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On extensions of Hermite–Hadamard type inclusions for interval-valued convex functions

Hasan Kara[a], Hüseyin Budak[a], Fatih Hezenci[a]

[a]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye*

Abstract. In this work, by using weighted Jensen inclusion, we establish some new weighted Hermite–Hadamard type inclusions involving two real parameters for interval-valued convex functions. In addition, some extensions of Hermite–Hadamard inclusion are obtained by special choices of parameters. Moreover, we give some examples to illustrate the main results of this work.

1. Introduction

Over the last century, integral inequalities have been attracted interest of many researchers because of the importance in applied and pure mathematics. For example, Hermite–Hadamard inequalities, based on convex functions, have an important place in many areas of mathematics, specifically optimization theory. These inequalities, introduced by C. Hermite and J. Hadamard, express that if $\phi : I \to \mathbb{R}$ is a convex mapping on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \phi(x)dx \leq \frac{\phi(a)+\phi(b)}{2}. \tag{1}$$

If ϕ is concave, both of the inequalities provide the opposite direction. The best known results associated with these inequalities are Midpoint and Trapezoid inequalities which are frequently used in special means and estimation errors (see [8, 14]). Afterwards, many authors obtained new results related to these inequalities under various conditions of the mappings. Also, some researchers examined generalizations, refinements and counterparts of the inequalities (1).

The weighted version of the inequality (1), which is also named Hermite–Hadamard-Fejér inequality, was established by Fejér in [9] as follows:

Theorem 1.1. *Let $\phi : [a, b] \to \mathbb{R}$ be a convex function and let $\varpi : [a, b] \to \mathbb{R}$ be a non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $\varpi(x) = \varpi(a+b-x)$). Then, we have the inequalities*

$$\phi\left(\frac{a+b}{2}\right)\int_a^b \varpi(x)dx \leq \int_a^b \phi(x)\varpi(x)dx \leq \frac{\phi(a)+\phi(b)}{2}\int_a^b \varpi(x)dx. \tag{2}$$

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Communicated by Dragan S. Djordjević
Email addresses: hasan64kara@gmail.com (Hasan Kara), hsyn.budak@gmail.com (Hüseyin Budak), fatihezenci@gmail.com (Fatih Hezenci)

Many mathematicians established some generalizations and new results involving fractional integrals regarding to the inequality (2) to obtain new bounds for the left and right sides of the inequality (2). We refer the reader to Refs. [1, 21–23] and the references therein. On the other side, interval analysis handled as one of the methods of solving interval uncertainty is an important material which is used in mathematical and computer models. Although this theory has a long history which may be dated back to Archimedes' calculation of the circumference of a circle, a considerable study was not published in this field until 1950s. The first book [17] about interval analysis was published by Ramon E. Moore known as the pioneer of interval calculus in 1966. In addition to this, a great many researchers started to investigate theories and applications of interval analysis. Recently, many authors have focused on integral inequalities obtained by using interval-valued functions. For example, Sadowska [20] established Hermite–Hadamard inequality for set-valued functions that is more general version of interval-valued mappings as follows:

Theorem 1.2. *[20] Assume that $F : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is interval–valued convex function so that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. Then, we have*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_{a}^{b} F(x)dx \supseteq \frac{F(a)+F(b)}{2}.$$

Well-known inequalities such as Ostrowski, Minkowski and Beckenbach and their some applications were provided by considering interval-valued functions in [5, 6, 10]. In addition, some inequalities involving interval-valued Riemann-Liouville fractional integrals were derived by Budak et al. in [2]. In [15], Liu et al. gave the definition of interval-valued harmonically convex functions, and so they obtain some Hermite–Hadamard type inequalities including interval-valued fractional integrals. On the other hand, Budak et al. prove some weighted Fejer type inclusions in [4]. For more details about this topic, one can refer to [3, 7, 11–13, 16, 24].

2. Preliminaries

In this section, we give some properties of one and two variables interval-valued functions.

2.1. Integral of interval-valued functions

The notion of integral of the interval-valued mappings is mentioned. Before we can understand the definition of integrals of interval-valued functions, we need to summarize some concepts in the following.

