Filomat 37:18 (2023), 5883–5894
https://doi.org/10.2298/FIL2318883S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some error bounds of Maclaurin's formula for convex functions in q-calculus

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Abstract. The main goal of this paper is to establish some error bounds for Maclaurin's formula which is three point quadrature formula using the notions of q-calculus. For this, we first prove a q-integral identity involving fist time q-differentiable functions. Then, by using the new established identity we find the error bounds for maclaurin's formula by using the convexity of fist time q-differentiable functions. It is also shown that the newly established inequalities are extension of some existing inequalities inside the literature.

1. Introduction

The Hermite-Hadamard inequality, named after Charles Hermite and Jacques Hadamard and commonly known as Hadamard's inequality, says that if a function $\Psi : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$ is convex, the following double inequality holds:

$$\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right) \leq \frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}} \int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}} \Psi(\tau)\, d\tau \leq \frac{\Psi\left(\widetilde{\lambda_1}\right)+\Psi\left(\widetilde{\lambda_2}\right)}{2}. \tag{1}$$

If Ψ is a concave mapping, the above inequality holds in the opposite direction. The inequality (1) can be proved using the Jensen inequlity. There has been much research done in the direction of Hermite-Hadamard for different kinds of convexities. For example, in [1, 2], the authors established some inequalities linked with midpoint and trapezoid formulas of numerical integration for convex functions.

There are many inequalities which gives us the error bounds of the quadrature rules, the error bound formula for the three point quadrature is that of Simpson's is given below:

$$\left|\frac{1}{6}\left[\Psi\left(\widetilde{\lambda_1}\right)+4\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+\Psi\left(\widetilde{\lambda_2}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}} \int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}} \Psi(\tau)\, d\tau\right| \leq \frac{\left(\widetilde{\lambda_2}-\widetilde{\lambda_1}\right)^4}{2880} \left\|\Psi^{(4)}\right\|_{\infty}, \tag{2}$$

2020 *Mathematics Subject Classification*. 26D10, 26A51, 26D15
Keywords. Maclaurin's formula, convex functions, q-calculus.
Received: 16 December 2022; Accepted: 05 February 2023
Communicated by Dragan S. Djordjević
This research was funded by National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-54. This work was also partially supported by National Natural Science Foundation of China (No. 11971241).
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where Ψ is four times continuous differentiable function over $\left(\widetilde{\lambda_1},\widetilde{\lambda_2}\right)$ and $\left\|\Psi^{(4)}\right\|_\infty=\sup_{\tau\in\left(\widetilde{\lambda_1},\widetilde{\lambda_2}\right)}\left|\Psi^{(4)}(\tau)\right|$.

There are more three point quadrature formulas like Bullen's, dual Simpson's and Maclaurin's formulas. For the following error bounds of maclaurin's formula, one can consult [3]:

$$\begin{aligned}&\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\,d\tau\right|\\ \le\ &\frac{7\left(\widetilde{\lambda_2}-\widetilde{\lambda_1}\right)^4}{51840}\left\|\Psi^{(4)}\right\|_\infty,\end{aligned}$$

where Ψ is with same properties as given in inequality (2).

On the other hand, quantum calculus is a very important branch of calculus and it has a wide range of applications in the fields of mathematics and physics. Because of the numerous applications of quantum calculus (shortly, q-calculus) without limit calculus, many researchers began working on it and applying its concepts in areas such as differential equations, integral equalitions, mathematical modeling, and integral inequalities.

In [4, 5], Alp et al. and Bermudo et al. used q-derivatives and integrals (defined in Section 2) to prove two different versions of q-Hermite-Hadamard inequalities and some estimates. The q-Hermite-Hadamard inequalities are described as:

Theorem 1.1. *[4, 5] For a convex mapping $\Psi:\left[\widetilde{\lambda_1},\widetilde{\lambda_2}\right]\to\mathbb{R}$, the following inequalities hold:*

$$\Psi\left(\frac{q\widetilde{\lambda_1}+\widetilde{\lambda_2}}{[2]_q}\right)\le\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}_{\widetilde{\lambda_1}}d_q\tau\le\frac{q\Psi\left(\widetilde{\lambda_1}\right)+\Psi\left(\widetilde{\lambda_2}\right)}{[2]_q},\tag{3}$$

$$\Psi\left(\frac{\widetilde{\lambda_1}+q\widetilde{\lambda_2}}{[2]_q}\right)\le\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\le\frac{\Psi\left(\widetilde{\lambda_1}\right)+q\Psi\left(\widetilde{\lambda_2}\right)}{[2]_q}.\tag{4}$$

