



The Ritz numerical method and hybrid functions (block-pulse functions and Legendre polynomials) for a class of two-dimensional time-delay optimal control problems

S. M. Hosseini^a, F. Soltanian^a, K. Mamehrashi^{a,b}

^aDepartment of Mathematics, Payame Noor University, Tehran, Iran

^bMathematics Unit, School of Science and Engineering, University of Kurdistan-Hewler, Erbil, Kurdistan Region, Iraq

Abstract. In this paper, we provided a numerical method to solve a class of two dimensional time-delay optimal control problems (2DTDOCPs) with quadratic cost functional using Ritz method and orthogonal Legendre Block-Pulse functions. First, the state and control vectors are approximated as a series of hybrid functions (block-pulse functions and Legendre polynomials) with unknown coefficients. Then, we derive an equation with unknown coefficients by substituting these approximations in the cost functional. A system of algebraic equations is obtained by applying the optimal conditions for this equation. Solving this system and substituting the coefficients into approximating the guessed functions, the state and control functions are obtained. By increasing the number of blocks, as well as the basic functions, we get more accurate solutions. The convergence of proposed method is discussed, and finally, we will present some examples to show the validity and applicability of proposed method, and evaluate its accuracy and efficiency. Moreover, our results are compared to previous results to show the superiority of this technique.

1. Introduction

Delays often occur in chemical, industrial, biological, electronic, and transportation systems as well as many other areas, such as population growth models, economic growth, and neural networks [1, 2]. Analysis, identification, and optimal control of delay systems were considered by many researchers which are very important from a practical point of view. In applied mathematics, solving delayed differential equations is important. The process of determining the response of delay systems is often very complex and in a wide range of these systems, determining the analytical response is difficult or impossible. For this reason, delay systems are among the most important categories of control systems and providing a suitable and efficient numerical method to solve them is of considerable importance.

In nature, many quantities are the functions of two independent variables, and two-dimensional systems and signals are used to model phenomena with two independent variables. Moreover, the importance and necessity of examining two-dimensional state delay optimal control systems is very important. Continuous-temporal dynamic equations describe many control systems. Considering lack of analytical solutions to

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Email addresses: mohsen.hssn419@gmail.com (S. M. Hosseini), f_soltanian@pnu.ac.ir (F. Soltanian), k_mamehrashi@pnu.ac.ir (K. Mamehrashi)

two-dimensional state delay, optimal control problems of continuous-temporal systems, numerical and semi-analytical methods are suggested to solve a number of such problems. Also, by using Pontryagin's maximum principle, the time-delay optimal control problem reduces to a system of coupled two-point boundary value problem involving both delay and advance terms whose exact solution can not be easily calculated except in very special cases [8]. Numerical computational methods which are currently used in science and engineering are very diverse, and a specific solution can be provided for each specific problem or special condition. Since it is very difficult and in some cases impossible to obtain the solution to the analysis of these types of problems, semi-analytical and numerical methods are used for solving them. In the following, some studies in delay systems are presented. In [3], Banks proposed the optimal control of nonlinear delay systems from a theoretical point of view and proposed an approximate method to solve them. In [4], Banks, Burns and Cliff, analyzed and identified delay systems. The authors of [5] proposed a method to solve multiple delay systems using integral and delay matrices corresponding to Taylor polynomials. Also, Lee presented another method to solve the optimal control of delay functions with non-equal constraints on state variables [6]. Another work, optimal control of a nonlinear time-delay system in batch fermentation process has been studied in [27]. In [28], Dadebo and Luus, investigated the use of iterative dynamic programming employing systematic region contraction and accessible grid points for the optimal control of time-delay systems. Also in [29], a symplectic local pseudospectral method for solving nonlinear state-delayed optimal control problems with inequality constraints is proposed by Wang et al. Peng et al. proposed an iterative Symplectic Pseudospectral Method to solve Nonlinear State-delayed Optimal Control Problems in [30].

In recent years, orthogonal functions have received special attention from the researchers for analyzing the optimal control systems [7]. As an example, a method for analyzing and solving time-delay optimal control problems using orthogonal hybrid functions was proposed in [8] by Marzban and Razzaghi. Hence, Kern, Maurer and Gllmann solved a series of one-dimensional time-delay optimal control problems [9]. Moreover, in [10], the authors studied and solved a range of nonlinear time-delay optimal control problems using fixed piecewise functions.

Two-dimensional systems along with their spatial models were first introduced as part of image processing by Roesser. [11]. Mamehrashi and Yousefi used a method to solve a class of two-dimensional control problems using the Ritz-Galerkin method [12]. In another work, Tsai, Li, and Shieh [13] transformed the Roesser type continuous-temporal state optimal control problem with a quadratic cost functional into a discrete two-dimensional control problem. Furthermore, Nemati and Yousefi used Ritz method to solve a class of two-dimensional control problems [14], and the problems of two-dimensional optimal control of deficit were studied by Rabiei, Ordokhani and Babolian [15].

In recent years, orthogonal Block-Pulse functions were used to solve many state optimal control problems, examples of which can be seen in the articles of researchers and scholars [16–19, 31, 32]. Rakhshan and Effati presented a class of deficit time-delay optimal control problems using an Euler-Lagrange approach [20]. In [21], Nouri, Nazari and Torkzadeh, transformed the problem of time-delay fractional differential equations into a system of nonlinear algebraic equations using hybrid functions. Moreover, Rafiei, Kaffash and Karbassi, solved a number of one-dimensional time-delay control problems using hybrid functions [22].

