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Variant of thin sets and their influence in convergence

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Abstract. A class of subsets designated as very thin subsets of natural numbers has been studied and seen that theory of convergence may be extended further if very thin sets are given to play main role instead of thin or finite sets which refines even statistical convergence. While developing the theory of very thin sets, concepts of super lacunary and very very thin sets are evolved spontaneously. Influence of very thin sets is reflected in various ways mainly in the BW property of the ideal consisting of very thin sets.

1. Introduction

Let's begin with the well known definition of asymptotic density **[1–4]** of subsets of set of natural numbers ω. For any $A \subset \omega$, $|A|$ denotes the cardinality of A and $A(n) = |\{m \in \omega : m \in A \cap \{1, 2, ..., n\}\}|$. The numbers

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \text{ and } \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called the lower and upper asymptotic density of A respectively. If $\underline{d}(A) = \overline{d}(A)$, then $d(A) = \overline{d}(A)$ is called asymptotic density or natural density of A. As in **[4]**, A is called thin subset of ω if $d(A) = 0$ otherwise A is nonthin. A lacunary sequence **[5, 6]** is a strictly increasing sequence $(k_n)_{n\in\omega}$ of natural numbers such that $k_n - k_{n-1} \to \infty$ as $n \to \infty$. A subset A of ω is called lacunary **[6]** if A is finite or A is the range of a lacunary sequence. It is seen that any lacunary subset of ω is thin.

The concept of statistical convergence **[3, 5, 7–10, 18]** of real sequences is a generalization of usual convergence based on asymptotic density, where thin subsets of ω play an important role. A sequence $(x_n)_{n\in\omega}$ of real numbers is statistically convergent to a real number a if for any $\epsilon > 0$ the set $\{n \in \omega :| x_n - a |\geqslant \epsilon\}$ is thin.

Consider a real sequence $(x_n)_{n\in\omega}$ where

$$x_n = \begin{cases} -1, & \text{if } n = 2^k + j, k \in \omega \text{ and } 0 \le j \le k-1 \\ 1, & \text{otherwise.} \end{cases} \qquad (\mathbf{N})$$

In this sequence -1 is repeated consecutively k times from $(2^k)^{th}$ term to $(2^k + (k-1))^{th}$ term for every natural number k. As k increases towards infinity, number of consecutive repetition of -1 is also increases towards

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infinity.
Let's consider another real sequence $(y_n)_{n\in\omega}$ where

$$y_n = \begin{cases} -1, & \text{if } n = 2^k, k \in \omega \\ 1, & \text{otherwise.} \end{cases} \tag{Y}$$

In this sequence -1 is appearing only at every $2^{k^{th}}$ place for each $k \in \omega$. As k increases towards infinity, gap between two consecutive appearance of -1 is also tending to infinity. According to the existing literature both $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ are statistically convergent to 1 and here is the where the present variant of thin sets play crucial role to distinguish $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ reasonably in regards of their convergence.

2. Variant of thin sets and their characterization

The notion of very thin, super lacunary and very very thin subsets of ω are introduced in this section. Then, characterization of very thin and very very thin sets and the relations among variant of thin sets are shown.
Suppose $A \subset \omega$ and $M \in \omega$. Define
$(A)_M$={1} ∪ {n: $n > 1$ and there exist n consecutive elements of A such that difference between any two consecutive among them is less than or equal to M}.

For example, if $A = \bigcup_{k\in\omega}\{2^k, 2^k+1, ..., 2^k+k\}$ then $(A)_M = \{n \in \omega : n \geqslant 1\}$ for all $M \geqslant 1$ and if $B = \{2^k : k \geqslant 1\}$ then $(B)_1 = \{1\}$ and $(B)_M = \{1, 2, ..., k+1\}$ for $2^k \leqslant M < 2^{k+1}$.

Definition 2.1. *A subset A of ω is very thin if there exist a sub-collection $\{A_n : n \in \omega\}$ of finite subsets of ω and $M \in \omega$ such that* $\max(A\backslash A_n)_n \leqslant M$ *for all n.*

Proposition 2.2. *A subset A of ω is very thin if and only if A is finite or A can be written as follows:*
$(V1)$ $A = \bigcup_{k\in\omega}\mathcal{A}_k$, *where* $1 \leqslant |\mathcal{A}_k| \leqslant \mathcal{M}$ *for some* $\mathcal{M} \in \omega$ *for all* $k \in \omega$,
$(V2)$ $\min(\mathcal{A}_{k+1}) - \max(\mathcal{A}_k) > 0$ *for all* $k \in \omega$,
$(V3)$ $\lim_{k\to\infty}(\min(\mathcal{A}_{k+1}) - \max(\mathcal{A}_k)) = \infty$.

Proof. For any finite subset A of ω, $\max(A)_n \leqslant |A|$ for all n. So finite subsets of ω are very thin.
Let A be an infinite very thin subset of ω. There exist a sub-collection $\{A_n : n \in \omega\}$ of non-empty finite subsets of ω and $M \in \omega$ such that $(\max(A_{n+1}) - \max(A_n)) > 2^n$ and $\max(A\backslash A_n)_n \leqslant M$ for all n. Let $B_k = A \cap \{\max(A_k)+1, ..., \max(A_{k+1})\}$, $k \geqslant 1$. If $|B_k| > 1$ then B_k can be decomposed as

$$B_k = \bigcup_{i=1}^{j_k} B_{ki}$$

such that $1 \leqslant |B_{ki}| \leqslant M$, $1 \leqslant i \leqslant j_k$ and

$$\min(B_{k(i+1)}) - \max(B_{ki}) > k, 1 \leqslant i \leqslant j_k - 1.$$

Thus, A can be expressed in such way that A satisfies $(V1)$, $(V2)$ and $(V3)$ where $\mathcal{M} = \max\{M, \max A_1\}$.
Conversely, let A be an infinite subset of ω so that A satisfies $(V1)$, $(V2)$ and $(V3)$. Then there exists an infinite subset $\{n_1 < n_2 < n_3 < ...\}$ of ω such that

$$(\min(\mathcal{A}_{m+1}) - \max(\mathcal{A}_m)) > k \text{ for all } m > n_k.$$

Let $A_k = \{1, ..., \min(\mathcal{A}_{n_k+1})\}$, $k \geqslant 1$. Then $\max(A\backslash A_k)_k \leqslant \mathcal{M}$ for all k. □