A function φ is said to be an interval-valued function of t on $[a, b]$ if it assigns a non-empty interval to each $t \in [a, b]$

$$\varphi(t) = \left[\underline{\varphi}(t), \overline{\varphi}(t)\right].$$

A partition of $[a, b]$ is any finite ordered subset D having the form

$$D : a = t_0 < t_1 < ... < t_n = b.$$

The mesh of a partition D is indicated by

$$mesh(D) = \max\{t_i - t_{i-1} : i = 1, 2, ..., n\}.$$

We denote by $D([a, b])$ the set of all partition of $[a, b]$. Suppose that $D(\delta, [a, b])$ is the set of all $D \in D([a, b])$ such that $mesh(D) < \delta$. We take an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $i = 1, 2, ..., n$, and we define the sum

$$S(\varphi, D, \delta) = \sum_{i=1}^{n} \varphi(\xi_i)\left[t_i - t_{i-1}\right].$$

Here, $\varphi : [a, b] \to \mathbb{R}_{\mathcal{I}}$. The sum $S(\varphi, D, \delta)$ is said to be a Riemann sum of φ corresponding to $D \in D(\delta, [a, b])$.

Definition 2.1. *([18],[19]) $\varphi : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is said to be an interval Riemann integrable function (IR-integrable) on $[a,b]$ if there exist $A \in \mathcal{P}$ and $\delta > 0$, for each $\varepsilon > 0$, so that*

$$d\left(S(\varphi, D, \delta), A\right) < \varepsilon$$

for every Riemann sum S of φ corresponding to each $D \in D\left(\delta, [a,b]\right)$ and independent of choice of $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$. In this case, A is called as the IR-integral of φ on $[a,b]$ and is denoted by

$$A = (IR)\int_a^b \varphi(t)dt.$$

The collection of all functions that are IR-integrable on $[a,b]$ will be denote by $\mathcal{IR}_{([a,b])}$.

The next theorem describes connection between IR-integrable and Riemann integrable (R-integrable):

Theorem 2.2. *Let $\varphi : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $\varphi(t) = \left[\underline{\varphi}(t), \overline{\varphi}(t)\right]$, $\varphi \in \mathcal{IR}_{([a,b])}$ if and only if $\underline{\varphi}(t), \overline{\varphi}(t) \in \mathcal{R}_{([a,b])}$ and*

$$(IR)\int_a^b \varphi(t)dt = \left[(R)\int_a^b \underline{\varphi}(t)dt, (R)\int_a^b \overline{\varphi}(t)dt\right].$$

Here, $\mathcal{R}_{([a,b])}$ is the all R-integrable function.

It is easy to see that if $\varphi(t) \subseteq \psi(t)$ for all $t \in [a,b]$, then $(IR)\int_a^b \varphi(t)dt \subseteq (IR)\int_a^b \psi(t)dt$.

3. Weighted Hermite–Hadamard type inclusions for interval-valued convex functions

In this section, we prove some weighted Hermite–Hadamard type inclusions for interval-valued convex functions. First, we need to following weighted Jensen inclusion:

Theorem 3.1 (Weighted Jensen Inclusion). *Suppose $g : [a,b] \to [a,b]$ is a function from $L^\infty[a,b]$ and suppose also $w : [a,b] \to \mathbb{R}$ is nonnegative functions from $L^1[a,b]$ such that $\int_a^b w(t)dt \neq 0$. If $F : [a,b] \to \mathbb{R}_I$ is an interval–valued convex function so that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have*

$$F\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)g(t)\,dt\right) \supseteq \frac{1}{\int_a^b w(t)\,dt}(IR)\int_a^b F(g(t))\,w(t)\,dt.$$

Proof. The proof can be easily seen by applying the classical Jensen inequality to convex function $\underline{F}$ and concave function $\overline{F}$. □

Theorem 3.2. *If $F : [a,b] \to \mathbb{R}_I$ is an interval–valued convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we obtain*

$$\begin{aligned} F\left(\frac{pa+qb}{p+q}\right) &\supseteq \frac{2}{(p+q)(b-a)^2}(IR)\int_a^b \left[(p-2q)a + (2p-q)b + 3(q-p)x\right]F(x)\,dx \\ &\supseteq \frac{pF(a)+qF(b)}{p+q} \end{aligned} \tag{3}$$

for $2q \geq p \geq \frac{q}{2} > 0$.

