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The maximal spectral radius of the uniform unicyclic hypergraphs with perfect matchings

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Abstract. Let $\mathcal{U}(n,k)$ and $\Gamma(n,k)$ be respectively the sets of the k-uniform connected linear and nonlinear unicyclic hypergraphs having perfect matchings with n vertices, where $n \geq k(k-1)$ and $k \geq 3$. By using some techniques of transformations and constructing the incidence matrices for the hypergraphs considered, we get the hypergraphs with the maximal spectral radii among three kinds of hypergraphs, namely $\mathcal{U}(n,k)$ with $n = 2k(k-1)$ and $n \geq 9k(k-1)$, $\Gamma(n,k)$ with $n \geq k(k-1)$, and $\mathcal{U}(n,k) \cup \Gamma(n,k)$ with $n \geq 2k(k-1)$, where $k \geq 3$.

1. Introduction

Let $G = (V, E)$ be a simple (i.e., no loops or multiple edges) hypergraph, where $V = V(G) = \{v_1, v_2, \cdots, v_n\}$ is the vertex set and $E = E(G) = \{e_1, e_2, \cdots, e_a\}$ is the edge set with $e_i \subseteq V(G)$ for $i = 1, \cdots, a$. e_i with $1 \leq i \leq a$ is called an edge of G. If $|e_i| = k$ for $1 \leq i \leq a$, then G is called a k-uniform hypergraph. If any two edges in G intersect on at most one common vertex, then G is called a linear hypergraph. Let $u, v \in V(G)$. A path between u and v is denoted by $P = (v_1, e_1, v_2, \ldots, v_p, e_p, v_{p+1})$, where $v_1 = u$, $v_{p+1} = v$, all v_i and all e_i are distinct, and $v_i, v_{i+1} \in e_i$ for $1 \leq i \leq p$. We say that u and v are connected if there exists a path in G between them. A hypergraph G is connected if every pair of vertices in $V(G)$ is connected. For $p \geq 2$, a cycle of length p of G is obtained from a path P of length p by identifying v_1 with v_{p+1}.

For a k-uniform hypergraph G, if $a(k-1) - n + \omega(G) = r(G)$, then we call G an $r(G)$-cyclic hypergraph [1], where a, n, $\omega(G)$, and $r(G)$ are the numbers of edges, vertices, components, and cyclomatics of G, respectively. If $r(G) = 1$, then G is a unicyclic hypergraph. In this paper, we consider k-uniform connected linear and nonlinear unicyclic hypergraphs.

For $u, v \in V(G)$ and $e \in E(G)$, if $\{u, v\} \subseteq e$, then we say that u and v are adjacent and v is incident with e. We denote by $d_G(v)$ the degree of v. Namely $d_G(v)$ is the number of the edges in G incident with v. If $d_G(v) = 0$, then we call v an isolated vertex. If $d_G(v) = 1$, then we call v a core vertex. If $d_G(v) \geq 2$, then we say that v is an intersection vertex. For $e = \{v_1, \ldots, v_r\} \in E(G)$, if $d_G(v_1) \geq 2$ and $d_G(v_i) = 1$ for $2 \leq i \leq r$, then e is called a pendent edge at v_1 of G.

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. A real tensor (or hypermatrix) $\mathcal{A} = (a_{i_1 i_2 \cdots i_r})$ of r-order and n-dimension is a multi-dimensional array with entries $a_{i_1 i_2 \cdots i_r}$ such that $a_{i_1 i_2 \cdots i_r} \in \mathbb{R}$,

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where $i_1, i_2, \cdots, i_r \in [n]$ with $[n] = \{1, 2, \cdots, n\}$. In 2005, Qi [2] and Lim [3] independently introduced the concept of tensor eigenvalues and the spectra of tensors as follows. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ be an n-dimensional complex column vector. Let $\boldsymbol{x}^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \cdots, x_n^{r-1})^{\mathrm{T}}$, where r is a positive integer. Then $\mathcal{A}\boldsymbol{x}$ is a vector in $\mathbb{C}^n$ whose i-th component is given by

$$(\mathcal{A}\boldsymbol{x})_i = \sum_{i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x_{i_2} \cdots x_{i_r}, \text{ for each } i \in [n]. \tag{1}$$

If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $\boldsymbol{x} \in \mathbb{C}^n$ such that $\mathcal{A}\boldsymbol{x} = \lambda \boldsymbol{x}^{[r-1]}$, then λ is called an eigenvalue of $\mathcal{A}$ and $\boldsymbol{x}$ is called an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ. The spectral radius of $\mathcal{A}$ is the largest modulus of the eigenvalues of $\mathcal{A}$, i.e., $\rho(\mathcal{A}) = \max\{|\lambda|\,|\, \lambda$ is an eigenvalue of $\mathcal{A}\}$.

For a hypergraph G, there are a few tensors associated with G. The most important tensor associated with G is the adjacency tensor which was proposed by Cooper and Dutle [4] in 2012 as follows. Let G be a k-uniform hypergraph with n vertices. The adjacency tensor of G is the k-ordered and n-dimensional adjacency tensor $\mathcal{A}(G) = (a_{i_1 i_2 \cdots i_k})$ whose $(i_1 i_2 \cdots i_k)$-entry is

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \dfrac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \cdots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The spectral radius of $\mathcal{A}(G)$ of a k-uniform hypergraph G, denoted by $\rho(G)$, is called the spectral radius of G. For a k-uniform hypergraph G with n vertices, if G is connected, then there exists a unique positive eigenvector $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ corresponding to $\rho(G)$ with $\Sigma_{i=1}^{n} x_i^k = 1$ [5, 6]. Such a positive eigenvector is called the principal eigenvector of G [5, 6]. In this paper, we will consider the principal eigenvector $\boldsymbol{x}$ as a mapping x: $V(G) \to \mathbb{R}^n$. The principal eigenvector $\boldsymbol{x}$ plays a key role in the spectral hypergraph theory.

The research on the spectra of hypergraphs via tensors has drawn increasingly extensive interest. In recent years, many interesting results about the characterization of the k-uniform hypergraphs with the extremal spectral radii have been obtained. Xiao and Wang [7] determined the hypergraphs with the maximal spectral radii among all the uniform supertrees and among all the uniform connected unicyclic hypergraphs with a given number of pendent edges. Fan et al. [1] characterized the hypergraph(s) with the maximal spectral radius over all unicyclic hypergraphs, linear or power unicyclic hypergraphs with a given girth, and linear or power bicyclic hypergraphs. Kang et al. [8] obtained the hypergraph with the maximal spectral radius among the linear bicyclic uniform hypergraphs. Ouyang et al. [9] deduced the first five hypergraphs with the maximal spectral radii among all unicyclic hypergraphs and the first three ones over all bicyclic hypergraphs. Among the set of supertrees [10–12] and the set of supertrees with given parameters, such as a fixed diameter [13], a given degree sequence [14], a perfect matching [15], a given number of pendent vertices [16], a given size of matching [17, 18], and two vertices of maximum degree [19], etc, the hypergraphs with the extremal spectral radii were also characterized.

A k-matching of G is a union of k independent edges in G, where $k \geq 0$. A perfect matching of G is a matching that covers $V(G)$. Namely, a set $\{S_1, S_2 \cdots, S_h\}$ of pairwise vertex disjoint edges of G with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_h$ is called a perfect matching of G. It is known that hypergraphs can be classified into two groups: one group which has a perfect matching and the other group which does not have perfect matchings. Since the hypergraphs with a perfect matching have many applications in graph theory, they are of great significance and attract a lot of people's attention. A large number of literatures are concerned with some properties of hypergraphs with a perfect matching. For example, some authors investigated the condition that ensure a perfect matching in hypergraphs [20–22].

Motivated by the preceding work on the hypergraphs with the extremal spectral radii, we consider, in this paper, the hypergraph with the maximal spectral radius among the set of the k-uniform connected unicyclic hypergraphs having perfect matchings.

Let G be a k-uniform connected unicyclic hypergraph having perfect matchings. Let $Q(G) = E(G) - M(G)$, where $M(G)$ is a perfect matching of G. An independent edge of $M(G)$ is called a perfect matching edge of G. Let $\widehat{G}$ be the hypergraph induced by $Q(G)$, that is, $\widehat{G} = G - M(G) - S_0$, where S_0 is the set of isolated vertices in $G - M(G)$. We call $\widehat{G}$ the capped hypergraph of G and G the original hypergraph of $\widehat{G}$. Let $|M(G)|$ and $|Q(G)|$ be the numbers of the edges in $M(G)$ and $Q(G)$, respectively.

We denote by $\mathcal{U}(n,k)$ the set of the k-uniform connected linear unicyclic hypergraphs having perfect matchings with n vertices, where $k \geq 3$. Let G be an arbitrary hypergraph in $\mathcal{U}(n,k)$. A vertex of G is saturated if it is incident with a perfect matching edge of G. Since each vertex of G is saturated, we have $|M(G)| = \frac{n}{k}$, where n is divisible by k and $k \geq 3$. Thus, it follows from $n = |E(G)|(k-1)$ that $|Q(G)| = |E(G)| - \frac{n}{k} = \frac{n}{k(k-1)}$, where n is divisible by $k(k-1)$. For simplicity, let $|Q(G)| = m$. Namely, m is the number of the edges of $\widehat{G}$. Thus, in $\mathcal{U}(n,k)$, we get $n = mk(k-1)$, where m is an integer not less than 2 and $k \geq 3$.

Let $\Gamma(n,k)$ be the set of the k-uniform connected nonlinear unicyclic hypergraphs having perfect matchings with n vertices, where $k \geq 3$. Obviously, for each $G \in \Gamma(n,k)$, we have $n = mk(k-1)$, where $m \geq 1$ and m is the number of the edges of $\widehat{G}$.

This paper is organized as follows. In Section 2, relevant notations and some lemmas which are useful for subsequent proofs are introduced. In Section 3, by using some transformations and constructing the incidence matrices for the hypergraphs considered, the hypergraph with the maximal spectral radius is derived among $\mathcal{U}(n,k)$ for $n \geq 2k(k-1)$ and $k \geq 3$. Furthermore, in Section 4, the hypergraphs with the maximal spectral radii are characterized among $\Gamma(n,k)$ with $n \geq k(k-1)$ and among $\mathcal{U}(n,k) \cup \Gamma(n,k)$ with $n \geq 2k(k-1)$, where $k \geq 3$.

2. Preliminaries

In Section 2, we introduce some relevant definitions and necessary lemmas which are useful for us to obtain the results.

Definition 2.1. *[10] Let $G = (V,E)$ be a hypergraph with $u \in V$ and $e_1, \ldots, e_r \in E$ such that $u \notin e_i$ for $i = 1, \ldots, r$, where $r \geq 1$. Suppose that $v_i \in e_i$ and write $e'_i = (e_i \setminus \{v_i\}) \cup \{u\}$ $(i = 1, \ldots, r)$. The vertices $v_1, \ldots, v_r$ need not be distinct. Let $G' = (V,E')$ be the hypergraph with $E' = (E \setminus \{e_1, \ldots, e_r\}) \cup \{e'_1, \ldots, e'_r\}$. Then we say that G' is obtained from G by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to u.*

In G, if there exist two edges (denoted by e and e') such that e and e' have the same vertices, then we say that e and e' are two multiple edges.

Lemma 2.2. *[10] Let G and G' be the two connected hypergraphs as defined in Definition 2.1. Suppose that G' contains no multiple edges. If $\boldsymbol{x}$ is the principal eigenvector of $\mathcal{A}(G)$ corresponding to $\rho(\mathcal{A}(G))$ and $x_u \geq \max_{1 \leq i \leq r}\{x_{v_i}\}$, then $\rho(\mathcal{A}(G')) > \rho(\mathcal{A}(G))$.*

Lemma 2.3. *[9] Let G be a connected k-uniform hypergraph having two adjacent vertices u_1 and u_2. Let G' be the hypergraph obtained from G by moving all incident edges of u_2 (except for all common edges shared by u_1 and u_2) from u_2 to u_1. If $G' \ncong G$, then $\rho(G') > \rho(G)$.*

Li et al. [10] proposed the edge-releasing operation for the k-uniform linear hypergraphs. In this paper, we generalize the edge-releasing operation to the k-uniform hypergraphs, which is shown in Definition 2.4.

