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Sign-changing solutions with prescribed number of nodes for elliptic equations with fast increasing weight

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Abstract. In this article, we study the problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u), \quad x \in \mathbb{R}^2,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a superlinear continuous function with exponential subcritical or exponential critical growth. The main results obtained in this paper are that for any given integer $k \geq 1$, there exists a pair of sign-changing radial solutions u_k^+ and u_k^- possessing exactly k nodes.

1. Introduction

In this paper, we are looking for a pair of sign-changing solutions for the following class of problems

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u) \ \ in \ \ \mathbb{R}^2. \tag{1}$$

In particular, we are interested in establishing two solutions of (1) which are nodal, namely with $u^+ \neq 0$ and $u^- \neq 0$ in $\mathbb{R}^2$, where

$$u^+(x) := \max\{u(x), 0\} \ \text{ and } \ u^-(x) := \min\{u(x), 0\}$$

and changing of sign k times, where $k \in \mathbb{N}$. Notice that, in this case, $u = u^+ + u^-$ and $|u| = u^+ - u^-$.

As observed by Escobedo and Kavian in [9], since the exponential-type weight $K(x) = \exp(|x|^2/4)$ verifies $\nabla K(x) = \frac{1}{2}xK(x)$, problem (1) can be written as

$$-\text{div}(K(x)\nabla u) = K(x)f(u) \ \ in \ \ \mathbb{R}^2. \tag{2}$$

Such classes of problems as (1) are related to evolution equations. Consider the parabolic equation

$$(P) \qquad v_t - \Delta v = |v|^{p-1}v \ \ in \ \ \mathbb{R}^N \times (0, +\infty),$$

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where $p > 1$ is a fixed parameter and $N \geq 1$. According to [15], a self-similar solution for (P) is a function $v(x,t) = t^{\frac{-1}{p-1}} u(xt^{-\frac{1}{2}})$. Note that v is a solution of (P) if, and only if, u is a solution of the problem

$$(PE) \qquad -\Delta u - \tfrac{1}{2}(x \cdot \nabla u) = \tfrac{1}{p-1}u + |u|^{p-1}u \ \ in \ \ \mathbb{R}^N.$$

In [15], Haraux and Weissler considered problem (PE) in order to prove some non-uniqueness results for the Cauchy problem associated to (P) in the case $N = 1$.

In this article, we consider the case of $N = 2$ and more general nonlinear terms. We will construct a pair of changing solutions u_k^+ and u_k^- possess exactly k nodes to a problem with a nonlinearity $f : \mathbb{R} \to \mathbb{R}$ superlinear continuous function with exponential subcritical or exponential critical growth. More precisely, the hypotheses on the continuous function $f : \mathbb{R} \to \mathbb{R}$ are the ones below.

($\mathbf{F_1}$) There exists $\alpha_0 \geq 0$ such that the function $f(t)$ satisfies

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = 0 \ \text{ for } \ \alpha > \alpha_0 \ \text{ and } \ \lim_{t\to\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = \infty \ \text{ for } \ \alpha < \alpha_0.$$

($\mathbf{F_2}$) There hold

$$\lim_{t\to 0} \frac{f(t)}{|t|} = 0.$$

($\mathbf{F_3}$) There exists $\theta > 2$ such that

$$0 < \theta F(t) \leq f(t)t \ \text{ for all } \ t \neq 0, \ \text{ where } \ F(t) = \int_0^t f(s)ds.$$

($\mathbf{F_4}$) The function $t \to f(t)/|t|$ is increasing in $\mathbb{R} \setminus \{0\}$.

($\mathbf{F_5}$) There exist $p > 2$ and $\tau^* > 0$ such that $sign(t)f(t) \geq \tau|t|^{p-1}$ for all $\tau > \tau^*$ and $t \neq 0$.

Let us denote by $X_{rad}(\mathbb{R}^2)$ the weighted Sobolev space of the radial functions, which is obtained as the closure of $C_{0,rad}^{\infty}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx\Big)^{1/2}.$$

The hypotheses ($\mathbf{F_1}$) – ($\mathbf{F_2}$) imply that the associated functional $I : X_{rad}(\mathbb{R}^2) \to \mathbb{R}$ of problem (2) given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx - \int_{\mathbb{R}^2} K(x)F(u)dx.$$

is well defined in $X_{rad}(\mathbb{R}^2)$.

The main results can be stated as following.

Theorem 1.1. *(Subcritical). Assume that* ($\mathbf{F_1}$) *with* $\alpha_0 = 0$, ($\mathbf{F_2}$), ($\mathbf{F_3}$) *and* ($\mathbf{F_4}$) *hold, then, for any given* $k \in \mathbb{N}$, *problem (2) admit a pair of nontrivial solutions* $u_k^{\pm}$ *with the following properties:*

(*i*) $u_k^-(0) < 0 < u_k^+(0)$.

(*ii*) $u_k^{\pm}$ *possess exactly* k *nodes* r_i *with* $0 < r_1^{\pm} < r_2^{\pm} < \cdots < r_k^{\pm} < \infty$ *and* $u_k^+(r_i^+) = u_k^-(r_i^-) = 0$ *for* $i = 1, 2, \cdots, k$.