Proof. It can be easily see that

$$\int_0^1 [2p - q + 3(q-p)t]\,dt = \frac{p+q}{2}, \tag{4}$$

$$\int_0^1 [2p - q + 3(q-p)t][(1-t)a + tb]\,dt = \frac{pa+qb}{2}, \tag{5}$$

and

$$\int_0^1 [2p - q + 3(q-p)t][(1-t)F(a) + tF(b)]\,dt = \frac{pF(a)+qF(b)}{p+q}. \tag{6}$$

By applying Theorem 3.1 with $g(t) = (1-t)a + tb$, $w(t) = 2p - q + 3(q-p)t$, $t \in [0,1]$ and $2q \geq p \geq \frac{q}{2} > 0$ and by using the equalities (4) and (5), we get

$$\begin{aligned}
F\left(\frac{pa+qb}{p+q}\right) &= F\left(\frac{\int_0^1 [2p - q + 3(q-p)t][(1-t)a + tb]\,dt}{\int [2p - q + 3(q-p)t]\,dt}\right) \\
&\supseteq \frac{(IR)\int_0^1 [2p - q + 3(q-p)t]\,F((1-t)a + tb)\,dt}{\int [2p - q + 3(q-p)t]\,dt} \\
&= \frac{2}{p+q}(IR)\int_0^1 [2p - q + 3(q-p)t]\,F((1-t)a + tb)\,dt \\
&= \frac{2}{(p+q)(b-a)^2}(IR)\int_a^b [(p-2q)a + (2p-q)b + 3(q-p)x]\,F(x)\,dx,
\end{aligned}$$

which gives the first inclusion in (3). On the other hand, by utilizing the interval-valued convexity of F and the inequality (6), we have

$$\begin{aligned}
&\frac{2}{p+q}(IR)\int_0^1 [2p - q + 3(q-p)t]\,F((1-t)a + tb)\,dt \\
\supseteq\ &\frac{2}{p+q}(IR)\int_0^1 [2p - q + 3(q-p)t][(1-t)F(a) + tF(b)]\,dt \\
=\ &\frac{pF(a)+qF(b)}{p+q}.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

Remark 3.3. *Let us consider $p = q = \frac{1}{2}$ in Theorem 3.2. Then, Theorem 3.2 reduces to Theorem 1.2.*

Corollary 3.4. *If we choose $q = 1$ and $p = 2$ in Theorem 3.2, then we have*

$$F\left(\frac{2a+b}{3}\right) \supseteq \frac{2}{(b-a)^2}(IR)\int_a^b (b-x)\,F(x)\,dx \supseteq \frac{2F(a)+F(b)}{3}. \tag{7}$$

Corollary 3.5. *Let us note that $q = 2$ and $p = 1$ in Theorem 3.2. Then, we obtain*

$$F\left(\frac{a+2b}{3}\right) \supseteq \frac{2}{(b-a)^2}(IR)\int_a^b (x-a)\,F(x)\,dx \supseteq \frac{F(a)+2F(b)}{3}. \tag{8}$$

Corollary 3.6. *Under the conditions of Theorem 3.2, we have*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{2}\left[F\left(\frac{2a+b}{3}\right)+F\left(\frac{a+2b}{3}\right)\right] \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{2}. \tag{9}$$

Proof. By adding (7) and (8), we can write

$$\frac{1}{2}\left[F\left(\frac{2a+b}{3}\right)+F\left(\frac{a+2b}{3}\right)\right] \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{2}.$$