We present an alternative numerical method to solve a class of two-dimensional time-delay optimal control problems. To the best of our knowledge, it is the first numerical method for 2DTDOCPs. The proposed method is a direct method based on approximating state and control variables via hybrid Block-Pulse functions and Legendre polynomials using the Ritz method [12, 14]. Considering the flexibility of Ritz method in the face of initial and boundary conditions, we used this approach in the proposed method. To calculate the dual integral in a standard function, we have used Gaussian method. By substituting the approximated functions in the constraints of the problem and using the suggested method, the optimal control problem is reduced to an unconstrained optimization problem which can be easily solved.

This article is organized as follows: Section 2 introduces the general problem of two-dimensional time-delay optimal control. Section 3 introduces the Ritz method and the hybrid Legendre Block-Pulse functions. In sections 4 and 5, we describe the proposed numerical method and discuss the convergence of the method, respectively. In section 6, the efficiency and accuracy of the proposed method are examined by providing

two examples. Furthermore, a conclusion is given in section 7.

2. Problem statement

Consider the following controllable and observable two-dimensional time delay system [23],

$$z'(x, t) = Az(x, t) + A_dz(x - \tau_1, t - \tau_2) + Bu(x, t) \tag{1}$$

where,

$$z'(x, t) = \begin{bmatrix} \frac{\partial z^h(x, t)}{\partial x} \\ \frac{\partial z^v(x, t)}{\partial t} \end{bmatrix}, z(x, t) = \begin{bmatrix} z^h(x, t) \\ z^v(x, t) \end{bmatrix}, z(x - \tau_1, t - \tau_2) = \begin{bmatrix} z^h(x - \tau_1, t) \\ z^v(x, t - \tau_2) \end{bmatrix}$$

$z^h(x, t)$ and $z^v(x, t)$ are the horizontal and vertical components of the space, A, B and A_d are constant matrices with appropriate dimensions, respectively. Also τ_1, τ_2 are constant delays of horizontal and vertical components. The boundary conditions are as follows,

$$z^h(a, t) = \begin{cases} f_a(t) & -\tau_1 \leq a \leq 0, 0 \leq t \leq t_f \\ 0 & -\tau_1 \leq a \leq 0, t \geq t_f. \end{cases} \tag{2}$$

$$z^v(x, b) = \begin{cases} g_b(x) & -\tau_2 \leq b \leq 0, 0 \leq x \leq x_f \\ 0 & -\tau_2 \leq b \leq 0, x \geq x_f. \end{cases}$$

which, x_f and t_f are fixed positive values, $f_a(t)$ and $g_b(x)$ are given vectors. The purpose of this paper is to determine the control vector $u(x, t)$ and the corresponding state vector $z(x, t)$ such that the following cost functional is minimized according to constraints (1) and (2),

$$J = \frac{1}{2} \xi^T(x_f, t_f) S \xi(x_f, t_f) + \frac{1}{2} \int_0^{t_f} \int_0^{x_f} [\xi^T(x, t) Q \xi(x, t) + u^T(x, t) R u(x, t)] dx dt, \tag{3}$$

where S and Q are semi-definite matrices and R is the positive definite matrix.

3. Function approximation

3.1. Ritz method

The Ritz method is a simple and efficient way to approximate the solution of an optimization problem. In this method, the solution of the functional minimization problem,

$$\min L[y(x)] = \int_a^b f(x, y, y') dx$$

with boundary conditions,

$$y(a) = a_0, y(b) = b_0$$

is considered as follows,

$$y_n(x) \approx \sum_{i=1}^n c_i \varphi_i(x) + \varphi_0(x). \tag{4}$$

We must select the basic functions $\varphi_i(x)$ that satisfy the following conditions,

$$\begin{aligned} \varphi_0(a) &= a_0, \quad \varphi_0(b) = b_0 \\ \varphi_i(a) &= \varphi_i(b) = 0, \quad i = 1, 2, \dots, n \end{aligned} \tag{5}$$

By substituting $y_n(x)$ into the problem and solving it, the unknown coefficients and the solution of $y_n(x)$ are obtained.

Suppose that $\varphi_i(x) = k(x)p_i(x)$, so Eq. (4) can be written as

$$y_n(x) \approx \sum_{i=1}^n k(x)c_i p_i(x) + \varphi_0(x) \tag{6}$$

where, $k(x)$ satisfies the homogeneous conditions and $p_i(x)$ are Legendre polynomials.

If we use the Ritz method to approximate the function $z(x, t)$, Eq. (6) is written as

$$z_{nm}(x, t) = \sum_{i=0}^m \sum_{j=0}^n k(x, t)c_{ij} p_i(x)p_j(t) + w(x, t), \tag{7}$$

where $k(x, t)$ and $w(x, t)$ satisfy the homogeneous and boundary conditions, respectively [24].