Remark 1.2. *It is very easy to observe that by adding (3) and (4), we have following q-Hermite-Hadamard inequality (see, [5]):*

$$\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\le\frac{1}{2\left(\widetilde{\lambda_2}-\widetilde{\lambda_1}\right)}\left[\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}_{\widetilde{\lambda_1}}d_q\tau+\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right]\le\frac{\Psi\left(\widetilde{\lambda_1}\right)+\Psi\left(\widetilde{\lambda_2}\right)}{2}.\tag{5}$$

Recently, Ali et al. [6] and Sitthiwirattham et al. [7] used new techniques to prove the following two different and new versions of Hermite-Hadamard type inequalities:

Theorem 1.3. *[6, 7] For a convex mapping $\Psi:\left[\widetilde{\lambda_1},\widetilde{\lambda_2}\right]\to\mathbb{R}$, the following inequalities hold:*

$$\begin{aligned}\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)&\le\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\left[\int_{\widetilde{\lambda_1}}^{\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}}\Psi(\tau)\ {}^{\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}}d_q\tau+\int_{\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}_{\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}}d_q\tau\right]\\&\le\frac{\Psi\left(\widetilde{\lambda_1}\right)+\Psi\left(\widetilde{\lambda_2}\right)}{2},\end{aligned}\tag{6}$$

$$\begin{aligned}\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)&\le\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\left[\int_{\widetilde{\lambda_1}}^{\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}}\Psi(\tau)\ {}_{\widetilde{\lambda_1}}d_q\tau+\int_{\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right]\\&\le\frac{\Psi\left(\widetilde{\lambda_1}\right)+\Psi\left(\widetilde{\lambda_2}\right)}{2}.\end{aligned}\tag{7}$$

Remark 1.4. *By setting the limit as $q \to 1^-$ in (3)-(7), we recapture the traditional Hermite-Hadamard inequality (1).*

There has been much research done in the direction of q-integral inequalities for different kind of convexities. For example, in [8], some new midpoint and trapezoidal type inequalities for q-integrals and q-differentiable convex functions were established. The authors of [9–13] used q-integral and established Simpson's type inequalities for q-differentiable convex and general convex functions. For more recent inequalities in q-calculus, one can consult [14–19].

Motivated by these ongoing studies, we are interested to find the error bounds for Maclaurin's formula in the framework of q-calculus. We can also obtain some error bounds for Maclaurin's formula in classical calculus by setting the limit as $q \to 1^-$ in the newly established results.

2. Preliminaries of q-Calculus and Some Inequalities

We shall recall some basics of quantum calculus in this section. For the sake of brevity, let $q \in (0,1)$ and we use the following notation (see, [20]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}.$$

Definition 2.1. *[19] The left quantum derivative or $q_{\widetilde{\lambda_1}}$-derivative of $\Psi : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$ at $\tau \in \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right]$ is expressed as:*

$$_{\widetilde{\lambda_1}}D_q\Psi(\tau) = \frac{\Psi(\tau) - \Psi\left(q\tau + (1-q)\widetilde{\lambda_1}\right)}{(1-q)\left(\tau - \widetilde{\lambda_1}\right)}, \ \tau \neq \widetilde{\lambda_1}. \tag{8}$$

Definition 2.2. *[5] The right quantum derivative or $q^{\widetilde{\lambda_2}}$-derivative of $\Psi : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$ at $\tau \in \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right]$ is expressed as:*

$$^{\widetilde{\lambda_2}}D_q\Psi(\tau) = \frac{\Psi\left(q\tau + (1-q)\widetilde{\lambda_2}\right) - \Psi(\tau)}{(1-q)\left(\widetilde{\lambda_2} - \tau\right)}, \ \tau \neq \widetilde{\lambda_2}.$$

Definition 2.3. *[19] The left quantum integral or $q_{\widetilde{\lambda_1}}$-integral of $\Psi : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$ at $\tau \in \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right]$ is defined as:*

$$\int_{\widetilde{\lambda_1}}^{\tau} \Psi(t) \ _{\widetilde{\lambda_1}}d_qt = (1-q)\left(\tau - \widetilde{\lambda_1}\right)\sum_{n=0}^{\infty} q^n \Psi\left(q^n\tau + (1-q^n)\widetilde{\lambda_1}\right).$$