Since F is an interval-valued convex function, we get

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &= F\left(\frac{1}{2}\left(\frac{2a+b}{3}\right)+\frac{1}{2}\left(\frac{a+2b}{3}\right)\right) \\ &\supseteq \frac{1}{2}\left[F\left(\frac{2a+b}{3}\right)+F\left(\frac{a+2b}{3}\right)\right], \end{aligned}$$

which completes the proof. □

Example 3.7. *Define a function $F : [0,1] \to \mathbb{R}_I^+$ by $F(t) = \left[t^2, 2-t^2\right]$, then $F(t)$ is an interval-valued convex function. By applying Corollary 3.6, the first expression of (9) becomes*

$$F\left(\frac{a+b}{2}\right) = F\left(\frac{1}{2}\right) = \left[\frac{1}{4}, \frac{7}{4}\right].$$

The second expression of (9) becomes

$$\begin{aligned} &\frac{1}{2}\left[F\left(\frac{2a+b}{3}\right)+F\left(\frac{a+2b}{3}\right)\right] \\ = &\frac{1}{2}\left[\left[\frac{1}{9}, \frac{17}{9}\right]+\left[\frac{4}{9}, \frac{14}{9}\right]\right] = \left[\frac{5}{18}, \frac{31}{18}\right]. \end{aligned}$$

By using the definition of integral for interval-valued function, the third expression of (9), we obtain

$$\begin{aligned} \frac{1}{b-a}(IR)\int_a^b F(t)\,dt &= \left[\int_0^1 t^2 dt, \int_0^1 \left(2-t^2\right)dt\right] \\ &= \left[\frac{1}{3}, \frac{5}{3}\right]. \end{aligned}$$

And then fourth expression of (9), we get

$$\frac{F(a)+F(b)}{2}=\frac{[0,2]+[1,1]}{2}=\left[\frac{1}{2},\frac{3}{2}\right].$$

It is clear that

$$\left[\frac{1}{4},\frac{7}{4}\right]\supseteq\left[\frac{5}{18},\frac{31}{18}\right]\supseteq\left[\frac{1}{3},\frac{5}{3}\right]\supseteq\left[\frac{1}{2},\frac{3}{2}\right],$$

which demonstrates the result described in Corollary 3.6.

Corollary 3.8. *With the help of the conditions of Theorem 3.2, we have*

$$F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{2}\left[F\left(\frac{5a+3b}{8}\right)+F\left(\frac{3a+5b}{8}\right)\right]\supseteq\frac{1}{b-a}(IR)\int_a^b F(x)dx\supseteq\frac{F(a)+F(b)}{2}. \tag{10}$$

Proof. If we assign $q=5$, $p=3$ and $q=3$, $p=5$ in Theorem 3.2, then we have

$$F\left(\frac{3a+5b}{8}\right)\supseteq\frac{1}{4(b-a)^2}\int_a^b(-7a+b+6x)F(x)\,dx\supseteq\frac{3F(a)+5F(b)}{8}, \tag{11}$$

and

$$F\left(\frac{5a+3b}{8}\right)\supseteq\frac{1}{4(b-a)^2}\int_a^b(-a+7b-6x)F(x)\,dx\supseteq\frac{5F(a)+3F(b)}{8}, \tag{12}$$

respectively. By adding (11) and (12), we get

$$\frac{1}{2}\left[F\left(\frac{5a+3b}{8}\right)+F\left(\frac{3a+5b}{8}\right)\right]\supseteq\frac{1}{b-a}(IR)\int_a^b F(x)dx\supseteq\frac{F(a)+F(b)}{2}.$$

By using interval-valued convexity of F, we get,

$$\begin{aligned}F\left(\frac{a+b}{2}\right) &= F\left(\frac{1}{2}\cdot\left(\frac{5a+3b}{8}\right)+\frac{1}{2}\cdot\left(\frac{3a+5b}{8}\right)\right)\\ &\supseteq \frac{1}{2}\left[F\left(\frac{5a+3b}{8}\right)+F\left(\frac{3a+5b}{8}\right)\right].\end{aligned}$$

This is the end of the proof of Corollary 3.8. □

Example 3.9. *Let us consider a function* $F:[0,1]\to\mathbb{R}_I^+$ *by* $F(t)=\left[t^3,3-t^3\right]$*. Then,* $F(t)$ *is an interval-valued convex function. By applying Corollary 3.8, the first expression of (10) becomes*

$$F\left(\frac{a+b}{2}\right)=F\left(\frac{1}{2}\right)=\left[\frac{1}{8},\frac{23}{8}\right].$$

