Filomat 37:17 (2023), 5785–5797
https://doi.org/10.2298/FIL2317785A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and regularity of solutions to unilateral nonlinear elliptic equation in Marcinkiewicz space with variable exponent

Benali Aharrouch[a], Ahmed Aberqi[b], Jaouad Bennouna[c]

[a]*Cadi Ayyad University, Multidisciplinary Faculty, Safi, Morocco*
[b]*Laboratory LAMA, Sidi Mohamed Ben Abdellah University, National School of Applied Sciences Fez, Morocco*
[c]*Laboratory LAMA, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, B.P 1796 Atlas Fez, Morocco*

Abstract. This manuscript proves the existence and regularity of solutions with respect to the summability of second member $g \in L^{m(\cdot)}(\Omega)$, to the obstacle problem associated to nonlinear elliptic equation

$$\begin{cases} -\mathrm{div}\, A(x, v, \nabla v) = g & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{1}$$

The arguments are based on the rearrangement techniques to obtain some priori estimates in Marcinkwicz spaces with variable exponents.

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^N, (N \geq 2)$. In this paper, we are concerned in proving the existence and regularity of entropy solutions to the following obstacle problems:

$$(\mathcal{P}) \begin{cases} \text{Find } v \text{ measurable function such that } v \geq \psi \text{ a.e. in } \Omega, \text{ and, for every } t > 0,\ T_t(u) \in W_0^{1,p(\cdot)}(\Omega) \text{ and} \\ \displaystyle\int_\Omega A(x, v, \nabla v)\nabla T_t(v - \varphi)\, dx \leq \int_\Omega g(x) T_t(v - \varphi)\, dx, \forall \varphi \in \mathcal{K}_\psi(\Omega) \cap L^\infty(\Omega), \end{cases}$$

where $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfy the following assumptions:

$$A(x, s, \xi)\xi \geq \alpha|\xi|^{p(x)}, \text{ and } (A(x, s, \xi) - A(x, s, \xi'))(\xi - \xi') > 0, \text{ for } \xi \neq \xi' \text{and } \alpha > 0, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 35J87; Secondary 35R35, 49J40
Keywords. Non-linear equation; Obstacle problem; Existence; Regularity; Marcinkwincz with variable exponents; Rearrangement techniques.
Received: 29 July 2022; Accepted: 17 September 2022
Communicated by Maria Alessandra Ragusa
Email addresses: bnaliaharrouch@gmail.com (Benali Aharrouch), aberqi_ahmed@yahoo.fr (Ahmed Aberqi), jbennouna@hotmail.com (Jaouad Bennouna)

$$|A(x,s,\xi)| \leq \beta\left(\varphi(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \tag{3}$$

where $\beta > 0$, and φ is a non-negative function that belong to $L^{p'(\cdot)}(\Omega)$.

and $\mathcal{K}_\psi = \left\{v \in W_0^{1,p(\cdot)}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\right\}$, $\qquad T_t(s) := \max\{-t, \min\{t,s\}\}, \quad s \in \mathbb{R}$

Our main novelty is to prove some regularity of entropy solutions to a problem $(\mathcal{P})$, when the second member g in $L^{m(\cdot)}(\Omega)$ and $m^- < (p_*^-)'$ in which we extend the results in the case of integrable datum $m = 1$ or in the dual space. Indeed we prove that entropy solutions $v \in M^{q_0(\cdot)}(\Omega)$ and $\nabla v \in M^{q_1(\cdot)}(\Omega)$, with $q_0(\cdot)$ and $q_1(\cdot)$ are the variable exponents defined by

$$\begin{cases} q_0(\cdot) = \dfrac{p_*(\cdot)}{\gamma}, \\ q_1(\cdot) = \dfrac{(m^-)'p(\cdot)p_*(\cdot)}{(m^-)'(p_*(\cdot)+\gamma) - (p_*)^-} \end{cases} \quad \text{and } \gamma = \frac{m^-p^- - (p_*)^-(m^- - 1)}{m^-(p^- - 1)}, \tag{4}$$

with all variable exponents are in $C_+(\overline{\Omega}) = \Big\{$log-Hölder continuous function $p : \overline{\Omega} \to \mathbb{R}$ with $1 < p^- \leq p^+ < N\Big\}$, and $p_* = \dfrac{Np}{N-p}$. The proof is based on a priori estimates obtained in Marcinkiewicz spaces with variable exponent $M^{p(\cdot)}(\Omega)$ introduced in first by the authors in [25].

Remark that for the case $m = 1$ in (4), we have $\gamma = (p^-)'$, $q_0(\cdot) = \dfrac{p_*(\cdot)}{(p^-)'}$ and $q_1(\cdot) = \dfrac{q_0(\cdot)}{q_0(\cdot)+1}p(\cdot)$ which are the same quantities obtained in [24] and [25] where the authors established the existence and uniqueness of an entropy solution to the obstacle problem for nonlinear elliptic equations with variable growth and a second member L^1.
Other results in the same context where the source is L^1−integrable, the reader can be refer to [2, 3, 28] in which the authors established the solvability of the equations with fractional Laplacian or balanced growth on the operators. Also see [1, 11, 12, 27], where the authors have proved the existence of solutions by using the rearrangement properties.

A nice overview of the recent work on such equations with the variable exponents and the second member is in the dual space can be found in in [21]. See also [4–8, 13, 15, 19, 20, 23, 29] for other research in Sobolev spaces with variable exponent and the references therein for more background.

Note that, our results can be seeing as a continuation for the regularity results obtained in [9, 16, 17] in classical cases, namely, when $p(\cdot) = p$ and $m(\cdot) = m$, the exponents $q_0(\cdot)$ and $q_1(\cdot)$ in (4) becomes $q_0(\cdot) = q_0 = \dfrac{Nm(p-1)}{N-mp}$ and $q_1(\cdot) = q_1 = \dfrac{Nm(p-1)}{N-m}$ which coincide with the regularities results in the above references.