Example 2.3. *Let* $A = \bigcup_{k\in\omega} A_k$*, where* $A_k = \{2^k, 2^k + 1, ..., 2^k + k\}$*. Then* $\frac{A(n)}{n} \leqslant \frac{(k+1)(k+2)}{2^k}$ *for* $2^k \leqslant n < 2^{k+1}$*. So A is thin. If* $A = \bigcup_{k\in\omega} B_k$ *where* $1 \leqslant |B_k| \leqslant M$ *for all* $k \in \omega$ *for some* $M \in \omega$ *and* $\min(B_{k+1}) - \max(B_k) > 0$ *for all* $k \in \omega$ *then there exists a subset* $\{n_1 < n_2 < n_3 < ...\}$ *of* ω *such that* $\lim_{k\to\infty}(\min(B_{n_k+1}) - \max(B_{n_k})) = 1$*. Therefore, A is not very thin.*

The Prime number theorem implies that set of prime numbers is thin (see in **[2]**). Whereas it can be shown that set of prime numbers is not very thin by using the Prime k-tuples conjecture which holds for a positive proportion of admissible k-tuples for each k in ω (see Theorem 1.2 in **[11]**) . As in **[11]**, a set $\mathcal{D} = \{d_1, ..., d_k\}$ consisting of non-negative integers is called admissible set if for any prime p, there is an integer b_p such that $b_p \not\equiv d \pmod p$ for all $d \in \mathcal{D}$.

Example 2.4. *Let* p_n *denotes the n-th prime. Then* $\{p_1, p_1p_2, ..., p_1p_2...p_k\}$ *is an admissible set.*

The statement of Prime k-tuples conjecture is given as in **[11]**:

Conjecture 2.5 (Prime k-tuples conjecture). *Let* $\mathcal{D} = \{d_1, ..., d_k\}$ *be an admissible set. Then there are infinitely many integers h such that* $\{h + d_1, ..., h + d_k\}$ *is a set of primes.*

Result 1. *Set of prime numbers is not very thin.*

Proof. Let P be the set of all primes and let $P = \bigcup_{k\in\omega} A_k$ where $1 \leqslant |A_k| \leqslant M$ for all $k \in \omega$ for some $M \in \omega$ and $\min(A_{k+1}) - \max(A_k) > 0$ for all $k \in \omega$. Then there exists an admissible set $\{d_1, ...d_{M+1}\}$.
Let $G = \max\{d_{i+1} - d_i : i = 1, ..., M\}$. By Prime k-tuples conjecture, there should be infinitely many $(M+1)$-tuple of primes $(p + d_1, ..., p + d_{M+1})$. Therefore, $(\min(A_{k+1}) - \max(A_k)) \leq G$ for infinitely many $k \in \omega$. So, P is not very thin. □

Definition 2.6. *A subset A of* ω *is super lacunary if A is finite or* $\sum_{k=1}^{\infty} \frac{1}{(n_{k+1} - n_k)} < \infty$ *if* $A = \{n_1 < n_2 < n_3 < ...\}$.

Definition 2.7. *A subset A of* ω *is very very thin if A is finite or A can be written as follows:*
(i) $A = \bigcup_{k\in\omega} \mathcal{A}_k$ *where* $1 \leqslant |\mathcal{A}_k| \leqslant \mathcal{M}$ *for some* $\mathcal{M} \in \omega$ *for all* $k \in \omega$,
(ii) $\min(\mathcal{A}_{k+1}) - \max(\mathcal{A}_k) > 0$ *for all* $k \in \omega$,
(iii) $\sum_{k=1}^{\infty} \frac{1}{(\min(\mathcal{A}_{k+1}) - \max(\mathcal{A}_k))} < \infty$.

Example 2.8. $\bigcup_{k\in\omega}\{2^k, 2^k + k\}$ *is lacunary (because* $2^k + k - 2^k = k \to \infty$ *and* $2^{k+1} - (2^k + k) = 2^k - k \to \infty$ *as* $k \to \infty$*) and very very thin (one may consider* $\mathcal{A}_k = \{2^k, 2^k + k\}$, $k \geqslant 1$ *then* $\min(\mathcal{A}_{k+1}) - \max(\mathcal{A}_k) = 2^k - k$ *and* $\sum_{k=1}^{\infty} \frac{1}{2^k - k} < \infty$*) but not super lacunary since the series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *is not convergent.*

Example 2.9. *Let* $X = \{b_1, b_2, b_3, ...\}$ *and* $Y = \{b_1, b_1 + 1, b_2, b_3, b_3 + 1, b_4, ...\}$ *where* $b_k = 1 + ... + k$*. Let* $X = \bigcup_{k\in\omega} X_k$ *where* $1 \leqslant |X_k| \leqslant M$ *for all* $k \in \omega$ *for some* $M \in \omega$*. Then* $(\min(X_{k+1}) - \max(X_k)) \leq (b_{l+1} - b_l) = l + 1$ *for some* $l \leq kM$*. Hence, for all* $k \geqslant 1$,

$$\frac{1}{k+1} \leq \frac{M}{\min(X_{k+1}) - \max(X_k)}.$$

Therefore, X is not very very thin but lacunary. Since $X \subset Y$*, Y is not very very thin but very thin.*

Lemma 2.10. *Union of two lacunary sets is very thin.*

Proof. Suppose $S = \{s_1 < s_2 < s_3 < ...\}$ and $T = \{t_1 < t_2 < t_3 < ...\}$ are lacunary subsets of ω. For each $i \in \omega$, construct a set containing s_i and the smallest number t of T such that $s_i \leqslant t < \frac{s_i+s_{i+1}}{2}$ if such a number t exists and the largest number t of T such that $\frac{s_{i-1}+s_i}{2} < t \leqslant s_i$ if such a number t exists. Leave all remaining elements of T as singleton. Then $S \cup T$ can be decomposed into the sets A_k such that
(i) $S \cup T = \bigcup_{k\in\omega} A_k$ where $1 \leqslant |A_k| \leqslant 3$ for all $k \in \omega$,
(ii) for all $k \in \omega$,

$$(\min(A_{k+1}) - \max(A_k)) \geqslant (t_{i+1} - t_i) \text{ for some } i \in \omega$$

or

$$(\min(A_{k+1}) - \max(A_k)) \geqslant \frac{(s_{j+1} - s_j)}{2} \text{ for some } j \in \omega.$$

Therefore, $S \cup T$ is very thin. □

Theorem 2.11. *Union of two super lacunary sets is very very thin.*

Proof. If S and T are two super lacunary subsets of ω, following the proof of Lemma 2.10 one will get in addition that

$$\sum_{k=1}^{\infty} \frac{1}{(\min(A_{k+1}) - \max(A_k))} \leqslant \sum_{i=1}^{\infty} \frac{1}{\frac{(s_{i+1}-s_i)}{2}} + \sum_{i=1}^{\infty} \frac{1}{(t_{i+1} - t_i)}.$$