The second expression of (10) becomes

$$\frac{1}{2}\left[F\left(\frac{5a+3b}{8}\right)+F\left(\frac{3a+5b}{8}\right)\right]$$

$$= \frac{1}{2}\left[\left[\frac{27}{512},\frac{1509}{512}\right]+\left[\frac{125}{512},\frac{1411}{512}\right]\right]=\left[\frac{19}{128},\frac{365}{128}\right].$$

By using the definition of integral for interval-valued function and the third expression of (10), we have

$$\begin{aligned}\frac{1}{b-a}(IR)\int_a^b F(t)\,dt &= \left[\int_0^1 t^3dt,\ \int_0^1\left(3-t^3\right)dt\right]\\ &= \left[\frac{1}{4},\frac{11}{4}\right].\end{aligned}$$

By using the fourth expression of (10), we get

$$\frac{F(a)+F(b)}{2}=\frac{[0,3]+[1,2]}{2}=\left[\frac{1}{2},2\right].$$

It is clear that

$$\left[\frac{1}{8},\frac{23}{8}\right]\supseteq\left[\frac{19}{128},\frac{365}{128}\right]\supseteq\left[\frac{1}{4},\frac{11}{4}\right]\supseteq\left[\frac{1}{2},2\right],$$

which demonstrates the result described in Corollary 3.8.

Theorem 3.10. *Let us note that $F:[a,b]\to\mathbb{R}_I$ is an interval–valued convex function so that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. Then, we have the inclusions*

$$\begin{aligned}&F\left(\frac{(3p+q)a+(p+3q)b}{4(p+q)}\right) \qquad (13)\\ \supseteq\ &\frac{1}{2}\left[F\left(\frac{(2p+q)a+qb}{2(p+q)}\right)+F\left(\frac{pa+(p+2q)b}{2(p+q)}\right)\right]\\ \supseteq\ &\frac{2}{(p+q)(b-a)^2}(IR)\int_a^{\frac{a+b}{2}}\left[(4p-5q)a+(2p-q)b+6(q-p)x\right]F(x)\,dx\\ &+\frac{2}{(p+q)(b-a)^2}(IR)\int_{\frac{a+b}{2}}^{b}\left[(p-2q)a+(5p-4q)b+6(q-p)x\right]F(x)\,dx\\ \supseteq\ &\frac{pF(a)+qF\left(\frac{a+b}{2}\right)+pF\left(\frac{a+b}{2}\right)+qF(b)}{2(p+q)}\\ \supseteq\ &\frac{(3p+q)F(a)+(p+3q)F(b)}{4(p+q)}\end{aligned}$$

for $2q\geq p\geq\frac{q}{2}>0$.

Proof. Since F is interval–valued convex function on $[a,b]$, then F is interval-valued convex function on the subintervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$. Then, by applying the Theorem 3.2 to a subinterval $\left[a,\frac{a+b}{2}\right]$, we have

$$\begin{aligned}&F\left(\frac{(2p+q)a+qb}{2(p+q)}\right) \qquad (14)\\ \supseteq\ &\frac{4}{(p+q)(b-a)^2}(IR)\int_a^{\frac{a+b}{2}}\left[(4p-5q)a+(2p-q)b+6(q-p)x\right]F(x)\,dx\end{aligned}$$

$$\begin{aligned} &\supseteq \frac{pF(a)+qF\left(\frac{a+b}{2}\right)}{p+q} \\ &\supseteq \frac{(2p+q)F(a)+qF(b)}{2(p+q)}. \end{aligned}$$

Similarly, by applying the Theorem 3.2 to a subinterval $\left[\frac{a+b}{2}, b\right]$, we get

$$\begin{aligned} & F\left(\frac{pa+(p+2q)b}{2(p+q)}\right) \\ \supseteq\ & \frac{4}{(p+q)(b-a)^2}(IR)\int_{\frac{a+b}{2}}^{b}\left[(p-2q)a+(5p-4q)b+6(q-p)x\right]F(x)\,dx \\ \supseteq\ & \frac{pF\left(\frac{a+b}{2}\right)+qF(b)}{p+q} \\ \supseteq\ & \frac{pF(a)+(p+2q)F(b)}{2(p+q)}. \end{aligned} \tag{15}$$

