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Exel's approximation property and the CBAP of crossed products

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Abstract. Let Γ be a countable discrete group that acts on a unital separable C^*-algebra A through an action α. Suppose that the C^*-dynamical system (A, Γ, α) has Exel's approximation property. Then A has the CBAP if and only if the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ has the CBAP.

1. Introduction

Throughout this paper, $A \subseteq B(\mathcal{H})$ is a unital separable C^*-algebra with the unit 1_A, Γ is a countable discrete group that acts on A through an action α. Let $CB(A)$ be the space of all completely bounded linear maps from A into A, and $C_c(\Gamma, A)$ the space of finitely supported functions on Γ with values in A. We denote by $A \rtimes_{\alpha,r} \Gamma$ the reduced crossed product of the C^*-dynamical system (A, Γ, α), and identify $A \subseteq A \rtimes_{\alpha,r} \Gamma$ and $\Gamma \subseteq A \rtimes_{\alpha,r} \Gamma$ through their canonical embeddings.

In 1989, Cowling and Haagerup [3] introduced weak amenability for groups and defined the Cowling-Haagerup constant. This constant for a great many of groups has been computed (see [2, 3]). Moreover, in [6], Haagerup introduced the completely bounded approximation property for C^*-algebras, which has been intensively studied in the literature (see [12, 15, 16]). It is known that a discrete group Γ is weakly amenable if and only if the reduced group C^*-algebra $C_r^*(\Gamma)$ has the completely bounded approximation property.

Before we present the definition of the completely bounded approximation property, let us first recall that an operator between two Banach spaces is said to have *finite rank* when its range is finite dimensional.

Definition 1.1. *We say a C^*-algebra A has the completely bounded approximation property (CBAP) if there exist a constant $C > 0$ and a net of finite-rank completely bounded maps $\Phi_i : A \to A$ such that*

$$\|\Phi_i(a) - a\| \to 0$$

for all $a \in A$ and $\sup\{\|\Phi_i\|_{cb}\} \le C$. The Haagerup constant $\Lambda(A)$ is the infimum of all C for which such a net $\{\Phi_i\}$ exists. If A does not have the CBAP, we set $\Lambda(A) = \infty$.

It is interesting to determine the behavior of the Haagerup constant under the classic constructions in operator algebra theory, such as the crossed product of a C^*-dynamical system. In 1996, Sinclair and Smith

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[15] considered various tensor products. In [16], Sinclair and Smith showed that $\Lambda(A \rtimes_{\alpha,r} \Gamma) = \Lambda(A)$, where Γ is an amenable group. In [12], we proved equality of the Haagerup constants for a C^*-algebra and its crossed product by an amenable action.

In [4], Exel introduced an approximation property for Fell bundles over discrete groups, which we call Exel's approximation property. Soon after, Exel and Ng [5] considered Exel's approximation property for Fell bundles over locally compact groups. The following is the definition of Exel's approximation property in our setting (see [1]).

Definition 1.2. *The C^*-dynamical system (A, Γ, α) is said to have Exel's approximation property, if there exist nets $\{H_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ in $C_c(\Gamma, A)$ such that*

(1) $\sup_i \| \sum_{s\in\Gamma} H_i^*(s)H_i(s)\| \sup_i \| \sum_{s\in\Gamma} T_i^*(s)T_i(s)\| < +\infty$,

(2) $\lim_i \| \sum_{s\in\Gamma} H_i^*(s)a\alpha_t(T_i(t^{-1}s)) - a\| = 0$, *for all $a \in A$ and $t \in \Gamma$.*

If one can choose $H_i = T_i$ for all $i \in I$, we will say that (A, Γ, α) has positive Exel's approximation property. Let

$$M_H := \sup_i \| \sum_{s\in\Gamma} H_i^*(s)H_i(s)\|, \quad M_T := \sup_i \| \sum_{s\in\Gamma} T_i^*(s)T_i(s)\|,$$

and $M_{H,T} = M_H M_T$. We define $\Delta(A, \Gamma, \alpha)$ as the infimum of all $M_{H,T}$ for which such nets $\{H_i\}$ and $\{T_i\}$ exist.