Definition 2.4. *Let G be a k-uniform hypergraph, where $k \geq 3$. Let e be a non-pendent edge of G and $\{e_1, \cdots, e_r\}$ be all the edges of G adjacent to e, where e_i and e share a common vertex which is denoted by v_i $(i = 1, \ldots, r)$. Let u be an arbitrary vertex of e. Let G' be the hypergraph obtained from G by moving edges $(e_1, \ldots, e_r)$ (except for all the edges which are incident with u) from $(v_1, \cdots, v_r)$ to u, where $v_i \neq u$ with $i = 1, \cdots, r$. Then G' is said to be obtained from G by the edge-releasing operation on e at u.*

Lemma 2.5. *Let G and G' be the two connected hypergraphs as defined in Definition 2.4. If G' does not have multiple edges, then we have $\rho(G') > \rho(G)$.*

Proof: Let G and G' be the two hypergraphs as defined in Definition 2.4. Since e is a non-pendent edge of G, there exist some vertices in e which have degrees not less than 2. We denote these vertices by $v_1, \cdots, v_r$, where $2 \le r \le k$. By repeatedly using Lemma 2.3, we get $\rho(G') > \rho(G)$. □

Let B_G be a weighted incidence matrix of a hypergraph G. We denote by $B_G(v,e)$ the entry of B_G corresponding to v and e, where $v \in V(G)$ and $e \in E(G)$.

Definition 2.6. *[23] A weighted incidence matrix B_G of a hypergraph G is a $|V| \times |E|$ matrix such that for any vertex $v \in V(G)$ and any edge $e \in E(G)$, the entry $B_G(v,e) > 0$ if $v \in e$ and $B_G(v,e) = 0$ if $v \notin e$.*

Let $E_G(v)$ be the set of the edges which are incident with v, where $v \in V(G)$.

Definition 2.7. *[23] A hypergraph G is α-normal if there exists a weighted incidence matrix B_G satisfying*
(i). $\sum_{e:e\in E_G(v)} B_G(v,e) = 1$, for any $v \in V(G)$.
(ii). $\prod_{v:v\in e} B_G(v,e) = \alpha$, for any $e \in E(G)$.
Moreover, the incidence matrix B_G is said to be consistent if for any cycle $v_0e_1v_1 \cdots v_l$ ($v_l = v_0$)

$$\prod_{i=1}^{l} \frac{B_G(v_i,e_i)}{B_G(v_{i-1},e_i)} = 1.$$

In this case, we say that G is consistently α-normal.

Definition 2.8. *[23] A hypergraph G is α-subnormal if there exists a weighted incidence matrix B_G satisfying*
(i). $\sum_{e:e\in E_G(v)} B_G(v,e) \le 1$, for any $v \in V(G)$.
(ii). $\prod_{v:v\in e} B_G(v,e) \ge \alpha$, for any $e \in E(G)$.
Moreover, G is strictly α-subnormal if it is α-subnormal but not α-normal.

Lemma 2.9. *[23] Let G be a connected k-uniform hypergraph.*
(i) G is consistently α-normal if and only if (iff) $\rho(G) = \alpha^{-\frac{1}{k}}$.
(ii) If G is α-subnormal, then $\rho(G) \le \alpha^{-\frac{1}{k}}$. Moreover, if G is strictly α-subnormal, then $\rho(G) < \alpha^{-\frac{1}{k}}$.

For the k-uniform connected linear and nonlinear unicyclic hypergraphs having perfect matchings with n vertices, we give a characterization of their perfect matchings as follows.

Property 2.10. *Let $G \in \mathcal{U}(n,k) \bigcup \Gamma(n,k)$, where $n \ge k(k-1)$ and $k \ge 3$. The perfect matching $M(G)$ of G is unique.*

Proof: Let $G \in \mathcal{U}(n,k) \bigcup \Gamma(n,k)$, where $n \ge k(k-1)$ and $k \ge 3$. Let C_l be the cycle contained in G, where $l \ge 2$. Since G has a perfect matching, obviously, at least one vertex in C_l of G is attached by a hypertree. Let u be an arbitrary vertex in C_l of G which is attached by a hypertree. We denote by T_u the hypertree attached at u. Since the perfect matching of T_u is unique, we get that the perfect matching edge incident with u is unique. Thus, for an arbitrary vertex in C_l of G (except for u), the perfect matching edge incident with it must be unique too. Therefore, Property 2.10 has been proved. □

3. The hypergraph with the maximal spectral radius among $\mathcal{U}(n,k)$

In Section 3, we will deduce the hypergraph with the maximal spectral radius among $\mathcal{U}(n,k)$, where $n = mk(k-1)$, $m \ge 2$ and $k \ge 3$. Some definitions are given first.

Let $\mathcal{U}(n,k,l)$ be a subset of $\mathcal{U}(n,k)$ in which each hypergraph has a cycle C_l, where l is an integer with $l \ge 3$. Let $C_l = v_1e_1v_2e_2v_3 \cdots v_le_lv_1$, where $e_i = \{v_i, v_{i,1}, \ldots, v_{i,k-2}, v_{i+1}\}$ with $1 \le i \le l-1$ and $e_l = \{v_l, v_{l,1}, \ldots, v_{l,k-2}, v_1\}$. Let $G \in \mathcal{U}(n,k,l)$ and $M(G)$ be a perfect matching of G. According to the fact whether C_l of G has at least one perfect matching edge or not, we classify $\mathcal{U}(n,k,l)$ into two subsets which are denoted by $\mathcal{U}_1(n,k,l)$ and $\mathcal{U}_2(n,k,l)$, where $\mathcal{U}_1(n,k,l)$ satisfies that each hypergraph G in it has no perfect matching edges on C_l of G and $\mathcal{U}_2(n,k,l)$ satisfies that each hypergraph G in it has at least one perfect matching edge on C_l of G. Obviously, $\mathcal{U}(n,k) = \bigcup_{l\ge3} \left(\mathcal{U}_1(n,k,l) \cup \mathcal{U}_2(n,k,l)\right)$.

Let $\bar{\mathcal{U}}_1(n,k,3)$ be a subset of $\mathcal{U}_1(n,k,3)$ in which each hypergraph satisfies two conditions: (i) each vertex in C_3 must be attached by a pendent edge; and (ii) at most one of the vertices in $\{v_1, v_2, v_3\}$ is attached by a hypertree which has at least k edges, where $k \geq 3$.

Let $\bar{\mathcal{U}}_2(n,k,3)$ be a subset of $\mathcal{U}_2(n,k,3)$ in which each hypergraph satisfies three conditions: (i) each vertex in $e_1 \setminus \{v_1, v_2\}$ of C_3 is a core vertex; (ii) each vertex in $(e_2 \cup e_3) \setminus \{v_1, v_2\}$ of C_3 must be attached by a pendent edge; and (iii) at most one of the vertices in $\{v_1, v_2, v_3\}$ of C_3 is attached by a hypertree which has at least $k \geq 3$ edges, and each vertex in $(e_2 \cup e_3) \setminus \{v_1, v_2, v_3\}$ is not attached by a hypertree which has at least $k \geq 3$ edges.

Let $\bar{\mathcal{U}}_{2,1}(n,k,3)$ (respectively $\bar{\mathcal{U}}_{2,2}(n,k,3)$) be a subset of $\bar{\mathcal{U}}_2(n,k,3)$ in which each hypergraph satisfies all the conditions for $\bar{\mathcal{U}}_2(n,k,3)$ and further satisfies that v_1 or v_2 (respectively v_3) of C_3 of G is attached by a hypertree which has at least k edges, where $k \geq 3$.

We denote by $S_{a,k}$ the k-uniform linear supertree obtained from a vertex u_0 by attaching a edges with k vertices at u_0, where $a \geq 1$. Namely, in $S_{a,k}$, all the a edges share a common vertex u_0. Let G and H be two hypergraphs whose vertex sets are disjoint with $v \in V(G)$ and $w \in V(H)$. We use $G(v,w)H$ to denote the hypergraph obtained by identifying the vertices v and w. For example, $C_3(v_1, u_0)S_{m-3,k}$ is shown in Fig. 1(a), where $C_3 = v_1e_1v_2e_2v_3e_3v_1$ is a cycle of length 3.

Let $A_{n,k}$ be the k-uniform linear unicyclic hypergraph obtained from $C_3(v_1, u_0)S_{m-3,k}$ by attaching one pendent edge at each vertex of $C_3(v_1, u_0)S_{m-3,k}$, where $n = mk(k-1)$ and $m, k \geq 3$. $A_{n,k}$ is shown in Fig. 1(b).

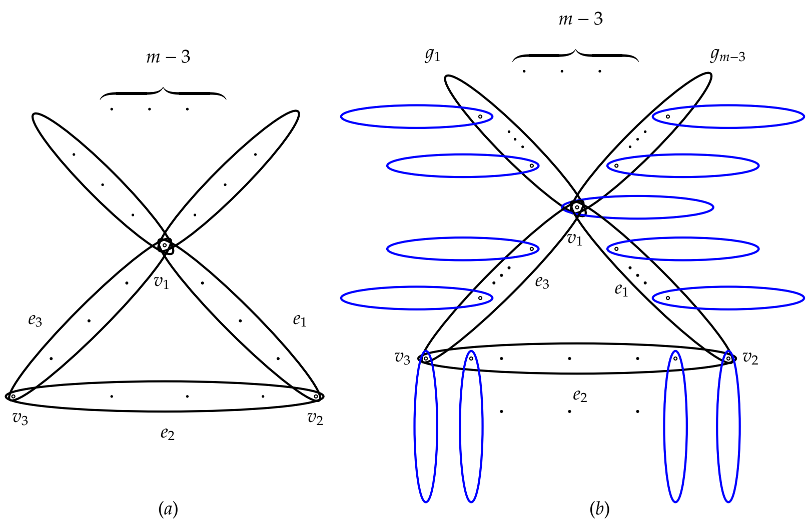

Figure 1: (a) $C_3(v_1, u_0)S_{m-3,k}$ and (b) $A_{n,k}$

Let $B_{n,k}$ (respectively $D_{n,k}$) be the k-uniform linear unicyclic hypergraph obtained from $C_3(v_1, u_0)S_{m-2,k}$ by attaching one pendent edge at each vertex of $C_3(v_1, u_0)S_{m-2,k}$ except for all the vertices which are incident with e_1 (respectively e_2), where $n = mk(k-1)$, $m \geq 2$ and $k \geq 3$. $B_{n,k}$ and $D_{n,k}$ are shown in Fig. 2.

Obviously, we have $A_{n,k} \in \bar{\mathcal{U}}_1(n,k,3)$, $B_{n,k} \in \bar{\mathcal{U}}_{2,1}(n,k,3)$, and $D_{n,k} \in \bar{\mathcal{U}}_{2,2}(n,k,3)$.

To obtain the hypergraph with the maximal spectral radius in $\mathcal{U}(n,k)$ (as shown in Theorem 3.13), several lemmas are introduced first. Lemmas 3.1–3.4 are introduced to get the hypergraph with the maximal spectral radius in $\mathcal{U}_1(n,k,l)$ (as shown in Corollary 3.5). Lemmas 3.6–3.11 are proposed to obtain the hypergraph with the maximal spectral radius in $\mathcal{U}_2(n,k,l)$ (as shown in Corollary 3.12).

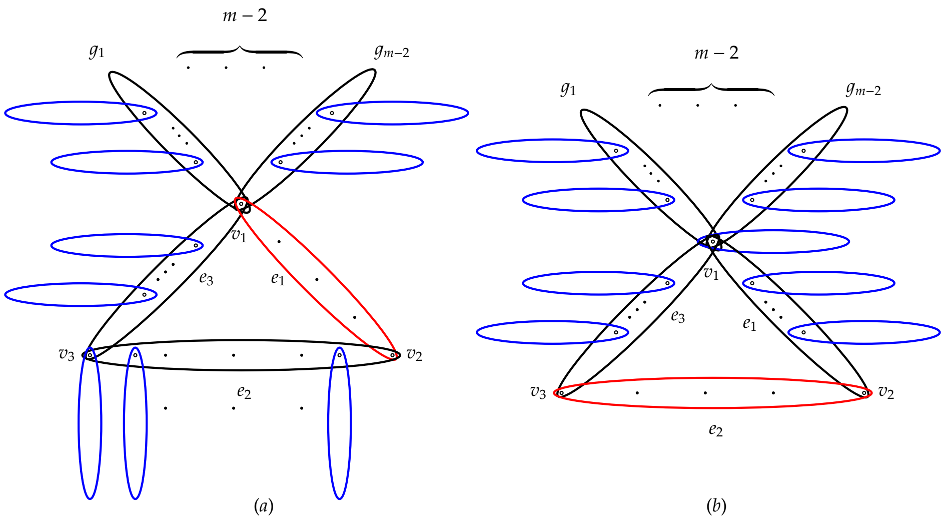

Figure 2: (a) $B_{n,k}$ and (b) $D_{n,k}$

Lemma 3.1. *Let $G \in \mathcal{U}(n,k,l)$, where $n \geq 3k(k-1)$, $k \geq 3$ and $l \geq 4$. Let e be a perfect matching edge of G and e is not a pendent edge. Let G_0 be the hypergraph obtained from G by applying the edge-releasing operation on e at a vertex of e such that e of G_0 is a pendent edge. We have $\rho(G_0) > \rho(G)$, where $G_0 \in \mathcal{U}(n,k)$.*

Proof: Let $n \geq 3k(k-1)$ and $k \geq 3$. Let G and G_0 be the two hypergraphs as defined in Lemma 3.1. Let $M(G)$ be the perfect matching of G and $e \in M(G)$. Then all the edges which are adjacent to e are edges of $Q(G)$. After applying the edge-releasing operation on e at a vertex of e, G_0 has the perfect matching $M(G)$ and $G_0 \in \mathcal{U}(n,k)$. Since G is linear, e of G_0 is a pendent edge. By Lemma 2.5, we have $\rho(G_0) > \rho(G)$. □

Lemma 3.2. *Let $G \in \mathcal{U}_1(n,k,l)$, where $n \geq 3k(k-1)$, $k \geq 3$ and $l \geq 4$. There exists a hypergraph $G^\star \in \mathcal{U}_1(n,k,3)$ such that $\rho(G^\star) > \rho(G)$.*

Proof: Let $n \geq 3k(k-1)$, $k \geq 3$ and $l \geq 4$. Let $G \in \mathcal{U}_1(n,k,l)$. We denote the perfect matching of G by $M(G)$. The cycle contained in G is denoted by $C_l = v_1e_1 \dots v_le_lv_1$. Bearing the definition of $\mathcal{U}_1(n,k,l)$ in mind, $e_i \notin M(G)$ and each vertex of e_i is incident with an edge in $M(G)$, where $1 \leq i \leq l$. Let $\boldsymbol{x}$ be the principal eigenvector of G corresponding to $\rho(G)$. Without loss of generality, we suppose $x_{v_1} \geq x_{v_2}$.

