deferred Cesàro means was first introduced by Agnew [1]. Agnew also proposed a necessary and sufficient condition so that deferred Cesàro means may include the $(C, 1)$ mean. Obviously, in the particular case of $p_n = 0$ and $q_n = n$ for all $n \in \mathbb{N}$, the corresponding deferred Cesàro mean is reduced to the $(C, 1)$ mean.

Due to its applications in approximation theory and summability theory, deferred Cesàro means has received increased attention from researchers in recent years. Combining the concepts of deferred Cesàro means and statistical convergence (see [17]), Küçükaslan and Yılmaztürk [24] defined the deferred statistical convergence of sequences and compared this new concept with the statistical convergence and strong deferred Cesàro means. Jena et al. [20] then introduced the statistical deferred Cesàro summability method and applied it to derive Korovkin type approximation theorems over a Banach space. Later, Srivastava et al. [40, 41] presented the notions of deferred weighted and deferred Nörlund statistical summability and established analogous approximation results. Paikray et al. [33] proved a Korovkin-type theorem for statistical deferred Cesàro summability based upon the (p, q)-integers. Recently, Sezer et al. [39] obtained some Tauberian theorems for the deferred Cesàro summability of real or complex sequences. Moreover, deferred Cesàro means have been studied by several researchers on time scales [9], sequence spaces of random variables [21], normed spaces [14], sequence spaces of uncertain [31] and fuzzy numbers [32]. For more recent studies in this direction, see [15, 23, 35].

In view of the above-mentioned studies, we investigate the deferred Cesàro means of fuzzy number-valued sequences and provide related Tauberian theorems in this paper.

1.1. *Fuzzy numbers*

A fuzzy set F (fuzzy subset of X) is a function $F : X \to [0, 1]$ where $F(x)$ is the membership grade of x to the fuzzy set X (Zadeh [49]). We mean by $\mathcal{F}(X)$ the family of all fuzzy subsets of X. Fuzzy sets may be regarded as generalizations of classical (crisp) sets, in which only total membership and certain non-membership are allowed. Consequently, fuzzy sets are good tools to represent ambiguous and imprecise expressions of natural language.

Another feasible and appropriate approach is to treat a fuzzy set as a nested family of classical subsets, through the concept of level set. For $F \in \mathcal{F}(X)$, the α-level set of F is given by $[F]_\alpha = \{x \in X \mid F(x) \geq \alpha\}$ for $\alpha \in (0, 1]$ and $[F]_0 = \mathrm{cl}\{x \in X \mid F(x) > 0\}$ where $\mathrm{cl}\{S\}$ shows the closure of the set S.

A fuzzy subset of the real line, $u : \mathbb{R} \to [0, 1]$, is called a fuzzy number if u is upper semi-continuous, fuzzy convex, normal and $[u]_0$ is compact (Dubois and Prade [12]). By $\mathbb{R}_{\mathcal{F}}$ the space of all fuzzy numbers is represented. The concept of a fuzzy number is fundamental to fuzzy analysis and is an extremely helpful tool in various applications of fuzzy logic and fuzzy sets (Bede [5]).

Obviously, if $u \in \mathbb{R}_{\mathcal{F}}$, then the level set $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$ is a bounded, closed interval for any $\alpha \in [0, 1]$. In this case, the following conditions are provided (Goetschel and Voxman [18]):

(i) $u^-(\alpha)$ and $u^+(\alpha)$ are right continuous functions at $\alpha = 0$.

(ii) $u^-(\alpha)$ is a left continuous non-decreasing and bounded function on $(0, 1]$.

(iii) $u^+(\alpha)$ is a left continuous non-increasing and bounded function on $(0,1]$.

(iv) $u^-(1) \leq u^+(1)$.

On the contrary, there exists a unique $u \in \mathbb{R}_{\mathcal{F}}$ which has $u^-(\alpha)$ and $u^+(\alpha)$ as endpoints of its α-level sets $[u]_\alpha$, if $u^-(\alpha)$ and $u^+(\alpha)$ hold the requirements (i)-(iv) above.

It is plain that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$, due to the fact that any $r \in \mathbb{R}$ may be thought of as a fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(x) = \begin{cases} 1, & x = r \\ 0, & \text{otherwise.} \end{cases}$$

Given $u, v \in \mathbb{R}_{\mathcal{F}}$ and $m \in \mathbb{R}$, the summation of two fuzzy numbers $u + v$ and the multiplication of a real and a fuzzy number mu are introduced as

$$u + v = w \quad \text{iff} \quad [w]_\alpha = [u]_\alpha + [v]_\alpha = \left[u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)\right],$$

$$[mu]_\alpha = m[u]_\alpha = \begin{cases} \left[mu^-(\alpha), mu^+(\alpha)\right], & m \geq 0 \\ \left[mu^+(\alpha), mu^-(\alpha)\right], & m < 0, \end{cases}$$

respectively. Also, the following algebraic properties of $\mathbb{R}_{\mathcal{F}}$ hold true (Dubois and Prade [13]):

(i) $u + v = v + u$ and $u + (v + w) = (u + v) + w$, for each $u, v, w \in \mathbb{R}_{\mathcal{F}}$.