In [4], Exel only considered the case when $H_i = T_i$, for all $i \in I$. Definition 1.2 is actually M-approximation property as in [5] and weak approximation property as in [1].

Herz-Schur multipliers have the large number of applications in operator algebra theory, it is natural to try to extend the notion to a more general setting. In [10], Mckee, Todorov and Turowska introduced Herz-Schur (A, Γ, α)-multipliers and Schur A-multipliers.

Definition 1.3. *[10, Definition 3.1] A bounded function $F : \Gamma \to CB(A)$ is called a Herz-Schur (A, Γ, α)-multiplier if the map $S_F : C_c(\Gamma, A) \to C_c(\Gamma, A)$ such that*

$$S_F(\sum_{s\in\Gamma} a_s s) = \sum_{s\in\Gamma} F(s)(a_s)s$$

is completely bounded. If this is the case, then S_F extends to a completely bounded map on $A \rtimes_{\alpha,r} \Gamma$, we still denote the extension by S_F. The set of all Herz-Schur (A, Γ, α)-multipliers is an algebra, we endow it with the norm $\|F\|_m = \|S_F\|_{cb}$. In the case when $A = \mathbb{C}$ and α is the trivial action, Herz-Schur (A, Γ, α)-multipliers are called Herz-Schur multipliers. We denote the algebra of all Herz-Schur multipliers by $B_2(\Gamma)$ and the Herz-Schur norm by $\| \cdot \|_{B_2}$.

Let $\varphi : \Gamma \times \Gamma \to CB(A)$ be a bounded function. We recall from [10, Theorem 2.6] that φ is a *Schur A-multiplier* if and only if there exist a separable Hilbert space $\mathcal{H}_\rho$, a non-degenerate $*$-representation $\rho : A \to B(\mathcal{H}_\rho)$, $V \in \ell^\infty(\Gamma, B(\mathcal{H}, \mathcal{H}_\rho))$, and $W \in \ell^\infty(\Gamma, B(\mathcal{H}, \mathcal{H}_\rho))$, such that

$$\varphi(s, t)(a) = W^*(t)\rho(a)V(s), \quad a \in A,$$

for all $(s, t) \in \Gamma \times \Gamma$.

For a function $F : \Gamma \to CB(A)$, we define $\mathcal{N}(F) : \Gamma \times \Gamma \to CB(A)$ by

$$\mathcal{N}(F)(s, t)(a) = \alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(a))),$$

for all $s, t \in \Gamma$, $a \in A$. It follows from [10, Theorem 3.8] that F is a Herz-Schur (A, Γ, α)-multiplier if and only if $\mathcal{N}(F)$ is a Schur A-multiplier.

It is shown that Herz-Schur multipliers can be used as a technical basis for approximation results (see [7, 9, 11, 13]). By Herz-Schur multipliers of a C^*-dynamical system, Mckee [8, Theorem 4.3] introduced the weak amenability of a C^*-dynamical system, and proved that the weak amenability of a C^*-dynamical system is equivalent to the CBAP of the associated crossed product.

Definition 1.4. *[8, Definition 4.1] A C^*-dynamical system (A, Γ, α) will be called weakly amenable, if there exists a net $\{F_i\}_{i\in I}$ of finitely supported Herz-Schur (A, Γ, α)-multipliers such that each $F_i(t)$ is a finite-rank completely bounded map on A,*

$$\|F_i(t)(a) - a\| \to 0$$

for all $a \in A$ and $t \in \Gamma$, and $\sup\{\|F_i\|_m\} = K < +\infty$. The infimum of all such K is denoted by $\Lambda(A, \Gamma, \alpha)$. If $(\mathbb{C}, \Gamma, \alpha)$ is weakly amenable when α is the trivial action, then Γ is said to be weakly amenable. In this case, the quantity $\Lambda(\Gamma) := \Lambda(\mathbb{C}, \Gamma, \alpha)$ is called the Cowling-Haagerup constant. When Γ is not weakly amenable, we set $\Lambda(\Gamma) = \infty$.