Let G_1 be the hypergraph obtained from G by moving e_2 from v_2 to v_1. It is noted that all the edges which are incident with v_2 of e_1 of G (except for e_2) remain unchanged. Therefore, $M(G)$ is the perfect matching of G_1 and G_1 contains a cycle C_{l-1}. Namely $G_1 \in \mathcal{U}_1(n,k,l-1)$. By Lemma 2.2, we have $\rho(G_1) > \rho(G)$. By repeatedly using the same procedure, we finally get a hypergraph $G^\star \in \mathcal{U}_1(n,k,3)$ such that $\rho(G^\star) > \rho(G)$. □

Lemma 3.3. *Let $G \in \mathcal{U}_1(n,k,3)$, where $n \geq 3k(k-1)$ and $k \geq 3$. There exists a hypergraph $G^\diamond \in \bar{\mathcal{U}}_1(n,k,3)$ such that $\rho(G^\diamond) \geq \rho(G)$ with the equality iff $G \cong G^\diamond$.*

Proof: Let $n = mk(k-1)$, $m \geq 3$ and $k \geq 3$. When $m = 3$, obviously Lemma 3.3 holds. Next, let $m \geq 4$. For $G \in \mathcal{U}_1(n,k,3)$, let $M(G)$ be the perfect matching of G. By the definition of $\mathcal{U}_1(n,k,3)$, each vertex in C_3 of G is incident with an edge (denoted by e) of $M(G)$ and e is not an edge on C_3 of G. By applying the edge-releasing operation on e at a vertex of e and by Lemma 3.1, we get a hypergraph G_2 such that $\rho(G_2) \geq \rho(G)$ with the equality iff $G \cong G_2$, where G_2 satisfies that (i) each vertex in C_3 must be attached by a pendent edge; (ii) there exists at least one vertex of C_3 in G_2 which is attached by a hypertree having at least $k \geq 3$ edges (since $m \geq 4$).

Let $\boldsymbol{x}$ be the principal eigenvector of G_2 corresponding to $\rho(G_2)$. Among all the vertices of C_3 in G_2 which are attached by hypertrees having at least $k \geq 3$ edges, we assume the vertex w at C_3 of G_2 has the largest component among the vector $\boldsymbol{x}$ and we denote it by x_w. Without loss of generality, we suppose w belongs to e_1 of C_3 of G_2. By Lemma 2.2, we get $\rho(G_3) \geq \rho(G_2)$ with the equality iff $G_2 \cong G_3$, where G_3 satisfies that each vertex in C_3 of G_3 is attached by a pendent edge, and only one vertex (namely w) at C_3 of G_3 is also attached by a hypertree having at least $k \geq 3$ edges. If w is one of v_1 and v_2, then we get Lemma 3.3. Next, we assume $w \neq v_1, v_2$. Let $\boldsymbol{y}$ be the principal eigenvector of G_3 corresponding to $\rho(G_3)$. Two cases are considered as follows.

Case (i). $y_{v_1} \geq y_w$.

Let G_4 be the hypergraph obtained from G_3 by moving all the edges which are incident with w (except for the edge e_1 and the pendent edge attached at w) from w to v_1. Obviously, $G_4 \in \bar{\mathcal{U}}_1(n,k,3)$. By Lemma 2.2, we obtain $\rho(G_4) > \rho(G_3)$.

Case (ii). $y_{v_1} < y_w$.

Let G_5 be the hypergraph obtained from G_3 by moving e_3 from v_1 to w. It is noted that the pendent edge attached at v_1 of C_3 in G_3 remains unchanged. Obviously, $G_5 \in \bar{\mathcal{U}}_1(n,k,3)$. By Lemma 2.2, we have $\rho(G_5) > \rho(G_3)$.

By combining the above proofs, we have Lemma 3.3. □

Lemma 3.4. *Let $G \in \bar{\mathcal{U}}_1(n,k,3)$, where $n \geq 3k(k-1)$ and $k \geq 3$. We have $\rho(A_{n,k}) \geq \rho(G)$ with the equality iff $G \cong A_{n,k}$.*

Proof: Let $G \in \bar{\mathcal{U}}_1(n,k,3)$ with $n = mk(k-1)$, $m \geq 3$ and $k \geq 3$. If $m = 3, 4$, then $\bar{\mathcal{U}}_1(n,k,3) = \{A_{n,k}\}$ and we have Lemma 3.4. Next, let $m \geq 5$. Bearing the definition of $\bar{\mathcal{U}}_1(n,k,3)$ in mind, we have that each vertex in C_3 of G is attached by a pendent edge. We assume v_1 of C_3 of G is attached by a hypertree (denoted by T) which has at least two edges belonging to $Q(G)$. If T has perfect matching edges which are not pendent edges, then let e be an arbitrary edge of those edges. By applying the edge-releasing operation on e at a vertex of e, we get a hypergraph G_6 such that $\rho(G_6) \geq \rho(G)$ with the equality iff $G \cong G_6$ (by Lemma 3.1), where $G_6 \in \bar{\mathcal{U}}_1(n,k,3)$ and all the perfect matching edges of G_6 are pendent edges.

If $G_6 \cong A_{n,k}$, then we get Lemma 3.4. Otherwise, we assume $G_6 \ncong A_{n,k}$. Obviously, v_1 of C_3 in G_6 is attached by a hypertree (denoted by T') which has at least two edges belonging to $Q(G_6)$. As $G_6 \ncong A_{n,k}$, in G_6, there exists an edge (denoted by $g = \{u_1, ..., u_k\}$) which satisfies the following conditions: (i) g belongs to $Q(G_6)$; (ii) g is not incident with v_1; and (iii) v_1 and u_1 are incident with a common edge. Let $\boldsymbol{x}$ be the principal eigenvector of G_6 corresponding to $\rho(G_6)$. Two cases are considered as follows.

Case (i). $x_{v_1} \geq x_{u_1}$.

Let G_7 be the hypergraph obtained from G_6 by moving g from u_1 to v_1. Obviously, $G_7 \in \bar{\mathcal{U}}_1(n,k,3)$. By Lemma 2.2, we obtain $\rho(G_7) > \rho(G_6)$.

Case (ii). $x_{v_1} < x_{u_1}$.

Let G_8 be the hypergraph obtained from G_6 by moving all the edges which are incident with v_1 (except for the pendent edge attached at v_1 and the common edge which is incident with v_1 and u_1) from v_1 to u_1. Obviously, $G_8 \in \bar{\mathcal{U}}_1(n,k,3)$. By Lemma 2.2, we obtain $\rho(G_8) > \rho(G_6)$.

By repeatedly using the same procedures as those in Cases (i) and (ii), we finally get $\rho(A_{n,k}) > \rho(G_6)$.

By combining the above proofs, we obtain $\rho(A_{n,k}) \geq \rho(G)$ with the equality iff $G \cong A_{n,k}$. □

By Lemmas 3.2–3.4, we get Corollary 3.5 as follows.

Corollary 3.5. *Let $G \in \mathcal{U}_1(n,k,l)$, where $n \geq 3k(k-1)$ and $k, l \geq 3$. We have $\rho(A_{n,k}) \geq \rho(G)$ with the equality iff $G \cong A_{n,k}$.*

Lemma 3.6. *Let $G \in \mathcal{U}_2(n,k,l)$, where $n \geq 2k(k-1)$, $k \geq 3$ and $l \geq 4$. There exists a hypergraph $\widetilde{G} \in \mathcal{U}_1(n,k,p) \cup \mathcal{U}_2(n,k,3)$ such that $\rho(\widetilde{G}) > \rho(G)$, where $3 \leq p \leq l-1$.*

Proof: Let $G \in \mathcal{U}_2(n,k,l)$, where $l \geq 4$. Let $M(G)$ be the perfect matching of G. By the definition of $\mathcal{U}_2(n,k,l)$, there exists an edge (denoted by e) on the cycle C_l of G which belongs to $M(G)$. Let v be an arbitrary vertex of

e. Let G_9 be the hypergraph obtained from G by applying the edge-releasing operation on e at v. It follows from Lemma 3.1 that $\rho(G_9) > \rho(G)$, e becomes a pendent edge in G_9 and the number of the perfect matching edges on the cycle contained in G_9 decreases by 1. Obviously, we have $G_9 \in \mathcal{U}_2(n, k, l-1)$. By repeatedly using the same procedure, we finally get a hypergraph $\widetilde{G} \in \mathcal{U}_1(n, k, p) \cup \mathcal{U}_2(n, k, 3)$ such that $\rho(\widetilde{G}) > \rho(G)$, where $3 \le p \le l-1$. □

Lemma 3.7. *Let $G \in \mathcal{U}_2(n, k, 3)$, where $n \ge 2k(k-1)$ and $k \ge 3$. There exists a hypergraph $G^* \in \bar{\mathcal{U}}_2(n, k, 3)$ such that $\rho(G^*) \ge \rho(G)$ with the equality iff $G \cong G^*$.*

Proof: Let $n \ge 2k(k-1)$ and $k \ge 3$. Let $G \in \mathcal{U}_2(n, k, 3)$ and $C_3 = v_1e_1v_2e_2v_3e_3v_1$ be the cycle contained in G. Let $M(G)$ be the perfect matching of G. According to the definition of $\mathcal{U}_2(n, k, 3)$, there exists one perfect matching edge on C_3. Without loss of generality, we suppose that $e_1 = \{v_1, v_{1,1}, \ldots, v_{1,k-2}, v_2\}$ is the perfect matching edge on C_3. If there exists a vertex in $e_1 \setminus \{v_1, v_2\}$ such that its degree is greater than 1, then we suppose this vertex is $v_{1,1}$. Let G_{10} be the hypergraph obtained from G by moving all the edges (except for e_1) which are incident with $v_{1,1}$ from $v_{1,1}$ to v_1. Obviously, G_{10} has the perfect matching $M(G)$ and the number of the core vertices of G_{10} increases by 1. By Lemma 2.3, $\rho(G_{10}) > \rho(G)$. By repeatedly using the same procedure, we finally get a hypergraph G_{11} such that $\rho(G_{11}) \ge \rho(G)$ with the equality iff $G \cong G_{11}$, where $G_{11} \in \mathcal{U}_2(n, k, 3)$, G_{11} has the perfect matching $M(G)$ and each vertex in $e_1 \setminus \{v_1, v_2\}$ of G_{11} is a core vertex.

Since e_2 and e_3 of C_3 in G_{11} are the edges of $Q(G_{11})$, each vertex in $(e_2 \cup e_3) \setminus \{v_1, v_2\}$ is incident with an edge in $M(G)$. By Lemma 3.1, we get a hypergraph G_{12} such that $\rho(G_{12}) \ge \rho(G_{11})$ with the equality iff $G_{12} \cong G_{11}$, where G_{12} satisfies the following three conditions: (i) each vertex in $e_1 \setminus \{v_1, v_2\}$ of C_3 is a core vertex; (ii) each vertex in $(e_2 \cup e_3) \setminus \{v_1, v_2\}$ of C_3 must be attached by a pendent edge; and (iii) there exists at least one vertex in $e_2 \cup e_3$ of C_3 which is attached by a hypertree having at least $k \ge 3$ edges (if $n \ge 3k(k-1)$).