□

Lemma 2.12. *Union of a lacunary and a very thin subset of ω is very thin.*

Proof. Let S be a very thin and $T = \{t_1 < t_2 < t_3 < ...\}$ be a lacunary subset of ω. Let $S = \bigcup_{k\in\omega} A_k$ where $1 \leqslant |A_k| \leqslant M$ for all $k \in \omega$ for some $M \in \omega$ and $\min(A_{k+1}) - \max(A_k) > 0$ for all $k \in \omega$ with $\lim_{k\to\infty}(\min(A_{k+1}) - \max(A_k)) = \infty$. For every $i \in \omega$, construct a set B_i containing A_i and the smallest number t of T such that $\max(A_i) \leqslant t < \frac{\max(A_i)+\min(A_{i+1})}{2}$ if such a number t exists and the largest number t of T such that $\frac{\min(A_{i-1})+\max(A_i)}{2} < t \leqslant \min(A_i)$ if such a number t exists and elements t of T such that $\min(A_i) < t < \max(A_i)$. Now each B_i can be decomposed as

$$B_i = \bigcup_{k=1}^{j(i)} B_{ik}$$

such that

$$\min(B_{i(k+1)}) - \max(B_{ik}) > 0, 1 \leqslant k \leqslant j(i) - 1,$$
$$\max(B_{ik}) \in T \text{ for } 1 \leqslant k < j(i) \text{ and } \min(B_{ik}) \in T \text{ for } 1 < k \leqslant j(i)$$

and there is no $r \in \omega$ so that $t_r, t_{r+1} \in B_{ik}$ if there does not exist any $s \in A_i$ satisfying $t_r < s < t_{r+1}$. Leave all remaining elements of T as singleton, $S \cup T$ can be decomposed into the sets D_k such that
(i) $S \cup T = \bigcup_{k\in\omega} D_k$ where $1 \leqslant |D_k| \leqslant 2M + 1$ for all $k \in \omega$,
(ii) for all $k \in \omega$,

$$(\min(D_{k+1}) - \max(D_k)) \geqslant (t_{i+1} - t_i) \text{ for some } i \in \omega$$

or

$$(\min(D_{k+1}) - \max(D_k)) \geqslant \frac{(\min(A_{j+1}) - \max(A_j))}{2} \text{ for some } j \in \omega.$$

Therefore, $S \cup T$ is very thin. □

Theorem 2.13. *Union of a super lacunary and a very very thin subset of ω is very very thin.*

Proof. If S is very very thin and T is super lacunary then using the proof of Lemma 2.12 one will obtain

$$\sum_{k=1}^{\infty} \frac{1}{(\min(D_{k+1}) - \max(D_k))} \leqslant \sum_{i=1}^{\infty} \frac{1}{\frac{(\min(A_{i+1}) - \max(A_i))}{2}} + \sum_{i=1}^{\infty} \frac{1}{(t_{i+1} - t_i)}$$

as well. □

Theorem 2.14. *A subset A of ω is very thin if and only if A can be expressed as a finite union of lacunary subsets of ω.*

Proof. Suppose A is a very thin subset of ω such that
(i) $A = \bigcup_{k \in \omega} A_k$ where $1 \leqslant |A_k| \leqslant M$ for all $k \in \omega$ for some $M \in \omega$,
(ii) $\min(A_{k+1}) - \max(A_k) > 0$ for all $k \in \omega$,
(iii) $\lim_{k \to \infty} (\min(A_{k+1}) - \max(A_k)) = \infty$.

Let $A_k = \{a_{k1} \leqslant a_{k2} \leqslant ... \leqslant a_{kM}\}, k \in \omega$. Define $B_i = \{a_{ki} : k \in \omega\}, 1 \leqslant i \leqslant M$. Then $A = \bigcup_{i=1}^{M} B_i$ and

$$(a_{(k+1)i} - a_{ki}) \geqslant (a_{k+1)1} - a_{kM}) = (\min(A_{k+1}) - \max(A_k))$$

Therefore, B_i is lacunary for $1 \leqslant i \leqslant M$.
Converse part follows directly from Lemma 2.10 and Lemma 2.12. □

Corollary 2.15. *Finite union of very thin subsets of ω is very thin.*

Corollary 2.16. *Very thin subsets of ω are thin.*

Theorem 2.17. *Any very thin set can be expressed as a finite intersection of thin but non very thin sets.*

Proof. Suppose $S = \{t_1 < t_2 < t_3 < ...\}$ is a lacunary subset of ω. Take two disjoint thin but not very thin sets A and B $\Big($ one may take $A = \bigcup_{k \geqslant 1} \{t_{n_k}, ..., t_{n_k} + k\}$ and $B = \bigcup_{k \geqslant 1} \{t_{n_k+1} - k, ..., t_{n_k+1}\}$ where $(n_k)_{k \geqslant 1}$ is a strictly increasing sequence of natural numbers such that $t_{n_1+1} - t_{n_1} > 2$ and $t_{n_k} > 2t_{n_{k-1}}$ and $t_{n_k+1} - t_{n_k} > 2k$ for $k \geq 2$ and let $u_k = \frac{(k+1)(k+2)}{t_{n_k}}, k \geqslant 1$ then for $t_{n_k} \leq r < t_{n_{k+1}}$, $d_r(T) \leq \frac{\sum_{i=1}^{k}(i+1)}{r} \leq \frac{(k+1)(k+2)}{t_{n_k}} = u_k$ where $T = A$ or B and so $\lim_{k \to \infty} u_k = 0$ as $\overline{\lim} \frac{u_{k+1}}{u_k} \leq \frac{1}{2}\Big)$. Let $A' = A \cup S$ and $B' = B \cup S$. Since S is lacunary, A' and B' are thin but not very thin with $A' \cap B' = S$. Hence any lacunary set can be expressed as intersection of two thin but non very thin sets. From Theorem 2.14, it follows that any very thin set can be expressed as a finite intersection of thin but non very thin sets. □

Theorem 2.18. *A subset A of ω is very very thin if and only if A can be expressed as a finite union of super lacunary subsets of ω.*

Proof. If A is very very thin then each B_i defined in Theorem 2.14 becomes super lacunary.
Theorem 2.11 and Theorem 2.13 together implies that finite union of super lacunary sets is very very thin. □