Definition 2.4. *[4] The right quantum integral or $q^{\widetilde{\lambda_2}}$-integral of $\Psi : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$ at $\tau \in \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right]$ is defined as:*

$$\int_{\tau}^{\widetilde{\lambda_2}} \Psi(t) \ ^{\widetilde{\lambda_2}}d_qt = (1-q)\left(\widetilde{\lambda_2} - \tau\right)\sum_{n=0}^{\infty} q^n \Psi\left(q^n\tau + (1-q^n)\widetilde{\lambda_2}\right).$$

Lemma 2.5. *[12] For continuous functions $\Psi, g : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$, the following equality is true:*

$$\int_0^c g(t) \ ^{\widetilde{\lambda_2}}D_q\Psi\left(t\widetilde{\lambda_1} + (1-t)\widetilde{\lambda_2}\right)d_qt$$
$$= \frac{1}{\widetilde{\lambda_2} - \widetilde{\lambda_1}} \int_0^c D_q g(t)\Psi\left(qt\widetilde{\lambda_1} + (1-qt)\widetilde{\lambda_2}\right)d_qt - \left.\frac{g(t)\Psi\left(t\widetilde{\lambda_1} + (1-t)\widetilde{\lambda_2}\right)}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}\right|_0^c.$$

Lemma 2.6. *[13] For continuous functions $\Psi, g : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$, the following equality is true:*

$$\begin{aligned} & \int_0^c g(t)\ {}_{\widetilde{\lambda_1}}D_q \Psi\left(t\widetilde{\lambda_2} + (1-t)\widetilde{\lambda_1}\right) d_q t \\ = & \left.\frac{g(t)\Psi\left(t\widetilde{\lambda_2} + (1-t)\widetilde{\lambda_1}\right)}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}\right|_0^c - \frac{1}{\widetilde{\lambda_2} - \widetilde{\lambda_1}} \int_0^c D_q g(t)\Psi\left(qt\widetilde{\lambda_2} + (1-qt)\widetilde{\lambda_1}\right) d_q t. \end{aligned}$$

3. Main Results

In this section, we establish some new Maclaurin's type inequalities for q-differentiable functions in q-calculus.

Lemma 3.1. *Let $\Psi : \left[\widetilde{\lambda_1}, \widetilde{\lambda_2}\right] \to \mathbb{R}$ be a q-differentiable function. If ${}^{\widetilde{\lambda_2}}D_q\Psi$ is q-integrable function, then we have the following equality:*

$$\begin{aligned} & \frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1} + \widetilde{\lambda_2}}{6}\right) + 2\Psi\left(\frac{\widetilde{\lambda_1} + \widetilde{\lambda_2}}{2}\right) + 3\Psi\left(\frac{\widetilde{\lambda_1} + 5\widetilde{\lambda_2}}{6}\right)\right] - \frac{1}{\widetilde{\lambda_2} - \widetilde{\lambda_1}} \int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}} \Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q \tau \quad (9) \\ = & \frac{\left(\widetilde{\lambda_2} - \widetilde{\lambda_1}\right)}{36} \sum_{j=1}^{4} I_j, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^1 (1-qt)\ {}^{\widetilde{\lambda_2}}D_q \Psi\left(\frac{5+t}{6}\widetilde{\lambda_1} + \frac{1-t}{6}\widetilde{\lambda_2}\right) d_q t, \\ I_2 &= \int_0^1 \left(\frac{3}{2} - 4qt\right) {}^{\widetilde{\lambda_2}}D_q \Psi\left(\frac{3+2t}{6}\widetilde{\lambda_1} + \frac{3-2t}{6}\widetilde{\lambda_2}\right) d_q t, \\ I_3 &= \int_0^1 \left(\frac{5}{2} - 4qt\right) {}^{\widetilde{\lambda_2}}D_q \Psi\left(\frac{1+2t}{6}\widetilde{\lambda_1} + \frac{5-2t}{6}\widetilde{\lambda_2}\right) d_q t, \\ I_4 &= \int_0^1 (-qt)\ {}^{\widetilde{\lambda_2}}D_q \Psi\left(\frac{t}{6}\widetilde{\lambda_1} + \frac{6-t}{6}\widetilde{\lambda_2}\right) d_q t. \end{aligned}$$