The contributions of the paper are as follows. In Sect. 2, we recall the most important and relevant properties and notations of Lebesgue spaces with variable exponents, and we review several rearrangement properties and Marcinkiewicz spaces with variable exponents, as well as their relationship to Lebesgue spaces. In Sect. 3, we show fourth new results (a priori estimates of entropy solutions and their gradients), and we establishes our main existence and regularity results.

2. Mathematical Background and Auxiliary Results

In what follows, we give some definitions and properties of Sobolev spaces and Marcinkiwks spaces, which we will use to prove our main results. For more details,[21, 29] and many references given therein.

2.1. Sobolev space with variable exponent

A real-valued continuous function $p(\cdot)$ is said to be log-Hölder continuous in Ω if

$$|p(x)-p(y)| \leq \frac{c}{|\log|x-y||} \quad \forall x,y \in \overline{\Omega} \text{ such that } |x-y| < \frac{1}{2},$$

where c is a constant. We denote

$$C_+(\overline{\Omega}) = \Big\{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p^+ < N\Big\},$$

where

$$p^- = \inf\{p(x),\ \ \forall x \in \Omega\} \quad \text{ and } \ p^+ = \sup\{p(x),\ \ \forall x \in \Omega\}.$$

For $p(\cdot)$ and $q(\cdot)$ in $C_+(\overline{\Omega})$ we means by $p(\cdot) \ll q(\cdot)$ that $\inf\limits_{\Omega}(q(\cdot)-p(\cdot)) > 0$.

For $p \in C_+(\overline{\Omega})$ the Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{v : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |v(x)|^{p(x)}\,dx < \infty\Big\},$$

the space $L^{p(\cdot)}(\Omega)$ under the norm

$$\|v\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{v(x)}{\lambda}\Big|^{p(x)}\,dx \leq 1\Big\}$$

is reflexive since it is uniformly convex. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for all $x \in \Omega$.

Proposition 1 (Hölder inequality [21, 29]). *(i) For any functions $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

(ii) For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ for all x in Ω, we have

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega),$$

moreover the embedding is continuous.

Proposition 2 ([21, 29]). *If we denote*

$$\rho(v) = \int_\Omega |v(x)|^{p(x)}\,dx \qquad \forall v \in L^{p(\cdot)}(\Omega),$$

then, the following assertions hold

1. $\|v\|_{p(\cdot)} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho(v) < 1$ *(resp.* $= 1, > 1$*);*
2. *We have the following implication*

$$\|v\|_{p(\cdot)} > 1 \ \textit{ implies } \ \|v\|_{p(\cdot)}^{p^-} \leq \rho(v) \leq \|v\|_{p(\cdot)}^{p^+},$$

$$\|v\|_{p(\cdot)} < 1 \ \textit{ implies } \ \|v\|_{p(\cdot)}^{p^+} \leq \rho(v) \leq \|v\|_{p(\cdot)}^{p^-};$$

3. $\|v\|_{p(\cdot)} \to 0$ *if and only if* $\rho(v) \to 0$*, and* $\|v\|_{p(\cdot)} \to \infty$ *if and only if* $\rho(v) \to \infty$.

We define Sobolev space with variable exponent by

$$W^{1,p(\cdot)}(\Omega) = \Big\{v \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla v| \in L^{p(\cdot)}(\Omega)\Big\},$$

with the norm

$$\|v\|_{1,p(\cdot)} = \|v\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot)} \quad \forall v \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \dfrac{Np(x)}{N-p(x)}$ with $p(x) < N$.

Proposition 3 ([21]). *(i) The spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) The embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact, if $q(x) < p^*(x), \forall x \in \Omega$.

(iii) (Poincaré inequality). For all $v \in W_0^{1,p(\cdot)}(\Omega)$ there exists a constant $c > 0$, such that $\|v\|_{p(\cdot)} \leq c\|\nabla v\|_{p(\cdot)}$.

(iv) (Sobolev-Poincaré inequality). For all $v \in W_0^{1,p(\cdot)}(\Omega)$ there exists a constant $c > 0$, such that $\|v\|_{p_*(\cdot)} \leq c\|\nabla v\|_{p(\cdot)}$.

Remark 1. We conclude that the norms $\|\nabla v\|_{p(\cdot)}$ and $\|v\|_{1,p(\cdot)}$ are equivalent in $W_0^{1,p(\cdot)}(\Omega)$ using (iii) of Theorem 3.

The truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(r) = \begin{cases} r & \text{if} \quad |r| \leq k \\ k.sign(r) & \text{if} \quad |r| > k. \end{cases}$$

Now, we define

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) := \Big\{v : \Omega \to \mathbb{R} \text{ measurable such that } T_k(v) \in W_0^{1,p(\cdot)}(\Omega)\Big\}.$$

Proposition 4. *([14]) Let $v \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$, there exists a unique measurable function $w : \Omega \to \mathbb{R}^N$ such that*

$$w\chi_{\{|v|\leq k\}} = \nabla T_k(v) \quad \text{for a.e. } x \in \Omega \text{ and for all } k > 0.$$

We will define the gradient of v as the function w, and we will denote it by $w = \nabla v$.

2.2. Marcinkiewicz spaces

In the following, we review several rearrangement properties and Marcinkiewicz spaces with variable exponents, as well as their relationship to Lebesgue spaces; for more details, see ([10], [22], [26]) and [25]. We recall some definitions about decreasing rearrangement of functions. Let Ω be a bounded open set of $\mathbb{R}^N$ and $v : \Omega \to \mathbb{R}$ a measurable function.