Furthermore, by the methods similar to the proofs for Lemma 3.3, we obtain a hypergraph G^* such that $\rho(G^*) \ge \rho(G_{12})$ with the equality iff $G^* \cong G_{12}$, where $G^* \in \bar{\mathcal{U}}_2(n, k, 3)$. Therefore, we get Lemma 3.7. □

Lemma 3.8. *For $n \ge 3k(k-1)$ and $k \ge 3$, we have $\rho(A_{n,k}) > \rho(B_{n,k})$, where $A_{n,k}$ and $B_{n,k}$ are the two hypergraphs as shown in Fig. 1(b) and Fig. 2(a), respectively.*

Proof: Let $n \ge 2k(k-1)$ and $k \ge 3$. Let $0 < \alpha < 1$. We construct a weighted incidence matrix $\mathcal{B}_{A_{n,k}}$ for $A_{n,k}$ as follows.

$$\mathcal{B}_{A_{n,k}}(v,e) = \begin{cases} 0 & v \notin e, \\ 1 & v \in e \text{ and } v \text{ is a core vertex}, \\ \alpha & v \in e,\ e \text{ is a pendent edge and } d_{A_{n,k}}(v) = 2, \\ 1-\alpha & v \in e,\ e \text{ is not a pendent edge and } d_{A_{n,k}}(v) = 2, \\ \dfrac{\alpha}{(1-\alpha)^{k-1}} & (v,e) = (v_1, g_i),\ \text{where } i = 1, \cdots, m-3, \text{ and } m \ge 3, \\ x_0 & (v,e) = (v_1, e_1), \\ y_0 & (v,e) = (v_2, e_1), \\ c_0 & (v,e) = (v_1, e_3), \\ d_0 & (v,e) = (v_2, e_2), \\ \dfrac{\alpha}{c_0(1-\alpha)^{k-2}} & (v,e) = (v_3, e_3), \\ \dfrac{\alpha}{d_0(1-\alpha)^{k-2}} & (v,e) = (v_3, e_2). \end{cases}$$

where x_0, y_0, c_0, d_0, and α satisfy the following five equations:

$$\begin{cases} x_0 + \alpha + c_0 + \dfrac{(m-3)\alpha}{(1-\alpha)^{k-1}} = 1, & (3)\\ y_0 + \alpha + d_0 = 1, & (4)\\ \dfrac{\alpha}{c_0(1-\alpha)^{k-2}} + \dfrac{\alpha}{d_0(1-\alpha)^{k-2}} + \alpha = 1, & \\ x_0 y_0 (1-\alpha)^{k-2} = \alpha, & (5)\\ \dfrac{x_0}{y_0} \cdot \dfrac{d_0^2}{c_0^2} = 1. & \end{cases}$$

We check that $\sum_{e:e\in E_{A_{n,k}}(v)} \mathcal{B}_{A_{n,k}}(v,e) = 1$ for any $v \in V(A_{n,k})$, $\prod_{v:v\in e} \mathcal{B}_{A_{n,k}}(v,e) = \alpha$ for any $e \in E(A_{n,k})$, and $\mathcal{B}_{A_{n,k}}$ is consistent. Thus, $A_{n,k}$ is consistently α-normal. By Lemma 2.9(i), we have $\rho(A_{n,k}) = \alpha^{-\frac{1}{k}}$.

We construct a weighted incidence matrix $\mathcal{B}_{B_{n,k}}$ for $B_{n,k}$ as follows. Let $\mathcal{B}_{B_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{B_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{B_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{B_{n,k}}(v) = 2$; $\mathcal{B}_{B_{n,k}}(v,e) = 1-\alpha$ if $v \in e$, $v \neq v_2$, e is not a pendent edge, and $d_{B_{n,k}}(v) = 2$; $\mathcal{B}_{B_{n,k}}(v_1, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-2$ and $m \geq 3$; $\mathcal{B}_{B_{n,k}}(v_1,e_1) = x_1$; $\mathcal{B}_{B_{n,k}}(v_2,e_1) = y_1$; $\mathcal{B}_{B_{n,k}}(v_1,e_3) = c_0$; $\mathcal{B}_{B_{n,k}}(v_2,e_2) = d_0$; $\mathcal{B}_{B_{n,k}}(v_3,e_3) = \dfrac{\alpha}{c_0(1-\alpha)^{k-2}}$; and $\mathcal{B}_{B_{n,k}}(v_3,e_2) = \dfrac{\alpha}{d_0(1-\alpha)^{k-2}}$, where x_1, y_1, c_0, d_0, and α satisfy (6) and (7) as follows:

$$\begin{cases} x_1 + c_0 + \dfrac{(m-2)\alpha}{(1-\alpha)^{k-1}} = 1, & (6)\\ y_1 + d_0 = 1. & (7) \end{cases}$$

We can verify that $\sum_{e:e\in E_{B_{n,k}}(v)} \mathcal{B}_{B_{n,k}}(v,e) = 1$ for any $v \in V(B_{n,k})$ and $\prod_{v:v\in e} \mathcal{B}_{B_{n,k}}(v,e) = \alpha$ for any $e \in E(B_{n,k})$ and $e \neq e_1$. Next, we prove $\prod_{v:v\in e_1} \mathcal{B}_{B_{n,k}}(v,e_1) > \alpha$.

We get

$$\begin{aligned} &\prod_{v:v\in e_1} \mathcal{B}_{B_{n,k}}(v,e_1) - \alpha = x_1 y_1 - \alpha \\ &= \Big(x_0 + \alpha - \frac{\alpha}{(1-\alpha)^{k-1}}\Big)(y_0+\alpha) - \alpha && (8)\\ &= \alpha(y_0 + \alpha - 1) + x_0 y_0 + x_0\alpha - \frac{\alpha}{(1-\alpha)^{k-1}}(y_0+\alpha) && (9)\\ &> \alpha(y_0+\alpha-1) + \frac{\alpha}{(1-\alpha)^{k-1}}(1-\alpha-y_0) && (10)\\ &= \alpha d_0\Big(\frac{1}{(1-\alpha)^{k-1}} - 1\Big) && (11)\\ &> 0. && (12) \end{aligned}$$

It is noted that (8) follows from $x_1 = x_0 + \alpha - \dfrac{\alpha}{(1-\alpha)^{k-1}}$ (by (3) and (6)) and $y_1 = y_0 + \alpha$ (by (4) and (7)). By (4) and $d_0 > 0$, we get $y_0 < 1-\alpha$. Furthermore, by $0 < y_0 < 1-\alpha < 1$ and (5), we obtain $x_0 = \dfrac{\alpha}{y_0(1-\alpha)^{k-2}} > \dfrac{\alpha}{(1-\alpha)^{k-1}}$. Substituting this inequality and (5) into (9), we have (10). By (4), we obtain (11). Since $0 < \alpha < 1$, we have (12).

By the above proofs, we get that $B_{n,k}$ is strictly α-subnormal. Therefore, by Lemma 2.9(ii), we have $\rho(B_{n,k}) < \alpha^{-\frac{1}{k}}$. Thus, by Lemma 2.9, we obtain $\rho(A_{n,k}) > \rho(B_{n,k})$, where $n \geq 3k(k-1)$ and $k \geq 3$. □

Lemma 3.9. *Let $G \in \bar{\mathcal{U}}_{2,1}(n,k,3)$, where $n \geq 3k(k-1)$ and $k \geq 3$. We have $\rho(A_{n,k}) > \rho(G)$.*

Proof: Let $n = mk(k-1)$ with $m \geq 3$ and $k \geq 3$. Let $G \in \bar{\mathcal{U}}_{2,1}(n,k,3)$. When $m = 3$, we get Lemma 3.9 since $G \cong B_{n,k}$ and $\rho(A_{n,k}) > \rho(B_{n,k})$ (by Lemma 3.8). Next, let $m \geq 4$.

Let $M(G)$ be the perfect matching of G. Since $m \geq 4$, bearing the definition of $\bar{\mathcal{U}}_{2,1}(n,k,3)$ in mind, we assume v_1 of C_3 in G is attached by a hypertree (denoted by T^*) which has at least two edges belonging to $Q(G)$. If T^* has perfect matching edges which are not pendent edges, then let e be an arbitrary edge of those edges. By applying the edge-releasing operation on e at a vertex of e, we get a hypergraph G_{13} such that $\rho(G_{13}) \geq \rho(G)$ with the equality iff $G \cong G_{13}$ (by Lemma 3.1), where $G_{13} \in \bar{\mathcal{U}}_{2,1}(n,k,3)$ and all the perfect matching edges of G_{13} are pendent edges except for e_1 contained in C_3 of G_{13}.

If $G_{13} \cong B_{n,k}$, then by Lemma 3.8, we get $\rho(A_{n,k}) > \rho(B_{n,k}) \geq \rho(G)$. Namely, Lemma 3.9 holds. Otherwise, we assume $G_{13} \ncong B_{n,k}$. Since $m \geq 4$, it is noted that v_1 of C_3 in G_{13} is attached by a hypertree (denoted by $T^\star$) which has at least two edges belonging to $Q(G_{13})$. As $G_{13} \ncong B_{n,k}$, in G_{13}, there exists an edge (denoted by $\dot{g} = \{w_1, ..., w_k\}$) which satisfies the following conditions: (i) $\dot{g} \in E(T^\star)$; (ii) $\dot{g}$ is not a pendent edge; (iii) v_1 and w_1 are incident with a common edge; and (iv) $\dot{g}$ is not incident with v_1. Let $\boldsymbol{x}$ be the principal eigenvector of G_{13} corresponding to $\rho(G_{13})$. Two cases are considered as follows.

Case (i). $x_{v_1} < x_{w_1}$.

Let G_{14} be the hypergraph obtained from G_{13} by moving all the edges which are incident with v_1 (except for e_1 of C_3 in G_{13} and the common edge which is incident with v_1 and w_1) from v_1 to w_1. Obviously, $G_{14} \in \mathcal{U}_2(n,k,4)$, where each vertex of $e_1 \setminus \{v_1, v_2\}$ of C_4 in G_{14} is a core vertex and only e_1 is the perfect matching edge on C_4 of G_{14}. By Lemma 2.2, we obtain $\rho(G_{14}) > \rho(G_{13})$. Let G_{15} be the hypergraph obtained from G_{14} by applying the edge-releasing operation on e_1 at v_1 in such a way that e_1 becomes a pendent edge. By Lemma 2.5, we have $\rho(G_{15}) > \rho(G_{14})$. Obviously, G_{15} does not have multiple edges and $G_{15} \in \bar{\mathcal{U}}_1(n,k,3)$. By Corollary 3.5, we get $\rho(A_{n,k}) \geq \rho(G_{15})$ with the equality iff $G_{15} \cong A_{n,k}$. By the above proofs, we obtain $\rho(A_{n,k}) > \rho(G)$ for $G \in \bar{\mathcal{U}}_{2,1}(n,k,3)$.

Case (ii). $x_{v_1} \geq x_{w_1}$.

Let G_{16} be the hypergraph obtained from G_{13} by moving $\dot{g}$ from w_1 to v_1 in such a way that G_{16} has the perfect matching $M(G)$. Obviously, $G_{16} \in \bar{\mathcal{U}}_{2,1}(n,k,3)$. By Lemma 2.2, we obtain $\rho(G_{16}) > \rho(G_{13})$. If $G_{16} \cong B_{n,k}$, then by Lemma 3.8 and the above proofs, we get $\rho(A_{n,k}) > \rho(B_{n,k}) > \rho(G)$ for $G \in \bar{\mathcal{U}}_{2,1}(n,k,3)$. Otherwise, we assume $G_{16} \ncong B_{n,k}$. By repeatedly using the same procedure as those in Cases (i) and (ii), we finally obtain $\max\{\rho(A_{n,k}), \rho(B_{n,k})\} > \rho(G_{16})$. Therefore, it follows from $\rho(A_{n,k}) > \rho(B_{n,k})$ (by Lemma 3.8) that $\rho(A_{n,k}) > \rho(G_{16})$. Thus, we get $\rho(A_{n,k}) > \rho(G)$ for $G \in \bar{\mathcal{U}}_{2,1}(n,k,3)$.

