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Orlicz mixed projection body

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Abstract. In the paper, our main aim is to generalize the mixed projection body $\mathbf{\Pi}(K_1,\ldots,K_{n-1})$ of $(n-1)$ convex bodies $K_1,\ldots,K_{n-1}$ to the Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric operation call it *Orlicz mixed projection body* $\mathbf{\Pi}_\varphi(K_1,\ldots,K_n)$ of n convex bodies $K_1,\ldots,K_n$. The new affine geometric quantity in special case yields the classical mixed projection body $\mathbf{\Pi}(K_1,\ldots,K_{n-1})$ and Orlicz projection body $\mathbf{\Pi}_\varphi K$ of convex body K, respectively. The related concept of L_p-mixed projection body of n convex bodies $\mathbf{\Pi}_p(K_1,\ldots,K_n)$ is also derived. An Orlicz Alesandrov-Fenchel inequality for the Orlicz mixed projection body is established, which in special case yields a new L_p-projection Alesandrov-Fenchel inequality. As an application, we establish a polar Orlicz Alesandrov-Fenchel inequality for the polar of Orlicz mixed projection body.

1. Introduction

If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then (see e.g. [6])

$$h(K,x)=\max\{x\cdot y: y\in K\},$$

for $x\in\mathbb{R}^n$, defines the support function $h(K,x)$ of K, where $x\cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function.

Associated with convex bodies (compact convex subsets with nonempty interiors) $K_1,\ldots,K_n$ is a Borel measure, $S(K_1,\ldots,K_{n-1};\cdot)$, on S^{n-1}, called the mixed surface area measure of $K_1,\ldots,K_{n-1}$, which has the property that for each compact convex subset K_n (see e.g [23]),

$$V(K_1,\ldots,K_n)=\frac{1}{n}\int_{S^{n-1}}h(K_n,u)dS(K_1,\ldots,K_{n-1};u). \tag{1.1}$$

In fact, the measure $S(K_1,\ldots,K_{n-1};\cdot)$, can be defined by the property that (1.1) holds for all K_n, and $V(K_1,\ldots,K_n)$ denotes the mixed volume of convex bodies $K_1,\ldots,K_n$. An important generalization of the Minkowski inequality is the Aleksandrov-Fenchel inequality:

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The Alesandrov-Fenchel inequality for mixed volumes. *If* $K_1,\ldots,K_n$ *are convex bodies and* $1\le r<n$, *then* (*see e.g.* [14])

$$V(K_1,\ldots,K_n)\ge\prod_{j=1}^{r}V(K_j,\ldots,K_j,K_{r+1},\ldots,K_n)^{1/r}. \tag{1.2}$$

Unfortunately, the equality conditions of this inequality are, in general, unknown.

If $K_1,\ldots,K_r$ are compact convex subsets and $\lambda_1,\ldots,\lambda_r\ge 0$, then the projection body of the Minkowski linear combination

$$\lambda_1K_1+\cdots+\lambda_rK_r,$$

(compact convex subset) can be written as a symmetric homogeneous polynomial of degree $(n-1)$ in the λ_i (see [14]).

$$\mathbf{\Pi}(\lambda_1K_1+\cdots+\lambda_rK_r)=\sum\lambda_{i_1}\cdots\lambda_{i_{n-1}}\mathbf{\Pi}_{i_1\cdots i_{n-1}}, \tag{1.3}$$

where the sum is a Minkowski sum of positive integers not exceeding r. The convex body $\mathbf{\Pi}_{i_1\cdots i_{n-1}}$ is uniquely determined by (1.3). It is called the *mixed projection body* of $K_{i_1},\ldots,K_{i_{n-1}}$ and is written as $\mathbf{\Pi}(K_1,\ldots,K_{n-1})$, and (see [15])

$$h(\mathbf{\Pi}(K_1,\ldots,K_{n-1}),x)=\frac{1}{2}\int_{S^{n-1}}|x\cdot u|dS(K_1,\ldots,K_{n-1};u), \tag{1.4}$$

for $x\in S^{n-1}$. One of the fundamental inequalities for the for mixed projection bodies is the following projection Alesandrov-Fenchel inequality:

The Alesandrov-Fenchel inequality for mixed projection bodies. *If* $K_1,\ldots,K_{n-1}$ *are compact convex subsets and* $1\le r<n$, *then* (*see* [14])

$$V(\mathbf{\Pi}(K_1,\ldots,K_{n-1}))\ge\prod_{j=1}^{r}V(\mathbf{\Pi}(K_j,\ldots,K_j,K_{r+1},\ldots,K_{n-1}))^{1/r}. \tag{1.5}$$

The study of projection bodies or zonoids in $\mathbb{R}^n$ had a long and complicated history. A extensive article that detail this is by Bolker [4]. After the appearance of Bolker's article, projection bodies have received considerable attention. Many recent excellent results have been discovered by Goodey and Weil [9] , Martini [21] and Schneider and Weil [24]. The definition and elementary properties of mixed projection bodies can be found in [5]. The support functions and brightness functions of mixed projection bodies were studied by Chakerian [7]. In 1988, a fascinating paper of Alexander [2] demonstrates a close relationship between the study of projection bodies and work on Hilbert's fourth Problem. Also, Lutwak had studied in systematize the the mixed projection bodies and their polars and obtained a number of elegant results [14], [15], [16], [17], [18] and [19]. Recent research on this subject can be found in the literature [1], [3], [11], [12], [13], [22], [25], [26], [27], [28], [29], [30], [31], [32] and [33].