Definition 1. *We define the distribution function of v as follows*

$$\mu_v(t) = meas\{x \in \Omega : |v(x)| > t\}, t \geq 0.$$

μ_v is right continuous and decreasing function.

Definition 2. *We define the decreasing rearrangement of v as follows*

$$v_*(s) := \sup\{t \geq 0 : \mu_v(t) > s\}, s \geq 0.$$

Definition 3. *A measurable function $v : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^p(\Omega)$(or weak-L^p) if*

$$\mu_v(t) \leq \frac{c}{t^r}, \quad \forall t > 0, \text{ or } v_*(s) \leq \frac{c}{s^{1/r}}, \quad \forall s > 0,$$

for some constant c.
The norm in $M^p(\Omega)$ is defined by

$$\|v\|_{M^p(\Omega)} = \sup_{s>0} v_*(s) s^{\frac{1}{r}}.$$

The Marcinkiewicz spaces are 'intermediate' between Lebesgue spaces, i.e. for any $1 \leq q < p$, we have

$$L^p(\Omega) \subset M^p(\Omega) \subset L^q(\Omega).$$

Let $q(\cdot)$ be a measurable function such that $q^- > 0$. We say that a measurable function v belongs to the Marcinkiewicz space $M^{q(\cdot)}$ if there exists a positive constant C such that

$$\int_{\{|v|>t\}} t^{q(x)}\, dx \leq C, \text{ for all } t > 0.$$

When $q(\cdot)$ is constant i.e. $q(\cdot) \equiv q$ this definition is coincides with the classical definition of the Marcinkiewicz space $M^q(\Omega)$. Moreover we have

$$\int_{\{|v|>t\}} t^{q(x)}\, dx \leq \int_{\Omega} |v|^{q(x)}\, dx, \quad \text{for all } t > 0.$$

Thus if $|v|^{q(\cdot)} \in L^1(\Omega)$, we have $v \in M^{q(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega) \subset M^{q(\cdot)}(\Omega)$, for all $q(\cdot) \geq 1$.
In the Marcinkiewicz space with constant exponent, if $v \in M^r(\Omega)$, then $|v|^q \in L^1(\Omega)$, for all $0 < q < r$.
This claim is extended to the non constant setting by the following Lemma, whose proof is given in [25].

Lemma 1. *Let $r(\cdot)$ and $q(\cdot)$ be bounded functions such that $0 \ll q(\cdot) \ll r(\cdot)$ and let $\epsilon := (r - q)^- > 0$. If $v \in M^{r(\cdot)}(\Omega)$, then*

$$\int_{\Omega} |v|^{q(x)}\, dx \leq 2|\Omega| + (r^+ - \epsilon)C/\epsilon,$$

where C is the constant appearing in the definition of $M^{r(\cdot)}(\Omega)$. In particular, $M^{r(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ for all $1 \leq q(\cdot) \ll r(\cdot)$.

The following are our main results, in which we obtain a priori estimates in Marcinkiewicz spaces with variable exponent for an entropy solution of the obstacle problem (($\mathcal{P}$), and we state our existence Theorem.

3. Main Results

Let $\psi \in W^{1,p(\cdot)}(\Omega)$, such that $\psi^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and define the convex subset $\mathcal{K}_\psi$ by

$$\mathcal{K}_\psi(\Omega) = \{v \in W_0^{1,p(\cdot)}(\Omega) \text{ such that } v \geq \psi \text{ a.e. in } \Omega\}$$

3.1. A priori estimates

Definition 4. *A measurable function v is an entropy solution for obstacle problem ($\mathcal{P}$) if, $v \geq \psi$ a.e. in Ω, for every $t > 0$, $T_t(v) \in W_0^{1,p(\cdot)}(\Omega)$ and*

$$\int_{\Omega} A(x, v, \nabla v) \nabla T_t(v - \varphi)\, dx \leq \int_{\Omega} g(x) T_t(v - \varphi)\, dx, \tag{5}$$

for all $\varphi \in \mathcal{K}_\psi(\Omega) \cap L^\infty(\Omega)$.

Remark 2. *For $\gamma > 0, v \in M^{\frac{p_*(\cdot)}{\gamma}}(\Omega)$ implies that $v \in M^{\frac{p_*^-}{\gamma}}(\Omega)$, witch gives $\mu_v(t) \leq c_1 t^{-\frac{p_*^-}{\gamma}}$, with c_1 is a positive constant.*

Remark 3. *Since $g \in L^{m(\cdot)}(\Omega) \subset M^{m(\cdot)}(\Omega)$ we have $g \in M^{m^-}(\Omega)$, witch gives $g_*(t) \leq c_2 t^{-\frac{1}{m^-}}$, with c_2 is a positive constant.*

Theorem 1. *Under assumptions (2), (3) and $g \in L^{m(\cdot)}(\Omega)$ with $m^- < (p_*^-)'$. If v is a solution in the sense of Definition 4 that belongs to $M^{\frac{p_*(\cdot)}{\gamma}}(\Omega)$, then there exists a positive constant C, depending only on $p(\cdot), N$, and Ω, such that*

$$\int_{\{|v|\leq t\}} |\nabla v|^{p(x)}\, dx \leq C t^{1-\frac{(p_*)^-}{\gamma(m^-)'}},$$

for all $t \geq \|\psi^+\|_\infty$ and $\gamma > \frac{(p_*)^-}{(m^-)'}$.