By summing the inequalities (14) and (15), we have

$$\begin{aligned} & \frac{1}{2}\left[F\left(\frac{(2p+q)a+qb}{2(p+q)}\right)+F\left(\frac{pa+(p+2q)b}{2(p+q)}\right)\right] \\ \supseteq\ & \frac{2}{(p+q)(b-a)^2}(IR)\int_{a}^{\frac{a+b}{2}}\left[(4p-5q)a+(2p-q)b+6(q-p)x\right]F(x)\,dx \\ & +\frac{2}{(p+q)(b-a)^2}(IR)\int_{\frac{a+b}{2}}^{b}\left[(p-2q)a+(5p-4q)b+6(q-p)x\right]F(x)\,dx \\ \supseteq\ & \frac{pF(a)+qF\left(\frac{a+b}{2}\right)+pF\left(\frac{a+b}{2}\right)+qF(b)}{2(p+q)} \\ \supseteq\ & \frac{(3p+q)F(a)+(p+3q)F(b)}{4(p+q)}. \end{aligned}$$

This gives the second, third, and fourth inequality in (13). The first inequality in (13) can be seen easily by interval-valued convexity of F. □

Corollary 3.11. *Considering $p=q=1$ in Theorem 3.10, we have the inclusions*

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &\supseteq \frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right] \\ &\supseteq \frac{1}{(b-a)}(IR)\int_{a}^{\frac{a+b}{2}}F(x)\,dx \\ &\supseteq \frac{F(a)+2F\left(\frac{a+b}{2}\right)+F(b)}{4} \\ &\supseteq \frac{F(a)+F(b)}{2}. \end{aligned} \tag{16}$$

Example 3.12. *Let us define a function* $F : [0, \pi] \to \mathbb{R}_I^+$ *by* $F(t) = [1 - \sin t, 2 + \sin t]$. *Then,* $F(t)$ *is an interval-valued convex function. By applying Corollary 3.11, the first expression of (16) is expressed as follows*

$$F\left(\frac{a+b}{2}\right) = F\left(\frac{\pi}{2}\right) = [0, 3].$$

The second expression of (16) becomes

$$\begin{aligned} & \frac{1}{2}\left[F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right)\right] \\ = & \frac{1}{2}\left[\left[1 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right] + \left[1 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right]\right] = \left[1 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right]. \end{aligned}$$

By using the definition of integral for interval-valued function and the third expression of (16), we obtain

$$\begin{aligned} \frac{1}{b-a}(IR)\int_a^b F(t)\,dt &= \frac{1}{\pi}\left[\int_0^\pi (1 - \sin t)\,dt, \int_0^\pi (2 + \sin t)\,dt\right] \\ &= \left[1 - \frac{2}{\pi}, 2 + \frac{2}{\pi}\right]. \end{aligned}$$

With the help of the fourth expression of (16), we get

$$\frac{F(a) + 2F\left(\frac{a+b}{2}\right) + F(b)}{4} = \frac{[1,2] + [0,6] + [1,2]}{4} = \left[1, \frac{5}{2}\right].$$

By using the fifth expression of (16), we have

$$\frac{F(a) + F(b)}{2} = \frac{[1,2] + [1,2]}{2} = [1, 2].$$

It is clear that

$$[0, 3] \supseteq \left[1 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right] \supseteq \left[1 - \frac{2}{\pi}, 2 + \frac{2}{\pi}\right] \supseteq \left[1, \frac{5}{2}\right] \supseteq [1, 2],$$

which demonstrates the result described in Corollary 3.11.