Proof. From Lemmas 2.5, we have

$$\begin{aligned} I_1 &= \int_0^1 (1-qt)\ {}^{\widetilde{\lambda_2}}D_q \Psi\left(\frac{5+t}{6}\widetilde{\lambda_1} + \frac{1-t}{6}\widetilde{\lambda_2}\right) d_q t \quad (10) \\ &= -\frac{6}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}(1-q)\Psi\left(\widetilde{\lambda_1}\right) + \frac{6}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}\Psi\left(\frac{5\widetilde{\lambda_1} + \widetilde{\lambda_2}}{6}\right) \\ &\quad - \frac{6q}{\widetilde{\lambda_2} - \widetilde{\lambda_1}} \int_0^1 \Psi\left(qt\widetilde{\lambda_1} + (1-qt)\frac{5\widetilde{\lambda_1} + \widetilde{\lambda_2}}{6}\right) d_q t \\ &= \frac{6}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}(q-1)\Psi\left(\widetilde{\lambda_1}\right) + \frac{6}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}\Psi\left(\frac{5\widetilde{\lambda_1} + \widetilde{\lambda_2}}{6}\right) \\ &\quad - \frac{6q}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}\left[(1-q)\sum_{n=0}^{\infty} q^n \Psi\left(q^{n+1}\widetilde{\lambda_1} + \left(1-q^{n+1}\right)\frac{5\widetilde{\lambda_1} + \widetilde{\lambda_2}}{6}\right)\right] \\ &= \frac{6}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}(q-1)\Psi\left(\widetilde{\lambda_1}\right) + \frac{6}{\widetilde{\lambda_2} - \widetilde{\lambda_1}}\Psi\left(\frac{5\widetilde{\lambda_1} + \widetilde{\lambda_2}}{6}\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\left[(1-q)\sum_{n=1}^{\infty}q^n\Psi\left(q^n\widetilde{\lambda}_1+(1-q^n)\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)\right]\\
=\;&\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}(q-1)\Psi\left(\widetilde{\lambda}_1\right)+\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\Psi\left(\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)\\
&-\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\left[(1-q)\sum_{n=0}^{\infty}q^n\Psi\left(q^n\widetilde{\lambda}_1+(1-q^n)\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)-(1-q)\Psi\left(\widetilde{\lambda}_1\right)\right]\\
=\;&\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\Psi\left(\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)-\frac{36}{\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)^2}(1-q)\\
&\times\left(\frac{\widetilde{\lambda}_2-\widetilde{\lambda}_1}{6}\right)\sum_{n=0}^{\infty}q^n\Psi\left(q^n\widetilde{\lambda}_1+(1-q^n)\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)\\
=\;&\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\Psi\left(\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)-\frac{36}{\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)^2}\int_{\widetilde{\lambda}_1}^{\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}}\Psi(\tau)\;{}^{\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}}d_q\tau,
\end{aligned}
$$

$$
\begin{aligned}
I_2 =\;&\int_0^1\left(\frac{3}{2}-4qt\right){}^{\widetilde{\lambda}_2}D_q\Psi\left(\frac{3+2t}{6}\widetilde{\lambda}_1+\frac{3-2t}{6}\widetilde{\lambda}_2\right)d_qt \qquad (11)\\
=\;&\frac{15}{2}\Psi\left(\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)+\frac{9}{2\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)}\Psi\left(\frac{\widetilde{\lambda}_1+\widetilde{\lambda}_2}{2}\right)\\
&-\frac{36}{\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)^2}\int_{\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}}^{\frac{\widetilde{\lambda}_1+\widetilde{\lambda}_2}{2}}\Psi(\tau)\;{}^{\frac{\widetilde{\lambda}_1+\widetilde{\lambda}_2}{2}}d_q\tau,
\end{aligned}
$$

$$
\begin{aligned}
I_3 =\;&\int_0^1\left(\frac{5}{2}-4qt\right){}^{\widetilde{\lambda}_2}D_q\Psi\left(\frac{1+2t}{6}\widetilde{\lambda}_1+\frac{5-2t}{6}\widetilde{\lambda}_2\right)d_qt \qquad (12)\\
=\;&\frac{9}{2\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)}\Psi\left(\frac{\widetilde{\lambda}_1+\widetilde{\lambda}_2}{2}\right)+\frac{15}{2\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)}\Psi\left(\frac{\widetilde{\lambda}_1+5\widetilde{\lambda}_2}{6}\right)\\
&-\frac{36}{\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)^2}\int_{\frac{\widetilde{\lambda}_1+\widetilde{\lambda}_2}{2}}^{\frac{\widetilde{\lambda}_1+5\widetilde{\lambda}_2}{6}}\Psi(\tau)\;{}^{\frac{\widetilde{\lambda}_1+5\widetilde{\lambda}_2}{6}}d_q\tau
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 =\;&\int_0^1(-qt)\;{}^{\widetilde{\lambda}_2}D_q\Psi\left(\frac{t}{6}\widetilde{\lambda}_1+\frac{6-t}{6}\widetilde{\lambda}_2\right)d_qt \qquad (13)\\
=\;&\frac{6}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\Psi\left(\frac{\widetilde{\lambda}_1+5\widetilde{\lambda}_2}{6}\right)-\frac{36}{\left(\widetilde{\lambda}_2-\widetilde{\lambda}_1\right)^2}\int_{\frac{\widetilde{\lambda}_1+5\widetilde{\lambda}_2}{6}}^{\widetilde{\lambda}_2}\Psi(\tau)\;{}^{\widetilde{\lambda}_2}d_q\tau.
\end{aligned}
$$