By combining the above proofs, we have Lemma 3.9. □

Lemma 3.10. *We have $\rho(D_{n,k}) > \rho(A_{n,k})$ for $n \geq 9k(k-1)$ and $k \geq 3$, where $A_{n,k}$ and $D_{n,k}$ are shown in Fig. 1(b) and Fig. 2(b), respectively.*

Proof: We construct a weighted incidence matrix $\mathcal{B}_{D_{n,k}}$ for $D_{n,k}$ as follows. Let $0 < \alpha < 1$. Let $\mathcal{B}_{D_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{D_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{D_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{D_{n,k}}(v) = 2$; $\mathcal{B}_{D_{n,k}}(v,e) = 1-\alpha$ if $v \in e$, $v \neq v_2, v_3$, e is not a pendent edge, and $d_{D_{n,k}}(v) = 2$; $\mathcal{B}_{D_{n,k}}(v_1, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-2$ and $m \geq 9$; $\mathcal{B}_{D_{n,k}}(v_1, e_1) = \mathcal{B}_{D_{n,k}}(v_1, e_3) = x_2$; $\mathcal{B}_{D_{n,k}}(v_2, e_1) = \mathcal{B}_{D_{n,k}}(v_3, e_3) = y_2$; and $\mathcal{B}_{D_{n,k}}(v_2, e_2) = \mathcal{B}_{D_{n,k}}(v_3, e_2) = c_1$, where y_2, c_1, x_2, and α satisfy (13)–(16) as follows:

$$\begin{cases} y_2 + c_1 = 1, & (13)\\ 2x_2 + \alpha + \dfrac{(m-2)\alpha}{(1-\alpha)^{k-1}} = 1, & (14)\\ x_2 y_2 (1-\alpha)^{k-2} = \alpha, & (15)\\ c_1^2 = \alpha. & (16)\end{cases}$$

We can verify that $\sum_{e: e \in E_{D_{n,k}}(v)} \mathcal{B}_{D_{n,k}}(v,e) = 1$ for any $v \in V(D_{n,k})$, $\prod_{v: v \in e} \mathcal{B}_{D_{n,k}}(v,e) = \alpha$ for any $e \in E(D_{n,k})$, and $\mathcal{B}_{D_{n,k}}$ is consistent. Thus, $D_{n,k}$ is consistently α-normal. By Lemma 2.9(i), we have $\rho(D_{n,k}) = \alpha^{-\frac{1}{k}}$.

We construct a weighted incidence matrix $\mathcal{B}_{A_{n,k}}$ for $A_{n,k}$ as follows. Let $0 < \alpha < 1$. Let $\mathcal{B}_{A_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{A_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{A_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{A_{n,k}}(v) = 2$;

$\mathcal{B}_{A_{n,k}}(v,e) = 1-\alpha$ if $v \in e$, e is not a pendent edge and $d_{A_{n,k}}(v) = 2$; $\mathcal{B}_{A_{n,k}}(v_1, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-3$ and $m \geq 9$; $\mathcal{B}_{A_{n,k}}(v_1, e_1) = \mathcal{B}_{A_{n,k}}(v_1, e_3) = x_3$; $\mathcal{B}_{A_{n,k}}(v_2, e_1) = \mathcal{B}_{A_{n,k}}(v_3, e_3) = y_3$; and $\mathcal{B}_{A_{n,k}}(v_2, e_2) = \mathcal{B}_{A_{n,k}}(v_3, e_2) = c_2$, where y_3, x_3, c_2, and α satisfy (17)–(19) as follows:

$$\begin{cases} y_3 + c_2 + \alpha = 1, & (17)\\ x_3 y_3 (1-\alpha)^{k-2} = \alpha, & (18)\\ c_2^2 (1-\alpha)^{k-2} = \alpha. & (19) \end{cases}$$

We can verify that $\sum_{e:e\in E_{A_{n,k}}(v)} \mathcal{B}_{A_{n,k}}(v,e) = 1$ for any $v \in V(A_{n,k}) \setminus \{v_1\}$ and $\prod_{v:v\in e} \mathcal{B}_{A_{n,k}}(v,e) = \alpha$ for any $e \in E(A_{n,k})$. Next, we will prove $\sum_{e:e\in E_{A_{n,k}}(v_1)} \mathcal{B}_{A_{n,k}}(v_1,e) < 1$.

Since $\alpha > 0$ and $1-\alpha > 0$, we have $0 < \alpha < 1$. It follows from (16) that $c_1 = \sqrt{\alpha}$. Combining (13) and $0 < \alpha < 1$, we obtain $y_2 = 1 - \sqrt{\alpha} < 1 - \alpha$. From $y_2 < 1-\alpha$, $y_2 > 0$ and (15), we have $x_2 = \dfrac{\alpha}{y_2(1-\alpha)^{k-2}} > \dfrac{\alpha}{(1-\alpha)^{k-1}}$. Substituting $x_2 > \dfrac{\alpha}{(1-\alpha)^{k-1}}$ into (14), we get $m\alpha < (1-\alpha)^k$. It follows from $m\alpha < (1-\alpha)^k$ and (19) that $c_2 < \frac{1-\alpha}{\sqrt{m}}$. Thus, for $m \geq 9$, we have

$$\begin{aligned} x_3 - x_2 &= \frac{\alpha}{(1-\alpha)^{k-2}}\left(\frac{1}{y_3} - \frac{1}{y_2}\right) && (20)\\ &= \frac{\alpha}{(1-\alpha)^{k-1}}\left(\frac{1-\alpha}{y_3} - \frac{1-\alpha}{y_2}\right)\\ &= \frac{\alpha}{(1-\alpha)^{k-1}}\left(\frac{1-\alpha}{1-c_2-\alpha} - \frac{1-\alpha}{1-c_1}\right) && (21)\\ &< \frac{\alpha}{(1-\alpha)^{k-1}}\left(\frac{\sqrt{m}}{\sqrt{m}-1} - \left(1+\sqrt{\alpha}\right)\right) && (22)\\ &< \frac{\alpha}{(1-\alpha)^{k-1}}\left(\frac{\sqrt{m}}{\sqrt{m}-1} - 1\right)\\ &= \frac{\alpha}{(1-\alpha)^{k-1}} \times \frac{1}{\sqrt{m}-1}\\ &\leq \frac{\alpha}{2\,(1-\alpha)^{k-1}}, && (23) \end{aligned}$$

where (20) follows from (15) and (18), (21) is deduced from (13) and (17), and (22) is obtained from $c_2 < \frac{1-\alpha}{\sqrt{m}}$ and $c_1 = \sqrt{\alpha}$. Therefore, it follows from (20)–(23) that $x_3 < x_2 + \dfrac{\alpha}{2(1-\alpha)^{k-1}}$ for $m \geq 9$. Thus, we obtain

$$\begin{aligned} \sum_{e:e\in E_{A_{n,k}}(v_1)} \mathcal{B}_{A_{n,k}}(v_1,e) &= 2x_3 + \alpha + \frac{(m-3)\alpha}{(1-\alpha)^{k-1}}\\ &< 2x_2 + \frac{\alpha}{(1-\alpha)^{k-1}} + \alpha + \frac{(m-3)\alpha}{(1-\alpha)^{k-1}}\\ &= 2x_2 + \alpha + \frac{(m-2)\alpha}{(1-\alpha)^{k-1}}\\ &= 1, && (24) \end{aligned}$$

where (24) follows from (14). Thus, for $m \geq 9$, $A_{n,k}$ is strictly α-subnormal. Therefore, by Lemma 2.9(ii), we have $\rho(A_{n,k}) < \alpha^{-\frac{1}{k}}$.

In conclusion, it follows from Lemma 2.9 that $\rho(D_{n,k}) > \rho(A_{n,k})$, where $n \geq 9k(k-1)$ and $k \geq 3$. □

It should be noted that, since (23) holds for $m \geq 9$, the methods proposed in Lemma 3.10 can not be used to compare the relationship between $\rho(D_{n,k})$ and $\rho(A_{n,k})$ when $n = mk(k-1)$ with $3 \leq m \leq 8$.

By the methods similar to those for Lemma 3.4, we have Lemma 3.11 as follows.

Lemma 3.11. *Let $G \in \bar{\mathcal{U}}_{2,2}(n,k,3)$, where $n \geq 2k(k-1)$ and $k \geq 3$. We have $\rho(D_{n,k}) \geq \rho(G)$ with the equality iff $G \cong D_{n,k}$.*

Corollary 3.12. *Let $G \in \mathcal{U}_2(n,k,l)$, where $n = mk(k-1)$ and $m,k,l \geq 3$. (i). If $3 \leq m \leq 8$, we have $\max\{\rho(A_{n,k}), \rho(D_{n,k})\} \geq \rho(G)$. (ii). If $m \geq 9$, we have $\rho(D_{n,k}) \geq \rho(G)$ with the equality iff $G \cong D_{n,k}$.*

Proof: Let $n = mk(k-1)$ and $m,k,l \geq 3$. By Corollary 3.5, we have $\rho(A_{n,k}) \geq \rho(G)$ with the equality iff $G \cong A_{n,k}$, where $G \in \mathcal{U}_1(n,k,l)$. By Lemmas 3.9 and 3.11, we have $\max\{\rho(A_{n,k}), \rho(D_{n,k})\} \geq \rho(G)$ for $G \in \bar{\mathcal{U}}_2(n,k,3)$ since $\bar{\mathcal{U}}_{2,1}(n,k,3) \cup \bar{\mathcal{U}}_{2,2}(n,k,3) = \bar{\mathcal{U}}_2(n,k,3)$. Furthermore, by Lemmas 3.6, 3.7 and Corollary 3.5, we obtain $\max\{\rho(A_{n,k}), \rho(D_{n,k})\} \geq \rho(G)$ for $G \in \mathcal{U}_2(n,k,l)$. Thus, when $3 \leq m \leq 8$, we get Corollary 3.12 (i). For $m \geq 9$, by Lemma 3.10, we have $\rho(D_{n,k}) > \rho(A_{n,k})$ for $n \geq 9k(k-1)$. Therefore, we get Corollary 3.12 (ii). □

By Corollaries 3.5 and 3.12, we obtain Theorem 3.13 as follows.

Theorem 3.13. *Let $G \in \mathcal{U}(n,k)$, where $n = mk(k-1)$, $m \geq 2$ and $k \geq 3$. (i). If $m = 2$, we have $G \cong B_{n,k} \cong D_{n,k}$ and $\rho(G) = \rho(B_{n,k}) = \rho(D_{n,k})$. (ii). If $3 \leq m \leq 8$, $\max\{\rho(A_{n,k}), \rho(D_{n,k})\} \geq \rho(G)$. (iii). If $m \geq 9$, $\rho(D_{n,k}) \geq \rho(G)$ with the equality iff $G \cong D_{n,k}$.*

4. The hypergraphs with the maximal spectral radii among $\Gamma(n,k)$ and $\mathcal{U}(n,k) \cup \Gamma(n,k)$

In Section 4, we will get the hypergraphs with the maximal spectral radii among $\Gamma(n,k)$ with $n \geq k(k-1)$ and among $\mathcal{U}(n,k) \cup \Gamma(n,k)$ with $n \geq 2k(k-1)$, where $k \geq 3$. Some necessary definitions are given as follows.

Let $G \in \Gamma(n,k)$. The unique cycle in G has two edges which share two common vertices. We denote the cycle in G by C_2 and the two edges contained in C_2 by $\widetilde{e_1}$ and $\widetilde{e_2}$, where $\widetilde{e_1} = \{u_1, u_{1,1}, ..., u_{1,k-2}, u_2\}$ and $\widetilde{e_2} = \{u_1, u_{2,1}, ..., u_{2,k-2}, u_2\}$. C_2 is shown in Fig. 3(a). According to the fact whether C_2 of G has one perfect matching edge or not, we classify $\Gamma(n,k)$ into two types: (1) C_2 has one perfect matching edge, and (2) $\widetilde{e_1}$ and $\widetilde{e_2}$ in C_2 are not perfect matching edges. The hypergraphs in $\Gamma(n,k)$ can be divided into two subsets according to Types (1) and (2). We denote $\Gamma(n,k) = \Gamma_1(n,k) \cup \Gamma_2(n,k)$, where all the hypergraphs in $\Gamma_1(n,k)$ and $\Gamma_2(n,k)$ have Types (1) and (2), respectively. Obviously, for all the hypergraphs in $\Gamma_1(n,k)$ and in $\Gamma_2(n,k)$, we have $m \geq 1$ and $m \geq 2$, respectively.