(*iii*) *The energy of* $u_k^{\pm}$ *is strictly increasing in* k, *i.e.* $I(u_{k+1}^{\pm}) > I(u_k^{\pm})$ *for all* $k \geq 0$ *and* $I(u_k^{\pm}) > (k+1)I(u_0^{\pm})$.

Theorem 1.2. *(Critical). Assume that* ($\mathbf{F_1}$) *with* $\alpha_0 > 0$, ($\mathbf{F_2}$), ($\mathbf{F_3}$), ($\mathbf{F_4}$) *and* ($\mathbf{F_5}$) *hold, then, for any given* $k \in \mathbb{N}$, *problem (2) admit a pair of nontrivial solutions* $u_k^{\pm}$ *with the following properties:*

(*i*) $u_k^-(0) < 0 < u_k^+(0)$.

(*ii*) $u_k^\pm$ *possess exactly k nodes* r_i *with* $0 < r_1^\pm < r_2^\pm < \cdots < r_k^\pm < \infty$ *and* $u_k^+(r_i^+) = u_k^-(r_i^-) = 0$ *for* $i = 1, 2, \cdots, k$.

(*iii*) *The energy of* $u_k^\pm$ *is strictly increasing in k, i.e.* $I(u_{k+1}^\pm) > I(u_k^\pm)$ *for all* $k \geq 0$ *and* $I(u_k^\pm) > (k+1)I(u_0^\pm)$.

Remark 1.3. *The results in Theorem 1.1 and Theorem 1.2 still hold for any rotationally symmetric domain. And compared with* $k = 0$*, the solutions* $u_k^\pm$ $(k \geq 1)$ *are the higher energy solutions.*

To our knowledge, the first article that appeared with this argument was that by Cerami, Solimini and Struwe [7]. They show the existence of solutions of changing sign for the classical problem studied by Brezis and Nirenberg [2] with $K = 1$.

Still with $K = 1$, Cao and Zhu [5] studied the case with subcritical polynomial growth and the case with exponential growth, considering the following hypothesis on the nonlinearity:

$$\lim_{t\to\infty} \frac{f(x,t)}{\exp(\gamma|t|)} = 0 \text{ for } 0 < \gamma < 2,$$

uniformly with respect to x. See also Bartsch and Willem [3] for independent work. In [17], Liu and Wang presented a different proof from [3, 5] and established various results on multiple solutions for superlinear elliptic equations with more natural super-quadratic condition.

These arguments were used for the version with system by Cao and Tang in [6], for the p-Laplacian operator by Deng, Guo and Wang in [8] and with the Laplacian operator and for an asymptotically linear nonlinearity by Liu in [16] , all these authors considering $K = 1$.

On the other hand, results on the existence of sign-changing solutions with $K(x) = \exp(|x|^2/4)$ were also studied. Qian and Chen in [19] show existence of sign-changing solutions for a problem with concave and convex nonlinearity with critical polynomial growth. These authors also studied a more general case in [20].

The version with the nonlinearity with exponential growth and the sign-changing solution with an unique node was studied by Figueiredo, Furtado and Ruviaro in [10]. Figueiredo and Montenegro also studied a more general case in [11]. For more discussions on the existence of sign-changing solutions for elliptic equations, we refer the readers to other references, such as [1, 14, 21] and so on.

The present work is strongly influenced by the articles above. Below we list what we believe that are the main contributions of our paper.

(1) Unlike [5], [6], [7], [8] and [16], we show existence of sign-changing solutions with $K(x) = \exp(|x|^2/4)$. Moreover, we also show the energy of $u_k^\pm$ is strictly increasing in k. This last result does not appear in those articles.

(2) We completed the studies done in [19] and [20] because in this paper we are considering nonlinearity with critical exponential growth.

(3) We complement the study that can be found in [10] and in [11] because, in our results, we show an arbitrary number of nodes.

This paper is organized as follows. In order to be able to deal variationally, in Section 2 we define some Function spaces and give radial solutions on rotationally symmetric domains. In Section 3, we prove the main results.

2. Function spaces and radial solutions on rotationally symmetric domains

In this section, we define the weighted Lebesgue spaces

$$L^s_K(\mathbb{R}^2) = \left\{ u \text{ measurable in } \mathbb{R}^2 : \|u\|^s_s = \int_{\mathbb{R}^2} K(x)|u|^s dx < \infty \right\}.$$

It follows from [12, Proposition 2.1] that the embedding $X_{rad}(\mathbb{R}^2) \hookrightarrow L^s_K(\mathbb{R}^2)$ is continuous and compact for $2 \le s < \infty$. Another interesting result is that $X_{rad}(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for any $s \ge 1$. Moreover, the following version of the Trudinger-Moser inequality holds, see[13, Theorem 1.1 and Corollary 1.2].