(ii) $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ is the zero element according to $+$, that is, $u + \tilde{0} = \tilde{0} + u = u$, for each $u \in \mathbb{R}_{\mathcal{F}}$.

(iii) None of $u \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ has an opposite element in $\mathbb{R}_{\mathcal{F}}$ with respect to $\tilde{0}$.

(iv) $(m + n)u = mu + nu$, for all $m, n \in \mathbb{R}$ such that $m, n \leq 0$ or $m, n \geq 0$ and all $u \in \mathbb{R}_{\mathcal{F}}$.

(v) $m(u + v) = mu + mv$, for each $m \in \mathbb{R}$ and $u, v \in \mathbb{R}_{\mathcal{F}}$.

(vi) $(mn)u = m(nu)$, for each $m, n \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathcal{F}}$.

As a conclusion, it is observed that $\mathbb{R}_{\mathcal{F}}$ is not a linear space over the real numbers.

The most utilized distance in $\mathbb{R}_{\mathcal{F}}$ is the supremum metric (Puri and Ralescu [34]) introduced as

$$d_\infty(u, v) = \sup_{\alpha \in [0,1]} \max \left\{ \left|u^-(\alpha) - v^-(\alpha)\right|, \left|u^+(\alpha) - v^+(\alpha)\right| \right\}$$

for all $u, v \in \mathbb{R}_{\mathcal{F}}$. For the supremum metric $d_\infty : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to [0, \infty)$, $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space and additionally, the features below are satisfied (Diamond and Kloeden [11]):

(i) $d_\infty(mu, mv) = |m| \, d_\infty(u, v)$, if $m \in \mathbb{R}$ and $u, v \in \mathbb{R}_{\mathcal{F}}$,

(ii) $d_\infty(u + w, v + w) = d_\infty(u, v)$, if $u, v, w \in \mathbb{R}_{\mathcal{F}}$,

(iii) $d_\infty(u + v, w + z) \leq d_\infty(u, w) + d_\infty(v, z)$, if $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$.

For $u, v \in \mathbb{R}_{\mathcal{F}}$, the partial ordering relation over $\mathbb{R}_{\mathcal{F}}$ is given by:
$u \preceq v$ if and only if $[u]_\alpha \preceq [v]_\alpha$, that is $u^-(\alpha) \leq v^-(\alpha)$ and $u^+(\alpha) \leq v^+(\alpha)$ for each $\alpha \in [0,1]$.

Assume that u, v, w, z are fuzzy numbers. From Li and Wu [25] and Talo and Başar [42], the properties below are known:

(i) If $u \preceq w$ and $v \preceq z$, then $u + v \preceq w + z$.

(ii) If $u \preceq v$ and $v \preceq w$, then $u \preceq w$.

(iii) If $u + w \preceq v + w$, then $u \preceq v$.

Besides, for given $\epsilon > 0$, Aytar et al. [3] and Aytar and Pehlivan [4] obtained that

(iv) If $u \preceq v + \tilde{\epsilon}$, then $u \preceq v$.

(v) $u - \tilde{\epsilon} \preceq v \preceq u + \tilde{\epsilon}$ if and only if $d_\infty(u, v) \leq \epsilon$.

A sequence $(u_n) \subset \mathbb{R}_\mathcal{F}$ is called convergent to $\mu \in \mathbb{R}_\mathcal{F}$, if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d_\infty(u_n, \mu) \leq \epsilon$ for every $n \geq n_0$. On the other hand, (u_n) converges to μ if and only if $(u_n^-(\alpha))$ and $(u_n^+(\alpha))$ are uniformly convergent to $\mu^-(\alpha)$ and $\mu^+(\alpha)$ on $[0, 1]$, respectively (Fang and Huang [16]). The sequence (u_n) is called bounded if there is some real number $C > 0$ such that $d_\infty(u_n, \tilde{0}) \leq C$. For the traditional sets of sequences of fuzzy numbers and associated topics, the reader may refer to Mursaleen and Başar [29] and the references therein.

1.2. Deferred Cesàro summability of fuzzy sequences

A fuzzy sequence (u_n) is said to be deferred Cesàro summable determined by (p_n) and (q_n), (or in short, $D[p, q]$-summable) to a fuzzy number μ if

$$\lim_{n\to\infty} D_n^{p,q} u = \mu. \tag{4}$$

In other words, for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d_\infty(D_n^{p,q} u, \mu) \leq \epsilon. \tag{5}$$

It is known from the Agnew's work that the $D[p, q]$-summability method is regular under the conditions (2) and (3). To be more specific, if the conditions (2) and (3) are fulfilled, any convergent sequence is $D[p, q]$-summable to the same limit. However, it is shown by the following example that convergence of a fuzzy sequence may not follows from its $D[p, q]$-summability.