Proof. We denote by C a constant that varies from line to line.
Since v is an entropy solution to the unilateral problem $(\mathcal{P})$, for all $\varphi \in \mathcal{K}_\psi(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega A(x, v, \nabla v)\nabla T_t(v-\varphi)\, dx \leq \int_\Omega g(x) T_t(v-\varphi)\, dx,$$

for $\varphi = T_{s+\|\psi^+\|_\infty}(v)$, we have

$$\int_{\{s+\|\psi^+\|_\infty<|v|\leq t+s+\|\psi^+\|_\infty\}} A(x, v, \nabla v)\nabla v\, dx \leq t \int_{\{|v|>s+\|\psi^+\|_\infty\}} |g(x)|\, dx. \tag{6}$$

Dividing (6) in the both sides by t, (2) gives

$$\frac{\alpha}{t}\int_{\{s+\|\psi^+\|_\infty<|v|\leq t+s+\|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\, dx \leq \int_{\{|v|>s+\|\psi^+\|_\infty\}} |g(x)|\, dx. \tag{7}$$

Passing to the limit in (7), for t goes to zero we have

$$\alpha \frac{d}{ds}\int_{\{|v|\leq s+\|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\, dx \leq \int_0^{\mu_v(s+\|\psi^+\|_\infty)} g_*(\tau)\, d\tau.$$

Integrating between 0 and t, we get

$$\alpha\Big(\int_{\{|v|\leq t+\|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\, dx - \int_{\{|v|\leq \|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\, dx\Big) \leq \int_0^t \int_0^{\mu_v(s+\|\psi^+\|_\infty)} g_*(\tau)\, d\tau\, ds, \tag{8}$$

which gives

$$\alpha \int_{\{|v|\leq t+\|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\, dx \leq \int_0^t \int_0^{\mu_v(s+\|\psi^+\|_\infty)} g_*(\tau)\, d\tau\ ds + \int_{\{|v|\leq \|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\, dx. \tag{9}$$

For the first term in the right hand side, by using Remark 2 and Remark 3 we have

$$
\begin{aligned}
\int_0^t \int_0^{\mu_v(s+\|\psi^+\|_\infty)} g_*(\tau)\, d\tau\, ds &\leq c_2 \int_0^t \int_0^{\mu_v(s+\|\psi^+\|_\infty)} \tau^{-\frac{1}{m^-}}\, d\tau\, ds \\
&\leq c_2 (m^-)' \int_0^t \mu_v(s+\|\psi^+\|_\infty)^{1-\frac{1}{m^-}}\, ds \\
&\leq c_2 (m^-)' c_1^{1-\frac{1}{m^-}} \int_0^t (s+\|\psi^+\|_\infty)^{-\frac{(p_*)^-}{(m^-)'\gamma}}\, ds \\
&\leq c_2 (m^-)' c_1^{1-\frac{1}{m^-}} \int_0^{t+\|\psi^+\|_\infty} s^{-\frac{(p_*)^-}{(m^-)'\gamma}}\, ds \\
&\leq \frac{c_2 (m^-)' c_1^{\frac{1}{(m^-)'}}}{1-\frac{(p_*)^-}{(m^-)'\gamma}} \left((t+\|\psi^+\|_\infty)^{1-\frac{(p_*)^-}{(m^-)'\gamma}} \right) \\
&=: c\,(t+\|\psi^+\|_\infty)^{1-\frac{(p_*)^-}{(m^-)'\gamma}}\,.
\end{aligned}
\tag{10}
$$

Now we estimate the second term in the right hand side of (9). Indeed by definition of entropy solution for obstacle problem $(\mathcal{P})$ we have

$$
\int_\Omega A(x,v,\nabla v)\nabla T_t(v-\varphi)\ dx \leq \int_\Omega g(x) T_t(v-\varphi)\ dx \leq t\|g\|_1, \tag{11}
$$

for all $t > 0$, and $\varphi \in \mathcal{K}_\psi(\Omega) \cap L^\infty(\Omega)$,
using assumptions (2)–(3) and Young's inequality, we have, for all $t > 0$,

$$
\begin{aligned}
\int_{\{|v-\varphi|\leq t\}} & A(x,v,\nabla v)\nabla(v-\varphi)\ dx \\
& \geq \alpha \int_{\{|v-\varphi|\leq t\}} |\nabla v|^{p(x)}\, dx \\
& -\beta \int_{\{|v-\varphi|\leq t\}} \left(\varphi(x) + |v|^{p(x)-1} + |\nabla v|^{p(x)-1}\right)|\nabla\varphi|\, dx \\
& \geq \alpha/2 \int_{\{|v-\varphi|\leq t\}} |\nabla v|^{p(x)}\, dx \\
& -C \int_{\{|v-\varphi|\leq t\}} \left(\varphi(x)^{p'(x)} + |\nabla\varphi|^{p(x)}\, dx\right) - (t+\|\varphi\|_\infty)^{p^\pm}|\Omega|,
\end{aligned}
\tag{12}
$$

by (11) and (12), for all $t > 0$ we have

$$
\begin{aligned}
\alpha/2 \int_{\{|v-\varphi|\leq t\}} |\nabla v|^{p(x)}\, dx \leq \quad & C \int_{\{|v-\varphi|\leq t\}} \left(\varphi(x)^{p'(x)} + |\nabla\varphi|^{p(x)}\, dx\right) \\
& + (t+\|\varphi\|_\infty)^{p^\pm}|\Omega| + t\|g\|_1,
\end{aligned}
$$

replacing t with $t+\|\varphi\|_\infty$ in the last inequality, we get

$$\begin{aligned}\int_{\{|v|\le t\}} |\nabla v|^{p(x)}\ dx &\le \int_{\{|v-\varphi|\le t+|\varphi\|_\infty\}} |\nabla v|^{p(x)}\ dx\\ &\le C\Big\{\int_{\{|v-\varphi|\le t+|\varphi\|_\infty\}} \Big(\varphi(x)^{p'(x)}+|\nabla\varphi|^{p(x)}\Big)\,dx\\ &+(t+2\|\varphi\|_\infty)^{p^\pm}|\Omega|+(t+\|\varphi\|_\infty)\|g\|_1\Big\}.\end{aligned}$$