In this paper, with the help of certain Herz-Schur multipliers of a C^*-dynamical system, we show that if the C^*-dynamical system (A, Γ, α) has Exel's approximation property, then A has the CBAP if and only if $A \rtimes_{\alpha,r} \Gamma$ has the CBAP. Since Exel's approximation property is weaker then the amenability of the action α, our result is a generalization of the main results in [12] and [16].

2. Main results

Using similar ideas of [9, Corollary 4.6] and [12, Theorem 2.2], we can get the main results of this paper.

Lemma 2.1. *If $A \rtimes_{\alpha,r} \Gamma$ has the CBAP, then A has the CBAP. In fact,*

$$\Lambda(A) \le \Lambda(A \rtimes_{\alpha,r} \Gamma).$$

Proof. Suppose that there exists a net $\{\Phi_i\}_{i\in I}$ of completely bounded maps on $A \rtimes_{\alpha,r} \Gamma$ witnessing the CBAP of $A \rtimes_{\alpha,r} \Gamma$. Let $\mathcal{E} : A \rtimes_{\alpha,r} \Gamma \to A$ be the canonical faithful conditional expectation, then $\{\mathcal{E} \circ \Phi_i|_A\}_{i\in I}$ shows the CBAP of A. Hence, we have $\Lambda(A) \le \Lambda(A \rtimes_{\alpha,r} \Gamma)$. □

Lemma 2.2. *Suppose that (A, Γ, α) has Exel's approximation property. If A has the CBAP, then $A \rtimes_{\alpha,r} \Gamma$ has the CBAP. In fact,*

$$\Lambda(A \rtimes_{\alpha,r} \Gamma) \le \sqrt{\Delta(A, \Gamma, \alpha)}\Lambda(A).$$

Proof. Assume that $\{\Phi_j\}_{j\in J}$ is a net of completely bounded maps on A witnessing the CBAP of A, $\{H_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ are nets as in Definition 1.2, where H_i and T_i are supported on D_i and F_i respectively. Let M_H, M_T and $M_{H,T}$ be as in Definition 1.2. For a fix $s \in \Gamma$, we define the maps $\Psi_{i,j}(s) : A \to A$ by

$$\Psi_{i,j}(s)(a) = \sum_{p\in\Gamma} H_i(p)^* \alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_s(T_i(s^{-1}p)),$$

for all $a \in A$. Note that the summation above is in fact over the finite set $D_i \cap sF_i$.

For all $s, t \in \Gamma$ and $a \in A$, we have

$$\begin{aligned}
\mathcal{N}(\Psi_{i,j})(s,t)(a) &= \alpha_{t^{-1}}(\Psi_{i,j}(ts^{-1})(\alpha_t(a))) \\
&= \sum_{p\in\Gamma} \alpha_{t^{-1}}(H_i(p)^* \alpha_p(\Phi_j(\alpha_p^{-1}(\alpha_t(a))))\alpha_{ts^{-1}}(T_i(st^{-1}p))) \\
&= \sum_{p\in\Gamma} \alpha_{t^{-1}}(H_i(p)^*)\alpha_{t^{-1}p}(\Phi_j(\alpha_{t^{-1}p}^{-1}(a)))\alpha_{s^{-1}}(T_i(st^{-1}p)) \\
&= \sum_{p\in\Gamma} \alpha_{t^{-1}}(H_i(tp)^*)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_{s^{-1}}(T_i(sp))
\end{aligned}$$

As each Φ_j is a completely bounded map on $A \subseteq B(\mathcal{H})$, it follows from [14, Theorem 8.4] that there exist a Hilbert space $\mathcal{H}_{p,j}$, a $*$-representations $\pi_{p,j} : A \to B(\mathcal{H}_{p,j})$ and bounded operators $W_{p,j}, V_{p,j} \in B(\mathcal{H}, \mathcal{H}_{p,j})$ such that

$$\|V_{p,j}\| = \|W_{p,j}\| = \|\alpha_p \circ \Phi_j \circ \alpha_p^{-1}\|_{cb}^{\frac{1}{2}} = \|\Phi_j\|_{cb}^{\frac{1}{2}}$$