In the paper, we consider convex function $\varphi:\mathbb{R}\to[0,\infty)$ with $\varphi(0)=0$. This means that φ must be decreasing on $(-\infty,0]$ and increasing on $[0.\infty)$. We will assume throughout that one of these is happening strictly so; namely, φ is either strictly decreasing on $(-\infty,0]$ or strictly increasing on $[0.\infty)$. Let Φ be the class of convex and strictly increasing functions $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(0)=0$, and C be the class of convex and strictly decreasing functions $\varphi:[-\infty,0)\to[0,\infty)$ such that $\varphi(0)=0$.

The *Orlicz mixed projection body* $\mathbf{\Pi}_\varphi(K_1,\ldots,K_n)$ of n convex bodies $K_1,\ldots,K_n$ is defined as the body whose support function (see Sec. 3 for the definition) is given by

$$h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),x)=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right)dV(K_1,\ldots,K_n;u)\le 1\right\}, \tag{1.6}$$

for $x\in S^{n-1}$, where $dV(K_1,\ldots,K_n;u)$ denotes mixed volume probability measure of convex bodies $K_1,\ldots,K_n$, and (see [34])

$$dV(K_1,\ldots,K_n;u)=\frac{1}{nV(K_1,\ldots,K_n)}h(K_n,u)dS(K_1,\ldots,K_{n-1};u). \tag{1.7}$$

For convex body K, and $u \in S^{n-1}$, let K^u denote the image of the orthogonal projection of K onto ξ_u, the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that is orthogonal to u. If $K_1, \ldots, K_{n-1}$ are convex bodies, then write $v(K_1^u, \ldots, K_{n-1}^u)$ for the mixed volume of the figures $K_1^u, \ldots, K_{n-1}^u$ in the space ξ_u. With $\varphi = \varphi_1(t) = |t|$, it turns out that for $u \in S^{n-1}$

$$h(\mathbf{\Pi}_{\varphi_1}(K_1, \ldots, K_n), u) = \frac{c_n}{V(K_1, \ldots, K_n)} v(K_1^u, \ldots, K_{n-1}^u).$$

where c_n denotes a constant depending only n. Further,

$$\mathbf{\Pi}_{\varphi_1}(K_1, \ldots, K_n) = \frac{c_n}{V(K_1, \ldots, K_n)} \mathbf{\Pi}(K_1, \ldots, K_{n-1}). \tag{1.8}$$

This shows the classical mixed projection body $\mathbf{\Pi}(K_1, \ldots, K_{n-1})$ is a special case of the Orlicz mixed projection body $\mathbf{\Pi}_\varphi(K_1, \ldots, K_n)$.

When $K_1 = \cdots = K_n = K$, it turns out that

$$\mathbf{\Pi}_\varphi(K, \ldots, K) = \mathbf{\Pi}_\varphi K, \tag{1.9}$$

where $\mathbf{\Pi}_\varphi K$ is the Orlicz projection body given by Lutwak, Yang and Zhang [20] as follows

$$h(\mathbf{\Pi}_\varphi K, x) = \inf\left\{\lambda > 0 : \int_{\partial K} \varphi\left(\frac{x \cdot v(y)}{\lambda y \cdot v(y)}\right) y \cdot v(y) d\mathcal{H}^{n-1}(y) \leq nV(K)\right\}, \tag{1.10}$$

for $x \in S^{n-1}$, where $v(y)$ is the outer unit normal of ∂K at $y \in \partial K$, $x \cdot v(y)$ denotes the inner product of x and $v(y)$, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

When $\varphi = \varphi_p(t) = |t|^p$, and $p \geq 1$,

$$\mathbf{\Pi}_{\varphi_p}(K_1, \ldots, K_n) = \frac{c_{n,p}}{V(K_1, \ldots, K_n)^{1/p}} \mathbf{\Pi}_p(K_1, \ldots, K_n), \tag{1.11}$$

where $c_{n,p}$ denotes a constant depending only n and p, and $\mathbf{\Pi}_{\varphi_p}(K_1, \ldots, K_n)$ is a new mixed projection body, and call it *L_p-mixed projection body* of convex bodies $K_1, \ldots, K_n$, defined as the convex body whose support function is given by

$$h(\mathbf{\Pi}_{\varphi_p}(K_1, \ldots, K_n), x) = \left(\int_{S^{n-1}} |x \cdot u|^p h(K_n, u)^{1-p} dS(K_1, \ldots, K_{n-1}; u)\right)^{1/p}, \tag{1.12}$$

for $x \in S^{n-1}$. Putting $K_1 = \cdots = K_n = K$ in (1.12), the L_p-mixed projection body $\mathbf{\Pi}_{\varphi_p}(K_1, \ldots, K_n)$ becomes the well-known L_p-projection body $\mathbf{\Pi}_p K$ of K, and (see [18])

$$h(\mathbf{\Pi}_{\varphi_p} K, x) = \left(\int_{S^{n-1}} |x \cdot u|^p h(K, u)^{1-p} dS(K; u)\right)^{1/p}.$$

Namely (see [20])

$$h(\mathbf{\Pi}_{\varphi_p} K, x) = \left(\int_{\partial K} |x \cdot v(y)|^p |y \cdot v(y)|^{1-p} d\mathcal{H}^{n-1}(y)\right)^{1/p}. \tag{1.13}$$

In this Section 4, we establish the following Alesandrov-Fenchel inequality for the Orlicz mixed projection bodies $\mathbf{\Pi}_\varphi(K_1, \ldots, K_n)$.