3.2. Two-dimensional Block-Pulse functions

A set of two dimensional Block-Pulse functions (2DBPFs) $\Phi_{i_1, i_2}(x, t)$ for $x \in [0, T_1], t \in [0, T_2]$ is defined as follows

$$\Phi_{i_1, i_2}(x, t) = \begin{cases} 1 & x \in \left[\frac{(i_1 - 1)T_1}{m_1}, \frac{i_1 T_1}{m_1} \right), t \in \left[\frac{(i_2 - 1)T_2}{m_2}, \frac{i_2 T_2}{m_2} \right) \\ 0 & \text{otherwise} \end{cases}$$

2DBPFs are disjointed with each other,

$$\Phi_{i_1, i_2}(x, t)\Phi_{j_1, j_2}(x, t) = \begin{cases} \Phi_{i_1, i_2}(x, t) & i_1 = j_1 \text{ and } i_2 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

and are orthogonal,

$$\int_0^{T_1} \int_0^{T_2} \Phi_{i_1, i_2}(x, t)\Phi_{j_1, j_2}(x, t)dxdt = \begin{cases} h_1 h_2 & i_1 = j_1 \text{ and } i_2 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

in the region $x \in [0, T_1)$ and $t \in [0, T_2)$, where

$$i_1, j_1 = 1, 2, \dots, m_1, i_2, j_2 = 1, 2, \dots, m_2, \quad h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}.$$

Also 2DBPFs are complete when both m_1 and m_2 approach infinity [26].

Since each 2DBPF takes only one value in its subregion, the 2DBPFs can be expressed as

$$\Phi_{i_1, i_2}(x, t) = \varphi_{i_1}(x)\Psi_{i_2}(t),$$

where $\varphi_{i_1}(x)$ and $\Psi_{i_2}(t)$ are one-dimensional Block-Pulse functions related to the variables x and t , respectively.

3.3. Hybrid functions of Block-Pulse and Legendre polynomials

The two-dimensional hybrid functions (block-pulse functions and Legendre polynomials) is defined as follows

$$\Psi_{i_1 j_1 i_2 j_2}(x, t) = \begin{cases} L_{j_1}(\frac{2N_1 x}{x_f} - 2i_1 + 1)L_{j_2}(\frac{2N_2 t}{t_f} - 2i_2 + 1) : (x, t) \in [\frac{i_1-1}{N_1}x_f, \frac{i_1}{N_1}x_f] \times [\frac{i_2-1}{N_2}t_f, \frac{i_2}{N_2}t_f] \\ 0 \text{ otherwise} \end{cases} \tag{8}$$

where, $i_1 = 1, 2, \dots, N_1, i_2 = 1, 2, \dots, N_2$ and $j_1, j_2 = 0, 1, \dots, M - 1$ are the order of Block-Pulse functions and Legendre polynomials, respectively and $L_{j_1}(x), L_{j_2}(t)$ are the well known Legendre polynomials [8].

4. Proposed state-control parameterization method

In this section, we present a numerical method based on approximating the state and control variables to solve a class of 2DTDOCPs.

Consider the following problem:

$$\text{Min } J = \frac{1}{2}z^T(x_f, t_f)Sz(x_f, t_f) + \frac{1}{2} \int_0^{t_f} \int_0^{x_f} [z^T(x, t)Qz(x, t) + u^T(x, t)Ru(x, t)]dxdt \tag{9}$$

subject to the time-delay system dynamics,

$$z'(x, t) = Az(x, t) + A_d z(x - \tau_1, t - \tau_2) + Bu(x, t) \tag{10}$$

with the following boundary conditions

$$z^h(a, t) = \begin{cases} f_a(t) & -\tau_1 \leq a \leq 0, 0 \leq t \leq t_f \\ 0 & -\tau_1 \leq a \leq 0, t \geq t_f. \end{cases} \tag{11}$$

$$z^v(x, b) = \begin{cases} g_b(x) & -\tau_2 \leq b \leq 0, 0 \leq x \leq x_f \\ 0 & -\tau_2 \leq b \leq 0, x \geq x_f. \end{cases} \tag{12}$$

Suppose that $Q \subset PC^2([0, x_f] \times [0, t_f])$ is a set of all continuous piecewise functions that satisfy the boundary condition (11)-(12). The cost functional J is a function of $z(x, t)$ and $u(x, t)$, so problem (9)-(12) can be considered as a problem of minimizing the value of J on Q . Suppose $Q_{N_1 N_2 (M-1)(M-1)} \subset Q$ is a set of the Legendre Block-Pulse hybrid functions consisting of $N_1 N_2$ polynomials and the degree of each polynomial is at most $(M - 1)(M - 1)$. The state variable is approximated using a finite number of the hybrid functions (block-pulse functions and Legendre polynomials) as

$$z(x, t) = \sum_{i_1=1}^{N_1} \sum_{j_1=0}^{M-1} \sum_{i_2=1}^{N_2} \sum_{j_2=0}^{M-1} f_{i_1, j_1, i_2, j_2} \Psi_{i_1, j_1, i_2, j_2}(x, t) \tag{13}$$

Let

$$\alpha = [f_{1,0,1,0}, f_{1,0,1,1}, \dots, f_{1,0,1,M-1}, \dots, f_{N_1, M-1, N_2, M-1}] \tag{14}$$

Now by selecting N_1, N_2 as

$$N_1 = \begin{cases} \frac{x_f}{\tau_1} & \frac{x_f}{\tau_1} \text{ is integer} \\ \left[\frac{x_f}{\tau_1} \right] + 1 & \text{otherwise} \end{cases} \tag{15}$$

$$N_2 = \begin{cases} \frac{t_f}{\tau_2} & \frac{t_f}{\tau_2} \text{ is integer} \\ \left[\frac{t_f}{\tau_2} \right] + 1 & \text{otherwise} \end{cases} \tag{16}$$

the interval $[0, x_f] \times [0, t_f]$ is converted to the following $N_1 \times N_2$ sub intervals

$$\left[0, \frac{1}{N_1}x_f\right] \times \left[0, \frac{1}{N_2}t_f\right], \left[0, \frac{1}{N_1}x_f\right] \times \left[\frac{1}{N_2}t_f, \frac{2}{N_2}t_f\right] \cdots \left[\frac{N_1-1}{N_1}x_f, x_f\right] \times \left[\frac{N_2-1}{N_2}t_f, t_f\right]$$