Lemma 2.1. *For any $q \ge 2$, $u \in X_{rad}(\mathbb{R}^2)$ and $\beta > 0$, we have that $K(x)|u|^q(e^{\beta u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover,if $\|u\| \le M$ and $\beta M^2 < 4\pi$, then there exists $C = C(M, \beta, q) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|u|^q(e^{\beta u^2} - 1)dx \le C(M, \beta, q)\|u\|^q.$$

The hypotheses $(\mathbf{F_1}) - (\mathbf{F_2})$ imply that, for any given $\epsilon > 0$, there exists C_ϵ such that

$$\max\{|f(t)t|, |F(t)|\} \le \epsilon|t|^2 + C_\epsilon|t|^q(\exp(\alpha t^2) - 1), \text{ for } q \ge 1, \text{and } t \in \mathbb{R}. \tag{3}$$

In particular, in this paper, we will use $q > 2$.

This inequality with $q = 2$ and Lemma 2.1 imply that the associated functional of problem (2) $I \in C^1(X_{rad}(\mathbb{R}^2), \mathbb{R})$. By using standard calculations we conclude that

$$I'(u)\phi = \int_{\mathbb{R}^N} K(x)\nabla u \nabla\phi dx - \int_{\mathbb{R}^N} K(x)f(u)\phi dx, \text{ for all } u, v \in X_{rad}(\mathbb{R}^2).$$

In [13, Lemma 4.3], the authors established a variant of the well-known Strauss inequality for the weighted Sobolev space $X_{rad}(\mathbb{R}^2)$ as follows, which is crucial in order to obtain multiple sign-changing solutions.

Lemma 2.2. *There exists $c > 0$ such that, for all $u \in X_{rad}(\mathbb{R}^2)$, there holds*

$$|u(x)| \le c|x|^{-\frac{1}{2}} e^{-\frac{|x|^2}{8}} \|u\|, \textit{ for all } x \in \mathbb{R}^2.$$

The following conclusion is crucial in the proof of our main results, which can be found in [13, inequality (2.4)].

Lemma 2.3. *For any $r \ge 1$ there exists $C = C(r)$ such that*

$$\left(\int_{\mathbb{R}^2} K(x)^r |u|^{2r} dx \right)^{\frac{1}{r}} \le C(r) \int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx, \textit{ for all } u \in X_{rad}(\mathbb{R}^2).$$

2.1. Radial solutions on rotationally symmetric domains

For any an open regular set $\Omega \subset \mathbb{R}^2$, we denote by $X_{0,rad}(\Omega)$ the closure of $C^\infty_{0,rad}(\overline{\Omega})$ with respect to the norm

$$\|u\| = \Big(\int_\Omega K(x)|\nabla u|^2 dx \Big)^{1/2}.$$

We also define the weighted Lebesgue spaces

$$L^s_K(\Omega) = \left\{ u \text{ measurable in } \Omega : \|u\|^s_s = \int_\Omega K(x)|u|^s dx < \infty \right\}.$$

In fact, by the same arguments can be found in [12, Proposition 2.1], we can prove that the embedding $X_{0,rad}(\Omega) \hookrightarrow L^s_K(\Omega)$ is continuous for $2 \le s \le \infty$, and compact for $2 \le s < \infty$.

In this subsection, we replace f by the odd continuous functions $f^{\pm}$, which are given by

$$f_+(t) = \begin{cases} f(t), t \ge 0, \\ -f(-t), t < 0, \end{cases} \text{ and } f_-(t) = \begin{cases} -f(-t), t > 0, \\ f(t), t \le 0. \end{cases}$$

Now, we consider respectively

$$-\operatorname{div}(K(x)\nabla u) = K(x)f_+(u), u \in X_{0,rad}(\Omega) \tag{4}$$

and

$$-\operatorname{div}(K(x)\nabla u) = K(x)f_-(u), u \in X_{0,rad}(\Omega), \tag{5}$$

where Ω is one of the following three kinds of rotationally symmetric domains:

$$\begin{aligned} &\text{Type one (ball centered at the origin)} : \Omega(0,\rho) := \{x \in \mathbb{R}^2 : |x| < \rho\}, \rho > 0; \\ &\text{Type two (annulus)} : \Omega(\rho,\sigma) := \{x \in \mathbb{R}^2 : \rho < |x| < \sigma\}, 0 < \rho < \sigma < \infty; \\ &\text{Type three (the exterior of a ball)} : \Omega(\sigma,\infty) := \{x \in \mathbb{R}^2 : |x| > \sigma\}, \sigma > 0. \end{aligned} \tag{6}$$

It is well known that the associated variational functional of (4) and (5)

$$I_{\pm}(u) = \frac{1}{2}\int_{\Omega} K(x)|\nabla u|^2 dx - \int_{\Omega} K(x)F_{\pm}(u)dx$$

are well-defined and $I_{\pm} \in C^1(X_{0,rad}(\Omega), \mathbb{R})$, where $F_{\pm}(t) = \int_0^t f_{\pm}(s)ds$. For fixed domain Ω, we define the corresponding Nehari's manifold as

$$\mathcal{N}^{\pm}(\Omega) = \left\{ u \in X_{0,rad}(\Omega) : u \neq 0, \int_{\Omega} K(x)|\nabla u|^2 = \int_{\Omega} K(x)f_{\pm}(u)u \right\}. \tag{7}$$

In what follows, by extending $u \in X_{0,rad}(\Omega)$ by zero outside Ω, we may assume that $u \in X_{rad}(\mathbb{R}^2)$.