Taking $t=\|\psi^+\|_\infty$ in the previous inequality, we obtain

$$\int_{\{|v|\le\|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\ dx\le C. \tag{13}$$

By combining (9), (10) and (13) we obtain

$$\int_{\{|v|\le t+\|\psi^+\|_\infty\}} |\nabla v|^{p(x)}\,dx\le C(t+\|\psi^+\|_\infty)^{1-\frac{(p_*)^-}{(m^-)'\gamma}}\quad \text{for all}\ \ t>0,$$

which implies

$$\int_\Omega |\nabla T_t(v)|^{p(x)}\,dx\le Ct^{1-\frac{(p_*)^-}{(m^-)'\gamma}}\quad \text{for all}\ \ t\ge\|\psi^+\|_\infty.$$

□

Theorem 2. *Under assumptions* (2), (3) *and* $g\in L^{m(\cdot)}(\Omega)$ *with* $m^-<(p_*^-)'$. *If* v *is a solution in the sense of Definition 4 there exists a constant* $c>0$, *such that*

1. $\displaystyle\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\,dx\le c,\forall t>0$, *with* $\gamma=\dfrac{m^-p^--(p_*)^-(m^--1)}{m^-(p^--1)}$,

2. *Let* $q_0(x)=\dfrac{p_*(x)}{\gamma}$, *for all* $q(\cdot)$ *such that* $0\ll q(\cdot)\ll q_0(\cdot)$, *we have* $|v|^{q(\cdot)}\in L^1(\Omega)$. *Moreover there exists a constant* $c_0>0$ *such that* $\displaystyle\int_\Omega |v|^{q(x)}\,dx\le c_0$.

Proof. 1). We find γ such that $v\in M^{\frac{p_*(\cdot)}{\gamma}}(\Omega)$.
By using Theorem 2 and Sobolev embedding we have

$$\begin{aligned}\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\,dx &= \int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\left|\frac{T_t(v)}{t}\right|^{p_*(x)}dx\\ &=\int_{\{|v|>t\}} \left|t^{(\frac{1}{\gamma}-1)}T_t(v)\right|^{p_*(x)}dx\\ &\le \left\|t^{(\frac{1}{\gamma}-1)}T_t(v)\right\|_{p_*(x)}^{\alpha_1}\le c\left\|\nabla(t^{(\frac{1}{\gamma}-1)}T_t(v))\right\|_{p(x)}^{\alpha_1}\\ &\le c\Big(\int_\Omega t^{(\frac{1}{\gamma}-1)p(x)}|\nabla T_t(v)|^{p(x)}\,dx\Big)^{\alpha_1/\alpha_2}\\ &\le c\Big(\int_\Omega t^{(\frac{1}{\gamma}-1)p(x)}\frac{|\nabla T_t(v)|^{p(x)}}{t^{1-\frac{(p_*)^-}{(m^-)'\gamma}}}t^{1-\frac{(p_*)^-}{(m^-)'\gamma}}\,dx\Big)^{\alpha_1/\alpha_2}\end{aligned} \tag{14}$$

for all $t \geq \|\psi^+\|_\infty$, with $\alpha_1 = (p_*)^\pm$ and $\alpha_2 = p^\pm$.
By choosing γ such that $(\frac{1}{\gamma} - 1)p(x) + 1 - \frac{(p_*)^-}{(m^-)'\gamma} \leq 0$ we have

$$\gamma \geq \frac{m^- p(x) - (p_*)^-(m^- - 1)}{m^-(p(x) - 1)}, \quad \forall x \in \Omega,$$

so it's enough to choose $\gamma = \left(\frac{m^- p(x) - (p_*)^-(m^- - 1)}{m^-(p(x) - 1)}\right)^+$. By the fact that $\frac{m^- p(x) - (p_*)^-(m^- - 1)}{m^-(p(x) - 1)}$ is non-increasing in $p(x)$ since $m^- < (p_*^-)'$ we have $\gamma = \frac{m^- p^- - (p_*)^-(m^- - 1)}{m^-(p^- - 1)}$. Then by Theorem 1, (14) becomes

$$\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}} \, dx \leq c.$$

For $t < \|\psi^+\|_\infty$ we have

$$\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}} \, dx \leq (\|\psi^+\|_\infty + 1)^{\frac{(p_*)^+}{\gamma}} |\Omega|.$$

2). Let $0 \ll q(\cdot) \ll q_0(\cdot)$ and $\epsilon = (q_0(\cdot) - q(\cdot))^- > 0$. By Theorem 1, we have

$$\int_{\{|v|>t\}} t^{q_0(x)} \, dx \leq c, \quad \text{for all } \ t > 0.$$

From Lemma 1, we get

$$\int_\Omega |v|^{q(x)} \, dx \leq 2|\Omega| + c\Big(\frac{q_0(\cdot) - \epsilon}{\epsilon}\Big)^+, \ \text{which give the results.}$$

□

Theorem 3. *Under assumptions* (2), (3) *and* $m^- < (p_*^-)'$, *if* v *is a solution in the sense of Definition 4 and there exists a positive constant* c *such that* $\int_{\{|v|>t\}} t^{q(x)} \, dx \leq c, \forall t > 0,$
then $|\nabla v|^{\alpha(\cdot)} \in M^{q(\cdot)}(\Omega)$, *where* $\alpha(\cdot) = \frac{\gamma(m^-)' p(\cdot)}{\gamma(m^-)'(q(\cdot) + 1) - (p_*)^-}$. *Moreover*

$$\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{q(x)} \, dx \leq C', \ \textit{for all } \ t > 0, \ \textit{with } C' \textit{ is a positive constant}.$$