and

$$\alpha_p(\Phi_j(\alpha_p^{-1}(a))) = W_{p,j}^*\pi_{p,j}(a)V_{p,j}$$

for all $p \in \Gamma$, $a \in A$. We set

$$\rho_j = \bigoplus_{p\in\Gamma} \pi_{p,j}.$$

Let $V_{i,j}(s) : \mathcal{H} \to \bigoplus_{p\in\Gamma} \mathcal{H}_{p,j}$ be the column operator $\{V_{p,j}\alpha_{s^{-1}}(T_i(sp))\}_{p\in\Gamma}$, and $W_{i,j}(t) : \mathcal{H} \to \bigoplus_{p\in\Gamma} \mathcal{H}_{p,j}$ the column operator $\{W_{p,j}\alpha_{t^{-1}}(H_i(tp))\}_{p\in\Gamma}$. Then, we get

$$\mathcal{N}(\Psi_{i,j})(s,t)(a) = W_{i,j}(t)^*\rho_j(a)V_{i,j}(s)$$

for all $s, t \in \Gamma, a \in A$. Moreover, for all $t \in \Gamma$,

$$\begin{aligned}
\|W_{i,j}(t)\|^2 &= \|\sum_{p\in\Gamma} \alpha_{t^{-1}}(H_i(tp))^* W_{p,j}^* W_{p,j}\alpha_{t^{-1}}(H_i(tp))\| \\
&\leq \|\sum_{p\in\Gamma} \alpha_{t^{-1}}(H_i(tp))^*\alpha_{t^{-1}}(H_i(tp))\|\|\Phi_j\|_{cb} \\
&= \|\sum_{p\in\Gamma} H_i(p)^*H_i(p)\|\|\Phi_j\|_{cb} \leq M_H\|\Phi_j\|_{cb}.
\end{aligned}$$

By the same argument, we can prove that

$$\|V_{i,j}(s)\|^2 \leq M_T\|\Phi_j\|_{cb}$$

for all $s \in \Gamma$. It follows from [10, Theorem 2.6] that each $\mathcal{N}(\Psi_{i,j})$ is a Schur A-multiplier. Hence, [10, Theorem 3.8] and the argument of [10, Theorem 2.6] show that each $\Psi_{i,j}$ is a Herz-Schur (A, Γ, α)-multiplier with

$$\|\Psi_{i,j}\|_m \leq \sup_{t\in\Gamma} \|W_{i,j}(t)\| \sup_{s\in\Gamma} \|V_{i,j}(s)\| \leq \sqrt{M_{H,T}}\|\Phi_j\|_{cb}.$$

Since each Φ_j is finite rank, we have that $\Psi_{i,j}(s)$ is finite rank for all $s \in \Gamma$, $i \in I$, $j \in J$. Furthermore,

$$\begin{aligned}
\|\Psi_{i,j}(s)(a) - a\| &= \|\sum_{p\in\Gamma} H_i(p)^*\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_s(T_i(s^{-1}p)) - a\| \\
&\leq \|\sum_{p\in\Gamma} H_i(p)^*(\alpha_p(\Phi_j(\alpha_p^{-1}(a))) - a)\alpha_s(T_i(s^{-1}p))\| \\
&+ \|\sum_{p\in\Gamma} H_i(p)^* a\alpha_s(T_i(s^{-1}p)) - a\| \to 0
\end{aligned}$$

for all $a \in A$. Hence, the C^*-dynamical system (A, Γ, α) is weakly amenable and

$$\Lambda(A, \Gamma, \alpha) \leq \sqrt{\Delta(A, \Gamma, \alpha)}\Lambda(A).$$

It follows from [8, Theorem 4.3] that the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ has the CBAP and $\Lambda(A \rtimes_{\alpha,r} \Gamma) \leq \sqrt{\Delta(A, \Gamma, \alpha)}\Lambda(A)$. □

Combining Lemma 2.1 and Lemma 2.2, we get the main theorem of this paper.