Thus, we obtain the required equality by adding equalities (10)-(13). □

Theorem 3.2. *Let all the conditions of Lemma 3.1 be hold. If* $\left|{}^{\widetilde{\lambda}_2}D_q\Psi\right|$ *is a convex function, then the following inequality holds:*

$$
\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda}_1+\widetilde{\lambda}_2}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda}_1+\widetilde{\lambda}_2}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda}_1+5\widetilde{\lambda}_2}{6}\right)\right]-\frac{1}{\widetilde{\lambda}_2-\widetilde{\lambda}_1}\int_{\widetilde{\lambda}_1}^{\widetilde{\lambda}_2}\Psi(\tau)\;{}^{\widetilde{\lambda}_2}d_q\tau\right| \qquad (14)
$$

$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{5q^2+5q+6}{6\left([4]_q+q[2]_q\right)}+\frac{5A_1(q)+3A_2(q)}{6}+\frac{3A_3(q)+A_4(q)}{6}+\frac{q}{6[3]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|\right.$$

$$\left.+\left(\frac{q}{6[3]_q}+\frac{A_1(q)+3A_2(q)}{6}+\frac{3A_3(q)+5A_4(q)}{6}+\frac{6q^3+5q^2+5q}{6\left([4]_q+q[2]_q\right)}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|\right.,$$

where

$$A_1(q) = \begin{cases} \frac{3-8q}{2[2]_q}, & 0<q<\frac{3}{8} \\ \frac{160q^2+160q-69}{64\left([4]_q+q[2]_q\right)}, & \frac{3}{8}<q<1, \end{cases}$$

$$A_2(q) = \begin{cases} \frac{3q}{2[2]_q}, & 0<q<\frac{3}{8} \\ \frac{160q^3-24q^2-24q+45}{64\left([4]_q+q[2]_q\right)}, & \frac{3}{8}<q<1 \end{cases}$$

$$A_3(q) = \begin{cases} \frac{5-3q^2-3q}{2\left([4]_q+q[2]_q\right)}, & 0<q<\frac{5}{8} \\ \frac{96q^2+96q-35}{64\left([4]_q+q[2]_q\right)}, & \frac{5}{8}<q<1 \end{cases}$$

$$A_4(q) = \begin{cases} \frac{5q+5q^2-3q^2}{2\left([4]_q+q[2]_q\right)}, & 0<q<\frac{5}{8} \\ \frac{-160q^3+40q^2+40q+331}{64\left([4]_q+q[2]_q\right)}, & \frac{5}{8}<q<1. \end{cases}$$

Proof. Taking modulus in (9), we have

$$\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right| \tag{15}$$
$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[|I_1|+|I_2|+|I_3|+|I_4|\right].$$

Now from convexity, we have

$$\begin{aligned}
|I_1| &\leq \int_0^1 (1-qt)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5+t}{6}\widetilde{\lambda_1}+\frac{1-t}{6}\widetilde{\lambda_2}\right)\right|d_qt \qquad (16)\\
&= \int_0^1 (1-qt)\left[\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(t\widetilde{\lambda_1}+(1-t)\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right|\right]d_qt\\
&\leq \int_0^1 (1-qt)\left[t\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|+(1-t)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right|\right]d_qt\\
&= \frac{\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|}{[4]_q+q[2]_q}+\frac{q\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right|}{[3]_q}\\
&\leq \frac{\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|}{[4]_q+q[2]_q}+\frac{q\left(\frac{5\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|+\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{6}\right)}{[3]_q}\\
&= \frac{\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|}{[4]_q+q[2]_q}+\frac{q\left(5\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|+\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|\right)}{6[3]_q}\\
&= \left(\frac{5q^2+5q+6}{6\left([4]_q+q[2]_q\right)}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|+\frac{q}{6[3]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|,
\end{aligned}$$