Example 1.1. Given the fuzzy sequence (u_n) for each $x \in \mathbb{R}$ by

$$u_n(x) = \begin{cases} \left.\begin{matrix} 1 + \dfrac{x}{2}, & x \in [-2, 0] \\ 0, & \text{otherwise} \end{matrix}\right\} \text{if } n \text{ is even} \\ \left.\begin{matrix} 1 - \dfrac{x^2}{4}, & x \in [0, 2] \\ 0, & \text{otherwise} \end{matrix}\right\} \text{if } n \text{ is odd.} \end{cases}$$

For any $\alpha \in [0, 1]$, α-level set of the deferred Cesàro means of (u_n), specified by the sequences $p_n = 2n - 1$ and $q_n = 4n$, follows as

$$[D_n^{p,q} u]_\alpha = \left[\frac{2(n+1)}{2n+1}(\alpha - 1), \frac{2n}{2n+1}\sqrt{1-\alpha}\right].$$

Then, $(D_n^{p,q} u)$ converges to $[\mu]_\alpha = \left[\alpha - 1, \sqrt{1-\alpha}\right]$ where

$$\mu(x) = \begin{cases} 1 + x, & x \in [-1, 0] \\ 1 - x^2, & x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, (u_n) is $D[p, q]$-summable to $\mu \in \mathbb{R}_\mathcal{F}$. However, it is not convergent.

In the next section, the converse implication is studied. An answer to the question, "Under which conditions does convergence occur from the $D[p, q]$-summability method?" is examined. In the literature, such a condition is known as a Tauberian condition, and the subsequent theorem is called a Tauberian theorem.

2. Tauberian theorems for the deferred Cesàro summability

In the sequel, it is assumed that (q_n) is strictly increasing and γ_n represents the integer part of the product γn. Before stating the main results, it is necessary to give the following lemmas.

Lemma 2.1. (Móricz and Rhoades [28]) *Assume that (q_n) is a non-decreasing sequence of positive numbers. The assertions*

$$\liminf_{n\to\infty} \frac{q_{\gamma_n}}{q_n} > 1 \quad \textit{for every} \quad \gamma > 1 \tag{6}$$

and

$$\liminf_{n\to\infty} \frac{q_n}{q_{\gamma_n}} > 1 \quad \textit{for every} \quad 0 < \gamma < 1 \tag{7}$$

are equivalent.

Lemma 2.2. *Let* (6) *be satisfied. If a sequence (u_n) of fuzzy numbers is $D[p,q]$-summable to $\mu \in \mathbb{R}_{\mathcal{F}}$, then for all $\gamma > 1$,*

$$\lim_{n\to\infty} \frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k = \mu, \tag{8}$$

and for all $0 < \gamma < 1$,

$$\lim_{n\to\infty} \frac{1}{q_n - q_{\gamma_n}} \sum_{k=q_{\gamma_n}+1}^{q_n} u_k = \mu. \tag{9}$$

Proof. Let $\gamma > 1$. For all large enough n such that $q_{\gamma_n} > q_n$,

$$\begin{aligned}
\frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q_\gamma} u + D_n^{p,q} u &= \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} \left(\frac{1}{q_{\gamma_n} - p_n} \sum_{k=p_n+1}^{q_{\gamma_n}} u_k \right) + \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} u_k \\
&= \frac{1}{q_{\gamma_n} - q_n} \left(\sum_{k=p_n+1}^{q_n} u_k + \sum_{k=q_n+1}^{q_{\gamma_n}} u_k \right) + \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} u_k,
\end{aligned}$$

whence it follows

$$\frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q_\gamma} u + D_n^{p,q} u = \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q} u + \frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k, \tag{10}$$

where $D_n^{p,q_\gamma} u = \dfrac{1}{q_{\gamma_n} - p_n} \displaystyle\sum_{k=p_n+1}^{q_{\gamma_n}} u_k$.

Then, by (10)

$$\begin{aligned}
d_\infty\left(\frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k, \mu \right) &= d_\infty\left(\frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q} u + \frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k, \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q} u + \mu \right) \\
&= d_\infty\left(\frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q_\gamma} u + D_n^{p,q} u, \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} D_n^{p,q} u + \mu \right),
\end{aligned}$$

and so

$$d_\infty\left(\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}u_k,\mu\right)\leq\frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n}d_\infty\left(D_n^{p,q_\gamma}u,D_n^{p,q}u\right)+d_\infty\left(D_n^{p,q}u,\mu\right). \tag{11}$$

Considering (6) gives

$$\limsup_{n\to\infty}\frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n}\leq\left\{1-\left(\liminf_{n\to\infty}\frac{q_{\gamma_n}}{q_n}\right)^{-1}\right\}^{-1}<\infty. \tag{12}$$

Hence, (8) is obtained from (11), (12) and assumed $D[p,q]$-summability of (u_n).