The Orlicz projection Alesandrov-Fenchel inequality. *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $1 \leq r < n$, $\varphi \in \Phi \cup \mathcal{C}$, then*

$$V(\mathbf{\Pi}_\varphi(K_1, \ldots, K_n)) \geq \left(\frac{2}{nc_\varphi V(K_1, \ldots, K_n)}\right)^n \prod_{j=1}^{r} V(\mathbf{\Pi}(K_j, \ldots, K_j, K_{r+1}, \ldots, K_{n-1}))^{1/r}, \tag{1.14}$$

where c_φ is in as (2.3).

Putting $\varphi(t) = |t|^p$ and $p \geq 1$ in (1.14), a new L_p projection Alesandrov-Fenchel inequality is also derived (see Sec. 4). Obviously, the classical projection Alesandrov-Fenchel inequality (1.5) is also a special case of (1.14).

As an application, we establish the following polar Orlicz Alesandrov-Fenchel inequality for polar of Orlicz mixed projection body.

The Orlicz polar projection Alesandrov-Fenchel inequality. *If $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $1 \leq r < n$ and $\varphi \in \Phi \cup C$, then*

$$V(\mathbf{\Pi}^*_\varphi(K_1, \ldots, K_n)) \leq \left(\frac{2V(K_1, \ldots, K_n)}{nc_\varphi}\right)^n \cdot \prod_{j=1}^{r} V(\mathbf{\Pi}^*(K_j, \ldots, K_j, K_{r+1}, \ldots, K_{n-1}))^{1/r}, \tag{1.15}$$

where $\mathbf{\Pi}^*(K_1, \ldots, K_{n-1})$ denotes the polar of mixed projection body $\mathbf{\Pi}(K_1, \ldots, K_{n-1})$ and $\mathbf{\Pi}^*_\varphi(K_1, \ldots, K_n)$ the polar of Orlicz mixed body $\mathbf{\Pi}_\varphi(K_1, \ldots, K_n)$ (see Section 2).

2. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. We write $\mathcal{K}^n$ for the set of convex bodies (compact convex subsets with nonempty interiors) of $\mathbb{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Support function is homogeneous of degree 1, that is,

$$h(K, rx) = rh(K, x), \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$.

2.1 Basics regarding convex bodies

For $\phi \in \mathrm{GL}(n)$ write ϕ^t for the transpose of ϕ and ϕ^{-t} for the inverse of the transpose of ϕ. Write $|\phi|$ for the absolute value of the determinant of ϕ. Observe that from the definition of the support function it follows immediately that for $\phi \in \mathrm{GL}(n)$ the support function of the image $\phi K = \{\phi y : y \in K\}$ is given by

$$h(\phi K, x) = h(K, \phi^t x), \tag{2.2}$$

Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$. Define c_φ by

$$c_\varphi = \min\{c > 0 : \max\{\varphi(c), \varphi(-c)\} \leq 1\} \tag{2.3}$$

We say that the sequence $\{\varphi_i\}$, where the $\varphi_i \in \Phi \cup C$, is such that $\varphi_i \to \varphi_0 \in \Phi \cup C$ provided

$$|\varphi_i - \varphi_0|_I := \max_{t \in I} |\varphi_i(t) - \varphi_0(t)| \to 0, \tag{2.4}$$

for every compact interval $I \subset \mathbb{R}$.

The classical Aleksandrov-Fenchel-Jessen surface area measure, $S(K, \cdot)$, of the convex body K can be defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u) dS(K, u) = \int_{\partial K} f(v_K(y)) d\mathcal{H}^{n-1}(y), \tag{2.5}$$

for each continuous $f : S^{n-1} \to \mathbb{R}$. Hence, for $K \in \mathcal{K}_o^n$

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K, u) = \frac{1}{n} \int_{\partial K} y \cdot v_K(y) d\mathcal{H}^{n-1}(y). \tag{2.6}$$

If $K \in \mathcal{K}_o^n$, then the polar body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \text{ for all } y \in K\}.$$

It is easy to verify that

$$(K^*)^* = K.$$

Let $\rho(K, \cdot) = \mathbb{R}^n \backslash \{0\} \to [0, \infty)$ denote radial function of $K \in \mathcal{K}_o^n$, i.e.

$$\rho(K, x) = \max\{\lambda > 0 : \lambda x \in K\}.$$

It is easily verified that

$$h(K^*, x) = 1/\rho(K, x) \text{ and } \rho(K^*, x) = 1/h(K, x). \tag{2.7}$$

2.2 Mixed volumes

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [14])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.8}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.8), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_1, \ldots, K_n)$. Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}; \cdot)$ on S^{n-1}, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$ (see e.g. [8], p.353),

$$V(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}; u). \tag{2.9}$$

In fact, the measure $S(K_1, \ldots, K_{n-1}; \cdot)$ can be defined by the propter that (2.9) holds for all $K \in \mathcal{K}^n$. Let $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \ldots, K_{n-1}; \cdot)$ is written as $S_i(K, L; \cdot)$. When $L = B$, $S_i(K, L; \cdot)$ is written as $S_i(K, \cdot)$ and called as ith mixed surface area measure. A fundamental inequality for mixed volume $V(K_1, \ldots, K_n)$ is the following Alesandrov-Fenchel inequality: If $K_1, \ldots, K_{n-1}$ are convex bodies and $1 \le r < n$, then

$$V(K_1, \ldots, K_n) \ge \prod_{j=1}^{r} V(K_j, \ldots, K_j, K_{r+1}, \ldots, K_n)^{1/r}. \tag{2.10}$$

Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V_i(K, L)$. When $i = 1$, $V_i(K, L)$ becomes the classical mixed volume $V_1(K, L)$ of K and L, and

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K, u). \tag{2.11}$$

A fundamental inequality for mixed volume $V_1(K, L)$ is the following Minkowski inequality: For $K, L \in \mathcal{K}^n$,

$$V_1(K, L)^n \ge V(K)^{n-1} V(L), \tag{2.12}$$

with equality if and only if K and L are homothetic.

2.3 Mixed projection body

The projection body $\mathbf{\Pi} K$ of a convex body $K \in \mathcal{K}^n$, is defined as the convex body whose support function is given by (see [14])

$$h(\mathbf{\Pi} K, u) = v(K^u), \quad u \in S^{n-1}, \tag{2.13}$$

where $v(K^u)$ denote the $(n-1)$-dimensional volume of $K|\xi_u$.

If $K_1,\ldots,K_{n-1}\in\mathcal{K}^n$, then the mixed projection body of $K_1,\ldots,K_{n-1}$ is denoted by $\mathbf{\Pi}(K_1,\ldots,K_{n-1})$, and whose support function is given, for $u\in S^{n-1}$, by (see [14])

$$h(\mathbf{\Pi}(K_1,\ldots,K_{n-1}),u)=v(K_1^u,\ldots,K_{n-1}^u). \tag{2.14}$$

Thus

$$h(\mathbf{\Pi}(K_1,\ldots,K_{n-1}),u)=nV(K_1,\ldots,K_{n-1},\bar{u}). \tag{2.15}$$

where $\bar{u}$ denotes the closed line segment connecting $-u$ and u.

If $K_1,\ldots,K_{n-1}\in\mathcal{K}^n$ and $\phi\in\mathrm{SL}(n)$, then

$$\mathbf{\Pi}(\phi K_1,\ldots,\phi K_{n-1})=\phi^{-t}\mathbf{\Pi}(K_1,\ldots,K_{n-1}). \tag{2.16}$$

The mixed projection operator is monotone nondecreasing with respect to set inclusion; i.e., if K_i,L_i $(i=1,2,\ldots,n-1)\in\mathcal{K}^n$ and $K_i\subset L_i$, then

$$\mathbf{\Pi}(K_1,\ldots,K_{n-1})\subset\mathbf{\Pi}(K_1,\ldots,K_{n-1}). \tag{2.17}$$

An important fact is the following:

$$h(\mathbf{\Pi}(K_1,\ldots,K_{n-1}),u)=\frac{1}{n}\int_{S^{n-1}}|u\cdot v|dS(K_1,\ldots,K_{n-1};v). \tag{2.18}$$

For the polar of mixed projection body $\mathbf{\Pi}(K_1,\ldots,K_{n-1})$ we will simply write $\mathbf{\Pi}^*(K_1,\ldots,K_{n-1})$ not $(\mathbf{\Pi}(K_1,\ldots,K_{n-1}))^*$. An important inequality on the polar of mixed projection body is the following polar Aleksandrov-Fenchel inequality for the polar of mixed projection body.

The polar Alesandrov-Fenchel inequality. *If $K_1,\ldots,K_{n-1}\in\mathcal{K}_o^n$ and $1\le r<n$, then*

$$V(\mathbf{\Pi}^*(K_1,\ldots,K_{n-1}))\le\prod_{j=1}^{r}V(\mathbf{\Pi}^*(K_j,\ldots,K_j,K_{r+1},\ldots,K_{n-1}))^{1/r}, \tag{2.19}$$

with equality if $K_1,\ldots,K_{n-1}$ are homothetic (*see* [11]).

3. Orlicz mixed projection body

We first give the definition of Orlicz mixed projection body of $(n+1)$ convex bodies as follows.

Definition 3.1 Let $K_1,\ldots,K_n\in\mathcal{K}^n$ and $\varphi\in\Phi\cup\mathcal{C}$, the Orlicz mixed projection body of $K_1,\ldots,K_n$, denoted by $\mathbf{\Pi}_\varphi(K_1,\ldots,K_n)$, defined by

$$h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),x):=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right)dV(K_1,\ldots,K_n;u)\le1\right\}. \tag{3.1}$$

Since $\varphi\in\Phi\cup\mathcal{C}$, it follows that the function:

$$\lambda\to\int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right)dV(K_1,\ldots,K_n;u)$$

is also strictly decreasing in $(0,\infty)$. This yields that

Lemma 3.2 *If $K_1,\ldots,K_n\in\mathcal{K}_o^n$, $\varphi\in\Phi\cup\mathcal{C}$ and $x_o\in\mathbb{R}^n/\{0\}$, then*

$$\int_{S^{n-1}}\varphi\left(\frac{x_o\cdot u}{\lambda_o h(K_n,u)}\right)dV(K_1,\ldots,K_n;u)=1$$

if and only if

$$h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x_o)=\lambda_o.$$

In the following, we show that the Orlicz mixed projection body $\mathbf{\Pi}_\varphi(K_1,\dots,K_n)$ is indeed a convex body containing the origin in its interior.