$$
\begin{aligned}
|I_2| &\leq \int_0^1 \left|\frac{3}{2}-4qt\right| \left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{3+2t}{6}\widetilde{\lambda_1}+\frac{3-2t}{6}\widetilde{\lambda_2}\right)\right| d_qt \\
&\leq \int_0^1 \left|\frac{3}{2}-4qt\right| \left[t\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right| + (1-t)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\right|\right] d_qt \\
&= A_1(q)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right| + A_2(q)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\right| \\
&\leq A_1(q)\left[\frac{5\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + \left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{6}\right] + A_2(q)\left[\frac{\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + \left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{2}\right] \\
&= \left[\frac{5A_1(q)+3A_2(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + \left[\frac{A_1(q)+3A_2(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
|I_3| &\leq \int_0^1 \left|\frac{5}{2}-4qt\right| \left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{1+2t}{6}\widetilde{\lambda_1}+\frac{5-2t}{6}\widetilde{\lambda_2}\right)\right| d_qt \\
&\leq \int_0^1 \left|\frac{5}{2}-4qt\right| \left[t\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\right| + (1-t)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right|\right] dt \\
&= A_3(q)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\right| + A_4(q)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right| \\
&\leq A_3(q)\left[\frac{\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + \left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{2}\right] + A_4(q)\left[\frac{\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + 5\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{6}\right] \\
&= \left[\frac{3A_3(q)+A_4(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + \left[\frac{3A_3(q)+5A_4(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
|I_4| &\leq \int_0^1 qt\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{t}{6}\widetilde{\lambda_1}+\frac{6-t}{6}\widetilde{\lambda_2}\right)\right| d_qt \\
&\leq \int_0^1 qt\left[(1-t)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right| + t\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right|\right] \\
&= \frac{q^3\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{[4]_q+q[2]_q} + \frac{q\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right|}{[3]_q} \\
&\leq \frac{q^3\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|}{[4]_q+q[2]_q} + \left[\frac{q\left(\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right| + 5\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|\right)}{6[3]_q}\right] \\
&= \left[\frac{6q^3+5q^2+5q}{6\left([4]_q+q[2]_q\right)}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right| + \frac{q}{6[3]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|.
\end{aligned} \tag{19}
$$

Thus, by using (16)-(19) in (15), we get the resultant inequality (14). □

Remark 3.3. *If we set limit as $q \to 1^-$ in Theorem 3.2, then we obtain the following inequality:*

$$
\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]\right. \tag{20}
$$

$$-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\,d\tau\Bigg|$$

$$\leq \frac{25\left(\widetilde{\lambda_2}-\widetilde{\lambda_1}\right)}{576}\left[\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|+\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|\right].$$

This inequality can be found as a special case of [3].

Theorem 3.4. *Let all the conditions of Lemma 3.1 be hold. If* $\left|{}^{\widetilde{\lambda_2}}D_q\Psi\right|^r$, $r \geq 1$ *is a convex function, then the following inequality holds:*

$$\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right|$$

$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left(\left(\frac{5q^2+5q+6}{6\left([4]_q+q[2]_q\right)}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{q}{6[3]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}\right.$$

$$+\left(A_5(q)\right)^{1-\frac{1}{r}}\left(\left[\frac{5A_1(q)+3A_2(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\left[\frac{A_1(q)+3A_2(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}$$

$$+\left(A_6(q)\right)^{1-\frac{1}{r}}\left(\left[\frac{3A_3(q)+A_4(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\left[\frac{3A_3(q)+5A_4(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}$$

$$\left.+\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{r}}\left(\left[\frac{6q^3+5q^2+5q}{6\left([4]_q+q[2]_q\right)}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r+\frac{q}{6[3]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r\right)^{\frac{1}{r}}\right],$$

where $A_1(q)-A_4(q)$ *are same as defined in the Theorem 3.2 and*

$$A_5(q)=\begin{cases}\frac{3-5q}{2[2]_q}, & 0<q<\frac{3}{8}\\ \frac{20q-3}{8[2]_q}, & \frac{3}{8}<q<1,\end{cases}$$

$$A_6(q)=\begin{cases}\frac{5-3q}{2[2]_q}, & 0<q<\frac{5}{8}\\ \frac{37-20q}{8[2]_q}, & \frac{5}{8}<q<1.\end{cases}$$