Corollary 2.19. *Finite union of very very thin subsets of ω is very very thin.*

From Corollary 2.16, it follows that very thin sets are thin. Since zero uniform density **([13, 14])** subsets of ω are thin, it is natural to arise a question that whether very thin sets have uniform density zero or zero uniform density sets are very thin. Let $B \subset \omega$. For $h \geqslant 0$ and $k \geqslant 1$, let $A(h+1, h+k) = |\{n \in B : h+1 \leqslant n \leqslant h+k\}|$. The existence of the following limits are proved in **[13]**:

$$\underline{u}(B) = \lim_{k\to\infty} \frac{1}{k} \liminf_{h\to\infty} A(h+1, h+k)\ ,\ \overline{u}(B) = \lim_{k\to\infty} \frac{1}{k} \limsup_{h\to\infty} A(h+1, h+k)$$

$\underline{u}(B)$ and $\overline{u}(B)$ are called lower and upper uniform density of B respectively. If $\underline{u}(B) = \overline{u}(B)$ then $u(B) = \overline{u}(B) = \underline{u}(B)$ is called uniform density of B. From now on call B is uniformly thin if $u(B) = 0$. Since $\underline{u}(B) \leqslant \underline{d}(B) \leqslant \overline{d}(B) \leqslant \overline{u}(B)$, B is thin if B is uniformly thin, but the converse is not true (see in **[13]**).

Theorem 2.20. *Any very thin subset of ω is uniformly thin.*

Proof. Suppose A be a very thin subset of ω such that $A = \bigcup_{k\in\omega} A_k$ where $1 \leqslant |A_k| \leqslant M$ for all $k \in \omega$ for some $M \in \omega$, $\min(A_{k+1}) - \max(A_k) > 0$ for all $k \in \omega$ and $\lim_{k\to\infty}(\min(A_{k+1}) - \max(A_k)) = \infty$. Let $n_k = \min(A_{k+1}) - \max(A_k)$, $k \in \omega$.
Let k_1 = the least element of ω such that $n_{k_1} \leqslant n_k$ for all $k \in \omega$ and let
k_{r+1} = the least element of $\omega - \{k_1, ..., k_r\}$ such that $n_{k_{r+1}} \leqslant n_k$ for all $k \in \omega - \{k_1, ..., k_r\}$, $r > 1$.
Then $\lim_{r\to\infty} n_{k_r} = \infty$ and so $\lim_{r\to\infty} \frac{r}{n_{k_1} + ... + n_{k_r}} = 0$.
By induction, it can be shown that for any $l \geqslant 1$ and for any l elements $m_1, ..., m_l$ of $\omega - \{k_1, ..., k_l\}$,

$$n_{k_1} + ... + n_{k_l} \leqslant n_{m_1} + ... + n_{m_l}.$$

Let $s_l = n_{k_1} + ... + n_{k_l}$. Then for any l, $\{n \in A : t+1 \leqslant n \leqslant t + s_l + 1\}$ can intersect at most l consecutive A_k for eventually many $t \in \omega$. Therefore,

$$\limsup_{t\to\infty} |\{n \in A : t+1 \leqslant n \leqslant t + s_l + 1\}| \leqslant Ml$$

and so

$$\lim_{l\to\infty} \frac{1}{s_l + 1} \limsup_{t\to\infty} |\{n \in A : t+1 \leqslant n \leqslant t + s_l + 1\}| \leqslant \lim_{l\to\infty} \frac{Ml}{s_l + 1} = 0$$

Hence $\overline{u}(A) = 0$ i.e. $u(A) = 0$. □

Example 2.21 shows that uniformly thin set may not be very thin.

Example 2.21. *Let $a_1 = 1$ and let $a_r = a_{r-1} + 2(1^3 + ... + (r-1)^3) + 1, r \in \omega$ and $r \geqslant 2$.*
Define $A_s = \{a_s, a_s + 1^3, a_s + 1^3 + 2^3, ..., a_s + 1^3 + 2^3 + ... + s^3\}$, $s \geqslant 1$. Let

$$A = \bigcup_{s=1}^{\infty} A_s$$

Let $b_n = 1^3 + 2^3 + ... + n^3$ and $s_{m,n} = |\{r \in A : m+1 \leqslant r \leqslant m + b_n + 1\}|$, $n \geqslant 1$, $m \geqslant 0$.
Suppose $k \in \omega$. Then

$$s_{m,n} \leq (n+1) \text{ if } a_{n+k} + 1^3 + ... + (n+k)^3 \leqslant m \leqslant a_{n+k+1}.$$

If $1 \leqslant i \leqslant k$ and $a_{n+k} + 0^3 + 1^3 + ... + (i-1)^3 \leqslant m < a_{n+k} + 1^3 + ... + (i)^3$ then $s_{m,n} < (n+1)$ for $m + b_n + 1 < a_{n+k} + 1^3 + ... + (n+i)^3$.
Again

$$s_{m,n} < (n+1) \text{ if } a_{n+k} + 1^3 + ... + k^3 \leqslant m < a_{n+k} + 1^3 + ... + (n+k)^3$$

because in this case $m + b_n + 1 < a_{n+k+1}$. *Thus,*

$$s_{m,n} \leqslant (n+1) \text{ if } a_n \leqslant m.$$

Also, $s_{m,n} \leqslant |A_1| + ... + |A_n| \leqslant \frac{(n+1)(n+2)}{2}$ *if* $0 \leqslant m < a_n$.
Therefore,

$$\lim_{n\to\infty} \frac{1}{b_n+1} \limsup_{t\to\infty} |\{r \in A : m+1 \leqslant r \leqslant m + b_n + 1\}| = 0.$$

Hence, A is uniformly thin.
Suppose $A = \bigcup_{k\in\omega} B_k$ *where* $1 \leqslant |B_k| \leqslant M$ *for all* $k \in \omega$ *for some* $M \in \omega$, $\min(B_{k+1}) - \max(B_k) > 0$ *for all* $k \in \omega$ *and* $\lim_{k\to\infty}(\min(B_{k+1}) - \max(B_k)) = \infty$. *Let* $T \in \omega$ *such that* $T > M$. *Then there is* $l \in \omega$ *so that* $(\min(B_{k+1}) - \max(B_k)) > T^3$ *for all* $k \geqslant l$. *So there exists a* $k \geqslant l$ *such that* B_k *contains first* $(T+1)$ *elements of* A_r *for some* $r > T+1$ *which contradicts* $|B_k| \leqslant M$. *Therefore, A is not very thin.*