Remark 2.4. *The result in Lemma 2.1 also holds for $X_{0,rad}(\Omega)$.*

In the next result we show that $\mathcal{N}^{\pm}(\Omega)$ is not empty.

Lemma 2.5. *For each $u \in X_{0,rad}(\Omega) \setminus \{0\}$, there exists a unique $t > 0$ such that $tu \in \mathcal{N}^{\pm}(\Omega)$.*

Proof. Given $u \in X_{0,rad}(\Omega) \setminus \{0\}$, we define the function $\gamma_u(t) := I(tu)$ on $[0,\infty)$. Then $tu \in \mathcal{N}^{\pm}(\Omega)$ if and only if $\gamma_u'(t) = 0$. Using (3) with ϵ small enough and the embedding inequality, we have

$$\gamma_u(t) \ge \left(\frac{1}{2} - \epsilon\frac{C}{2}\right)t^2\|u\|^2 - t^q C_\epsilon \int_{\Omega} K(x)|u|^q(\exp(\alpha|tu|^2) - 1)dx,$$

for some $C > 0$. By Lemma 2.1, there exists $C_1 := C_1(\|u\|, q) > 0$ such that

$$\gamma_u(t) \ge \left(\frac{1}{2} - \epsilon\frac{C}{2}\right)t^2\|u\|^2 - t^q C_\epsilon C_1\|u\|^q,$$

for any $0 \le t < t^* := \sqrt{4\pi/\alpha\|u\|^2}$. Since $q > 2$, there is $0 < t_* \le t^*$ such that $\gamma_u(t) > 0$ for all $0 < t < t_*$.

Moreover, from $(\mathbf{F_2})$ and $(\mathbf{F_3})$, there exist $C_2 > 0$ and $C_3 > 0$ such that

$$\gamma_u(t) \le \frac{t^2}{2}\|u\|^2 - t^\theta C_2|u|_\theta^\theta + t^2C_3|u|_2^2.$$

Therefore, since $\theta > 2$, we conclude that $\lim_{t\to+\infty}\gamma_u(t) = -\infty$. Consequently, there exists at least one $t := t(u) > 0$ such that $\gamma_u'(t) = 0$, i.e. $tu \in \mathcal{N}^\pm(\Omega)$. Note, in particular, that

$$\frac{\gamma_u'(t)}{t} = \|u\|^2 - \int_\Omega K(x)\frac{f_\pm(u)}{t}udx.$$

Then, it follows from $(\mathbf{F_4})$ that $\frac{\gamma_u'(t)}{t}$ is decreasing, and so we get the uniqueness. The lemma is proved. □

In the next results we prove that sequences in $\mathcal{N}^\pm(\Omega)$ cannot converge to 0.

Lemma 2.6. *For any $u \in \mathcal{N}^\pm(\Omega)$, there exists $C > 0$ such that $\|u\| \ge C$.*

Proof. We prove it by contradiction. Suppose that there is $u_n \in \mathcal{N}^+(\Omega)$ such that $u_n \to 0$ in $X_{0,rad}(\Omega)$. It follows from (3) and Sobolev inequality that

$$\begin{aligned}\|u_n\|^2 &= \int_\Omega K(x)f_+(u_n)u_ndx\\ &\le \epsilon\int_\Omega K(x)|u_n|^2dx + C_\epsilon\int_\Omega K(x)|u_n|^q(\exp(\alpha u_n^2)-1)dx\\ &\le C\epsilon\|u_n\|^2 + C_\epsilon\int_\Omega K(x)|u_n|^q(\exp(\alpha u_n^2)-1)dx,\end{aligned}$$

that is,

$$(1-C\epsilon)\|u_n\|^2 \le C_\epsilon\int_\Omega K(x)|u_n|^q(\exp(\alpha u_n^2)-1)dx.$$

Since $u_n \to 0$ in $X_{0,rad}(\Omega)$, there exists $n_0 \in \mathbb{N}$ such that $\|u_n\| \le M$ with $\alpha M^2 < 4\pi$ for all $n \ge n_0$ and some $M > 0$. Then, it follows from Lemma 2.1 that

$$\int_\Omega K(x)|u_n|^q(\exp \alpha {u_n}^2 - 1)dx \le C(M,\alpha,q)\|u_n\|^q.$$

Therefore, we have

$$(1-C\epsilon)\|u_n\|^2 \le C_\epsilon C(M,\alpha,q)\|u_n\|^q,$$

which implies

$$\frac{1-C\epsilon}{C_\epsilon C(M,\alpha,q)} \le \|u_n\|^{q-2}. \tag{8}$$

Since $q > 2$, the above inequality contradicts the fact that $u_n \to 0$ in $X_{0,rad}(\Omega)$ and the lemma is proved. □

The following proposition shows that the minimizer of $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$ and $\inf_{\mathcal{N}^-(\Omega)} I_-(u)$ are solutions.