Proof. Taking modulus in (9) and using power mean inequality, we have

$$\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right|$$

$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[|I_1|+|I_2|+|I_3|+|I_4|\right]$$

$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\int_0^1(1-qt)\,d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1(1-qt)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5+t}{6}\widetilde{\lambda_1}+\frac{1-t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}\right.$$

$$+\left(\int_0^1\left|\frac{3}{2}-4qt\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1\left|\frac{3}{2}-4qt\right|\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{3+2t}{6}\widetilde{\lambda_1}+\frac{3-2t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1\left|\frac{5}{2}-4qt\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1\left|\frac{5}{2}-4qt\right|\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{1+2t}{6}\widetilde{\lambda_1}+\frac{5-2t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1 qt\,d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1 qt\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{t}{6}\widetilde{\lambda_1}+\frac{6-t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}\Bigg].$$

From convexity of $\left|{}^{\widetilde{\lambda_2}}D_q\Psi\right|^r$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right|\\
\leq\ &\frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{r}}\left(\left(\frac{5q^2+5q+6}{6\left([4]_q+q[2]_q\right)}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{q}{6[3]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}\right.\\
&+\left(A_5(q)\right)^{1-\frac{1}{r}}\left(\left[\frac{5A_1(q)+3A_2(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\left[\frac{A_1(q)+3A_2(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}\\
&+\left(A_6(q)\right)^{1-\frac{1}{r}}\left(\left[\frac{3A_3(q)+A_4(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\left[\frac{3A_3(q)+5A_4(q)}{6}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}\\
&\left.+\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{r}}\left(\left[\frac{6q^3+5q^2+5q}{6\left([4]_q+q[2]_q\right)}\right]\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r+\frac{q}{6[3]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Thus, the proof is completed. □

Remark 3.5. *If we set the limit as $q\to 1^-$ in Theorem 3.4, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\,d\tau\right|\\
\leq\ &\frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\frac{1}{2}\left\{\left(\frac{8\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{9}\right)^{\frac{1}{r}}+\left(\frac{\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+8\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{9}\right)^{\frac{1}{r}}\right\}\right.\\
&\left.+\frac{17}{16}\left\{\left(\frac{1726\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+722\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{9}\right)^{\frac{1}{r}}+\left(\frac{722\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+1726\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{9}\right)^{\frac{1}{r}}\right\}\right].
\end{aligned}$$

This inequality can be found as a special case of [3].

Theorem 3.6. *Let all the conditions of Lemma 3.1 be hold. If $\left|{}^{\widetilde{\lambda_2}}D_q\Psi\right|^r$, $r>1$ is a convex function, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right|\\
\leq\ &\frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{1-(1-q)^{s+1}}{q(s+1)}\right)^{\frac{1}{s}}\left(\left(\frac{6+5q}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{q}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}\right.\\
&+\left(A_7(q,s)\right)^{\frac{1}{s}}\left(\left(\frac{5+3q}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{1+3q}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}
\end{aligned}$$

$$+\left(A_8(q,s)\right)^{\frac{1}{s}}\left(\left(\frac{3+q}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{3+5q}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}$$
$$+\left(\frac{q^s}{(s+1)}\right)^{\frac{1}{s}}\left(\left(\frac{6q+5}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r+\frac{1}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r\right)^{\frac{1}{r}}\Bigg],$$

where $r^{-1}+s^{-1}=rs$ *and*

$$A_7(q,s)=\begin{cases}\frac{3^{s+1}-(3-8q)^{s+1}}{2^{s+3}q(s+1)}, & 0<q<\frac{3}{8}\\ \frac{3^{s+1}-(8q-3)^{s+1}}{2^{s+3}q(s+1)}, & \frac{3}{8}<q<1,\end{cases}$$

$$A_8(q,s)=\begin{cases}\frac{5^{s+1}-(5-8q)^{s+1}}{2^{s+3}q(s+1)}, & 0<q<\frac{5}{8}\\ \frac{5^{s+1}-(8q-5)^{s+1}}{2^{s+3}q(s+1)}, & \frac{5}{8}<q<1.\end{cases}$$