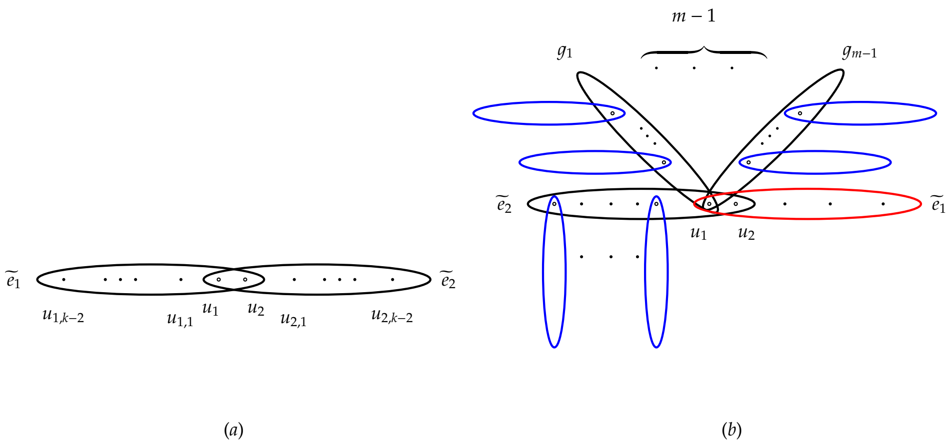

Figure 3: (a) C_2 and (b) $I_{n,k}$

Let $\bar{\Gamma}_1(n,k)$ be a subset of $\Gamma_1(n,k)$ in which each hypergraph satisfies three conditions: (i) each vertex in $\widetilde{e_1} \setminus \{u_1, u_2\}$ of C_2 is a core vertex; (ii) each vertex in $\widetilde{e_2} \setminus \{u_1, u_2\}$ of C_2 must be attached by a pendent edge;

and (iii) at most one of the vertices (denoted by v) in $\widetilde{e_2}$ of C_2 is attached by a hypertree which has at least $k \geq 3$ edges. We further classify $\bar{\Gamma}_1(n,k)$ into two subsets which are denoted by $\bar{\Gamma}_{1,1}(n,k)$ and $\bar{\Gamma}_{1,2}(n,k)$, where the hypergraphs in $\bar{\Gamma}_{1,1}(n,k)$ satisfy that $v = u_1$ or $v = u_2$ and the hypergraphs in $\bar{\Gamma}_{1,2}(n,k)$ satisfy that v is one of the vertices in $\{u_{2,1}, \cdots, u_{2,k-2}\}$.

Let $\bar{\Gamma}_2(n,k)$ be a subset of $\Gamma_2(n,k)$ in which each hypergraph satisfies two conditons: (i) each vertex in C_2 must be attached by a pendent edge, and (ii) at most one of the vertices in $\widetilde{e_1} \cap \widetilde{e_2} = \{u_1, u_2\}$ of C_2 is attached by a hypertree which has at least k edges, where $k \geq 3$.

Let $I_{n,k}$ be the hypergraph obtained from $C_2(u_1,u_0)S_{m-1,k}$ by attaching one pendent edge at each vertex of $C_2(u_1,u_0)S_{m-1,k}$ (except for all the vertices of $\widetilde{e_1}$), where $m \geq 1$ and $k \geq 3$. Let $J_{n,k}$ be the hypergraph obtained from $C_2(u_{2,1},u_0)S_{m-1,k}$ by attaching one pendent edge at each vertex of $C_2(u_{2,1},u_0)S_{m-1,k}$ (except for all the vertices of $\widetilde{e_1}$), where $m \geq 1$ and $k \geq 3$. Obviously, when $n = k(k-1)$, $I_{n,k} \cong J_{n,k}$. $I_{n,k}$ and $J_{n,k}$ are shown in Fig. 3(b) and Fig. 4(a), respectively. Let $L_{n,k}$ be the hypergraph obtained from $C_2(u_1,u_0)S_{m-2,k}$ by attaching one pendent edge at each vertex of $C_2(u_1,u_0)S_{m-2,k}$, where $m \geq 2$ and $k \geq 3$. $L_{n,k}$ is shown in Fig. 4(b). Obviously, $I_{n,k} \in \bar{\Gamma}_{1,1}(n,k)$, $J_{n,k} \in \bar{\Gamma}_{1,2}(n,k)$, and $L_{n,k} \in \bar{\Gamma}_2(n,k)$.

To obtain the hypergraph with the maximal spectral radius in $\Gamma(n,k) = \Gamma_1(n,k) \cup \Gamma_2(n,k)$ (as shown in Theorem 4.12), we introduce several lemmas first. We propose Lemmas 4.1–4.7 to get the hypergraph with the maximal spectral radius in $\Gamma_1(n,k)$ (as shown in Corollary 4.8). Lemmas 4.9 and 4.10 are deduced to obtain the hypergraph with the maximal spectral radius in $\Gamma_2(n,k)$ (as shown in Corollary 4.11).

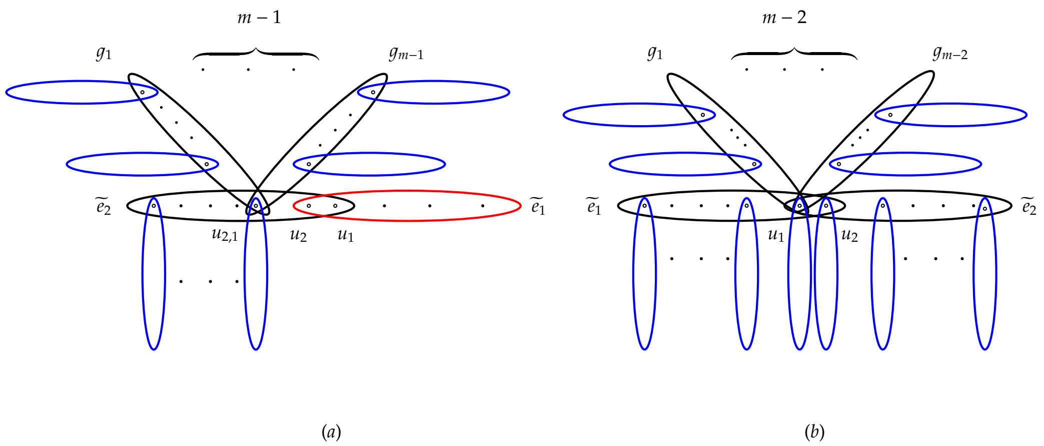

Figure 4: (a) $J_{n,k}$ and (b) $L_{n,k}$

By the methods similar to those for Lemma 3.1, we have Lemma 4.1 as follows.

Lemma 4.1. *Let $G \in \Gamma_i(n,k)$, where $i = 1,2$, $n \geq k(k-1)$ and $k \geq 3$. Let e be a perfect matching edge of G, and e is neither an edge on C_2 of G nor a pendent edge. Let G'_0 be the hypergraph obtained from G by applying the edge-releasing operation on e at a vertex of e such that e of G'_0 is a pendent edge. We have $\rho(G'_0) > \rho(G)$, where $G'_0 \in \Gamma_i(n,k)$ and $i = 1,2$.*

Lemma 4.2. *Let $G \in \Gamma_1(n,k)$, where $n \geq k(k-1)$ and $k \geq 3$. There exists a hypergraph $\ddot{G} \in \bar{\Gamma}_1(n,k)$ such that $\rho(\ddot{G}) \geq \rho(G)$ with the equality iff $G \cong \ddot{G}$.*

Proof: Let $n \geq k(k-1)$ and $k \geq 3$. Let $G \in \Gamma_1(n,k)$. According to the definition of $\Gamma_1(n,k)$, we suppose that $\widetilde{e_1}$ of G is a perfect matching edge. In $\widetilde{e_1} \setminus \{u_1,u_2\}$, if there exists a vertex having degree not less than 2, without loss of generality, we suppose that this vertex is $u_{1,1}$. By the methods similar to those for the first paragraph in Lemma 3.7, we finally obtain a hypergraph (denoted by G_{17}) in $\Gamma_1(n,k)$ satisfying $\rho(G_{17}) > \rho(G)$ and each vertex in $\widetilde{e_1}\setminus\{u_1,u_2\}$ of G_{17} is a core vertex.

If $G_{17} \in \bar{\Gamma}_1(n,k)$, then we get Lemma 4.2. Otherwise, by Lemma 4.1, we obtain a hypergraph $\tilde{G}$ satisfying that $\rho(\tilde{G}) \geq \rho(G_{17})$ with the equality iff $G_{17} \cong \tilde{G}$, where $\tilde{G}$ satisfies three conditions: (i) each vertex in

$\widetilde{e_1} \setminus \{u_1, u_2\}$ of C_2 is a core vertex; (ii) each vertex in $\widetilde{e_2} \setminus \{u_1, u_2\}$ of C_2 must be attached by a pendent edge; and (iii) at least one vertex in $\widetilde{e_2}$ of C_2 is attached by a hypertree which has at least $k \geq 3$ edges (if $n \geq 2k(k-1)$).

If $\tilde{G} \in \bar{\Gamma}_1(n,k)$, then we get Lemma 4.2. Otherwise, we assume $\tilde{G} \notin \bar{\Gamma}_1(n,k)$. By the definition of $\bar{\Gamma}_1(n,k)$, there exist two vertices (denoted by $\tilde{u}_1$ and $\tilde{u}_2$) in $\widetilde{e_2}$ of $\tilde{G}$ which have degrees not less than 3. Namely, $\tilde{u}_1$ and $\tilde{u}_2$ are attached by hypertrees which have at least $k \geq 3$ edges. Let $f_1^1, \cdots, f^1_{d_G(\tilde{u}_1)-2}$ and $f_1^2, \cdots, f^2_{d_G(\tilde{u}_2)-2}$ be all the edges which are incident with $\tilde{u}_1$ and $\tilde{u}_2$, respectively. It is noted that $f_1^1, \cdots, f^1_{d_G(\tilde{u}_1)-2}$ and $f_1^2, \cdots, f^2_{d_G(\tilde{u}_2)-2}$ are not perfect matching edges and all of them do not contain $\widetilde{e_2}$.

Let $\boldsymbol{x}$ be the principal eigenvector of $\tilde{G}$ corresponding to $\rho(\tilde{G})$. Without loss of generality, we suppose $x_{\tilde{u}_1} \geq x_{\tilde{u}_2}$. Let G_{18} be the hypergraph obtained from $\tilde{G}$ by removing $f_1^2, \cdots, f^2_{d_G(\tilde{u}_2)-2}$ from $\tilde{u}_2$ to $\tilde{u}_1$. By Lemma 2.2, we obtain $\rho(G_{18}) > \rho(\tilde{G})$. By repeatedly using the same procedure as above, we can find a hypergraph $\ddot{G} \in \bar{\Gamma}_1(n,k)$ such that $\rho(\ddot{G}) \geq \rho(G_{18}) > \rho(G)$ with the equality iff $\ddot{G} \cong G_{18}$. Therefore, we get Lemma 4.2. □

To obtain the hypergraph with the maximal spectral radius in $\bar{\Gamma}_{1,1}(n,k)$ with $n \geq 2k(k-1)$ and $k \geq 3$ (as shown in Lemma 4.5), we introduce Lemmas 4.3 and 4.4 first.

Lemma 4.3. *We have $\rho(L_{n,k}) > \rho(A_{n,k})$ and $\rho(L_{n,k}) > \rho(D_{n,k})$, where $n \geq 2k(k-1)$ and $k \geq 3$.*

Proof: Let $n \geq 2k(k-1)$ and $k \geq 3$. Let $\boldsymbol{x}$ be the principal eigenvector corresponding to $\rho(A_{n,k})$. In $A_{n,k}$, if $x_{v_1} \geq x_{v_3}$, then let H_1 be the hypergraph obtained from $A_{n,k}$ by removing the edge e_2 of $A_{n,k}$ from v_3 to v_1. By Lemma 2.2, we have $\rho(H_1) > \rho(A_{n,k})$. In $A_{n,k}$, if $x_{v_3} > x_{v_1}$, then let H_2 be the hypergraph obtained from $A_{n,k}$ by removing all the edges of $e_1, g_1, \cdots, g_{m-3}$ ($m \geq 3$) from v_1 to v_3. By Lemma 2.2, we obtain $\rho(H_2) > \rho(A_{n,k})$. Obviously, $H_1 \cong H_2 \cong L_{n,k}$. Therefore, we get $\rho(L_{n,k}) > \rho(A_{n,k})$ for $n \geq 3k(k-1)$ and $k \geq 3$.