Lemma 3.3 *If $K_1,\dots,K_n\in\mathcal{K}^n_o$, and $\varphi\in\Phi\cup C$, then the function $h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x)$ is homogeneous of degree one, sub-additive and positive.*

Proof First, for any $\gamma>0$, and noticing that $dV(K_1,\dots,K_n;u)$ is a probability measure on S^{n-1}, we obtain

$$\begin{aligned} h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),\gamma x) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{\gamma(x\cdot u)}{\lambda h(K_n,u)}\right)dV(K_1,\dots,K_n;u)\le 1\right\}\\ &= \gamma\inf\left\{\mu>0:\int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\mu h(K_n,u)}\right)dV(K_1,\dots,K_n;u)\le 1\right\}\\ &= \gamma h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x),\end{aligned}$$

where $\mu=\dfrac{\lambda}{\gamma}$.

Next, we prove that $h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x)$ is sub-additive.

Let $h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x_1)=\lambda_1$ and $h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x_2)=\lambda_2$, then

$$\int_{S^{n-1}}\varphi\left(\frac{x_1\cdot u}{\lambda_1 h(K_n,u)}\right)dV(K_1,\dots,K_n;u)=1, \tag{3.2}$$

and

$$\int_{S^{n-1}}\varphi\left(\frac{x_2\cdot u}{\lambda_2 h(K_n,u)}\right)dV(K_1,\dots,K_n;u)=1. \tag{3.3}$$

Combining the convexity of the function $s\to\varphi(s/h(K_n,u))$, we obtain

$$\begin{aligned} 1 &= \frac{\lambda_1}{\lambda_1+\lambda_2}\int_{S^{n-1}}\varphi\left(\frac{x_1\cdot u}{\lambda_1 h(K_n,u)}\right)dV(K_1,\dots,K_n;u)+\frac{\lambda_2}{\lambda_1+\lambda_2}\int_{S^{n-1}}\varphi\left(\frac{x_2\cdot u}{\lambda_2 h(K_n,u)}\right)dV(K_1,\dots,K_n;u)\\ &\ge \int_{S^{n-1}}\varphi\left(\frac{x_1\cdot u+x_2\cdot u}{(\lambda_1+\lambda_2)h(K_n,u)}\right)dV(K_1,\dots,K_n;u)\\ &= \int_{S^{n-1}}\varphi\left(\frac{(x_1+x_2)\cdot u}{(\lambda_1+\lambda_2)h(K_n,u)}\right)dV(K_1,\dots,K_n;u)\end{aligned}$$

Hence

$$\begin{aligned} h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x_1+x_2) &\le \lambda_1+\lambda_2\\ &= h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x_1)+h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x_2).\end{aligned}$$

Moreover, for $x\ne 0$, obviously $h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x)>0$. □

This shows also that $h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x)$ is a support function of a convex body $\mathbf{\Pi}_\varphi(K_1,\dots,K_n)$ that contains the origin in its interior.

In the following, we prove that the Orlicz projection operator $\mathbf{\Pi}_\varphi(K_1,\dots,K_n):\underbrace{\mathcal{K}^n\times\cdots\times\mathcal{K}^n}_{n-1}\to\mathcal{K}^n$ is continuous.

Lemma 3.4 *If $K_1,\dots,K_n\in\mathcal{K}^n_o$, and $\varphi\in\Phi\cup C$, then the Orlicz mixed projection operator $\mathbf{\Pi}_\varphi(K_1,\dots,K_n):\underbrace{\mathcal{K}^n\times\cdots\times\mathcal{K}^n}_{n}\to\mathcal{K}^n$ is continuous.*

Proof To see this, indeed, let $K_{ij}\in\mathcal{S}^n$, $i\in\mathbb{N}\cup\{0\}$, $j=1,\dots,n$, be such that $K_{ij}\to K_{0j}$ as $i\to\infty$. Noting that

$$h(\mathbf{\Pi}_\varphi(K_{i1},\dots,K_{in}),x)$$

$$= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_{in},u)}\right) dV(K_{i1},\ldots,K_{i(n-1)};u) \le 1\right\}$$

$$= \inf\left\{\lambda > 0 : \frac{1}{nV(K_{i1},\ldots,K_{in})}\int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_{in},u)}\right) h(K_{in},u)dS(K_{i1},\ldots,K_{i(n-1)};u) \le 1\right\}$$

Since the mixed area measures is weakly continuous, i.e.

$$dS(K_{i1},\ldots,K_{i(n-1)};u) \to dS(K_{01},\ldots,K_{0(n-1)};u) \text{ weakly on } S^{n-1}.$$

Since $h(K_{in},u) \to h(K_{0n},u)$, uniform on S^{n-1}, and φ is continuous, implies that

$$\varphi\left(\frac{x\cdot u}{\lambda h(K_{in},u)}\right) \to \varphi\left(\frac{x\cdot u}{\lambda h(K_{0n},u)}\right).$$

Further

$$\int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_{in},u)}\right) dV(K_{i1},\ldots,K_{i(n-1)};u) \to \int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_{0n},u)}\right) dV(K_{01},\ldots,K_{0(n-1)};u).$$

Hence

$$\begin{aligned}\lim_{i\to\infty} h(\mathbf{\Pi}_\varphi(K_{i1},\ldots,K_{in}),x) &= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_{0n},u)}\right) dV(K_{01},\ldots,K_{0(n-1)};u) \le 1\right\}\\ &= h(\mathbf{\Pi}_\varphi(K_{01},\ldots,K_{0n}),x).\end{aligned}$$