As a result, the state variable (13) is written as

$$\hat{z}(x, t) = \begin{cases} \hat{z}_{11}(x, t) = \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2} \psi_{1,j_1,1,j_2} & : (x, t) \in \left[0, \frac{1}{N_1}x_f\right] \times \left[0, \frac{1}{N_2}t_f\right] \\ \vdots \\ \hat{z}_{N_1 N_2}(x, t) = \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} f_{N_1,j_1,N_2,j_2} \psi_{N_1,j_1,N_2,j_2} & : (x, t) \in \left[\frac{N_1-1}{N_1}x_f, x_f\right] \times \left[\frac{N_2-1}{N_2}t_f, t_f\right] \end{cases} \tag{17}$$

To approximate $z(x - \tau_1, t - \tau_2)$ without interfering the whole discussion, we explain the steps for $z^v(x, t - \tau_2)$. Eq.(12) yields

$$z^v(x, t) = g(x, t) \quad 0 \leq x \leq x_f, -\tau_2 \leq t \leq 0.$$

If $s = t + \frac{1}{N_2}t_f$, then $t = s - \frac{1}{N_2}t_f$ and considering the definition of N_2 , we can write $t = s - \tau_2$, therefore, Eq. (12) becomes as

$$z_{11}^v(x, s - \tau_2) = g(x, s - \tau_2) \quad 0 \leq s \leq \tau \tag{18}$$

Also, $z^v(x, s - \tau_2)$ for

$$(x, t) \in \left[0, \frac{1}{N_1}x_f\right] \times \left[\frac{1}{N_2}t_f, \frac{2}{N_2}t_f\right]$$

can be defined as

$$z_{11}^v(x, t) = \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2} \psi_{1,j_1,1,j_2}(x, t).$$

Considering $s = t + \frac{1}{N_2}t_f$, we can write

$$z_{12}^v(x, s - \tau_2) = \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2} \psi_{1,j_1,1,j_2}(x, s - \tau_2) \quad , (x, s) \in \left[0, \frac{1}{N_1}x_f\right] \times \left[\frac{1}{N_2}t_f, \frac{2}{N_2}t_f\right].$$

Now, according to the definition of $\Psi_{i_1 j_1 i_2 j_2}(x, t)$, we have

$$\psi_{1,j_1,1,j_2}(x, s - \tau_2) = \psi_{1,j_1,1,j_2}(x, s)$$

therefore,

$$z_{12}^v(x, s - \tau_2) = \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2} \psi_{1,j_1,2,j_2}(x, s) \quad : (x, s) \in [0, \frac{1}{N_1}x_f] \times [\frac{1}{N_2}t_f, \frac{2}{N_2}t_f].$$

Similarly, $z^v(x, s - \tau_2)$ can be specified for other intervals and as a result, it is written as

$$z^v(x, s - \tau_2) = \begin{cases} g(x, t) & (x, t) \in [0, \frac{1}{N_1}x_f] \times [0, \frac{1}{N_2}t_f] \\ \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2} \psi_{1,j_1,1,j_2}(x, t) & (x, t) \in [0, \frac{1}{N_1}x_f] \times [\frac{1}{N_2}t_f, \frac{2}{N_2}t_f] \\ \vdots \end{cases} \quad (19)$$

The same procedure is applied to approximate the other component of the state function $z^h(x - \tau_1, t)$. Now, by substituting the values of $z(x, t)$ and $z(x - \tau_1, t - \tau_2)$ in (10), the control variable $u(x, t)$ is approximated. Then, substituting the approximations of $z(x, t)$ and $u(x, t)$ into the cost functional (9) an unconstrained optimization problem is obtained as

$$\text{Min } J(\alpha), \quad \alpha = [f_{1,0,1,0}, f_{1,0,1,1}, \dots, f_{N_1, M-1, N_2, M-1}].$$

To minimize this, we solve the following system of equations using Newton’s iterative method

$$\frac{\partial J[\alpha]}{\partial f_{i_1 j_1 i_2 j_2}} = 0, \quad i_1 = 1, 2, \dots, N_1, \quad i_2 = 1, 2, \dots, N_2, \quad j_1, j_2 = 1, 2, \dots, M - 1. \quad (20)$$

By solving the above algebraic system for α , the unknown coefficients $f_{i_1 j_1 i_2 j_2}$ are achieved. Then, by determining $f_{i_1 j_1 i_2 j_2}$, we can find the approximate value of $z^h(x, t)$, $z^v(x, t)$ and $u(x, t)$ from (13) and (10), respectively.

5. Convergence analysis

In this section, we recall two theorems and present a lemma which ensure the convergence analysis of suggested method. Here, the approximation convergence of a function is derived with respect to the Legendre Block-Pulse bases.