Let $0<\gamma<1$. For all large enough n such that $q_{\gamma_n}<q_n$,

$$\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}D_n^{p_\gamma,q}u+D_n^{p_\gamma,q_\gamma}u=\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}D_n^{p_\gamma,q_\gamma}u+\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}u_k, \tag{13}$$

where $D_n^{p_\gamma,q}u=\dfrac{1}{q_n-p_{\gamma_n}}\displaystyle\sum_{k=p_{\gamma_n}+1}^{q_n}u_k$.

Then, by (13)

$$\begin{aligned}d_\infty\left(\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}u_k,\mu\right)&=d_\infty\left(\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}D_n^{p_\gamma,q_\gamma}u+\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}u_k,\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}D_n^{p_\gamma,q_\gamma}u+\mu\right)\\&=d_\infty\left(\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}D_n^{p_\gamma,q}u+D_n^{p_\gamma,q_\gamma}u,\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}D_n^{p_\gamma,q_\gamma}u+\mu\right),\end{aligned}$$

and so

$$d_\infty\left(\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}u_k,\mu\right)\leq\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}d_\infty\left(D_n^{p_\gamma,q}u,D_n^{p_\gamma,q_\gamma}u\right)+d_\infty\left(D_n^{p_\gamma,q_\gamma}u,\mu\right). \tag{14}$$

Now, considering (7) gives

$$\limsup_{n\to\infty}\frac{q_n-p_{\gamma_n}}{q_n-q_{\gamma_n}}\leq\left\{1-\left(\liminf_{n\to\infty}\frac{q_n}{q_{\gamma_n}}\right)^{-1}\right\}^{-1}<\infty. \tag{15}$$

Therefore, (9) follows from (14), (15) and assumed $D[p,q]$-summability of (u_n). □

The main results are now ready to be presented.

Theorem 2.3. *Suppose that* (6) *is satisfied. If a sequence* (u_n) *of fuzzy numbers is* $D[p,q]$*-summable to* $\mu\in\mathbb{R}_{\mathcal{F}}$*, then* (u_{q_n}) *is convergent to* μ*, if and only if for given* $\epsilon>0$*, there exists* $\gamma>1$*, as close to 1 as we want, such that for all sufficiently large n*

$$\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}u_k\succeq u_{q_n}-\tilde{\epsilon}, \tag{16}$$

and there exists another γ *with* $0<\gamma<1$*, as close to 1 as we want, such that for all sufficiently large n*

$$\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}u_k\preceq u_{q_n}+\tilde{\epsilon}. \tag{17}$$

Proof. *Necessity.* Assume the convergence of both (u_{q_n}) and $(D_n^{p,q}u)$ to μ. Given any $\gamma > 1$, from Lemma 2.2

$$\lim_{n\to\infty} d_\infty\left(\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}} u_k, u_{q_n}\right) \leq \lim_{n\to\infty} d_\infty\left(\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}} u_k, \mu\right) + \lim_{n\to\infty} d_\infty\left(u_{q_n}, \mu\right) = 0,$$

which is, even stronger condition than (16).

Similarly, for any given $0 < \gamma < 1$,

$$\lim_{n\to\infty} d_\infty\left(\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n} u_k, u_{q_n}\right) \leq \lim_{n\to\infty} d_\infty\left(\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n} u_k, \mu\right) + \lim_{n\to\infty} d_\infty\left(u_{q_n}, \mu\right) = 0,$$

that is stronger than (17).

Sufficiency. Conversely, let conditions (16) and (17) be satisfied. Take any $\epsilon > 0$. Considering (16), $\gamma > 1$ may be chosen as close to 1 as wished such that

$$\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}} u_k \succeq u_{q_n} - \frac{\tilde{\epsilon}}{3} \tag{18}$$

for all large enough n.

Taking (12) and the assumed convergence of $(D_n^{p,q}u)$ into account gives

$$\lim_{n\to\infty} d_\infty\left(D_n^{p,q}u, \mu\right) = 0, \tag{19}$$

and so

$$\lim_{n\to\infty} d_\infty\left(\frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n} D_n^{p,q_\gamma}u, \frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n} D_n^{p,q}u\right) = 0$$

which implies that

$$\mu - \frac{\tilde{\epsilon}}{3} \preceq D_n^{p,q}u \preceq \mu + \frac{\tilde{\epsilon}}{3}, \tag{20}$$

and

$$\frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n} D_n^{p,q}u - \frac{\tilde{\epsilon}}{3} \preceq \frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n} D_n^{p,q_\gamma}u \preceq \frac{q_{\gamma_n}-p_n}{q_{\gamma_n}-q_n} D_n^{p,q}u + \frac{\tilde{\epsilon}}{3} \tag{21}$$

respectively, for all sufficiently large n.