This shows that the Orlicz projection operator $\mathbf{\Pi}_\varphi(K_1,\ldots,K_n)$ is continuous. □

Lemma 3.5 *If $K_1,\ldots,K_n \in \mathcal{K}^n_o$, and $\varphi_i \in \Phi \cup C$, then*

$$\varphi_i \to \varphi \Rightarrow h(\mathbf{\Pi}_{\varphi_i}(K_1,\ldots,K_n),x) \to h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),x). \tag{3.4}$$

Proof Noting that $\varphi_i \to \varphi \in \Phi \cap C$ and $dV(K_1,\ldots,K_n;u)$ is a probability measure on S^{n-1}, implies that

$$\varphi_i\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right) \to \varphi\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right) \in \Phi.$$

Further

$$\int_{S^{n-1}} \varphi_i\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right) dV(K_1,\ldots,K_n;u) \to \int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right) dV(K_1,\ldots,K_n;u).$$

Hence

$$\begin{aligned}\lim_{i\to\infty} h(\mathbf{\Pi}_{\varphi_i}(K_1,\ldots,K_n),x) &= \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{x\cdot u}{\lambda h(K_n,u)}\right) dV(K_1,\ldots,K_n;u) \le 1\right\}\\ &= h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),x).\end{aligned}$$

□

Lemma 3.6 *If $K_1,\ldots,K_n \in \mathcal{K}^n_o$, and $\varphi \in \Phi \cup C$, then the Orlicz mixed projection operator $\mathbf{\Pi}_\varphi(K_1,\ldots,K_n) : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n} \to \mathcal{K}^n$ is bounded.*

Proof Choosing c such that $h(\mathbf{\Pi}(K_1,\ldots,K_{n-1}),u) \ge c > 0$. Let $x \in S^{n-1}$, and suppose that

$$h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),x) = \lambda_0.$$

From the definition (2.3), we may let $\varphi(c_\varphi)=1$. Hence from the fact that φ in nonnegative and $\varphi(0)=0$, and (2.5), Lemma 3.2 and Jensen's inequality, we obtain

$$\begin{aligned}1 &= \varphi(c_\varphi)\\ &= \int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda_0 h(K_n,u)}\right)dV(K_1,\ldots,K_n;u)\\ &\geq \varphi\left(\int_{S^{n-1}}\frac{x\cdot u}{\lambda_0 h(K_n,u)}dV(K_1,\ldots,K_n;u)\right)\\ &= \varphi\left(\frac{1}{n\lambda_0 V(K_1,\ldots,K_n)}\int_{S^{n-1}}x\cdot u dS(K_1,\ldots,K_{n-1};u)\right)\\ &\geq \varphi\left(\frac{2c}{n\lambda_0 V(K_1,\ldots,K_n)}\right).\end{aligned}$$

Noticing the fact that φ is increasing on $[0,\infty)$, we have

$$\lambda_0 \geq \frac{2c}{nc_\varphi V(K_1,\ldots,K_n)}. \tag{3.5}$$

Next, we give the upper estimate. From (2.3), together with the fact that the function $t\to\max\{\varphi(t),\varphi(-t)\}$ is increasing on $[0,\infty)$ and noticing that $dV(K_1,\ldots,K_n;u)$ is a probability measure on S^{n-1}, it yields that

$$\begin{aligned}1 &= \max\{\varphi(c_\varphi),\varphi(-c_\varphi)\}\\ &= \int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda_0 h(K_n,u)}\right)dV(K_1,\ldots,K_n;u)\\ &\leq \int_{S^{n-1}}\max\left\{\varphi\left(\frac{|x\cdot u|}{\lambda_0 h(K_n,u)}\right),\varphi\left(\frac{-|x\cdot u|}{\lambda_0 h(K_n,u)}\right)\right\}dV(K_1,\ldots,K_n;u)\\ &\leq \int_{S^{n-1}}\max\left\{\varphi\left(\frac{1}{\lambda_0 \min_{u\in S^{n-1}}h(K_n,u)}\right),\varphi\left(\frac{-1}{\lambda_0 \min_{u\in S^{n-1}}h(K_n,u)}\right)\right\}dV(K_1,\ldots,K_n;u).\end{aligned}$$

Hence

$$\lambda_0 \leq \frac{1}{\varphi(c_\varphi)\min_{u\in S^{n-1}}h(K_n,u)}. \tag{3.6}$$

This completes the proof. □

We easy find that the volume of Orlicz mixed projection body $\mathbf{\Pi}_\varphi(K_1,\ldots,K_n)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 3.7 *If $K_1,\ldots,K_n\in\mathcal{K}_o^n$, $\phi\in\mathrm{SL}(n)$, then*

$$V(\mathbf{\Pi}_\varphi(\phi K_1,\ldots,\phi K_n)) = V(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n)). \tag{3.7}$$