Consider the restriction of the cost functional J to $Q_{N_1 N_2 (M-1)(M-1)} \subset Q$

$$J[\hat{z}(x, t)] = J[\sum_{i_1=1}^{N_1} \sum_{j_1=0}^{M-1} \sum_{i_2=1}^{N_2} \sum_{j_2=0}^{M-1} f_{i_1, j_1, i_2, j_2} \Psi_{i_1, j_1, i_2, j_2}(x, t)] \quad (21)$$

as a function of $N_1 N_2 (M - 1)(M - 1)$ variables. The coefficients $f_{i_1 j_1 i_2 j_2}$ are chosen in such a way as to minimize (21). Let $\beta_{N_1 N_2 (M-1)(M-1)}$ indicate the minimum value of J restricted to $Q_{N_1 N_2 (M-1)(M-1)}$. The following theorem is a remarkable result of the Weierstrass famous theorem for two-dimensional space.

Theorem 5.1. For any $\hat{z}(x, t) \in Q \subset PC^2([0, x_f] \times [0, t_f])$, there exists a sequence of polynomials $\{\Psi_{i_1 j_1 i_2 j_2}(x, t)\}_{i_1, j_1, i_2, j_2=0}^\infty \in Q$ that converges uniformly to $\hat{z}(x, t)$.

Proof. See [33]. \square

The convergence of the proposed method is provided by the following Lemma. We use the facts that have been mentioned in Theorem (5.1).

Lemma 5.2. *If*

$$\beta_{N_1N_2(M-1)(M-1)} = \inf_{Q_{N_1N_2(M-1)(M-1)}} J, \quad \text{for } N_1, N_2, M = 1, 2, 3, \dots,$$

where $Q_{N_1N_2(M-1)(M-1)}$ is a subset of Q including the Legendre Block-Pulse hybrid functions involving N_1N_2 polynomials of degree at most $(M - 1)(M - 1)$, then

$$\lim_{N_1, N_2, M \rightarrow \infty} \beta_{N_1N_2(M-1)(M-1)} = \inf_Q J.$$

Proof. Let

$$\beta_{N_1N_2(M-1)(M-1)} = \min_{\alpha_{N_1N_2(M-1)(M-1)}} J(\alpha_{N_1N_2(M-1)(M-1)}),$$

then,

$$\beta_{N_1N_2(M-1)(M-1)} = J(\alpha^*_{N_1N_2(M-1)(M-1)}),$$

where

$$\alpha^*_{N_1N_2(M-1)(M-1)} \in \text{Argmin}\{J(\alpha_{N_1N_2(M-1)(M-1)}) : \alpha_{N_1N_2(M-1)(M-1)} \in R^{2N_1N_2M}\}.$$

Now, let

$$(z^*_{N_1N_2(M-1)(M-1)}(x, t), u^*_{N_1N_2(M-1)(M-1)}(x, t)) \in \text{Argmin}\{J(z(x, t), u(x, t)) : (z(x, t), u(x, t)) \in Q_{N_1N_2(M-1)(M-1)}\},$$

then

$$J(z^*_{N_1N_2(M-1)(M-1)}(x, t), u^*_{N_1N_2(M-1)(M-1)}(x, t)) = \min_{(z(x,t), u(x,t)) \in Q_{N_1N_2(M-1)(M-1)}} J(z(x, t), u(x, t)),$$

where $Q_{N_1N_2(M-1)(M-1)}$ is a class of combinations of the continuous hybrid functions (block-pulse functions and Legendre polynomials) involving N_1N_2 polynomials of degree at most $(M - 1)(M - 1)$, so

$$\beta_{N_1N_2(M-1)(M-1)} = J(z^*_{N_1N_2(M-1)(M-1)}(x, t), u^*_{N_1N_2(M-1)(M-1)}(x, t)).$$

Furthermore, according to $Q_{N_1N_2(M-1)(M-1)} \subset Q_{N_1N_2MM}$, we have

$$\min_{(z(x,t), u(x,t)) \in Q_{N_1N_2MM}} J(z(x, t), u(x, t)) \leq \min_{(z(x,t), u(x,t)) \in Q_{N_1N_2(M-1)(M-1)}} J(z(x, t), u(x, t)).$$

Thus, $\beta_{N_1N_2MM} \leq \beta_{N_1N_2(M-1)(M-1)}$ is achieved which means $\beta_{N_1N_2(M-1)(M-1)}$ is a non-increasing sequence. Also, this sequence is lower bounded, so its infimum is the limit. Due to the continuity J and by taking the limit when $N_1, N_2, M \rightarrow \infty$, we can write,

$$\lim_{N_1, N_2, M \rightarrow \infty} \beta_{N_1N_2(M-1)(M-1)} = \min_{(z(x,t), u(x,t)) \in Q} J(z(x, t), u(x, t)).$$

which completes the proof. \square

6. Numerical results

In this section, three optimal control examples are considered to illustrate the theoretical result. The cost functional for different values of τ is computed. If $\tau = 0$, the 2DTDOCP reduces to two-dimensional quadratic optimal control problem was studied by other researchers. The numerical results were computed using Maple 2018 programming.