Let $\boldsymbol{y}$ be the principal eigenvector corresponding to $\rho(D_{n,k})$. By the symmetry of the vertices of $D_{n,k}$, we have $y_{v_2} = y_{v_3}$. Let H_3 be the hypergraph obtained from $D_{n,k}$ by removing e_3 from v_3 to v_2. Obviously, $H_3 \cong L_{n,k}$. Therefore, by Lemma 2.2, we get $\rho(L_{n,k}) > \rho(D_{n,k})$ for $n \geq 2k(k-1)$ and $k \geq 3$. □

Lemma 4.4. *We have $\rho(L_{n,k}) > \rho(I_{n,k})$ for $n \geq 2k(k-1)$ and $k \geq 3$, where $I_{n,k}$ and $L_{n,k}$ are shown in Fig. 3(b) and Fig. 4(b), respectively.*

Proof: Let $n \geq 2k(k-1)$ and $k \geq 3$. We construct a weighted incidence matrix $\mathcal{B}_{L_{n,k}}$ for $L_{n,k}$ as follows. Let $0 < \alpha < 1$. Let $\mathcal{B}_{L_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{L_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{L_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{L_{n,k}}(v) = 2$; $\mathcal{B}_{L_{n,k}}(v,e) = 1-\alpha$ if $v \in e$, e is not a pendent edge and $d_{L_{n,k}}(v) = 2$; $\mathcal{B}_{L_{n,k}}(u_1, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-2$ and $m \geq 2$; $\mathcal{B}_{L_{n,k}}(u_1, \widetilde{e_1}) = \mathcal{B}_{L_{n,k}}(u_1, \widetilde{e_2}) = x_4$; and $\mathcal{B}_{L_{n,k}}(u_2, \widetilde{e_1}) = \mathcal{B}_{L_{n,k}}(u_2, \widetilde{e_2}) = y_4$, where x_4, y_4 and α satisfy (25)–(27) as follows:

$$\begin{cases} 2x_4 + \alpha + \dfrac{(m-2)\alpha}{(1-\alpha)^{k-1}} = 1, & (25) \\ 2y_4 + \alpha = 1, & (26) \\ x_4 y_4 (1-\alpha)^{k-2} = \alpha. & (27) \end{cases}$$

We can check that $\sum_{e: e \in E_{L_{n,k}}(v)} \mathcal{B}_{L_{n,k}}(v,e) = 1$ for any $v \in V(L_{n,k})$, $\prod_{v: v \in e} \mathcal{B}_{L_{n,k}}(v,e) = \alpha$ for any $e \in E(L_{n,k})$, and $\mathcal{B}_{L_{n,k}}$ is consistent. Thus, $L_{n,k}$ is consistently α-normal. By Lemma 2.9(i), we have $\rho(L_{n,k}) = \alpha^{-\frac{1}{k}}$.

We construct a weighted incidence matrix $\mathcal{B}_{I_{n,k}}$ for $I_{n,k}$ as follows. Let $\mathcal{B}_{I_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{I_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{I_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{I_{n,k}}(v) = 2$; $\mathcal{B}_{I_{n,k}}(v,e) = 1-\alpha$ if $v \in e$ ($v \neq u_2$), e is not a pendent edge and $d_{I_{n,k}}(v) = 2$; $\mathcal{B}_{I_{n,k}}(u_1, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-1$ and $m \geq 2$; $\mathcal{B}_{I_{n,k}}(u_1, \widetilde{e_1}) = x_5$; $\mathcal{B}_{I_{n,k}}(u_2, \widetilde{e_1}) = y_5$; $\mathcal{B}_{I_{n,k}}(u_1, \widetilde{e_2}) = x_4$; and $\mathcal{B}_{I_{n,k}}(u_2, \widetilde{e_2}) = y_4$, where x_4, y_4, x_5, y_5, and α satisfy (28) and (29) as follows:

$$\begin{cases} x_4 + x_5 + \dfrac{(m-1)\alpha}{(1-\alpha)^{k-1}} = 1, & (28) \\ y_4 + y_5 = 1. & (29) \end{cases}$$

We can check that $\sum_{e:e\in E_{I_{n,k}}(v)} \mathcal{B}_{I_{n,k}}(v,e) = 1$ for any $v \in V(I_{n,k})$ and $\prod_{v:v\in e} \mathcal{B}_{I_{n,k}}(v,e) = \alpha$ for any $e \in E(I_{n,k})$ and $e \neq \widetilde{e_1}$. Next, we prove $\prod_{v:v\in\widetilde{e_1}} \mathcal{B}_{I_{n,k}}(v,\widetilde{e_1}) > \alpha$. We have

$$\prod_{v:v\in\widetilde{e_1}} \mathcal{B}_{I_{n,k}}(v,\widetilde{e_1}) - \alpha = x_5y_5 - \alpha$$

$$= [x_4 + \alpha - \frac{\alpha}{(1-\alpha)^{k-1}}](y_4 + \alpha) - \alpha \tag{30}$$

$$= x_4y_4 + x_4\alpha + y_4\alpha + \alpha^2 - \frac{\alpha}{(1-\alpha)^{k-1}}y_4 - \frac{\alpha^2}{(1-\alpha)^{k-1}} - \alpha$$

$$= \frac{\alpha}{(1-\alpha)^{k-2}} + \frac{2\alpha^2}{(1-\alpha)^{k-1}} + \frac{\alpha-\alpha^2}{2} + \alpha^2 - \frac{\alpha}{2(1-\alpha)^{k-2}} - \frac{\alpha^2}{(1-\alpha)^{k-1}} - \alpha \tag{31}$$

$$= \frac{\alpha}{2(1-\alpha)^{k-2}} + \frac{\alpha^2}{(1-\alpha)^{k-1}} + \frac{\alpha^2}{2} - \frac{\alpha}{2}$$

$$> \frac{\alpha}{2} + \frac{\alpha^2}{(1-\alpha)^{k-1}} + \frac{\alpha^2}{2} - \frac{\alpha}{2} \tag{32}$$

$$> \frac{3\alpha^2}{2} \tag{33}$$

$$> 0. \tag{34}$$

It is noted that (30) follows from $x_5 = x_4 + \alpha - \frac{\alpha}{(1-\alpha)^{k-1}}$ (by (25) and (28)) and $y_5 = y_4 + \alpha$ (by (26) and (29)). Since $y_4 = \frac{1-\alpha}{2}$ (by (26)) and $x_4 = \frac{2\alpha}{(1-\alpha)^{k-1}}$ (by (27) and $y_4 = \frac{1-\alpha}{2}$), we get (31). Since $0 < 1-\alpha < 1$, we obtain (32)–(34). Thus, we get $\prod_{v:v\in\widetilde{e_1}} \mathcal{B}_{I_{n,k}}(v,\widetilde{e_1}) > \alpha$.

By the above proofs, we get that $I_{n,k}$ is strictly α-subnormal. Therefore, by Lemma 2.9(ii), we have $\rho(I_{n,k}) < \alpha^{-\frac{1}{k}}$. Thus, by Lemma 2.9, we obtain $\rho(L_{n,k}) > \rho(I_{n,k})$ for $n \geq 2k(k-1)$ and $k \geq 3$. □

Lemma 4.5. *Let $G \in \bar{\Gamma}_{1,1}(n,k)$, where $n \geq 2k(k-1)$ with $k \geq 3$. We have $\rho(L_{n,k}) > \rho(G)$.*

Proof: Let $n = mk(k-1)$ with $m \geq 2$ and $k \geq 3$. Let $G \in \bar{\Gamma}_{1,1}(n,k)$. When $m = 2$, we get Lemma 4.5 since $G \cong I_{n,k}$ and $\rho(L_{n,k}) > \rho(I_{n,k})$ (by Lemma 4.4). Next, let $m \geq 3$.

By the definition of $\bar{\Gamma}_{1,1}(n,k)$, each vertex in $\widetilde{e_2} \setminus \{u_1,u_2\}$ of G is attached by a pendent edge and is not attached by a hypertree which has at least k ($k \geq 3$) edges, and we assume u_1 of C_2 in G is attached by a hypertree (denoted by $\overline{T}$) which has at least two edges belonging to $Q(G)$ and $\overline{T}$ has perfect matching edges which are not pendent edges. By Lemma 4.1, we get a hypergraph G_{19} such that $\rho(G_{19}) > \rho(G)$, where $G_{19} \in \bar{\Gamma}_{1,1}(n,k)$ and all the perfect matching edges of G_{19} are pendent edges except for $\widetilde{e_1}$ contained in C_2 of G_{19}.

If $G_{19} \cong I_{n,k}$, then by Lemma 4.4, we get $\rho(L_{n,k}) > \rho(I_{n,k}) \geq \rho(G)$ and Lemma 4.5 holds. Otherwise, we assume $G_{19} \ncong I_{n,k}$. It is noted that u_1 of C_2 in G_{19} is attached by a hypertree (denoted by $\widetilde{T}$) which has at least two edges belonging to $Q(G_{19})$. As $G_{19} \ncong I_{n,k}$, in G_{19}, there exists an edge (denoted by $g' = \{z_1, ..., z_k\}$) which satisfies the following conditions: (i) $g' \in E(\widetilde{T})$; (ii) g' is not a pendent edge; (iii) u_1 and z_1 are incident with a common edge (denoted by g''); and (iv) g' is not adjacent to u_1. Let $\boldsymbol{x}$ be the principal eigenvector of G_{19} corresponding to $\rho(G_{19})$. Two cases are considered as follows.

Case (i). $x_{u_1} < x_{z_1}$.

Let G_{20} be the hypergraph obtained from G_{19} by moving all the edges which are incident with u_1 (except for $\widetilde{e_1}$ and g'' of G_{19}) from u_1 to z_1. Obviously, $G_{20} \in \mathcal{U}_2(n,k,3)$, where the cycle contained in G_{20} is denoted by $C'_3 = u_1\widetilde{e_1}u_2\widetilde{e_2}z_1g''u_1$, each vertex of $\widetilde{e_1} \setminus \{u_1,u_2\}$ of C'_3 of G_{20} is a core vertex and $\widetilde{e_1}$ is the perfect matching edge on C'_3 of G_{20}. By Lemma 2.2, we obtain $\rho(G_{20}) > \rho(G_{19})$. By Corollary 3.12, we

obtain $\max\{\rho(A_{n,k}), \rho(D_{n,k})\} \geq \rho(G_{20})$ for $3 \leq m \leq 8$ and $\rho(D_{n,k}) \geq \rho(G_{20})$ for $m \geq 9$, where $\rho(D_{n,k}) = \rho(G_{20})$ iff $G_{20} \cong D_{n,k}$. Therefore, we get $\rho(L_{n,k}) > \rho(G_{20})$ since $\rho(L_{n,k}) > \rho(A_{n,k})$ and $\rho(L_{n,k}) > \rho(D_{n,k})$ (by Lemma 4.3). By the above proofs, we obtain $\rho(L_{n,k}) > \rho(G)$ for $G \in \bar{\Gamma}_{1,1}(n,k)$. Therefore, we get Lemma 4.5 in Case (i).
Case (ii). $x_{u_1} \geq x_{z_1}$.

Let G_{21} be the hypergraph obtained from G_{19} by moving g' from z_1 to u_1 in such a way that G_{21} has a perfect matching. Obviously, $G_{21} \in \bar{\Gamma}_{1,1}(n,k)$. By Lemma 2.2, we obtain $\rho(G_{21}) > \rho(G_{19})$. If $G_{21} \cong I_{n,k}$, then by Lemma 4.4 and the above proofs, we get $\rho(L_{n,k}) > \rho(I_{n,k}) > \rho(G)$ for $G \in \bar{\Gamma}_{1,1}(n,k)$. Namely, Lemma 4.5 holds. Otherwise, we assume $G_{21} \ncong I_{n,k}$. By repeatedly using the same procedure as those in Cases (i) and (ii), we finally obtain $\max\{\rho(L_{n,k}), \rho(I_{n,k})\} > \rho(G_{21})$. Therefore, it follows from $\rho(L_{n,k}) > \rho(I_{n,k})$ (by Lemma 4.4) and the above proofs, we have Lemma 4.5. □

By the methods similar to those for the proofs of Lemma 3.4, we get Lemma 4.6 as follows.

Lemma 4.6. *Let $G \in \bar{\Gamma}_{1,2}(n,k)$, where $n \geq k(k-1)$ with $k \geq 3$. We have $\rho(J_{n,k}) \geq \rho(G)$ with the equality iff $G \cong J_{n,k}$.*

Lemma 4.7. *We have $\rho(I_{n,k}) > \rho(J_{n,k})$ for $n \geq 2k(k-1)$ and $k \geq 3$, where $I_{n,k}$ and $J_{n,k}$ are shown in Fig. 3(b) and Fig. 4(a), respectively.*

Proof: For $n \geq 2k(k-1)$, obviously $I_{n,k} \ncong J_{n,k}$. We construct a weighted incidence matrix $\mathcal{B}_{I_{n,k}}$ for $I_{n,k}$ as follows. Let $0 < \alpha < 1$. Let $\mathcal{B}_{I_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{I_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{I_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{I_{n,k}}(v) = 2$; $\mathcal{B}_{I_{n,k}}(v,e) = 1-\alpha$ if $v \in e$ ($v \neq u_2$), e is not a pendent edge and $d_{I_{n,k}}(v) = 2$; $\mathcal{B}_{I_{n,k}}(u_1, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-1$ and $m \geq 2$; $\mathcal{B}_{I_{n,k}}(u_1, \widetilde{e_2}) = x_6$; $\mathcal{B}_{I_{n,k}}(u_2, \widetilde{e_2}) = y_6$; $\mathcal{B}_{I_{n,k}}(u_1, \widetilde{e_1}) = c_3$; and $\mathcal{B}_{I_{n,k}}(u_2, \widetilde{e_1}) = d_3$, where x_6, y_6, c_3, d_3, and α satisfy (35)–(38) as follows:

$$\begin{cases} x_6 + c_3 + \dfrac{(m-1)\alpha}{(1-\alpha)^{k-1}} = 1, & (35)\\ y_6 + d_3 = 1, & (36)\\ c_3 d_3 = \alpha, & (37)\\ x_6 y_6 (1-\alpha)^{k-2} = \alpha, & (38)\\ \dfrac{x_6 d_3}{y_6 c_3} = 1. \end{cases}$$

We can check that $\sum_{e:e\in E_{I_{n,k}}(v)} \mathcal{B}_{I_{n,k}}(v,e) = 1$ for any $v \in V(I_{n,k})$, $\prod_{v:v\in e} \mathcal{B}_{I_{n,k}}(v,e) = \alpha$ for any $e \in E(I_{n,k})$, and $\mathcal{B}_{I_{n,k}}$ is consistent. Thus, $I_{n,k}$ is consistently α-normal. By Lemma 2.9(i), we have $\rho(I_{n,k}) = \alpha^{-\frac{1}{k}}$.