Proposition 2.7. *Assume that $\hat{u}$ and $\hat{v}$ are minima of $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$ and $\inf_{\mathcal{N}^-(\Omega)} I_-(u)$, then $|\hat{u}|$ and $-|\hat{v}|$ are positive and negative radial solutions of problems* (4) *and* (5)*, respectively.*

Proof. We first prove that if $\hat{u}$ is the minima of $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$, then $\hat{u}$ is a solution of (4). Suppose by contradiction, that $\hat{u}$ is not a weak solution of (4). Then one can find $\varphi \in X_{0,rad}(\Omega)$ such that

$$I'_+(\hat{u})\varphi = \int_\Omega K(x)\nabla\hat{u}\nabla\varphi - \int_\Omega K(x)f_+(\hat{u})\varphi \leq -1.$$

Choose $\varepsilon > 0$ very small such that

$$I'_+(t\hat{u} + \sigma\varphi)\varphi \leq -\frac{1}{2}, \text{ for all } |t-1| + |\sigma| \leq \varepsilon.$$

Let η be a cut-off function such that $\eta(t) = 1$, if $|t-1| \leq \frac{1}{2}\varepsilon$; $\eta(t) = 0$, if $|t-1| \geq \varepsilon$. In the following, we estimate $\sup_{t\geq 0} I_+(t\hat{u} + \varepsilon\eta(t)\varphi)$. If $|t-1| + |\sigma| \leq \varepsilon$, then

$$\begin{aligned} I_+(t\hat{u} + \varepsilon\eta(t)\varphi) &= I_+(t\hat{u}) + \int_0^1 I'_+(t\hat{u} + \sigma\varepsilon\eta(t)\varphi)\varepsilon\eta(t)\varphi d\sigma \\ &\leq I_+(t\hat{u}) - \frac{1}{2}\varepsilon\eta(t). \end{aligned}$$

For $|t-1| \geq \varepsilon$, $\eta(t) = 0$, the above inequality is trivial. Since $\hat{u} \in \mathcal{N}^+(\Omega)$, for $t \neq 1$, we have $I_+(t\hat{u} + \varepsilon\eta(t)\varphi) < I_+(\hat{u})$, hence

$$I_+(t\hat{u} + \varepsilon\eta(t)\varphi) \leq I_+(t\hat{u}) < I_+(\hat{u}) \text{ for } t \neq 1.$$

If $t = 1$, then $I_+(t\hat{u} + \varepsilon\eta(1)\varphi) \leq I_+(t\hat{u}) - \frac{1}{2}\varepsilon\eta(1) = I_+(\hat{u}) - \frac{1}{2}\varepsilon$. In any case, we have

$$I_+(t\hat{u} + \varepsilon\eta(t)\varphi) < I_+(\hat{u}) = \inf_{\mathcal{N}^+(\Omega)} I_+(u).$$

Therefore, we have

$$\sup_{t\geq 0} I_+(t\hat{u} + \varepsilon\eta(t)\varphi) := \hat{m} < \inf_{\mathcal{N}^+(\Omega)} I_+(u).$$

Now, we define $g(t) = I'_+(t\hat{u} + \varepsilon\eta(t)\varphi)(t\hat{u} + \varepsilon\eta(t)\varphi)$. By direct computation, one gets $g(1-\varepsilon) = I'_+((1-\varepsilon)\hat{u})((1-\varepsilon)\hat{u}) > 0$ and $g(1+\varepsilon) = I'_+((1+\varepsilon)\hat{u})((1+\varepsilon)\hat{u}) < 0$. Thus, By Miranda's theorem [18], there exists $\hat{t} \in (1-\varepsilon, 1+\varepsilon)$ such that $g(\hat{t}) = 0$, that is $\hat{t}\hat{u} + \varepsilon\eta(\hat{t})\varphi \in \mathcal{N}^+(\Omega)$ and so $I_+(\hat{t}\hat{u} + \varepsilon\eta(\hat{t})\varphi) < \inf_{\mathcal{N}^+(\Omega)} I_+(u)$, which is a contradiction. We have proved that $\hat{u}$ is a solution to equation (4).

Next we prove that $\hat{u}$ is constant-sign. Indeed, let $\hat{u} = \hat{u}^+ + \hat{u}^-$, we get $I_+(\hat{u}) = I_+(\hat{u}^+) + I_+(\hat{u}^-)$. If $\hat{u}^+ \neq 0$, $\hat{u}^- \neq 0$, it is easy to verify that $I_+(\hat{u}^+) > 0$, $I_+(\hat{u}^-) > 0$, $\hat{u}^+ \in \mathcal{N}^+(\Omega)$ and $\hat{u}^- \in \mathcal{N}^+(\Omega)$, which contradicts the definition of $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$. Thus, u remains non-positive or non-negative on Ω. By classical regularity elliptic theory, we can obtain that $\hat{u} \in C^2(\overline{\Omega})$. Since $f_+(u)$ is a odd function, then both $\hat{u}$ and $-\hat{u}$ attain $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$, and so we can deduce $|\hat{u}|$ is a positive solution of (4) by standard strong maximum principle.