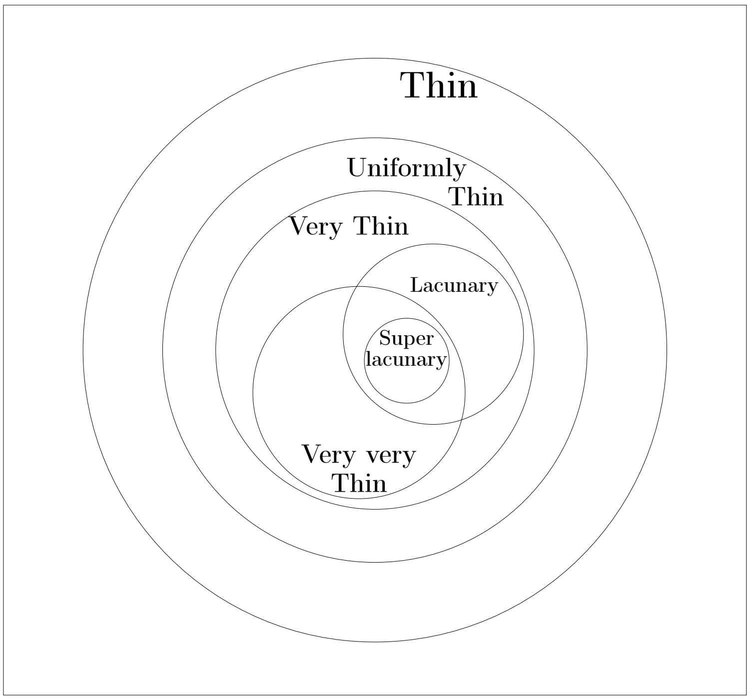

Figure 1:

Figure 1 shows the relation among various thin sets.

3. Influence of very thin sets in convergence

In this section mainly the Fin-BW property and BW property of the ideal $\mathcal{I}_v$ consisting of very thin subsets of ω are discussed. As in **[5, 12, 14–17, 19]**, an ideal $\mathcal{I}$ on ω is a family of subsets of natural numbers close under taking subsets and finite unions and a real sequence $(x_n)_{n\in A}$, $A \subset \omega$, is $\mathcal{I}$-convergent to a real number a if for any $\epsilon > 0$, $\{n \in A : |x_n - a| \geqslant \epsilon\} \in \mathcal{I}$. A real sequence $(a_n)_{n\in\omega}$ is $\mathcal{I}^*$ convergent **[16, 19]** to a real number a if there exists a subset K of ω such that $\omega \setminus K \in \mathcal{I}$ and $(a_n)_{n\in K}$ converges to a. An ideal $\mathcal{I}$ on ω has the Fin-BW property **[12, 14]** if for any bounded sequence $(x_n)_{n\in\omega}$ of real numbers there is $A \notin \mathcal{I}$ such that $(x_n)_{n\in A}$ is convergent and the BW property **[12, 14, 15]** if for any bounded sequence $(x_n)_{n\in\omega}$ of real numbers there is $A \notin \mathcal{I}$ such that $(x_n)_{n\in A}$ is $\mathcal{I}$-convergent. If an ideal $\mathcal{I}$ has Fin-BW property then it also has BW property.

Recall that $2^\omega, 2^{<\omega}$ and 2^n denote the set of all infinite sequences of zeros and ones, the set of all finite sequences of zeros and ones and the set of all sequences of zeros and ones of length n respectively. If $s \in 2^n$, then $s\hat{\ }i$ denotes the sequence of length $n+1$ which extends s by i for $i \in \omega$. If $x \in 2^\omega$ then $x \upharpoonright n = (x(0), x(1), ..., x(n-1))$ for $n \in \omega$. From Proposition 3.3 in **[12]** the following characterizations of BW property and Fin-BW property can be obtained which is also given in **[14, 15]**.

Proposition 3.1. *An ideal $\mathcal{I}$ has the BW property (the Fin-BW property) if and only if for every family of sets $\{A_s : s \in 2^{<\omega}\}$ satisfying the following conditions*
$(S1) A_\emptyset = \omega$,
$(S2) A_s = A_{s\hat{\ }0} \cup A_{s\hat{\ }1}$,
$(S3) A_{s\hat{\ }0} \cap A_{s\hat{\ }1} = \emptyset$,
there exist $x \in 2^\omega$ and $B \subset \omega$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\upharpoonright n} \in \mathcal{I}$ ($B \setminus A_{x\upharpoonright n}$ is finite respectively) for all n.

Example 3.2. *Define a family of sets $\{A_s : s \in 2^{<\omega}\}$ as follows:*

$$A_\emptyset = \omega,$$

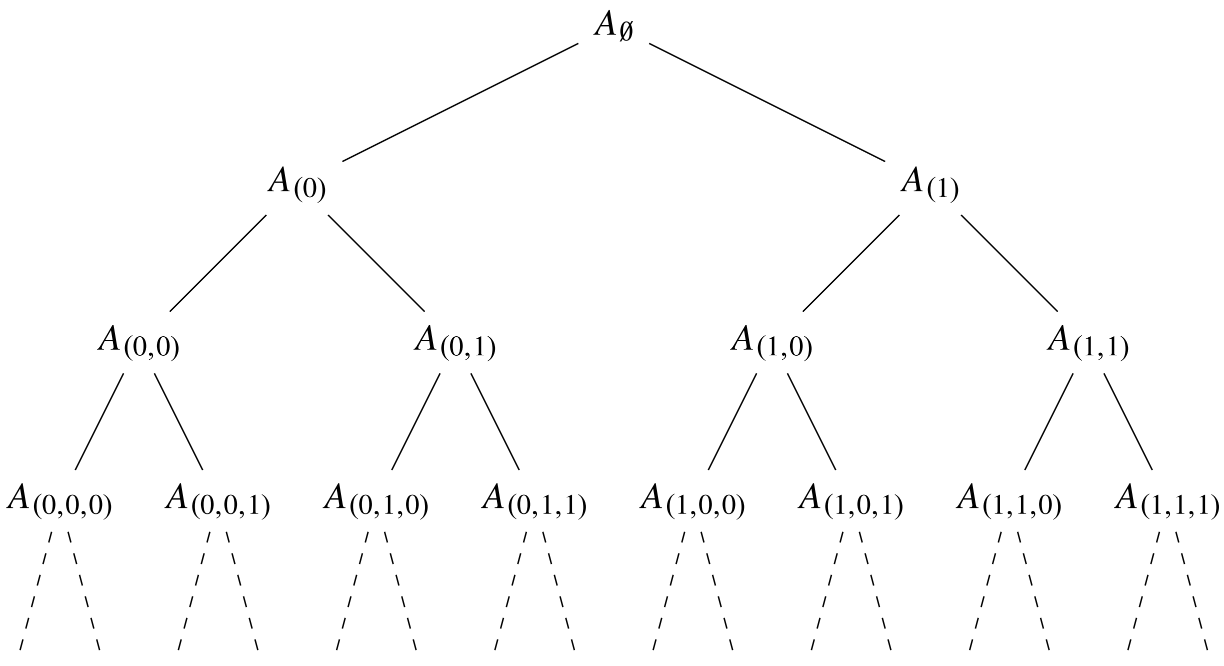

Figure 2:

$$A_{(0)} = 2\omega,\ A_{(1)} = 2\omega - 1,$$

$$A_{(0,0)} = 2^2\omega,\ A_{(0,1)} = 2^2\omega - 2,\ A_{(1,0)} = 2^2\omega - 1\ ,A_{(1,1)} = 2^2\omega - 3,$$

$$A_{(0,0,0)} = 2^3\omega,\ A_{(0,0,1)} = 2^3\omega - 4,\ A_{(0,1,0)} = 2^3\omega - 2,\ A_{(0,1,1)} = 2^3\omega - 6,$$
$$A_{(1,0,0)} = 2^3\omega - 1,\ A_{(1,0,1)} = 2^3\omega - 5,\ A_{(1,1,0)} = 2^3\omega - 3,\ A_{(1,1,1)} = 2^3\omega - 7$$

and so on.
So if $s \in 2^n$ *then* $A_s = 2^n\omega - i$, $0 \leqslant i \leqslant 2^n - 1$ *and*

$$A_{s^\frown 0} = 2^n(2\omega) - i,\ A_{s^\frown 1} = 2^n(2\omega - 1) - i.$$

Let $x \in 2^\omega$. *Then* $\bigcap_{n\in\omega} A_{x\upharpoonright n} = \emptyset$ *and* $(A_{x\upharpoonright n} \setminus A_{x\upharpoonright n+1})_{n\in\omega}$ *is a collection of mutually disjoint sets so that numbers in each* $A_{x\upharpoonright n} \setminus A_{x\upharpoonright n+1}$ *are in arithmetic progressions.*
Let $N = \{n_0 < n_1 < n_2 < n_3 < ...\}$ *be an infinite subset of* ω *and let* a_{n_k} *be the* n_k^{th} *element in the arithmetic progression formed by elements of* $A_{x\upharpoonright k} \setminus A_{x\upharpoonright k+1}$, $k \geqslant 0$.
Let $Ar_N = \bigcup_{k\geqslant 0}\{$*first* n_k *numbers in the arithmetic progression of elements of* $A_{x\upharpoonright k} \setminus A_{x\upharpoonright k+1}\}$.
If a *and* b *are consecutive numbers in* Ar_N *where* $a < b$ *and* $a_{n_k} < a \leqslant a_{n_{k+1}}$ *then* $b - a = 2^{k+1}$, $k \geqslant 0$. *So,* Ar_N *is lacunary.*
Suppose $B \subset \omega$ *such that* $B \setminus A_{x\upharpoonright n}$ *is finite for all* n. *Since* $\bigcap_{n\in\omega} A_{x\upharpoonright n} = \emptyset$, $B = \bigcup_{n\in\omega} B \setminus A_{x\upharpoonright n} = \bigcup_{n\in\omega} B \cap (A_{x\upharpoonright n} \setminus A_{x\upharpoonright n+1})$.
Therefore, $B \subset Ar_N$ *for some infinite subset* N *of* ω *and so* B *is lacunary.*

Theorem 3.3. $\mathcal{I}_v$ *= Ideal comprising very thin subsets of* ω *satisfies the BW property.*

Proof. Let A be a subset of ω and $M \in \omega$. Define
$(A)_M = \{1\} \cup \{n \in \omega$: $n > 1$ and there exist n consecutive elements of A such that difference between any two consecutive among them is less than or equal to $M\}$. Then A is very thin implies $(A)_M$ is finite for all $M \in \omega$. Let $\{A_s : s \in 2^{<\omega}\}$ be a family of sets satisfying the three conditions $S1, S2$ and $S3$ given in Proposition 3.1. This theorem can be proved in the following cases.

Case 1: Suppose there exist $x \in 2^\omega$ and $M \in \omega$ such that $(A_{x\upharpoonright n})_M$ is infinite for all $n \in \omega$.
Let $B_0 = \{1\}$ and define B_{n+1} to be the set of $(n+1)$ consecutive elements of $A_{x\upharpoonright n+1}$ such that difference between each two consecutive among them is $\leqslant M$ so that $\max(B_n) < \min(B_{n+1})$, $n \in \omega$.
Let $B = \bigcup_{n\in\omega} B_n$. Then B is not very thin and $B \setminus A_{x\upharpoonright n} \subset \bigcup_{i=0}^{n-1} B_i$ is finite for $n > 0$.
Case 2: Suppose that for every $x \in 2^\omega$ and $M \in \omega$ there exists $n \in \omega$ such that $(A_{x\upharpoonright n})_M$ is finite.
So there exist $k_0 > 0$ and $s_0 \in 2^{k_0}$ such that A_{s_0} is not very thin and $(A_{s_0})_1$ is finite. Then A_s is infinite for some $s \in 2^{k_0} \setminus \{s_0\}$ and let $M_0 = \max(A_{s_0})_1$.
As $\max(A_{s_0})_1 = M_0$, due to the hypothesis there exist $k_1 \in \omega$ with $k_0 < k_1$ and $s_1 \in 2^{k_1}$ such that (see Figure 2)
(i) A_{s_0} and A_{s_1} are disjoint and $(A_{s_1})_{M_0+1}$ is finite,
(ii) there is an infinite subset $\mathcal{A}_1$ of A_{s_0} such that if $p \in \mathcal{A}_1$ then $p + i \in A_{s_1}$ for some i, $1 \leqslant i \leqslant M_0$ and if $p < q < p + i$ then $q \in A_{s_0}$,
(iii) $\max(A_{s_0} \cup A_{s_1})_1 \leqslant (M_0 + 1)M_1 + M_0 = N_1$ where $M_1 = \max(A_{s_1})_{M_0+1}$.
Let $B_1 = \{a_1^0, a_1^1\}$ where $a_1^i \in A_{s_i}$, $i \in \{0, 1\}$ and $(a_1^1 - a_1^0) \leqslant M_0$.