Proof. Using Theorem 1, and the definition of $\alpha(\cdot)$, for $t \geq \|\psi^+\|_\infty$ we have

$$\begin{aligned}
\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{q(x)} \, dx &\leq \int_{\{|\nabla v|^{\alpha(\cdot)}>t\} \cap \{|v| \leq t\}} t^{q(x)} \, dx + \int_{\{|\nabla v|^{\alpha(\cdot)}>t\} \cap \{|v|>t\}} t^{q(x)} \, dx \\
&\leq \int_{\{|v| \leq t\}} t^{q(x)} \Big(\frac{|\nabla v|^{\alpha(x)}}{t}\Big)^{p(x)/\alpha(x)} \, dx + c \\
&= \int_{\{|v| \leq t\}} t^{q(x)+1-\frac{(p_*)^-}{\gamma(m^-)'} - \frac{p(x)}{\alpha(x)}} \frac{|\nabla T_t(v)|^{p(x)}}{t^{1-\frac{(p_*)^-}{\gamma(m^-)'}}} \, dx + c \\
&= \int_{\{|v| \leq t\}} \frac{|\nabla T_t(v)|^{p(x)}}{t^{1-\frac{(p_*)^-}{\gamma(m^-)'}}} \, dx + c \leq C'.
\end{aligned}$$

where c and C' are positive constants,
for $t < \|\psi^+\|_\infty$ we have

$$\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{q(x)}\, dx \leq (\|\psi^+\|_\infty + 1)^{q^+}|\Omega|.$$

□

Theorem 4. *Under assumptions* (2), (3) *and* $g \in L^{m(\cdot)}(\Omega)$ *with* $m^- < (p_*^-)'$, $q_0(\cdot)$ *be as defined in Theorem 2 and* $q_1(\cdot) = q_0(\cdot)\alpha(\cdot)$.
If v is an entropy solution of problem ($\mathcal{P}$), *then* $|\nabla v|^{q(\cdot)} \in L^1(\Omega)$, *for all* $q(\cdot)$ *such that* $0 \ll q(\cdot) \ll q_1(\cdot)$. *Moreover there exists a constant C such that*

$$\int_\Omega |\nabla v|^{q(x)}\, dx \leq C.$$

Proof. By Theorem 3, we have

$$|\nabla v|^{\alpha(\cdot)} \in M^{q_0(\cdot)}(\Omega), \quad with \quad \alpha(\cdot) = \frac{p(\cdot)}{(q_0(\cdot)+1) - \frac{(p_*)^-}{\gamma(m^-)'}}.$$

Let $0 \ll q(\cdot) \ll q_1(\cdot)$ and $r(\cdot) = q(\cdot)/\alpha(\cdot) \ll q_0(\cdot)$.
Using the Lemma 1 we obtain

$$\int_\Omega |\nabla v|^{q(x)}\, dx = \int_\Omega |\nabla v|^{\alpha(x)r(x)}\, dx \leq C.$$

□

3.2. Existence of Solutions

In this subsection we prove the existence and regularity of solutions in the sense of Definition 4, extending some results known in the constant exponent case.

Theorem 5. *Under assumptions* (2), (3) *and* $g \in L^{m(\cdot)}(\Omega)$, *with* $m^- < (p_*^-)'$. *Then there exists a solution v in the sense of Definition 4. Moreover*

1. $|v|^{q(\cdot)} \in L^1(\Omega)$ *for* $0 \ll q(\cdot) \ll q_0(\cdot)$, *with* $q_0(\cdot) = \dfrac{p_*(\cdot)}{\gamma}$.
2. $|\nabla v|^{q(\cdot)} \in L^1(\Omega)$ *for* $0 \ll q(\cdot) \ll q_1(\cdot)$, *with* $q_1(\cdot) = \dfrac{(m^-)'p(\cdot)p_*(\cdot)}{(m^-)'(p_*(\cdot)+\gamma) - (p_*)^-}$,

where

$$\gamma = \frac{m^-p^- - (p_*)^-(m^- - 1)}{m^-(p^- - 1)}.$$

Remark 4. *In the case $p(\cdot) = p$ and $m(\cdot) = m$, the exponents $q_0(\cdot)$ and $q_1(\cdot)$ are respectively of the form* $\dfrac{Nm(p-1)}{N-mp}$ *and* $\dfrac{Nm(p-1)}{N-m}$.

Let $(g_n)_n \subset L^\infty(\Omega)$ a sequence that converge strongly to g in $L^{m(\cdot)}(\Omega)$, and $\|g_n\|_{m(\cdot)} \leq \|g\|_{m(\cdot)}$, for all n.
Let $(\mathcal{P}_n)$ the approximate problem defined by

$$(\mathcal{P}_n)\begin{cases} -\mathrm{div}\, A(x, v_n, \nabla v_n) = g_n, & \text{in} \quad \Omega, \\ v_n = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

The problem $(\mathcal{P}_n)$ has a weak energy solution $v_n \in \mathcal{K}_\psi \cap W_0^{1,p(\cdot)}(\Omega)$ as a result of a standard modification of the arguments in [21]. Our goal is to prove that v_n tend to a measurable function v as n tend to infinity, and we prove that v is solution in the sense of Definition 4. We will divide the proof in two steps and we employ the a priori estimates for v_n and its gradient derived in the preceding section as our main tool. We follow the standard method used in the several paper as [9], [14] and [24].

We prove in first step the almost everywhere convergence of the gradient.