Utilizing the identity (10) and combining (18), (20) and (21) gives

$$u_{q_n} \preceq \mu + \tilde{\epsilon}.$$

Now, it is needed to show that

$$\mu - \tilde{\epsilon} \preceq u_{q_n}.$$

Since from (17), $0 < \gamma < 1$ may be choosen as close to 1 as wished such that

$$\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n} u_k \preceq u_{q_n} + \frac{\tilde{\epsilon}}{3} \tag{22}$$

for all large enough n.

Further, by virtue of (15) and (19), it is observed that

$$\lim_{n\to\infty} d_\infty\left(\frac{q_n - p_{\gamma_n}}{q_n - q_{\gamma_n}} D_n^{p_\gamma,q} u, \frac{q_n - p_{\gamma_n}}{q_n - q_{\gamma_n}} D_n^{p_\gamma,q_\gamma} u\right) = 0$$

which indicates

$$\frac{q_n - p_{\gamma_n}}{q_n - q_{\gamma_n}} D_n^{p_\gamma,q_\gamma} u - \frac{\tilde{\epsilon}}{3} \preceq \frac{q_n - p_{\gamma_n}}{q_n - q_{\gamma_n}} D_n^{p_\gamma,q} u \preceq \frac{q_n - p_{\gamma_n}}{q_n - q_{\gamma_n}} D_n^{p_\gamma,q_\gamma} u + \frac{\tilde{\epsilon}}{3} \tag{23}$$

for all sufficiently large n.

Using the identity (13) and putting (20), (22) and (23) together gives the desired result. □

Remark 2.4. The above theorem remains valid if (16) and (17) are replaced by the following ones, respectively:

$$\frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k \preceq u_{q_n} + \tilde{\epsilon}, \tag{24}$$

and

$$\frac{1}{q_n - q_{\gamma_n}} \sum_{k=q_{\gamma_n}+1}^{q_n} u_k \succeq u_{q_n} - \tilde{\epsilon}. \tag{25}$$

Remark 2.5. Following Talo and Başar [42], it is said that $(u_n) \subset \mathbb{R}_{\mathcal{F}}$ is deferred slowly decreasing, if for each $\epsilon > 0$ there exists $\gamma > 1$, as close to 1 as wished, such that for all large enough n

$$u_k \succeq u_{q_n} - \tilde{\epsilon} \text{ whenever } q_n < k \leq q_{\gamma_n}. \tag{26}$$

In the case of $0 < \gamma < 1$, an equivalent definition of a deferred slowly decreasing sequence is given as follows: for each $\epsilon > 0$ there exists $0 < \gamma < 1$, as close to 1 as wished, such that for all large enough n

$$u_k \preceq u_{q_n} + \tilde{\epsilon} \text{ whenever } q_{\gamma_n} < k \leq q_n. \tag{27}$$

It is plain that (26) and (27) yield the conditions (16) and (17), respectively. Thus, via Theorem 2.3 the following Tauberian result is established.

Theorem 2.6. *Let* (6) *be satisfied. If a sequence* (u_n) *of fuzzy numbers is* $D[p,q]$*-summable to* $\mu \in \mathbb{R}_{\mathcal{F}}$ *and if* (u_n) *is deferred slowly decreasing, then* (u_n) *is convergent to* μ.

Proof. Take any $\epsilon > 0$ and assume that hypotheses (26) and (27) are satisfied. Since (26) and (27) imply (16) and (17), respectively, it is obtained using Theorem 2.3 that

$$d_\infty(u_{q_n}, \mu) \leq \epsilon \tag{28}$$

for all large enough n. Obviously, (28) is equivalent to

$$\mu - \tilde{\epsilon} \preceq u_{q_n} \preceq \mu + \tilde{\epsilon}. \tag{29}$$

Now, taking (26) and (29) into account gives

$$\mu - 2\tilde{\epsilon} \preceq u_k \tag{30}$$

whenever $k \geq n$ and n is sufficiently large. Further, considering (27) and (29) gives

$$u_k \preceq \mu + 2\tilde{\epsilon} \tag{31}$$

whenever $k \geq n$ and n is sufficiently large.

Since $\epsilon > 0$ is arbitrary, combining (30) and (31) gives the desired result. □

Remark 2.7. A sequence of positive numbers (q_n) is said to be regularly varying of index ϑ, $\vartheta \in \mathbb{R}$, if

$$\lim_{n\to\infty} \frac{q_{\gamma_n}}{q_n} = \gamma^{\vartheta} \quad \text{for every } \gamma > 0. \tag{32}$$

It is easy to verify that (6) is satisfied by those regularly varying sequences of index $\vartheta > 0$ (see Chen and Hsu [7]).