Because of $\max(A_{s_0} \cup A_{s_1})_1 \leqslant N_1$ and the supposition, there exist a $k_2 \in \omega$ with $k_1 < k_2$ and $s_2 \in 2^{k_2}$ such that
(i) A_{s_0}, A_{s_1} and A_{s_2} are mutually disjoint and $(A_{s_2})_{N_1+1}$ is finite,
(ii) there is an infinite subset $\mathcal{A}_2$ of $\mathcal{A}_1$ such that if $p \in \mathcal{A}_2$ then

$$p + i \in A_{s_1} \text{ and } p + i + j \in A_{s_2} \text{ for some } i, j,\ 1 \leqslant i \leqslant M_0 \text{ and } 2 \leqslant i + j \leqslant N_1$$

so that if $p < q < p + i$ then $q \in A_{s_0}$ and if $p + i < q < p + i + j$ then $q \in A_{s_0} \cup A_{s_1}$,
(iii) $\max(A_{s_0} \cup A_{s_1} \cup A_{s_2})_1 \leqslant (N_1 + 1)M_2 + N_1 = N_2$ where $M_2 = \max(A_{s_2})_{N_1+1}$.
Let

$$B_2 = \{a_2^0, a_2^1, a_2^2\} \text{ where } a_2^i \in A_{s_i},\ i \in \{0, 1, 2\} \text{ and } (a_2^1 - a_2^0) \leqslant M_0,\ (a_2^2 - a_2^1) \leqslant N_1 \text{ with } (a_2^0 - a_1^1) > 2.$$

Continuing in this way an infinite subset $\{k_0 < k_1 < k_2 < ...\}$ of ω, a sequence $(s_i)_{i\in\omega}$ such that $s_i \in 2^{k_i}$, mutually disjoint sets $\{A_{s_i} : i \in \omega\}$, a collection of infinite sets $\{A_{s_0} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset ...\}$, a collection of finite sets $\{B_i : i \geqslant 1\}$, a sequence $(M_i)_{i\in\omega}$ and an infinite subset $\{M_0 = N_0 < N_1 < N_2 < ...\}$ of ω will be obtained such that for $n \in \omega$
(i) $M_{n+1} = \max(A_{s_{n+1}})_{N_n+1}$,

(ii) $\mathcal{A}_{n+1}$ infinite subset of $\mathcal{A}_n$ such that if $p \in \mathcal{A}_{n+1}$ then $p + i_0 + i_1 + ... + i_k \in A_{s_{k+1}}$ for some i_k with $k + 1 \leqslant i_0 + i_1 + ... + i_k \leqslant N_k$, $0 \leqslant k \leqslant n$ so that if $p < q < p + i_0$ then $q \in A_{s_0}$ and if $p + i_0 + i_1 + ... + i_{k-1} < q < p + i_0 + i_1 + ... + i_k$ then $q \in A_{s_0} \cup A_{s_1} \cup ... \cup A_{s_k}$ where $1 \leqslant k \leqslant n$,

(iii) $\max(A_{s_0} \cup A_{s_1} \cup ... \cup A_{s_{n+1}})_1 \leqslant (N_n + 1)M_{n+1} + N_n = N_{n+1}$ and

(iv) $B_{n+1} = \{a^0_{n+1}, a^1_{n+1}, ..., a^{n+1}_{n+1}\}$ where $a^i_{n+1} \in A_{s_i}$ and $(a^{i+1}_{n+1} - a^i_{n+1}) \leqslant N_i$ for $0 \leqslant i \leqslant n$ and for $n \geqslant 1$, $a^0_{n+1} - a^n_n > 2^n$.

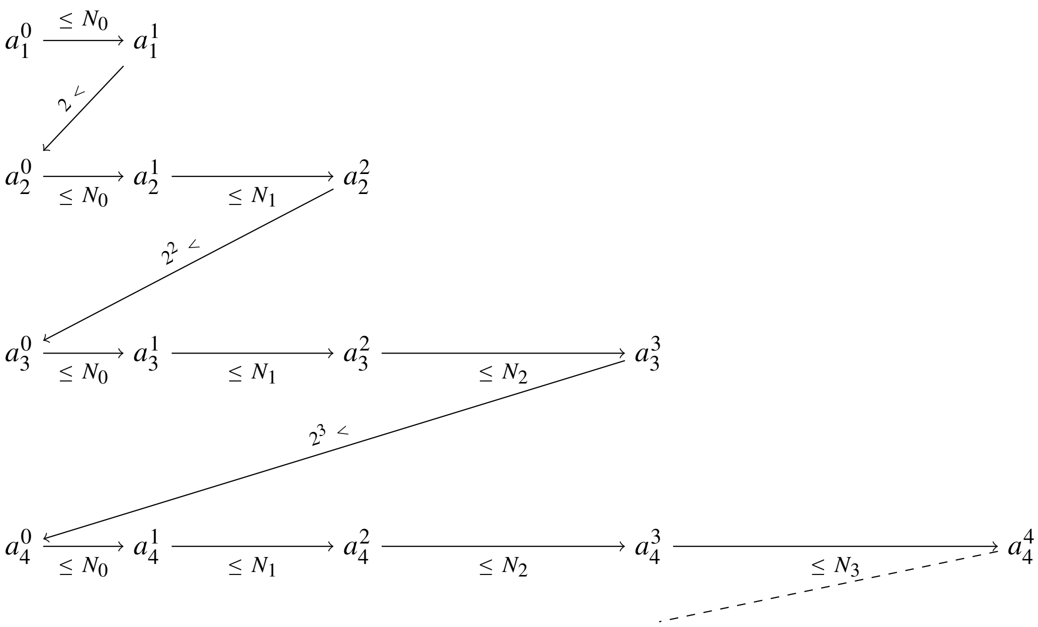

Figure 3:

Let $B = \bigcup_{n=1}^{\infty} B_n$. Then B is not very thin and $B \cap A_{s_i}$ is lacunary for all $i \in \omega$. Moreover, if M is an infinite subset of $\{s_i : i \in \omega\}$ then $B_M = \bigcup_{s\in M} B \cap A_s$ is not very thin (see Figure 3).
Define $x \in 2^\omega$ such that there are infinitely many $i \in \omega$ so that $A_{s_i} \subset A_{x\restriction n}$, $n \in \omega$. If there exists $m \in \omega$ such that there is no $i \in \omega$ so that $A_{s_i} \subset A_{x\restriction n} \backslash A_{x\restriction n+1}$ for any $n \geqslant m$ then one can take $M = \{s_i : A_{s_i} \subset A_{x\restriction m+1}\}$. If there does not exist such m, construct M by taking least s_i such that $A_{s_i} \subset A_{x\restriction n} \backslash A_{x\restriction n+1}$ (if such s_i exists) for $n \in \omega$. In both cases M is infinite and so B_M is not very thin with $B_M \backslash A_{x\restriction n}$ is very thin for all n. □

Remark 3.4. *Let*

$$r_n = 0 + 1 + ... + n, n \in \omega,$$
$$p_n = N_0 + N_1 + ... + N_{r_n}, n \in \omega \text{ and}$$
$$q_0 = N_0 \text{ and } q_n = q_{n-1} + p_n, n \geqslant 1$$

Define a set $D = \bigcup_{i=0}^{\infty} D_i$ *as follows:*

$$D_0 = \bigcup_{i=1}^{q_0} B_i$$

and for $n \geqslant 1$,

$$D_n = \bigcup_{i=q_{n-1}+1}^{q_n} B_i \Big\backslash \bigcup_{i=0}^{n-1} A_{s_i}$$

Suppose $D = \bigcup_{k\in\omega} \mathcal{D}_k$ *where* $1 \leqslant |\mathcal{D}_k| \leqslant L$ *for all* $k \in \omega$ *for some* $M \in \omega$, $\min(\mathcal{D}_{k+1}) - \max(\mathcal{D}_k) > 0$ *for all* $k \in \omega$.