Example 6.1. Consider the following 2DTDOCP

$$\text{Min } J = \int_0^3 \int_0^3 \left\{ [z^h(x, t)]^2 + [z^v(x, t)]^2 + [u(x, t)]^2 \right\} dxdt, \tag{22}$$

s.t

$$\begin{aligned} \frac{\partial z^h(x, t)}{\partial x} &= -3z^h(x, t) + 3.2z^v(x, t) + 0.3u(x, t) \\ \frac{\partial z^v(x, t)}{\partial t} &= z^h(x, t) - z^v(x, t - \tau) \\ z^v(x, 0) &= e^{-3x} \cos(2\pi x) \\ z^h(0, t) &= -e^{-2t} \\ z^v(x, \theta) &= 1, -\tau \leq \theta \leq 0. \end{aligned} \tag{23}$$

According to the suggested method in section 4, based on the Ritz method and using two-dimensional Block-Pulse functions, the state variable $z^v(x, t)$ is approximated as

$$z^v(x, t) = \sum_{i_1=1}^{N_1} \sum_{j_1=0}^{M-1} \sum_{i_2=1}^{N_2} \sum_{j_2=0}^{M-1} t x f_{i_1, j_1, i_2, j_2} \Psi_{i_1, j_1, i_2, j_2}(x, t) + e^{-3x-2t} \cos(2\pi x),$$

where N_1, N_2 are the numbers of blocks and M is the number of Legendre polynomials. For simplicity, we assume that $N_1 = N_2$. In each subinterval

$$(x, t) \in \left[\frac{i_1 - 1}{N}, \frac{i_1}{N} \right] \times \left[\frac{i_2 - 1}{N}, \frac{i_2}{N} \right],$$

$z^v(x, t)$ can be written for $i_1, i_2 = 1, 2, \dots, N$ and $j_1, j_2 = 0, 1, \dots, M - 1$ as

$$z^v [i_1, i_2] = \sum_{j_1=0}^{M-1} \sum_{j_2=0}^{M-1} L_{j_1} \left(\frac{2Nx}{x_f} - 2i_1 + 1 \right) L_{j_2} \left(\frac{2Nt}{t_f} - 2i_2 + 1 \right) f_{i_1, j_1, i_2, j_2} \tag{24}$$

The approximation of $z^v(x, t - \tau)$ is also obtained using (19).

Now from (28), we have

$$\begin{aligned} z^h(x, t) &= \frac{\partial z^v(x, t)}{\partial t} + z^v(x, t - \tau) \\ u(x, t) &= \frac{10}{3} \left[\frac{\partial z^h(x, t)}{\partial x} + 3z^h(x, t) - 3.2z^v(x, t) \right] \end{aligned} \tag{25}$$

According to (25), $z^h[i_1, i_2]$ and $u[i_1, i_2]$ can be easily obtained in any corresponding subinterval. By substituting the approximated values $z^v(x, t)$, $z^h(x, t)$ and $u(x, t)$ in (27), the 2DTDOPCP reduces to an optimization problem which can be easily solved using existing optimization methods. By solving obtained optimization problem, the unknown coefficients f_{i_1, j_1, i_2, j_2} are determined. Using f_{i_1, j_1, i_2, j_2} , we can determine the approximate value of $z^h(x, t)$, $z^v(x, t)$ and $u(x, t)$. The obtained vertical, horizontal and control variables from proposed method for this example are shown in Fig. 1,2 and 3, respectively. The cost functional values for $\tau = 1, \tau = \frac{1}{2}, \tau = \frac{1}{3}$ and different values of M are shown in table 1. As seen from the results reported in this table, by increasing M we can get better solutions for the cost functional J . If $\tau = 0$, the example 6.1 reduces to two-dimensional optimal control problem without time-delay that has been studied by other researchers. The comparison of the estimated values of J by different methods is shown in Table 2. By comparing the results for similar values of M and N , it is clear that the results of the proposed method are better.

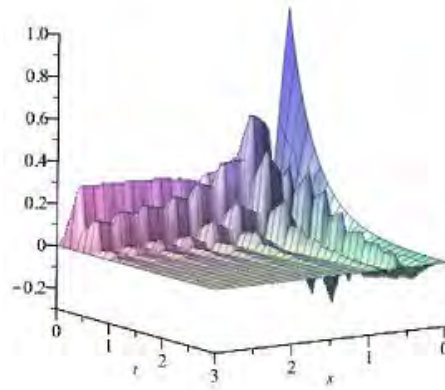


Figure 1: Approximate solution of state vertical component $z^v(x, t)$ with $\tau = 1$ for Example 6.1

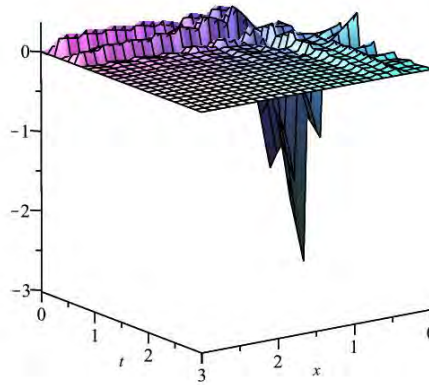


Figure 2: Approximate solution of state horizontal component $z^h(x, t)$ with $\tau = 1$ for Example 6.1

Table 1: Estimated values of J for various values of N, M and τ

	$N = 3, \tau = 1$	$N = 6, \tau = \frac{1}{2}$	$N = 9, \tau = \frac{1}{3}$
$M = 2$	9.9091	4.7208	2.1969
$M = 3$	6.6079	1.7672	1.2483
$M = 7$	2.5895	1.1256	0.7543

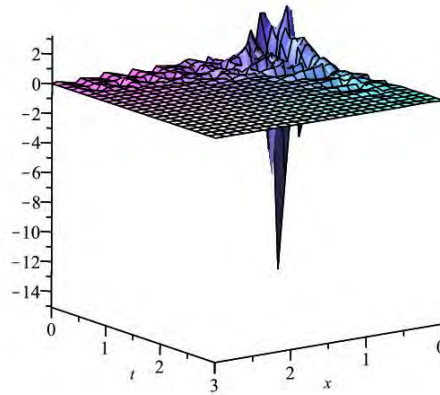


Figure 3: Approximate solution of control variable $u(x, t)$ with $\tau = 1$ for Example 6.1