First we prove that the sequence $(v_n)_n$ of solutions to problem $(\mathcal{P}_n)$ converges in measure to a measurable function v.
Define the $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ sets as follows

$$\mathcal{E}_1 = \{|v_n| > t\}, \;\; \mathcal{E}_2 = \{|v_m| > t\}, \;\text{ and }\; \mathcal{E}_3 = \{|T_t(v_n) - T_t(v_m)| > s\},$$

for $s > 0$ and $t \geq \|\psi^+\|_\infty$. Since

$$\{|v_n - v_m| > s\} \subset \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3,$$

it follows that

$$\text{meas}\{|v_n - v_m| > s\} \leq \text{meas}(\mathcal{E}_1) + \text{meas}(\mathcal{E}_2) + \text{meas}(\mathcal{E}_3).$$

Let $\epsilon > 0$, by Theorem 1, v_n is uniformly bounded sequence, thus there exists t_ϵ, such that for $t \geq t_\epsilon$ we have

$$\text{meas}(\mathcal{E}_1) \leq \epsilon/3 \;\text{ and }\; \text{meas}(\mathcal{E}_2) \leq \epsilon/3.$$

In the approximate problem $(\mathcal{P}_n)$, we take $T_t(v_n)$ as test function and following the outlines of Theorem 1, we get

$$\int_\Omega |\nabla T_t(v_n)|^{p(x)}\,dx \leq c t^{1-\frac{(p_*)^-}{\gamma(m^-)'}}, \;\text{ for all }\; n \geq 0 \;\text{ and }\; t \geq \|\psi^+\|_\infty.$$

Sobolev embedding imply that there exists a subsequence still denoted by $(T_t(v_n))_n$ such that

$$\begin{aligned} &T_t(v_n) \rightharpoonup T_t(v) \;\text{ weakly in }\; W_0^{1,p(\cdot)}(\Omega),\\ &T_t(v_n) \to T_t(v) \;\text{ strongly in }\; L^{q(\cdot)}(\Omega), \;\text{ for }\; 1 \leq q(\cdot) < p_*(\cdot),\\ &T_t(v_n) \to T_t(v) \;\text{ a.e. in }\; \Omega, \end{aligned} \tag{15}$$

for all $t \geq \|\psi^+\|_\infty$. Thus,

$$\text{meas}(\mathcal{E}_3) \leq \int_\Omega \Big(\frac{|T_t(v_n) - T_t(v_m)|}{s}\Big)^{q(x)} dx \leq \epsilon/3, \;\text{ for all }\; n \geq n_0(s,\epsilon).$$

Finely we have

$$\text{meas}\{|v_n - v_m| > s\} \leq \epsilon, \;\text{ for all }\; n, m \geq n_0(s,\epsilon) \;\text{ i.e. }\; (v_n)_n \;\text{ is a Cauchy sequence in measure.}$$

Following the standard argument as in [18], proving that $(\nabla v_n)_n$ is a Cauchy sequence in measure is an easy task.
In the second step we passing to the limit.
Let v_n be a solution of approximate unilateral problem $(\mathcal{P}_n)$, for $\varphi \in \mathcal{K}_\psi(\Omega)$ we have

$$\int_\Omega A(x, v_n, \nabla v_n)\nabla(v_n - \varphi)\,dx \leq \int_\Omega g_n(x)(v_n - \varphi)\,dx.$$

Taking $\varphi = v_n - T_t(v_n - w)$ with $w \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ we get

$$\int_\Omega A(x, v_n, \nabla v_n)\nabla T_t(v_n - w)\,dx \leq \int_\Omega g_n(x) T_t(v_n - w)\,dx.$$

For the term in the right hand side, since g_n converge strongly to g in $L^1(\Omega)$ and $T_t(v_n - w)$ converge weakly-* to $T_t(v - w)$ in $L^\infty(\Omega)$, and a.e. in Ω we have

$$\int_\Omega g_n(x) T_t(v_n - w)\, dx \longrightarrow \int_\Omega g(x) T_t(v - w)\, dx.$$

For the left hand side we have

$$\begin{aligned}\int_\Omega A(x, v_n, \nabla v_n) \cdot \nabla T_t(v_n - w)\, dx &= \int_{\{|v_n - w| \le t\}} A(x, v_n, \nabla v_n) \cdot \nabla v_n\, dx \\ &\quad - \int_{\{|v_n - w| \le t\}} A(x, v_n, \nabla v_n) \cdot \nabla w\, dx \\ &= \int_{\{|v_n - w| \le t\}} A(x, v_n, \nabla v_n) \cdot \nabla v_n\, dx \\ &\quad - \int_{\{|v_n - w| \le t\}} A(x, T_s(v_n), \nabla T_s(v_n)) \cdot \nabla w\ d\,x,\end{aligned}$$

with $s = t + \|w\|_\infty$.
By (3) and (15), we can prove that $A(x, T_s(v_n), \nabla T_s(v_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$, and converges weakly to $A(x, T_s(v), \nabla T_s(v))$ in $(L^{p'(\cdot)}(\Omega))^N$. Therefore we have

$$\int_{\{|v_n - w| \le t\}} A(x, v_n, \nabla v_n) \cdot \nabla w\, dx \longrightarrow \int_{\{|v - w| \le t\}} A(x, v, \nabla v) \cdot \nabla w\, dx. \tag{16}$$

Since $A(x, v_n, \nabla v_n) \cdot \nabla v_n \to A(x, v, \nabla v) \cdot \nabla v$ a.e. in Ω, by Fatou's lemma we have

$$\liminf_n \int_{\{|v_n - w| \le t\}} A(x, v_n, \nabla v_n) \cdot \nabla v_n\, dx \ge \int_{\{|v - w| \le t\}} A(x, v, \nabla v) \cdot \nabla v\, dx. \tag{17}$$

By (16) and (17), for all $w \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ we have

$$\int_\Omega A(x, v, \nabla v) \nabla T_t(v - w)\, dx \le \int_\Omega g(x) T_t(v - w)\ d\,x.$$

Arguing as in Theorem 2 and Theorem 4 we obtain the results of regularity.