By a similar argument, we can obtain that $-|\hat{v}|$ is a negative solution of (5). The proof is completed. □

In the following, we verify that $\inf_{\mathcal{N}^-(\Omega)} I_-(u)$ and $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$ are achieved.

2.2. The Subcritical Case

Proposition 2.8. *(Subcritical). Suppose that* $(\mathbf{F_1})$ *with* $\alpha_0 = 0$, $(\mathbf{F_2}) - (\mathbf{F_4})$ *hold, then* $\inf_{\mathcal{N}^\pm(\Omega)} I_\pm(u)$ *can be achieved by some* $v \in \mathcal{N}^\pm(\Omega)$.

Proof. We only give the proof for $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$ since the other case is similar and we omit it here. By **(F₃)**, if $u \in \mathcal{N}^+(\Omega)$, then

$$I_+(u) = I_+(u) - \frac{1}{\theta}I'_+(u)u \geq (\frac{1}{2} - \frac{1}{\theta}) \int_\Omega K(x)|\nabla u|^2 dx.$$

Since $\theta > 2$, then $I_+(u)$ is bounded from below. Therefore, the minimizing sequence (u_n) of $\inf_{\mathcal{N}^+(\Omega)} I_+(u)$ is bounded in $X_{0,rad}(\Omega)$. Hence, up to a subsequence, still denoted by u_n, there exists $u \in X_{0,rad}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X_{0,rad}(\Omega)$ and $u_n \to u$ a.e. in Ω.

We claim that $u \not\equiv 0$. Indeed, if $u \equiv 0$ then, from [11, Lemma 3.1] that

$$\int_\Omega K(x) f_+(u_n)u_n dx \to \int_\Omega K(x) f_+(u)u dx, \tag{9}$$

$$\int_\Omega K(x) F_+(u_n) dx \to \int_\Omega K(x) F_+(u) dx, \tag{10}$$

which implies

$$\|u_n\|^2 = \int_\Omega K(x) f_+(u_n)u_n dx \to 0,$$

contradicting Lemma 2.6. By Lemma 2.5, there exists $t > 0$ such that $v := tu \in \mathcal{N}^+$. From (10), we obtain

$$\inf_{\mathcal{N}^+(\Omega)} I_+(u) \leq I_+(v) \leq \liminf_{n\to\infty} I_+(tu_n).$$

Since $u_n \in \mathcal{N}^+(\Omega)$, from Lemma 2.5 again, we conclude that $\max_{t\geq 0} I_+(tu_n) = I_+(u_n)$. Therefore, $\liminf_{n\to\infty} I_+(tu_n) \leq \liminf_{n\to\infty} \max_{t\geq 0} I_+(tu_n) = \liminf_{n\to\infty} I_+(u_n) = \inf_{\mathcal{N}^+(\Omega)} I_+(u)$. The equality $I'(v) = 0$ is a consequence of Proposition 2.8. □

In what follows, we consider the critical case.

2.3. The Critical Case

Proposition 2.9. *(Critical).Suppose that* **(F₁)** *with* $\alpha_0 > 0$, **(F₂)** – **(F₅)** *hold, then* $\inf_{\mathcal{N}^\pm(\Omega)} I_\pm(u)$ *can be achieved by some* $u \in \mathcal{N}^\pm(\Omega)$.

To prove Proposition 2.9, we first consider the following auxiliary equation

$$-\operatorname{div}\left(K(x)\nabla u\right) = K(x)|u|^{p-2}u, \quad x \in \Omega. \tag{11}$$

where $p > 2$. The functional associated with auxiliary problem (11) is given by

$$I_p(u) = \frac{1}{2}\int_\Omega K(x)|\nabla u|^2 dx - \frac{1}{p}\int_\Omega K(x)|u|^p dx.$$

Define the Nehari's manifold

$$\mathcal{N}_p(\Omega) = \left\{u \in X_{0,rad}(\Omega) : u \neq 0, I'_p(u)u = 0\right\}.$$

It is not difficult to verify that there exists $u_p \in X_{0,rad}(\Omega)$ such that $I_p(u_p) = c_p$, $I'_p(u) = 0$ and

$$c_p = (\frac{p-2}{2p}) \int_\Omega K(x)|u_p|^p,$$

where $c_p = \inf_{\mathcal{N}_p(\Omega)} I_p$. We have the following results.

Lemma 2.10. *There holds* $\inf\limits_{\mathcal{N}^{\pm}(\Omega)} I_{\pm}(u) \leq \frac{c_p}{\tau^{2/(p-2)}}$.

Lemma 2.11. *If* $(u_n) \subset \mathcal{N}^{\pm}(\Omega)$ *is a minimizing sequence for* $\inf\limits_{\mathcal{N}^{\pm}(\Omega)} I_{\pm}(u)$, *then there holds* $\limsup_{n\to\infty} \|u_n\|^2 \leq \frac{2\pi}{\alpha_0}$.