Theorem 2.3. *Suppose that (A, Γ, α) has Exel's approximation property, then A has the CBAP if and only if $A \rtimes_{\alpha,r} \Gamma$ has the CBAP. Moreover, we have that*

$$\Lambda(A) \leq \Lambda(A \rtimes_{\alpha,r} \Gamma) \leq \sqrt{\Delta(A, \Gamma, \alpha)}\Lambda(A).$$

Let $Z(A)$ be the center of A and $Z(A)^+$ be the cone of positive elements in $Z(A)$. We recall from [2, Definition 4.3.1] that the action α is amenable if there exists a net $\{T_i\}_{i\in I}$ of finitely supported functions $T_i : \Gamma \to Z(A)^+$ such that $\sum_{s\in\Gamma} T_i(s)^2 = 1_A$ and

$$\|\sum_{s\in\Gamma}(T_i(s) - \alpha_t(T_i(t^{-1}s)))^*(T_i(s) - \alpha_t(T_i(t^{-1}s)))\| \to 0$$

for all $t \in \Gamma$.

Remark 2.4. *It follows from [2, Lemma 4.3.2] that if the action α is amenable, then (A, Γ, α) has positive Exel's approximation property with*

$$\Delta(A, \Gamma, \alpha) = 1.$$

Hence, Theorem 2.3 is a generalization of [12, Theorem 2.2].

Corollary 2.5. *Suppose that (A, Γ, α) has Exel's approximation property and A has an α-invariant state τ. If A has the CBAP, then Γ is weakly amenable. In fact,*

$$\Lambda(\Gamma) \le \Lambda(A \rtimes_{\alpha,r} \Gamma) \le \sqrt{\Delta(A, \Gamma, \alpha)}\Lambda(A).$$

Proof. It follows from [8, Theorem 4.3] and Theorem 2.3 that (A, Γ, α) is weakly amenable and

$$\Lambda(A, \Gamma, \alpha) = \Lambda(A \rtimes_{\alpha,r} \Gamma) \le \sqrt{\Delta(A, \Gamma, \alpha)}\Lambda(A).$$

Suppose we are given a net $\{F_i\}_{i\in I}$ as in Definition 1.4. For each $i \in I$, set

$$\omega_i(t) = \tau(F_i(t)(1_A))$$

for all $t \in \Gamma$. Since F_i is finitely supported, ω_i has finite support. Moreover,

$$|\omega_i(t) - 1| = |\tau(F_i(t)(1_A) - 1_A)| \le \|F_i(t)(1_A) - 1_A\| \to 0$$

for all $t \in \Gamma$. Define $m_{\omega_i} : C_r^*(\Gamma) \to C_r^*(\Gamma)$ such that

$$m_{\omega_i}(\sum_{s\in\Gamma} \theta_s s) = \sum_{s\in\Gamma} \omega_i(s)\theta_s s$$

for all $\sum_{s\in\Gamma} \theta_s s \in C_c(\Gamma, \mathbb{C})$, and $M_\tau : A \rtimes_{\alpha,r} \Gamma \to C_r^*(\Gamma)$ such that

$$M_\tau(\sum_{s\in\Gamma} a_s s) = \sum_{s\in\Gamma} \tau(a_s)\lambda_s$$

for all $\sum_{s\in\Gamma} a_s s \in C_c(\Gamma, A)$. Since τ is an α-invariant state, it follows from [2, Exercise 4.1.4] that M_τ is unital and completely positive. Hence,

$$m_{\omega_i} = M_\tau \circ S_{F_i}|_{C_r^*(\Gamma)}$$

is completely bounded and $\|m_{\omega_i}\|_{cb} \le \|S_{F_i}\|_{cb}$. It follows that $\omega_i \in B_2(\Gamma)$ and $\|\omega_i\|_{B_2} \le \Lambda(A, \Gamma, \alpha)$. These prove the weak amenability of Γ. □

We conclude this article with a special example.