Lemma 2.8. *Let* (32) *be satisfied. If* $(u_n) \subset \mathbb{R}_{\mathcal{F}}$ *satisfies*

$$nu_n \geq nu_{n-1} - \tilde{C} \tag{33}$$

for some $C > 0$ *and each* $n \in \mathbb{N}$*, then* (u_n) *is deferred slowly decreasing.*

Proof. It is assumed that $(u_n) \subset \mathbb{R}_{\mathcal{F}}$ satisfies (33). Thus, for every $\alpha \in [0, 1]$

$$u_n^-(\alpha) - u_{n-1}^-(\alpha) \geq -\frac{C}{n} \quad \text{and} \quad u_n^+(\alpha) - u_{n-1}^+(\alpha) \geq -\frac{C}{n},$$

for some $C > 0$. Therefore, for every $\alpha \in [0, 1]$ and $q_n < k \leq q_{\gamma_n}$

$$\begin{aligned} u_k^-(\alpha) - u_{q_n}^-(\alpha) &= \sum_{i=q_n+1}^{k} [u_i^-(\alpha) - u_{i-1}^-(\alpha)] \\ &\geq -C \sum_{i=n+1}^{k} \frac{1}{i} \geq -C\left(\frac{q_{\gamma_n}}{q_n} - 1\right). \end{aligned}$$

Since (q_n) is regularly varying of index $\vartheta > 0$, it follows that

$$u_k^-(\alpha) - u_{q_n}^-(\alpha) \geq -C\left(\gamma^{\vartheta} - 1\right)$$

for all large enough n.

Taking any $\epsilon > 0$ and $1 < \gamma \leq (1 + \epsilon/C)^{1/\vartheta}$, the inequality

$$u_k^-(\alpha) - u_{q_n}^-(\alpha) \geq -\epsilon \tag{34}$$

immediately follows.

Similarly, for all $\alpha \in [0, 1]$ and $q_n < k \leq q_{\gamma_n}$ it is found that

$$u_k^+(\alpha) - u_{q_n}^+(\alpha) \geq -\epsilon. \tag{35}$$

Putting (34) and (35) together yields $u_k \geq u_{q_n} - \tilde{\epsilon}$ which completes the proof. □

In consequence of Theorem 2.6 and Lemma 2.8, the following corollary is apparent.

Corollary 2.9. *Suppose that* (32) *is satisfied. If a sequence* (u_n) *of fuzzy numbers is* $D[p, q]$*-summable to* $\mu \in \mathbb{R}_{\mathcal{F}}$ *and satisfies* (33)*, then* (u_n) *is convergent to* μ*.*

Lemma 2.10. *Assume that*

$$\frac{n}{q_n - p_n} = O_L(1). \tag{36}$$

If $(u_n) \subset \mathbb{R}_{\mathcal{F}}$ *satisfies* (33)*, then the sequence* $(D_n^{p,q}u)$ *of deferred Cesàro means of* (u_n) *satisfies*

$$nD_n^{p,q}u \geq nD_n^{p,q-1}u - \tilde{C} \tag{37}$$

for some $C > 0$ *and each* $n \in \mathbb{N}$*.*

Proof. Assume (36), that is, assume

$$\frac{n}{q_n - p_n} \geq -C$$

for some $C > 0$ and all n. Hence, for any $\alpha \in [0,1]$

$$\begin{aligned} n\left[(D_n^{p,q}u)^-(\alpha) - (D_n^{p,q-1}u)^-(\alpha)\right] &= \frac{n}{q_n - p_n}\sum_{j=p_n+1}^{q_n} u_j^-(\alpha) - \frac{n}{q_n - p_n - 1}\sum_{j=p_n+1}^{q_n-1} u_j^-(\alpha) \\ &= \frac{n}{q_n - p_n}u_{q_n}^-(\alpha) - \frac{n}{(q_n - p_n)(q_n - p_n - 1)}\sum_{j=p_n+1}^{q_n-1} u_j^-(\alpha) \\ &= \frac{n}{(q_n - p_n)(q_n - p_n - 1)}\sum_{j=p_n+1}^{q_n-1}\left[u_{q_n}^-(\alpha) - u_j^-(\alpha)\right] \\ &= \frac{n}{(q_n - p_n)(q_n - p_n - 1)}\sum_{j=p_n+1}^{q_n-1}\sum_{i=j+1}^{q_n}\left[u_i^-(\alpha) - u_{i-1}^-(\alpha)\right] \\ &= \frac{n}{(q_n - p_n)(q_n - p_n - 1)}\sum_{i=p_n+2}^{q_n}(i - (p_n + 1))\left[u_i^-(\alpha) - u_{i-1}^-(\alpha)\right]. \end{aligned}$$

By virtue of (36) and (37), it follows from the above identity that

$$n\left[(D_n^{p,q}u)^-(\alpha) - (D_n^{p,q-1}u)^-(\alpha)\right] \geq -C \tag{38}$$

for some $C > 0$. Using the similar argument above, it also follows

$$n\left[(D_n^{p,q}u)^+(\alpha) - (D_n^{p,q-1}u)^+(\alpha)\right] \geq -C. \tag{39}$$

Considering (38) and (39), it is concluded that

$$nD_n^{p,q}u \succeq nD_n^{p,q-1}u - \tilde{C}$$

which completes the proof. □

By Corollary 2.9 and Lemma 2.10, the following corollary is obvious.