Using Lemma 2.10 and Lemma 2.11, we have the following compactness properties of minimizing sequences.

Lemma 2.12. *If* $(u_n) \subset \mathcal{N}^{\pm}(\Omega)$ *is a minimizing sequence for* $\inf\limits_{\mathcal{N}^{\pm}(\Omega)} I_{\pm}(u)$, *then*

$$\int_{\Omega} K(x) f_{+}(u_n) u_n dx \to \int_{\Omega} K(x) f_{+}(u) u dx, \tag{12}$$

$$\int_{\Omega} K(x) F(u_n) dx \to \int_{\Omega} K(x) F(u) dx. \tag{13}$$

The proof of lemma 2.10, lemma 2.11 and lemma 2.12 are similar to those in [11]. Here we omit the details.

Proof of Proposition 2.9.

Combine lemma 2.10, lemma 2.11 and lemma 2.12, and recall the proof of Proposition 2.8, we can obtain the results immediately.

3. Proof of Main Results

In this section, we will give the proof Theorems 1.1 and 1.2. We fix some integer $k \geq 1$ and want to find a pair of radial solutions u_k^+ and u_k^- of problem (2) having k nodes with $u_k^-(0) < 0 < u_k^+(0)$. Here a nodal $\rho > 0$ is such that $u(\rho) = 0$. Recall that radial solutions of problem (2) correspond to critical points of the energy functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} K(x) |\nabla u|^2 dx - \int_{\mathbb{R}^2} K(x) F(u) dx.$$

We will work on the Nehari manifold

$$\mathcal{N} = \left\{ u \in X_{rad}(\mathbb{R}^2) : u \neq 0, \int_{\mathbb{R}^2} K(x) |\nabla u|^2 = \int_{\mathbb{R}^2} K(x) f(u) u \right\}.$$

If we replace $\mathbb{R}^2$ with $\Omega(\rho, \sigma)$ and $X_{rad}(\mathbb{R}^2)$ with $X_{0,rad}(\Omega(\rho, \sigma))$, where $0 \leq \rho < \sigma \leq \infty$. The Nehari manifold $\mathcal{N}$ is replaced by $\mathcal{N}(\Omega(\rho, \sigma))$, for simplicity, we denote it briefly by $\mathcal{N}_{\rho,\sigma}$. By extending $u(x) = 0$ for $x \notin (\rho, \sigma)$ if $u \in X_{0,rad}(\Omega(\rho, \sigma))$, we understand that $X_{0,rad}(\Omega(\rho, \sigma)) \subset X_{rad}(\mathbb{R}^2)$ and $\mathcal{N}_{\rho,\sigma} \subset \mathcal{N}$. For positive integer k fixed, we define a Nehari type set

$$\begin{aligned} \mathcal{N}_k^{\pm} := \Big\{ & u \in X_{rad}(\mathbb{R}^2) \,\Big|\, u \neq 0, \text{there exist } 0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := \infty \\ & \text{such that } \pm(-1)^j u|_{\Omega(r_j, r_{j+1})} \geq 0 \text{ and } u|_{\Omega(r_j, r_{j+1})} \in \mathcal{N}_{r_j, r_{j+1}}, j = 0, 1, \cdots, k. \Big\} \end{aligned}$$

and

$$c_k^{\pm} := \inf_{\mathcal{N}_k^{\pm}} I(u).$$

Lemma 3.1. *For each positive integer k, there are* $u_k^{\pm} \in \mathcal{N}_k^{\pm}$ *such that* $I(u_k^{\pm}) = c_k^{\pm}$.

Proof. We only prove the case for u_k^+ and leave the other case to reader. It follows from Proposition 2.8 (subcritical case) and Proposition 2.9 (critical case) that $c^+(\rho,\sigma) := \inf_{\mathcal{N}^+_{\rho,\sigma}} I^+(u)$ is achieved by some $u \in \mathcal{N}^+_{\rho,\sigma}$. Since I^+ is a even functional, $|u|$ is also the a minimizer and from the strong maximum principle that $|u| > 0$, then we may assume that the minimizer u is a positive solution of problem (4).

Therefore, the minimizer $u > 0$ is a solution of problem

$$\begin{cases} -\operatorname{div}(K(x)\nabla u) = K(x)f(u), & \text{in } \Omega(\rho,\sigma), \\ u = 0, & \text{on } \partial\Omega(\rho,\sigma). \end{cases} \tag{14}$$

Similarly, the infimum $c^-(\rho,\sigma) := \inf_{\mathcal{N}^-_{\rho,\sigma}} I^-(u)$ is also achieved by some $u \in \mathcal{N}^-_{\rho,\sigma}$, which are negative solutions of (14).