Example 2.6. *It follows from [2, Corollary 12.3.5] that the Cowling-Haagerup constant of special linear group* $\mathrm{SL}(2, \mathbb{Z})$ *is 1. Let* $C_r^*(\mathrm{SL}(2, \mathbb{Z}))$ *be the reduced group C^*-algebra of* $\mathrm{SL}(2, \mathbb{Z})$*, then*

$$\Lambda(C_r^*(\mathrm{SL}(2, \mathbb{Z}))) = 1.$$

Moreover, [2, Theorem 5.1.7] shows that the left translation action lt of $\mathrm{SL}(2, \mathbb{Z})$ *on the Stone-Čech compactification $\beta\mathrm{SL}(2, \mathbb{Z})$ is amenable. We also denote the induced action on $C(\beta\mathrm{SL}(2, \mathbb{Z}))$ by lt. If γ is an action of* $\mathrm{SL}(2, \mathbb{Z})$ *on* $C_r^*(\mathrm{SL}(2, \mathbb{Z}))$*, then the action $lt \otimes \gamma$ of* $\mathrm{SL}(2, \mathbb{Z})$ *on* $C(\beta\mathrm{SL}(2, \mathbb{Z})) \otimes C_r^*(\mathrm{SL}(2, \mathbb{Z}))$ *is amenable. Since* $C(\beta\mathrm{SL}(2, \mathbb{Z}))$ *is nuclear, it follows from [15] that*

$$\Lambda(C(\beta\mathrm{SL}(2, \mathbb{Z})) \otimes C_r^*(\mathrm{SL}(2, \mathbb{Z}))) = \Lambda(C(\beta\mathrm{SL}(2, \mathbb{Z})))\Lambda(C_r^*(\mathrm{SL}(2, \mathbb{Z}))) = 1.$$

Hence, Theorem 2.3 and Remark 2.4 show that

$$\Lambda((C(\beta\mathrm{SL}(2, \mathbb{Z})) \otimes C_r^*(\mathrm{SL}(2, \mathbb{Z}))) \rtimes_{lt\otimes\gamma,r} \mathrm{SL}(2, \mathbb{Z})) = 1.$$

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References

[1] E. Bédos, R. Conti, *On discrete twisted C^*-dynamical systems, Hilbert C^*-modules and regularity*, Mönster J. Math. **5** (2012), 183-208.
[2] N. P. Brown, N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Grad. Stud. Math. 88, Amer. Math. Soc., Providence, RI, 2008.
[3] M. Cowling, U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. **96** (1989), 507-549.
[4] R. Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. **492** (1997), 41-73.
[5] R. Exel, C. K. Ng, *Approximation property of C^*-algebraic bundles*, Math. Proc. Camb. Philos. Soc. **132**(3) (2002), 509-522.
[6] U. Haagerup, *Group C^*-algebras without the completely bounded approximation property*, J. Lie Theory **26** (2016), 861-887.
[7] S. Knudby, *The weak Haagerup property*, Trans. Amer. Math. Soc. **368** (2016), 3469-3508.
[8] A. Mckee, *Weak amenability for dynamical systems*, Stud. Math. **258**(1) (2021), 53-70.
[9] A. Mckee, A. Skalski, I. G. Todorov, L. Turowska, *Positive Herz-Schur multipliers and approximation properties of crossed products*, Math. Proc. Camb. Philos. Soc. **165**(3) (2018), 511-532.
[10] A. Mckee, I. G. Todorov, L. Turowska, *Herz-Schur multipliers of dynamical systems*, Adv. Math. **331** (2018), 387-438.
[11] A. Mckee, L. Turowska, *Exactness and SOAP of crossed products via Herz-Schur multipliers*, J. Math. Anal. Appl. **496** (2021), 124812.
[12] Q. Meng, *On the completely bounded approximation property of crossed products*, P. Indian AS-Math. Sci. **131** (2021), 26.
[13] Q. Meng, L. G. Wang, *Weak Haagerup property of dynamical systems*, Linear Multilinear A. **67**(7) (2019), 1294-1307.
[14] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press, Cambridge, 2003.
[15] A. M. Sinclair, R. R. Smith. *The Haagerup invariant for tensor products of operator spaces*, Math. Proc. Camb. Philos. Soc. **120** (1996), 147-153.
[16] A. M. Sinclair, R. R. Smith, *The completely bounded approximation property for discrete crossed products*, Indiana U. Math. J. **46**(4) (1997), 1311-1322.