Authors' contributions

The authors declare that their contributions are equal. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to thank the referees for their suggestions and helpful comments which have improved the presentation of the original manuscript.

References

[1] A. Aberqi, A. Benkirane and M. Elmassoudi, On some nonlinear degenerate elliptic equations having a lower term in Musielak spaces. Adv. Oper. Theory 7, 14 (2022). https://doi.org/10.1007/s43036-021-00182-x
[2] A. Aberqi, J. Bennouna, O. Benslimane, and M. A. Ragusa, Existence Results for Double Phase Problem in Sobolev–Orlicz Spaces with Variable Exponents in Complete Manifold, Mediterr. J. Math. 19, 158 (2022). https://doi.org/10.1007/s00009-022-02097-0

[3] A. Aberqi, O. Benslimane, M. Elmassoudi and M. A. Ragusa, Nonnegative solution of a class of double phase problems with logarithmic nonlinearity. Bound Value Probl 2022, 57 (2022). https://doi.org/10.1186/s13661-022-01639-5

[4] E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$-Laplacian system, J. reine angew. Math. 584 (2005), 117–148.

[5] B. Aharrouch, and J. Bennouna, Existence and Regularity Results for Nonlinear and Nonhomogeneous Elliptic Equation, Journal of Elliptic and Parabolic Equations 7 (2), (2021), 961–975.

[6] B. Aharrouch, J. Bennouna and B. El hamdaoui, Existence of weak and renormalized solutions of degenerated elliptic equation, Afrika mathematika Volume 30, Pages 755–776, (2019).

[7] B. Aharrouch and J. Bennouna, Existence and uniqueness of solution for a unilateral problem in Sobolev paces with variable exponent, Applicationes Mathematicae 46 (2019), 175–189.

[8] T. Ahmedatt, A. Aberqi, A. Touzani and C. Yazough. On some nonlinear hyperbolic p(x,t)-Laplacian equations, Journal of Applied Analysis, vol. 24, no. 1, 2018, pp. 55-69. https://doi.org/10.1515/jaa-2018-0006

[9] A. Alvino, L. Boccardo, V. Ferone, L. Orsina and G. Trombetti, Existence results for non-linear elliptic equations with degenerate coercivity, Ann. Mat. Pura Appl. 182 (2003), 53–79.

[10] C. Bandle, Isoperimetric inequalities and applications, in: Monographs and Studies in Math., Pitman, London, (1980).

[11] A. Benali and B. Jaouad, Nonlinear degenerate elliptic equations in weighted Sobolev spaces, Electronic Journal of Differential Equations 2020 (105), 1–15.

[12] A. Benali and B. Jaouad, Existence of solutions for an elliptic problem with degenerate coercivity, Nonlinear Studies Vol. 28, No. 1, (2021), 237–252.

[13] M. Bendahmane, K. H. Karlsen and M. Saad, Nonlinear anisotropic elliptic and parabolic equations with variable exponents and L^1 data, communications on pure and applied analysis Vol. 12, No. 3, (2013), 1201–1220.

[14] Ph. Benilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J.L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 22 (1995), 241–273.

[15] O. Benslimane, A. Aberqi and J. Bennouna, Existence results for double phase obstacle problems with variable exponents. J Elliptic Parabol Equ 7, 875–890 (2021). https://doi.org/10.1007/s41808-021-00122-z

[16] L. Boccardo and T. Gallouet, Nonlinear elliptic equations with right-hand side measures, Comm. Partial Diferential Equations 17 (1992), 641–655.

[17] L. Boccardo and T. Gallouet, Nonlinear Elliptic and Parabolic Equations Involving Measure Data, Journal of Functional Analysis 87, (1989), 149–169.

[18] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal., 19 (1992), 581–597.

[19] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration, SIAMJ. Appl. Math. 66 (2006), no. 4, 1383–1406.

[20] L. Diening, P. Harjulehto, P. Hasto, and M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Mathematics, 28 (4), (2011), 741–808.

[21] X.-L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Analysis 52 (2003), 1843–1852.

[22] G. H. Hardy, J. L. Littlewood and G. Pólya, Inequalities, Cambridge Univ. Press, 1964.

[23] M. A. Ragusa, A. Razani, and F. Safari, Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem, Adv Differ Equ 2021, 215 (2021). https://doi.org/10.1186/s13662-021-03369-x

[24] J. F. Rodrigues, M. Sanchon, and J. M. Urbano, The obstacle problem for nonlinear elliptic equations with variable growth and L^1-data. Monatshefte for Mathematik, 154, (2008), 303–322.

[25] M. Sanchon and J. M. Urbano, Entropy solutions for the $p(x)$-Laplace equation. Transactions of the american mathematical society, 361(12), (2009), 6387–6405.

[26] G. Talenti, Linear Elliptic P.D.E.'s: Level sets, rearrangements and a priori estimates of solutions, Boll. Unione Mat. Ital.; B (6) 4 (1985) 917–949.

[27] X. Wang, J. Zhang, Existence and regularity of positive solutions of a degenerate fourth order elliptic problem, Topological Methods in Nonlinear Analysis, (2022),https://doi.org/10.12775/TMNA.2021.019

[28] P. Wu, Y. Huang, and Y. Zhou, Existence and regularity of solutions for a class of fractional Laplacian problems, Journal of Differential Equations, vol 318, (2022), 480–501

[29] D. Zhao, W. J. Qiang and X. L. Fan, On generalized Orlicz spaces $L^{p(\cdot)}(\Omega)$, J. Gansu Sci. 9(2), (1997), 1–7.