Proof For $x\in\mathbb{R}^n$, let $\bar{x}$ denote the closed line segment connecting $-x$ and x. From (2.2) and (3.1), we have, for $\phi\in\mathrm{SL}(n)$,

$$\begin{aligned}h(\mathbf{\Pi}_\varphi(\phi K_1,\ldots,\phi K_n),x) &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda h(\phi K_n,u)}\right)dV(\phi K_1,\ldots,\phi K_{n-1};u)\leq 1\right\}\\ &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(\bar{x},u)}{\lambda h(K_n,\phi^t u)}\right)dV(K_1,\ldots,K_{n-1};\phi^t u)\leq 1\right\}\\ &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{h(\phi^{-1}\bar{x},\phi^t u)}{\lambda h(K_n,\phi^t u)}\right)dV(K_1,\ldots,K_{n-1};\phi^t u)\leq 1\right\}\\ &= \inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\left(\frac{\phi^{-1}x\cdot\phi^t u}{\lambda h(K_n,\phi^t u)}\right)dV(K_1,\ldots,K_{n-1};\phi^t u)\leq 1\right\}\\ &= h(\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),\phi^{-1}x)\\ &= h(\phi^{-t}\mathbf{\Pi}_\varphi(K_1,\ldots,K_n),x).\end{aligned}$$

Hence

$$\mathbf{\Pi}_\varphi(\phi K_1,\dots,\phi K_n)=\phi^{-t}\mathbf{\Pi}_\varphi(K_1,\dots,K_n).$$

Since $|\det(\phi^{-1})|=1$, it follows

$$V(\mathbf{\Pi}_\varphi(\phi K_1,\dots,\phi K_n))=V(\mathbf{\Pi}_\varphi(K_1,\dots,K_n)).$$

This completes the proof. □

4. The Orlicz projection Alesandrov-Fenchel inequality

Lemma 4.1 (Jensen's inequality) *Let μ be a probability measure on a space X and $g: X\to I\subset\mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\psi: I\to\mathbb{R}$ is a convex function, then*

$$\int_X\psi(g(x))d\mu(x)\geq\psi\left(\int_X g(x)d\mu(x)\right).$$

If ψ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x\in X$ (see [10, p.165]).

Next, we establish the following Orlicz Alesandrov-Fenchel inequality for the Orlicz mixed projection body $\mathbf{\Pi}_\varphi(K_1,\dots,K_n)$.

Theorem 4.2 (Orlicz projection Alesandrov-Fenchel inequality) *If $K_1,\dots,K_n\in\mathcal{K}^n_o$, $1\leq r<n$, $\varphi\in\Phi\cup C$, then*

$$V(\mathbf{\Pi}_\varphi(K_1,\dots,K_n))\geq\left(\frac{2}{nc_\varphi V(K_1,\dots,K_n)}\right)^n\prod_{j=1}^{r}V(\mathbf{\Pi}(K_j,\dots,K_j,K_{r+1},\dots,K_{n-1}))^{1/r}.\tag{4.1}$$

Proof For $\varphi\in\Phi$, there must be a real number $0<c_\varphi<\infty$ such that $\varphi(c_\varphi)=1$, and let

$$h(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),x)=\lambda_0.\tag{4.2}$$

From (1.7), Lemma 3.2 and Lemma 4.1, we obtain

$$\begin{aligned}1&=\varphi(c_\varphi)\\&=\int_{S^{n-1}}\varphi\left(\frac{x\cdot u}{\lambda_0 h(K_n,u)}\right)dV(K_1,\dots,K_n;u)\\&\geq\varphi\left(\int_{S^{n-1}}\frac{x\cdot u}{\lambda_0 h(K_n,u)}dV(K_1,\dots,K_n;u)\right)\\&=\varphi\left(\frac{1}{n\lambda_0 V(K_1,\dots,K_n)}\int_{S^{n-1}}x\cdot u\,dS(K_1,\dots,K_{n-1};u)\right).\end{aligned}$$

From (1.4) and in view of the monotonicity of the function φ, we obtain

$$\lambda_0\geq\frac{2}{nc_\varphi}\cdot\frac{1}{V(K_1,\dots,K_n)}\cdot h(\mathbf{\Pi}(K_1,\dots,K_{n-1}),x).\tag{4.3}$$

From (2.11), (2.12) and (4.3), we have

$$\begin{aligned}V(\mathbf{\Pi}_\varphi(K_1,\dots,K_n))&\geq\frac{2}{nc_\varphi}\cdot\frac{1}{V(K_1,\dots,K_n)}V_1(\mathbf{\Pi}_\varphi(K_1,\dots,K_n),\mathbf{\Pi}(K_1,\dots,K_{n-1}))\\&\geq\frac{2}{nc_\varphi}\cdot\frac{1}{V(K_1,\dots,K_n)}V(\mathbf{\Pi}_\varphi(K_1,\dots,K_n))^{(n-1)/n}V(\mathbf{\Pi}(K_1,\dots,K_{n-1}))^{1/n}.\end{aligned}$$

That is

$$V(\mathbf{\Pi}_{\varphi}(K_1,\ldots,K_n)) \geq \left(\frac{2}{nc_{\varphi}}\right)^n \cdot \frac{1}{V(K_1,\ldots,K_n)^n} V(\mathbf{\Pi}(K_1,\ldots,K_{n-1}))$$
$$\geq \left(\frac{2}{nc_{\varphi}}\right)^n \cdot \frac{1}{V(K_1,\ldots,K_n)^n} \prod_{j=1}^{r} V(\mathbf{\Pi}(K_j,\ldots,K_j,K_{r+1},\ldots,K_{n-1}))^{1/r}.$$

On the other hand, the case where $\varphi \in \mathcal{C}$ (i.e. $\varphi(-c_{\varphi}) = 1$) is handled the same way and gives the same result.