Proof. Taking modulus in (9) and using Hölder's inequality, we have

$$\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right|$$
$$\leq\quad\frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[|I_1|+|I_2|+|I_3|+|I_4|\right]$$
$$\leq\quad\frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\int_0^1(1-qt)^s\,d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5+t}{6}\widetilde{\lambda_1}+\frac{1-t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}\right.$$
$$+\left(\int_0^1\left|\frac{3}{2}-4qt\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{3+2t}{6}\widetilde{\lambda_1}+\frac{3-2t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\left|\frac{5}{2}-4qt\right|^s d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{1+2t}{6}\widetilde{\lambda_1}+\frac{5-2t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}$$
$$\left.+\left(\int_0^1(qt)^s\,d_qt\right)^{\frac{1}{s}}\left(\int_0^1\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{t}{6}\widetilde{\lambda_1}+\frac{6-t}{6}\widetilde{\lambda_2}\right)\right|^r d_qt\right)^{\frac{1}{r}}\right].$$

Now from convexity, we have

$$\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\ {}^{\widetilde{\lambda_2}}d_q\tau\right|$$
$$\leq\quad\frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{1-(1-q)^{s+1}}{q(s+1)}\right)^{\frac{1}{s}}\left(\frac{1}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{q}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right|^r\right)^{\frac{1}{r}}\right.$$
$$+\left(A_7(q,s)\right)^{\frac{1}{s}}\left(\frac{1}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)\right|^r+\frac{q}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\right|^r\right)^{\frac{1}{r}}$$
$$+\left(A_8(q,s)\right)^{\frac{1}{s}}\left(\frac{1}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)\right|^r+\frac{q}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right|^r\right)^{\frac{1}{r}}$$
$$\left.+\left(\frac{q^s}{s+1}\right)^{\frac{1}{s}}\left(\frac{q}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r+\frac{1}{[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right|^r\right)^{\frac{1}{r}}\right]$$

$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{1-(1-q)^{s+1}}{q(s+1)}\right)^{\frac{1}{s}}\left(\left(\frac{6+5q}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{q}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}\right.$$

$$+\left(A_7(q,s)\right)^{\frac{1}{s}}\left(\left(\frac{5+3q}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{1+3q}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}$$

$$+\left(A_8(q,s)\right)^{\frac{1}{s}}\left(\left(\frac{3+q}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r+\frac{3+5q}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r\right)^{\frac{1}{r}}$$

$$\left.+\left(\frac{q^s}{s+1}\right)^{\frac{1}{s}}\left(\left(\frac{6q+5}{6[2]_q}\right)\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_2}\right)\right|^r+\frac{1}{6[2]_q}\left|{}^{\widetilde{\lambda_2}}D_q\Psi\left(\widetilde{\lambda_1}\right)\right|^r\right)^{\frac{1}{r}}\right].$$

Thus, the proof is completed. □

Remark 3.7. *If we set the limit as $q \to 1^-$ in Theorem 3.6, then we have the following inequality:*

$$\left|\frac{1}{8}\left[3\Psi\left(\frac{5\widetilde{\lambda_1}+\widetilde{\lambda_2}}{6}\right)+2\Psi\left(\frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{2}\right)+3\Psi\left(\frac{\widetilde{\lambda_1}+5\widetilde{\lambda_2}}{6}\right)\right]-\frac{1}{\widetilde{\lambda_2}-\widetilde{\lambda_1}}\int_{\widetilde{\lambda_1}}^{\widetilde{\lambda_2}}\Psi(\tau)\,d\tau\right|$$

$$\leq \frac{\widetilde{\lambda_2}-\widetilde{\lambda_1}}{36}\left[\left(\frac{1}{s+1}\right)^{\frac{1}{s}}\left\{\left(\frac{11\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{12}\right)^{\frac{1}{r}}+\left(\frac{\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+11\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{12}\right)^{\frac{1}{r}}\right\}\right.$$

$$\left.+\frac{1}{2}\left(\frac{3^{s+1}+5^{s+1}}{8}\right)^{\frac{1}{s}}\left\{\left(\frac{2\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{3}\right)^{\frac{1}{r}}+\left(\frac{\left|\Psi'\left(\widetilde{\lambda_1}\right)\right|^r+2\left|\Psi'\left(\widetilde{\lambda_2}\right)\right|^r}{3}\right)^{\frac{1}{r}}\right\}\right].$$

This inequality can be found as a special case of [3].

4. Conclusion

In this work, some inequalities were proved for q-differentiable functions in the setting of q-calculus. The newly established inequalities were showing the error bounds for Maclaurin's formula which is the three step quadrature formula. The inequalities proved here are very useful in the field of analysis because these can be used in defining the error bound for Maclaurin's formula in numerical integration formulas. It is an intersecting and new problem that the new researchers can obtain similar inequalities for coordinated functions or other different kind of convexity.

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