Table 2: Comparison of estimated value of J in case $\tau = 0$ for different methods

Methods	J
Method of [13]	
$X = 0.1, T = 0.1$	0.7348
$X = 0.05, T = 0.05$	0.5510
$X = 0.03, T = 0.03$	0.4760
Method of [12]	
$M = 7, N = 8$	0.6202
$M = 8, N = 3$	0.2792
$M = 8, N = 8$	0.2026
Method of [25]	
$M = 7, N = 6$	0.3094
$M = 9, N = 8$	0.0951
$M = 10, N = 8$	0.0608
Present Method	
$M = 7, N = 8$	0.0075
$M = 8, N = 3$	0.0659
$M = 8, N = 8$	0.0072
$M = 7, N = 6$	0.0693
$M = 9, N = 8$	0.0069
$M = 10, N = 8$	0.0051

Example 6.2. Consider the following problem

$$\text{Min } J = \frac{1}{2} \int_0^5 \int_0^5 10^7 [z^v(x, t) - \sin(x + t)]^2 + [u(x, t)]^2 dx dt,$$

$$\frac{\partial z^h(x, t)}{\partial x} = -3z^h(x, t) + 3.2z^v(x, t) + 0.3u(x, t)$$

$$\frac{\partial z^v(x, t)}{\partial t} = z^h(x, t) - z^v(x, t - \tau)$$

$$z^v(x, 0) = e^{-3x} \cos(2\pi x)$$

$$z^h(0, t) = -e^{-2t}$$

$$z^v(x, \theta) = 1 : -\tau \leq \theta \leq 0.$$

(26)

Following the mentioned methodology to solve 2DTDOCPs in section 4, we approximate the function $z^v(x, t)$ in terms of Legendre Block-Pulse basis. We obtain $z^h(x, t)$ and $u(x, t)$ by applying the method similar to Example 6.1. The achieved vertical, horizontal and control variables by solving the mentioned example are shown in Fig. 4, 5 and 6, respectively. The cost functional values for $\tau = 3, \tau = 2, \tau = 1$ and different values of M are shown in Table 3. If $\tau = 0$, similar to example 6.1, comparing the estimated values of J by different methods is shown in Table 4. As seen from the reported results in Table 4, our solution is better compared with the methods presented in [12] and [25]. By comparing the results for similar values of M and N , it is clear that the results of the proposed method are better.

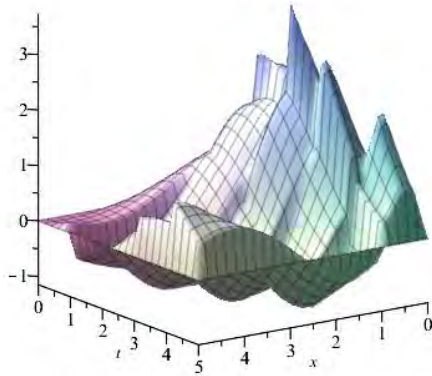


Figure 4: Approximate solution of state vertical component $z^v(x, t)$ with $\tau = 1$ for Example 6.2

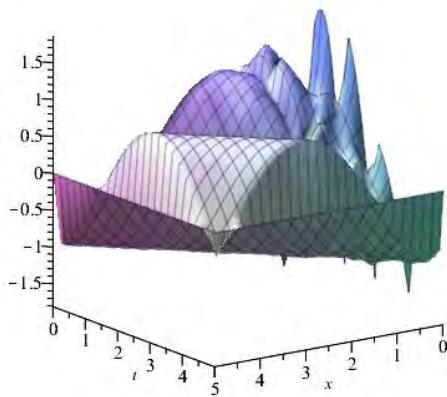


Figure 5: Approximate solution of state horizontal component $z^h(x, t)$ with $\tau = 1$ for Example 6.2

Example 6.3. Consider the following 2DTDOCP

$$\text{Min } J = \int_0^3 \int_0^3 \left\{ [z^h(x, t)]^2 + [z^v(x, t)]^2 + [u(x, t)]^2 \right\} dxdt, \tag{27}$$

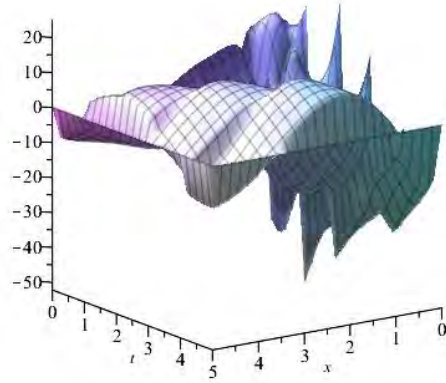


Figure 6: Approximate solution of control variable $u(x, t)$ with $\tau = 1$ for Example 6.2

Table 3: Estimated values of J for various values of N, M and τ

	$N = 2, \tau = 3$	$N = 3, \tau = 2$	$N = 5, \tau = 1$
$M = 3$	11.86268×10^6	8.18034×10^6	4.42061×10^6
$M = 5$	7.54674×10^6	5.42092×10^6	2.23495×10^6
$M = 7$	5.16402×10^6	3.70614×10^6	0.75272×10^6

Table 4: Comparison of estimated value of J in case $\tau = 0$ for different methods