Let $n_k = \min(\mathcal{D}_{k+1}) - \max(\mathcal{D}_k)$, $k \in \omega$. *There exists* $K \in \omega$ *such that there are more than L numbers in each* $B_i \Big\backslash \bigcup_{i=0}^{n-1} A_{s_i}$, $q_{n-1} + 1 \leqslant i \leqslant q_n$ *for all* $n \geqslant K$.

So there exists a sequence $(\mathcal{E}_n)_{n\in\omega}$ *of mutually disjoint subsets of* ω *such that* $|\mathcal{E}_n| = p_{K+n}$ *and* $\frac{1}{N_{L+K-1+n}} \leqslant \frac{1}{n_k}$ *for all* $k \in \mathcal{E}_n$, $n \in \omega$. *Since*

$$\frac{p_{K+n}}{N_{L+K-1+n}} > 1 \text{ eventually},$$

D is not very very thin and also $D \cap A_{s_i}$ *is finite for all* $i \in \omega$. *Similarly, one can construct a non very very thin set* D_M *such that* $D \cap A_s$ *is finite for all* $s \in M$ *and* $D_M \subset B_M$ *where M is an infinite subset of* $\{s_i : i \in \omega\}$.

Replacing B_M *by* D_M *and defining same* $x \in 2^\omega$ *as in the last case of Theorem 3.3 , it can be shown that* $\mathcal{I}_{vv}$ = *ideal of very very thin subsets of* ω *has the Fin-BW property.*

4. Conclusions

- Like the ideal Fin = collection of all finite subsets of ω, from Theorem 3.3 it follows that $\mathcal{I}_v$ also satisfies the BW property. But $\mathcal{I}_d$ = ideal of all thin subsets of ω and $\mathcal{I}_u$ = ideal of all uniformly thin subsets of ω do not satisfy BW and so Fin-BW property (see Example 4 in **[7]** and Corollary 1 in **[14]**).

- The sequence $(x_n)_{n\in\omega}$ defined by $x_0 = 0$ and $x_n = \frac{1}{n-k!}$ whenever $k! < n \leqslant (k+1)!$ shows that $\mathcal{I}_u$-convergence does not imply $\mathcal{I}_u{}^*$-convergence. As in case of $\mathcal{I}_u$-convergence, $\mathcal{I}_v$-convergence does not imply $\mathcal{I}_v{}^*$-convergence unlike $\mathcal{I}_d$-convergence(see Lemma 1.1 in **[3]**) which is shown by the example given below:
  Consider the sequence $(a_n)_{n\in\omega}$ where

  $$a_n = \begin{cases} \frac{1}{k}, & \text{if } n = \frac{r(r+1)}{2} + k, r \geqslant k \geqslant 1 \\ 0, & \text{if } n = \frac{r(r+1)}{2}, r \geqslant 0 \end{cases}$$

  and $(a_n)_{n\in\omega}$ is $\mathcal{I}_v$-convergent to 0.

- It follows from Figure 1 that in the family of $\mathcal{I}_v$-convergent sequences the family of convergent sequences is totally included, but the family of statistically convergent sequences is partially included. Now the question is 'which of the statistically convergent sequences are excluded from the family of $\mathcal{I}_v$-convergent sequences ?' It can be observed that the family of $\mathcal{I}_v$-convergent sequences successfully excludes the statistically convergent sequences of type described in (**N**) and it carefully includes all the statistically convergent sequences of type (**Y**).

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References

[1] P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, On statistical limit points, Proceedings of the American Mathematical Society 129(2001) 2647–2654.
[2] I. Niven, The asymptotic density of sequences, Bulletin of the American Mathematical Society 57(1951) 420–434.
[3] T. Šalát, On statistically convergent sequences of real numbers, Mathematica Slovaca 30(1980) 139–150.
[4] H. I. Miller, L. Miller-Van Wieren, Some statistical cluster point theorems, Hacettepe Journal of Mathematics and Statistics 44(2015) 1405–1409.
[5] U. Ulusu, E. Dündar, $\mathcal{I}$-lacunary statistical convergence of sequences of sets, Filomat 28(2014) 1567–1574.
[6] J. J. Sember, A. R. Freedman, On summing sequences of 0's and 1's, Rocky Mountain Journal of Mathematics (1981) 419–425.
[7] J. A. Fridy, Statistical limit points, Proceedings of the American Mathematical Society 118(1993) 1187–1192.
[8] G. Di Maio, L. D. R. Kočinac, Statistical convergence in topology, Topology and its Applications 156(2008) 28–45.
[9] J. A. Fridy, On statistical convergence, Analysis 5(1985) 301–313.
[10] A. Caserta, L. D. R. Kočinac, On statistical exhaustiveness, Applied Mathematics Letters 25(2012) 1447–1451.
[11] J. Maynard, Small gaps between primes, Annals of Mathematics 181(2015) 383–413.
[12] R. Filipów, N. Mrożek, I. Recław, P. Szuca, Ideal convergence of bounded sequences, The Journal of Symbolic Logic 72(2007) 501–512.
[13] G. Grekos, V. Toma, J. Tomanová, A note on uniform or Banach density, Annales Mathématiques Blaise Pascal 17(2010) 153–163.
[14] P. Barbarski, R. Filipów, N. Mrożek, P. Szuca, Uniform density u and $\mathcal{I}_u$-convergence on a big set, Mathematical Communications 16(2011) 125–130.
[15] R. Filipów, On Hindman spaces and the Bolzano-Weierstrass property, Topology and its Applications 160(2013) 2003–2011.
[16] P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-convergence, Real Analysis Exchange 26(2000/01) 669–685.
[17] M. Mursaleen, S. Debnath, D. Rakshit, I-statistical limit superior and I-statistical limit inferior, Filomat 31(2017) 2103–2108.
[18] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, Journal of Mathematical Analysis and Applications 245 (2000) 513–527.
[19] P. Kostyrko, M. M. Mačaj, T. Šalát, M. Sleziak. $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, Math Slovaca 55 (4) (2005) 443–464.