Corollary 2.11. *Let* (32) *and* (36) *be satisfied. If the sequence* $(D_n^{p,q}u)$ *of deferred Cesàro means of* $(u_n) \subset \mathbb{R}_{\mathcal{F}}$ *is* $D[p,q]$*-summable to* $\mu \in \mathbb{R}_{\mathcal{F}}$ *and satisfies* (37)*, then* $(D_n^{p,q}u)$ *is convergent to* μ*.*

3. Conclusion

The deferred Cesàro means of fuzzy number-valued sequences are investigated in this study. Any convergent sequence is known to be $D[p,q]$-summable to the same limit under the restrictions (2) and (3), i.e., $D[p,q]$-summability is regular if these conditions are met. To show that the converse of this implication is not generally true, an example is provided . Later, various Tauberian theorems are proved to recover the convergence of a fuzzy number-valued sequence from its $D[p,q]$-summability.

The concept of summability of sequences and Tauberian theorems find applications in various branches of science. Researchers working on fuzzy analogues of existing results can benefit from this newly studied approach to summability of fuzzy number-valued sequences.

References

[1] R.P. Agnew, On deferred Cesàro means, Ann. Math. 33 (1932) 413–421.
[2] Y. Altin, M. Mursaleen, H. Altinok, Statistical summability $(C, 1)$ for sequences of fuzzy real numbers and a Tauberian theorem, J. Intell. Fuzzy Syst. 21 (2010) 379–384.
[3] S. Aytar, M. Mammadov, S. Pehlivan, Statistical limit inferior and limit superior for sequences of fuzzy numbers, Fuzzy Sets Syst. 157 (2006) 976–985.
[4] S. Aytar, S. Pehlivan, Statistical cluster and extreme limit points of sequences of fuzzy numbers, Inf. Sci. 177 (2007) 3290–3296.
[5] B. Bede, Mathematics of Fuzzy Sets and Fuzzy Logic, Springer, Berlin/Heidelberg, 2013.
[6] N.L. Braha, M. Et, Tauberian theorems for the Euler-Nörlund mean-convergent sequences of fuzzy numbers, Iran. J. Fuzzy Syst. 14 (2017) 79–92.
[7] C.P. Chen, J. M. Hsu, Tauberian theorems for the weighted means of double sequences, Anal. Math. 26 (2000) 243–262.
[8] İ. Çanak, Some conditions under which slow oscillation of a sequence of fuzzy numbers follows from Cesàro summability of its generator sequence, Iran. J. Fuzzy Syst. 11 (2014) 15–22.
[9] M. Çınar, E. Yılmaz, M. Et, Deferred statistical convergence on time scales, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. 22 (2021) 301–307.
[10] A.A. Das, S.K. Paikray, T. Pradhan, H. Dutta, Statistical $(C, 1)(E, \mu)$-summability and associated fuzzy approximation theorems with statistical fuzzy rates, Soft Comput. 24 (2020) 10883–10892.
[11] P. Diamond, P. Kloeden, Metric Spaces of Fuzzy Sets, Theory and Applications, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
[12] D. Dubois, H. Prade, Fuzzy Sets and Systems, Theory and Applications, Academic Press, New York, 1980.
[13] D. Dubois, H. Prade, Fuzzy numbers: An overview. In: The Analysis of Fuzzy Information, vol. 1: Mathematics and Logic. CRC Press, Boca Raton, 1987.
[14] S. Ercan, Y. Altin, M. Et, V.K. Bhardwaj, On deferred weak statistical convergence, J. Anal. 28 (2020) 913–922.
[15] M. Et, On some generalized deferred Cesàro means of order beta, Math. Methods Appl. Sci. 44 (2021) 7433–7441.
[16] J. Fang, H. Huang, On the level convergence of a sequence of fuzzy numbers, Fuzzy Sets Syst. 147 (2004) 417–435.
[17] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[18] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst. 18 (1986) 31–43.
[19] J. Gogoi, H. Dutta, Weighted (λ, μ)-statistical convergence and statistical summability methods of double sequences of fuzzy numbers with application to Korovkin-type fuzzy approximation theorem, Soft Comput. 25 (2021) 7645–7656.
[20] B.B. Jena, S.K. Paikray, U. Misra, Statistical deferred Cesàro summability and its applications to approximation theorems, Filomat 32 (2018) 2307–2319.
[21] B.B. Jena, S.K. Paikray, H. Dutta, On various new concepts of statistical convergence for sequences of random variables via deferred Cesàro mean, J. Math. Anal. Appl. 487 (2020) 123950.