This completes the proof. □

What's interesting is that a new L_p projection Alesandrov-Fenchel inequality is derived as follows:

Corollary 4.3 (L_p projection Alesandrov-Fenchel inequality) *If $K_1,\ldots,K_n \in \mathcal{K}_o^n$, $1 \leq r < n$, $p \geq 1$, then*

$$\frac{V(\mathbf{\Pi}_{\varphi_p}(K_1,\ldots,K_n))}{V(K_1,\ldots,K_n)^{n(1-p)/p}} \geq \left(\frac{2}{nc_{n,p}}\right)^n \prod_{j=1}^{r} V(\mathbf{\Pi}(K_j,\ldots,K_j,K_{r+1},\ldots,K_{n-1}))^{1/r}. \tag{4.4}$$

Proof This yields immediately from (1.11) and Theorem 4.2 with $\varphi(t) = |t|^p$ and $p \geq 1$. □

Putting $K_1 = \cdots = K_n = K$ in (4.1), it yields the following an interesting ratio.

If $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cup \mathcal{C}$, then

$$\frac{V(\mathbf{\Pi}_{\varphi}K)}{V(\mathbf{\Pi}K)} \geq \left(\frac{2}{nc_{\varphi}V(K)}\right)^n.$$

From above the ratio and the Petty conjecture inequality ([23, p.415, (7.4.2)]), we get that

$$V(\mathbf{\Pi}_{\varphi}K)V(K) \geq \omega_n^2 \left(\frac{2\omega_{n-1}}{nc_{\varphi}\omega_n}\right)^n.$$

As an application, we establish also the following polar Orlicz Alesandrov-Fenchel inequality for the polar of Orlicz mixed projection body.

Theorem 4.4 (Orlicz polar projection Alesandrov-Fenchel inequality) *If $K_1,\ldots,K_n \in \mathcal{K}_o^n$, $1 \leq r < n$ and $\varphi \in \Phi \cup \mathcal{C}$, then*

$$V(\mathbf{\Pi}^*_{\varphi}(K_1,\ldots,K_n)) \leq \left(\frac{2V(K_1,\ldots,K_n)}{nc_{\varphi}}\right)^n \cdot \prod_{j=1}^{r} V(\mathbf{\Pi}^*(K_j,\ldots,K_j,K_{r+1},\ldots,K_{n-1}))^{1/r}. \tag{4.5}$$

Proof From (2.7) and (4.3), for $x \in S^{n-1}$, we have

$$\rho(\mathbf{\Pi}^*_{\varphi}(K_1,\ldots,K_n),x) \leq \frac{2V(K_1,\ldots,K_n)}{nc_{\varphi}} \cdot \rho(\mathbf{\Pi}^*(K_1,\ldots,K_{n-1}),x). \tag{4.6}$$

Hence

$$V(\mathbf{\Pi}^*_{\varphi}(K_1,\ldots,K_n)) \leq \left(\frac{2V(K_1,\ldots,K_n)}{nc_{\varphi}}\right)^n \cdot V(\mathbf{\Pi}^*(K_1,\ldots,K_{n-1})). \tag{4.7}$$

From (2.19) and (4.7), we obtain

$$V(\mathbf{\Pi}^*_{\varphi}(K_1,\ldots,K_n)) \leq \left(\frac{2V(K_1,\ldots,K_n)}{nc_{\varphi}}\right)^n \cdot \prod_{j=1}^{r} V(\mathbf{\Pi}^*(K_j,\ldots,K_j,K_{r+1},\ldots,K_{n-1}))^{1/r}.$$

This completes the proof. □

Corollary 4.5 (L_p polar projection Alesandrov-Fenchel inequality) *If* $K_1, \ldots, K_n \in \mathcal{K}_o^n$, $1 \leq r < n$, $p \geq 1$, *then*

$$\frac{V(\mathbf{\Pi}^*_{\varphi_p}(K_1, \ldots, K_n))}{V(K_1, \ldots, K_n)^{n(p-1)/p}} \leq \left(\frac{2c_{n,p}}{n}\right)^n \prod_{j=1}^{r} V(\mathbf{\Pi}^*(K_j, \ldots, K_j, K_{r+1}, \ldots, K_{n-1}))^{1/r}. \tag{4.8}$$

Proof From (1.11), (2,7) and in view of the definition of the polar body, we have

$$\mathbf{\Pi}^*_{\varphi_p}(K_1, \ldots, K_n) = \frac{V(K_1, \ldots, K_n)^{1/p}}{c_{n,p}} \mathbf{\Pi}^*_p(K_1, \ldots, K_n), \tag{4.9}$$

This yields immediately from (4.5) and (4.9) with $\varphi(t) = |t|^p$ and $p \geq 1$. □

Putting $K_1 = \cdots = K_n = K$ in (4.5), it follows the following an interesting ratio.

If $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cup C$, then

$$\frac{V(\mathbf{\Pi}^*_\varphi K)}{V(\mathbf{\Pi}^* K)} \leq \left(\frac{2V(K)}{nc_\varphi}\right)^n.$$

From above the ratio and the well-known Petty conjecture inequality ([23, p.415, (7.4.5)]), we get that

$$\frac{V(\mathbf{\Pi}^*_\varphi K)}{V(K)} \leq \left(\frac{2\omega_n}{nc_\varphi \omega_{n-1}}\right)^n.$$

In fact, the polar projection Alesandrov-Fenchel inequality (2.19) is also a special case of (4.5).

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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