Methods	J
Method of [12]	
$M = 4, N = 7$	2.56765×10^6
$M = 6, N = 6$	2.03080×10^6
$M = 7, N = 5$	2.15360×10^6
Method of [25]	
$M = 3, N = 4$	4.090680×10^6
$M = 5, N = 6$	2.398277×10^6
$M = 8, N = 9$	1.199138×10^6
Present Method	
$M = 4, N = 7$	0.000586×10^6
$M = 6, N = 6$	0.410512×10^6
$M = 7, N = 5$	0.610010×10^6
$M = 3, N = 4$	0.909921×10^6
$M = 5, N = 6$	0.423217×10^6
$M = 8, N = 9$	0.000062×10^6

s.t

$$\frac{\partial z^h(x, t)}{\partial x} = -3z^h(x - \tau, t) + 3.2z^v(x, t) + 0.3u(x, t)$$

$$\frac{\partial z^v(x, t)}{\partial t} = z^h(x, t) - z^v(x, t - \tau)$$

$$z^v(x, 0) = e^{-3x} \cos(2\pi x)$$

$$z^h(0, t) = -e^{-2t}$$

$$z^v(x, \theta) = 1, -\tau \leq \theta \leq 0.$$

$$z^h(\beta, t) = 1, -\tau \leq \beta \leq 0.$$

(28)

Following the mentioned methodology for solving the 2DTDOCPs in section 4, we approximate the function $z^v(x, t)$ in terms of the Legendre Block-Pulse basis. We obtain $z^h(x, t)$ and $u(x, t)$ by applying the method similar to Example 6.1.

The achieved vertical, horizontal and control variables by solving the mentioned example are shown in Fig. 7, 8 and 9, respectively. The cost functional values for $\tau = \frac{1}{2}$, $\tau = \frac{1}{3}$, $\tau = \frac{1}{4}$ and different values of M are shown in Table 5. If $\tau = 0$, the problem such as example 6.1 reduces to the two-dimensional optimal control problem without time-delay that has been studied by other researchers. The comparison of the estimated values of J by different methods is shown in Table 2.

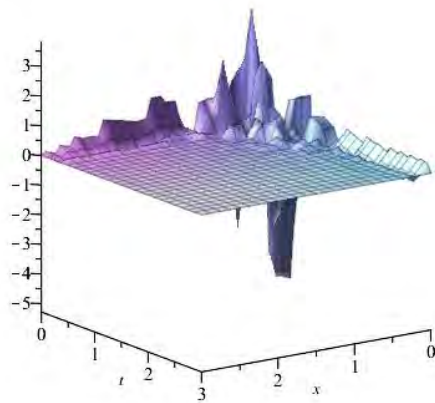


Figure 7: Approximate solution of state vertical component $z^v(x, t)$ with $\tau = \frac{1}{3}$ for Example 6.3

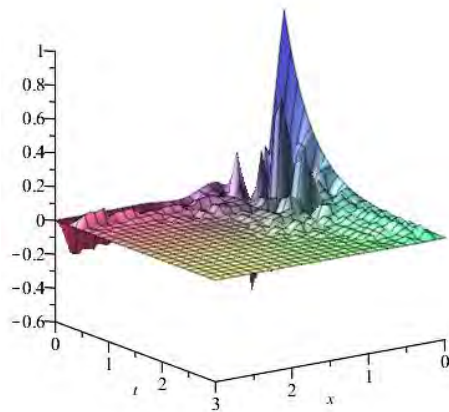


Figure 8: Approximate solution of state horizontal component $z^h(x, t)$ with $\tau = \frac{1}{3}$ for Example 6.3

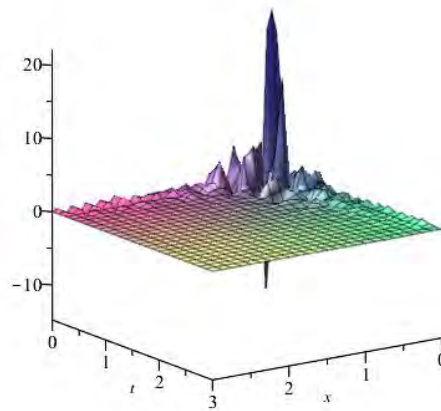


Figure 9: Approximate solution of control variable $u(x, t)$ with $\tau = \frac{1}{3}$ for Example 6.3

Table 5: Estimated values of J for various values of N, M and τ

	$N = 6, \tau = \frac{1}{2}$	$N = 9, \tau = \frac{1}{3}$	$N = 12, \tau = \frac{1}{4}$
$M = 2$	26.0203	22.9754	13.6724
$M = 3$	11.2012	10.1130	4.1256
$M = 5$	5.4121	4.1283	1.7811

7. Conclusion

This paper presents an efficient method to solve a class of 2DTDOCPs. The state and control variables are approximated using Ritz method and the hybrid functions (block-pulse functions and Legendre polynomials). Using the proposed method, without using derivative and multiplicative functional matrices, the optimal control problem reduces to an optimization problem which can be easily solved. We described the proposed numerical method and discuss the convergence of the method. The obtained results showed that our method gives a good approximate of the solution and these results are more accurate than other results that have been obtained from existing methods while $\tau = 0$. Moreover, we obtained satisfactory results only in a small number of polynomial orders compared to other published methods in literatures. To solve 2DTDOCPs, we introduced a building block which can be extended. The problems with inequality constraints and Multiple Time-Varying Delays are under consideration as our future research work.

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