[22] U. Kadak, M. Mursaleen, S.A. Mohiuddine, Statistical weighted matrix summability of fuzzy mappings and associated approximation results, J. Intell. Fuzzy Syst. 36 (2019) 3483–3494.
[23] V.A. Khan, B. Hazarika, I.A. Khan, U. Tuba, I-deferred strongly Cesàro summable and μ-deferred I-statistically convergent sequence spaces, Ricerche Mat. (2021). https://doi.org/10.1007/s11587-021-00619-8.
[24] M. Küçükaslan, M. Yılmaztürk, On deferred statistical convergence of sequences, Kyungpook Math. J. 56 (2016) 357–366.
[25] H. Li, C. Wu, The integral of a fuzzy mapping over a directed line, Fuzzy Sets Syst. 158 (2007) 2317–2338.
[26] M. Matloka, Sequences of fuzzy numbers, Busefal 28 (1986) 28–37.
[27] S.A. Mohiuddine, A. Asim, B. Hazarika, Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, Int. J. Gen. Syst. 48 (2019) 492–506.
[28] F. Móricz, B.E. Rhoades, Necessary and sufficient Tauberian conditions for certain weighted mean methods of summability II, Acta Math. Hungar. 102 (2004) 279–285.
[29] M. Mursaleen, F. Başar, Sequence Spaces: Topics in Modern Summability Theory, CRC Press, Boca Raton, FL, 2020.
[30] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets Syst. 33 (1989) 123–126.
[31] L. Nayak, B.C. Tripathy, P. Baliarsingh, On deferred-statistical convergence of uncertain fuzzy sequences, Int. J. Gen. Syst. 51 (2022) 631–647.
[32] L. Nayak, M. Mursaleen, P. Baliarsingh, On deferred statistical A-convergence of fuzzy sequence and applications, Iran. J. Fuzzy Syst. 19 (2022) 119–131.
[33] S.K. Paikray, B.B. Jena, U. Misra, Statistical deferred Cesàro summability mean based on (p, q)-integers with application to approximation theorems, Advances in Summability and Approximation Theory. Springer: 203-222, 2018.
[34] M.L. Puri, D.A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.
[35] K. Saini, K. Raj, M. Mursaleen, Deferred Cesàro and deferred Euler equi-statistical convergence and its applications to Korovkin-type approximation theorem, Int. J. Gen. Syst. 50 (2021) 567–579.
[36] E. Savaş, Generalized $\mathcal{I}$-statistical convergence of weight g for fuzzy numbers, J. Intell. Fuzzy Syst. 37 (2019) 2197–2202.
[37] S.A. Sezer, İ. Çanak, Power series methods of summability for series of fuzzy numbers and related Tauberian theorems, Soft Comput. 21 (2017) 1057–1064.
[38] S.A. Sezer, Logarithmic means of sequences of fuzzy numbers and a Tauberian theorem, Soft Comput. 24 (2020) 367–374.
[39] S.A. Sezer, İ. Çanak, H. Dutta, Conditions under which convergence of a sequence or its certain subsequences follows from deferred Cesàro summability, Filomat 36 (2022) 921–931.
[40] H.M. Srivastava, B.B. Jena, S.K. Paikray, U. Misra, A certain class of weighted statistical convergence and associated Korovkin-type approximation theorems involving trigonometric functions, Math. Methods Appl. Sci. 41 (2018) 671–683.

[41] H.M. Srivastava, B.B. Jena, S.K. Paikray, Statistical Deferred Nörlund summability and Korovkin-type approximation theorem, Mathematics 8 (2020) 636.
[42] Ö. Talo, F. Başar, On the slowly decreasing sequences of fuzzy numbers, Abstr. Appl. Anal. 7 (2013) 891986.
[43] Ö. Talo, C. Çakan, On the Cesàro convergence of sequences of fuzzy numbers, Appl. Math. Lett. 25 (2012) 676–681.
[44] Ö. Talo, On the almost everywhere statistical convergence of sequences of fuzzy numbers, Filomat 33 (2019) 2683–2693.
[45] B.C. Tripathy, A. Baruah, Nörlund and Riesz mean of sequences of fuzzy real numbers, Appl. Math. Lett. 23 (2010) 651–655.
[46] E. Yavuz, H. Coşkun, On the Borel summability method of sequences of fuzzy numbers, J. Intell. Fuzzy Syst. 30 (2016) 2111–2117.
[47] E. Yavuz, Ö. Talo, Abel summability of sequences of fuzzy numbers, Soft Comput. 20 (2016) 1041–1046.
[48] E. Yavuz, Tauberian theorems for statistical summability methods of sequences of fuzzy numbers, Soft Comput. 23 (2019) 5659–5665.
[49] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.