We construct a weighted incidence matrix $\mathcal{B}_{J_{n,k}}$ for $J_{n,k}$ as follows. Let $\mathcal{B}_{J_{n,k}}(v,e) = 0$ if $v \notin e$; $\mathcal{B}_{J_{n,k}}(v,e) = 1$ if $v \in e$ and v is a core vertex; $\mathcal{B}_{J_{n,k}}(v,e) = \alpha$ if $v \in e$, e is a pendent edge and $d_{J_{n,k}}(v) = 2$; $\mathcal{B}_{J_{n,k}}(v,e) = 1-\alpha$ if $v \in e$ ($v \neq u_1, u_2$), e is not a pendent edge and $d_{J_{n,k}}(v) = 2$; $\mathcal{B}_{J_{n,k}}(u_{2,1}, g_i) = \dfrac{\alpha}{(1-\alpha)^{k-1}}$ for $i = 1, \cdots, m-1$ and $m \geq 2$; $\mathcal{B}_{J_{n,k}}(u_2, \widetilde{e_2}) = x_7$; $\mathcal{B}_{J_{n,k}}(u_1, \widetilde{e_2}) = y_7$; $\mathcal{B}_{J_{n,k}}(u_2, \widetilde{e_1}) = c_3$; $\mathcal{B}_{J_{n,k}}(u_1, \widetilde{e_1}) = d_3$; and $\mathcal{B}_{J_{n,k}}(u_{2,1}, \widetilde{e_2}) = 1 - \alpha - \dfrac{(m-1)\alpha}{(1-\alpha)^{k-1}}$, where x_7, y_7, c_3, and d_3 satisfy (39) and (40) as follows:

$$\begin{cases} x_7 + c_3 = 1, & (39)\\ y_7 + d_3 = 1. & (40) \end{cases}$$

We can check that $\sum_{e:e\in E_{J_{n,k}}(v)} \mathcal{B}_{J_{n,k}}(v,e) = 1$ for any $v \in V(J_{n,k})$ and $\prod_{v:v\in e} \mathcal{B}_{J_{n,k}}(v,e) = \alpha$ for any $e \in E(J_{n,k})$ and $e \neq \widetilde{e_2}$. Next, we prove $\prod_{v:v\in \widetilde{e_2}} \mathcal{B}_{J_{n,k}}(v, \widetilde{e_2}) > \alpha$.

Let $\dfrac{x_6}{y_6} = \dfrac{c_3}{d_3} = b$. We have $x_6 = by_6$ and $c_3 = bd_3$. Substituting $c_3 = bd_3$ into (37), we get $\alpha = bd_3^2$. Substituting $x_6 = by_6$, $\alpha = bd_3^2$ and (36) into (38), we get $d_3 = \dfrac{(1-\alpha)^{k/2-1}}{1+(1-\alpha)^{k/2-1}}$. Therefore, by (36), we have

$$y_6 = 1 - d_3 = \frac{1}{1+(1-\alpha)^{k/2-1}}. \tag{41}$$

Substituting $x_6 = by_6$ and $c_3 = bd_3$ into (35) and bearing (36) (namely $y_6 + d_3 = 1$), $\alpha = bd_3^2$ and $d_3 = \dfrac{(1-\alpha)^{k/2-1}}{1+(1-\alpha)^{k/2-1}}$ in mind, we get

$$\frac{(m-1)\alpha}{(1-\alpha)^{k-1}} = 1 - b\,(y_6 + d_3) = 1 - b = 1 - \alpha\frac{\left(1+(1-\alpha)^{k/2-1}\right)^2}{(1-\alpha)^{k-2}}. \tag{42}$$

We obtain

$$\prod_{v:v\in\widetilde{e_2}} \mathcal{B}_{J_{n,k}}(v,\widetilde{e_2}) - \alpha = x_7 y_7\left[1-\alpha-\frac{(m-1)\alpha}{(1-\alpha)^{k-1}}\right](1-\alpha)^{k-3} - \alpha$$

$$= y_6\left[x_6 + \frac{(m-1)\alpha}{(1-\alpha)^{k-1}}\right]\left[1-\alpha-\frac{(m-1)\alpha}{(1-\alpha)^{k-1}}\right](1-\alpha)^{k-3} - \alpha \tag{43}$$

$$= \left[\frac{\alpha}{(1-\alpha)^{k-2}} + \frac{(m-1)\alpha}{(1-\alpha)^{k-1}}y_6\right]\left[1-\alpha-\frac{(m-1)\alpha}{(1-\alpha)^{k-1}}\right](1-\alpha)^{k-3} - \alpha \tag{44}$$

$$= \frac{(m-1)\alpha}{1-\alpha}\left[y_6\left(1-\frac{(m-1)\alpha}{(1-\alpha)^{k-1}}\cdot\frac{1}{1-\alpha}\right) - \frac{\alpha}{(1-\alpha)^{k-1}}\right] \tag{45}$$

$$= \frac{(m-1)\alpha^2}{(1-\alpha)^2}\left[\frac{1}{(1-\alpha)^{k/2-1}} - \frac{1}{1+(1-\alpha)^{k/2-1}}\right] \tag{46}$$

$$> 0. \tag{47}$$

It is noted that (43) follows from $x_7 = x_6 + \dfrac{(m-1)\alpha}{(1-\alpha)^{k-1}}$ (by (35) and (39)) and $y_7 = y_6$ (by (36) and (40)). Substituting $x_6 y_6 = \alpha/(1-\alpha)^{k-2}$ (by (38)) into (43), we have (44). By calculation, we get (45). Substituting the expression of y_6 (namely (41)) and the expression of $\dfrac{(m-1)\alpha}{(1-\alpha)^{k-1}}$ (namely (42)) into (45), we have (46). Since $m > 1$ and $0 < \alpha < 1$, we get (47).

By the above proofs, we get that $J_{n,k}$ is strictly α-subnormal. Therefore, by Lemma 2.9(ii), we have $\rho(J_{n,k}) < \alpha^{-\frac{1}{k}}$. Thus, by Lemma 2.9, we obtain $\rho(I_{n,k}) > \rho(J_{n,k})$ for $n \geq 2k(k-1)$ and $k \geq 3$. □

The hypergraph with the maximal spectral radius in $\Gamma_1(n,k)$ are shown in Corollary 4.8.

Corollary 4.8. *Let $G \in \Gamma_1(n,k)$, where $n \geq 2k(k-1)$ with $k \geq 3$. We have $\rho(L_{n,k}) > \rho(G)$.*

Proof: Let $n \geq 2k(k-1)$ with $k \geq 3$. By Lemma 4.5, we have $\rho(L_{n,k}) > \rho(G)$ for $G \in \bar{\Gamma}_{1,1}(n,k)$. By Lemmas 4.4, 4.6 and 4.7, we obtain $\rho(L_{n,k}) > \rho(I_{n,k}) > \rho(J_{n,k}) > \rho(G)$ for $G \in \bar{\Gamma}_{1,2}(n,k)$. Therefore, we have $\rho(L_{n,k}) > \rho(G)$ for $G \in \bar{\Gamma}_1(n,k)$ since $\bar{\Gamma}_{1,1}(n,k) \cup \bar{\Gamma}_{1,2}(n,k) = \bar{\Gamma}_1(n,k)$. Furthermore, by Lemma 4.2, we get Corollary 4.8. □

Lemma 4.9. *Let $G \in \Gamma_2(n,k)$, where $n \geq 2k(k-1)$ and $k \geq 3$. There exists a hypergraph $\overline{G} \in \bar{\Gamma}_2(n,k)$ such that $\rho(\overline{G}) \geq \rho(G)$ with the equality iff $G \cong \overline{G}$.*

Proof: Let $G \in \Gamma_2(n,k)$ with $n \geq 2k(k-1)$ and $k \geq 3$. By Lemma 4.1, we get that there exists a hypergraph $\dot{G}$ such that $\rho(\dot{G}) \geq \rho(G)$ with the equality iff $G \cong \dot{G}$, where $\dot{G}$ satisfies two conditions: (i) each vertex in C_2 must be attached by a pendent edge, and (ii) at least one vertex in C_2 is attached by a hypertree which has at least $k \geq 3$ edges (if $n \geq 3k(k-1)$).

If $\dot{G} \in \bar{\Gamma}_2(n,k)$, then Lemma 4.9 holds. Next, suppose $\dot{G} \notin \bar{\Gamma}_2(n,k)$. In $\dot{G}$, there exist at least one vertex in $\widetilde{e_1} \cup \widetilde{e_2}$ which is attached by a hypertree having at least $k \geq 3$ edges. In $\dot{G}$, let $V_1(\dot{G})$ be the subset of $V(\dot{G})$ in which each vertex is attached by a hypertree having at least $k \geq 3$ edges, where $|V_1(\dot{G})| \geq 1$. Let w be a vertex in $V_1(\dot{G})$ and let $w \in \widetilde{e_1}$. Let $\boldsymbol{x}$ be the principal eigenvector of $\dot{G}$ corresponding to $\rho(\dot{G})$. Among all the vertices in $V_1(\dot{G})$, we suppose that w has the maximal component x_w among $\boldsymbol{x}$. By the methods similar to those for the second paragraph, Cases (i) and (ii) in Lemma 3.3, we obtain a hypergraph $\overline{G} \in \bar{\Gamma}_2(n,k)$ such that $\rho(\overline{G}) > \rho(\dot{G}) \geq \rho(G)$. Therefore, we have Lemma 4.9. □

By the methods similar to those for the proofs of Lemma 3.4, we get Lemma 4.10 as follows.

Lemma 4.10. *Let $G \in \bar{\Gamma}_2(n,k)$, where $n \geq 2k(k-1)$ with $k \geq 3$. We have $\rho(L_{n,k}) \geq \rho(G)$ with the equality iff $G \cong L_{n,k}$.*

By Lemmas 4.9 and 4.10, we have Corollary 4.11 as follows.

Corollary 4.11. *Let $G \in \Gamma_2(n,k)$, where $n \geq 2k(k-1)$ with $k \geq 3$. We have $\rho(L_{n,k}) \geq \rho(G)$ with the equality iff $G \cong L_{n,k}$.*

By Corollaries 4.8 and 4.11, we get Theorem 4.12.

Theorem 4.12. *Let $G \in \Gamma(n,k)$, where $n = mk(k-1)$ with $m \geq 1$ and $k \geq 3$. When $m = 1$, $G \cong I_{n,k} \cong J_{n,k}$ and $\rho(G) = \rho(I_{n,k}) = \rho(J_{n,k})$. When $m \geq 2$, $\rho(L_{n,k}) \geq \rho(G)$ with the equality iff $G \cong L_{n,k}$.*

For $n = k(k-1)$ with $k \geq 3$, there is only one hypergraph $I_{n,k}$ (namely $J_{n,k}$) among $\mathcal{U}(n,k) \cup \Gamma(n,k)$. For $n \geq 2k(k-1)$ with $k \geq 3$, we get the hypergraph with the maximal spectral radius among $\mathcal{U}(n,k) \cup \Gamma(n,k)$, which is shown in Theorem 4.13.

Theorem 4.13. *Let $G \in \mathcal{U}(n,k) \cup \Gamma(n,k)$ with $n \geq 2k(k-1)$ and $k \geq 3$, we have $\rho(L_{n,k}) \geq \rho(G)$ with the equality iff $G \cong L_{n,k}$.*

Proof: Let $n \geq 2k(k-1)$ and $k \geq 3$. By Theorem 3.13 and Lemma 4.3, we have $\rho(L_{n,k}) > \rho(G)$ for $G \in \mathcal{U}(n,k)$. By Theorem 4.12, we have $\rho(L_{n,k}) \geq \rho(G)$ for $G \in \Gamma(n,k)$ with the equality iff $G \cong L_{n,k}$. Therefore, we get Theorem 4.13. □

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