References

[1] H. Budak and Y. Bakış, *On Fejér type inequalities for products two convex functions*, Note di Matematica, 40(1), (2020), 27–44.
[2] H. Budak, T. Tunç, and M. Z. Sarikaya, *Fractional Hermite–Hadamard-type inequalities for interval-valued functions*, Proceedings of the American Mathematical Society, 148(2), (2020), 705–718.
[3] H. Budak, H. Kara, M. A. Ali, S. Khan, and Y. Chu *Fractional Hermite–Hadamard type inequalities for interval-valued co-ordinated convex functions*, Open Mathematics, 19(1), (2021), 1081–1097.
[4] H. Budak, H. Kara, and S. Erden, *On Fejér type Inequalities for products of Interval-valued convex functions*, 2021, Filomat, 35(14), (2021), 4937–4955.
[5] Y. Chalco-Cano, A. Flores-Franulic, and H. Roman-Flores, *Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative*, Comput. Appl. Math., 31, (2012), 457–472
[6] Y. Chalco-Cano, W. A. Lodwick, and W. Condori-Equice, *Ostrowski type inequalities and applications in numerical integration for interval-valued function*s, Soft Comput., 19, (2015), 3293–3300.
[7] T. M. Costa, *Jensen's inequality type integral for fuzzy interval-valued functions*, Fuzzy Sets and Systems, 327, (2017), 31–47.
[8] S. S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., 11(5), (1998), 91–95.
[9] L. Fejér, *Über die Fourierreihen*, II. Math. Naturwiss. Anz Ungar. Akad. Wiss., 24, (1906), 369–390. (Hungarian).
[10] H. Roman-Flores, Y. Chalco-Cano, and W. A. Lodwick, *Some integral inequalities for interval-valued functions*, Comput. Appl. Math., 37, (2018), 1306–1318.

[11] H. Roman-Flores, Y. Chalco-Cano, and G. N. Silva, *A note on Gronwall type inequality for interval-valued functions*, IFSA World Congress and NAFIPS Annual Meeting IEEE, 35, (2013), 1455–1458.
[12] R. Osuna-Gómez, M. D. Jiménez-Gamero, Y. Chalco-Cano, M. A. Rojas-Medar, *Hadamard and Jensen inequalities for s-convex fuzzy processes,* In: Soft Methodology and Random Information Systems, Springer, Berlin, (2004), 645–652.
[13] H. Kara, M. A. Ali, and H. Budak, *Hermite–Hadamard-type inequalities for interval-valued co-ordinated convex functions involving generalized fractional integrals*, Mathematical Methods in the Applied Sciences, 44(1), (2021), 104–123.
[14] U. S. Kırmacı, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., 147, (2004), 137–146.
[15] X. Liu, G. Ye, D. Zhao, and W. Liu, *Fractional Hermite–Hadamard type inequalities for interval-valued functions*. Journal of Inequalities and Applications, 2019(1), (2019), 1–11.
[16] F. C. Mitroi, N. Kazimierz, and W. Szymon, *Hermite–Hadamard inequalities for convex set-valued functions*, Demonstratio Mathematica, XLVI(4), (2013), 655-662.
[17] R. E. Moore, *Interval analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., (1966).
[18] B. Piatek, *On the Riemann integral of set-valued functions*, Zeszyty Naukowe. Matematyka Stosowana/Politechnika Slaska, (2012).
[19] B. Piatek, *On the Sincov functional equation*, Demonstratio Mathematica 38(4), (2005), 875–882.
[20] E. Sadowska, *Hadamard inequality and a refinement of Jensen inequality for set-valued functions*, Results in Mathematics, 32, (1997), 332-337.
[21] M. Z. Sarikaya and S. Erden, *On The Hermite- Hadamard-Fejér Type Integral Inequality for Convex Function*, Turkish J.of Anal. and Number Theory, 2(3), 2014, 85–89.
[22] M. Z. Sarikaya and S. Erden, *On the weigted integral inequalities for convex functions*, Acta Universitatis Sapientiae Mathematica, 6(2), (2014), 194–208.
[23] S. Wu, *On the weighted generalization of the Hermite–Hadamard inequality and its applications*, The Rocky Mountain Journal of Mathematics, (2009), 1741-1749.
[24] D. Zhao et al, *Some inequalities for interval-valued functions on time scales*, Soft Computing (2018), 1–11.