Let (u_n) be minimizing sequence of c_k^+. By the some arguments as in the proof of Proposition 2.8, we can prove that (u_n) is bounded. Since $(u_n) \in \mathcal{N}_k^+$, then there exist $0 =: r_0^n < r_1^n < \cdots < r_k^n < r_{k+1}^n := \infty$ such that $\pm(-1)^j u_n|_{\Omega(r_j^n, r_{j+1}^n)} \geq 0$ and $u_n|_{\Omega(r_j^n, r_{j+1}^n)} \in \mathcal{N}_{r_j^n, r_{j+1}^n}$, $j = 0, 1, \cdots, k$. Note that

$$\|u_n|_{\Omega(r_j^n, r_{j+1}^n)}\|^2 = \int_{\Omega(r_j^n, r_{j+1}^n)} K(x) f_+(u_n) u_n dx.$$

Using (3) and embedding inequality, we have

$$\begin{aligned} &\int_{\Omega(r_j^n, r_{j+1}^n)} K(x) f_+(u_n) u_n dx \\ \leq &\epsilon \int_{\Omega(r_j^n, r_{j+1}^n)} K(x)|u_n|^2 dx + C_\epsilon \int_{\Omega(r_j^n, r_{j+1}^n)} K(x)|u_n|^q(\exp(\alpha u_n^2) - 1) dx \\ \leq &C\epsilon \|u_n|_{\Omega(r_j^n, r_{j+1}^n)}\|^2 + C_\epsilon \int_{\Omega(r_j^n, r_{j+1}^n)} K(x)|u_n|^q(\exp(\alpha u_n^2) - 1) dx. \end{aligned} \tag{15}$$

Let $p_i > 1, i = 1, 2, 3$, be such that $1/p_1 + 1/p_2 + 1/p_3 = 1$ and $(q-2)p_2 \geq 3$. By Hölder's inequality we have

$$\begin{aligned} &\int_{\Omega(r_j^n, r_{j+1}^n)} K(x)|u_n|^q(\exp(\alpha u_n^2) - 1) dx \\ \leq &\left(\int_{\Omega(r_j^n, r_{j+1}^n)} K(x)^{p_1}|u_n|^{2p_1}\right)^{1/p_1} \left(\int_{\Omega(r_j^n, r_{j+1}^n)} |u_n|^{(q-2)p_2}\right)^{1/p_2} \left(\int_{\Omega(r_j^n, r_{j+1}^n)} (\exp(\alpha u_n^2) - 1)^{p_3}\right)^{1/p_3} \\ \leq &C(p_1)\|u_n|_{\Omega(r_j^n, r_{j+1}^n)}\|^2 \left(\int_{\Omega(r_j^n, r_{j+1}^n)} |u_n|^{(q-2)p_2}\right)^{1/p_2} \left(\int_{\Omega(r_j^n, r_{j+1}^n)} (\exp(p_3 \alpha u_n^2) - 1)\right)^{1/p_3}, \end{aligned} \tag{16}$$

where the last inequality we used the result in Lemma 2.3 and the following fact

$$(e^s - 1)^r \leq e^{rs} - 1 \text{ for all } r \geq 1,\ s \geq 0.$$

In the subcritical case, we can prove that (u_n) is bounded by using exactly the same arguments as in the proof of Proposition 2.8, that is, there exists $M_1 > 0$ such that $\|u_n\| \leq M_1$. Choosing $\alpha < \frac{4\pi}{p_3 M_1^2}$, we conclude by the classical Trudinger-Moser inequality (see [4]) that

$$\int_{\Omega(r_j^n, r_{j+1}^n)} (\exp(\alpha p_3 u_n^2) - 1) \leq \int_{\Omega(r_j^n, r_{j+1}^n)} \left(\exp\left(\alpha p_3 M_1^2 \left(\frac{|u_n|}{\|u_n\|}\right)^2\right) - 1\right) \leq C(M_1, \alpha), \tag{17}$$

for some $C(M_1, \alpha) > 0$.

In the critical case, from lemma 2.11, we have

$$\limsup_{n\to\infty}\|u_n\|^2 \le \frac{2\pi}{\alpha_0}.$$

Let p_3 close to 1, choosing $\alpha > \alpha_0$ and close to α_0, then $\alpha p_3\|u_n\|^2 < 4\pi$. Thus, we conclude the same inequality (17).

Therefore, it follows from (15),(16) and (17) that

$$\frac{(1-C\epsilon)}{C_\epsilon C(p_1)C(M_1,\alpha)} \le \left(\int_{\Omega(r_j^n,r_{j+1}^n)}|u_n|^{(q-2)p_2}\right)^{1/p_2}. \tag{18}$$

Considering Hölder inequality again and by embedding inequality, there exists $\overline{C} > 0$ such that

$$\begin{aligned}\frac{(1-C\epsilon)}{C_\epsilon C(p_1)C(M_1,\alpha)} &\le \left(\int_{\Omega(r_j^n,r_{j+1}^n)}|u_n|^{(q-2)p_2-2}\right)^{\frac{1}{((q-2)p_2-2)p_2}}\left((r_{j+1}^n)^2-(r_j^n)^2\right)^{\frac{1}{p_2}\left(1-\frac{1}{(q-2)p_2-2}\right)}\\ &\le \overline{C}\|u_n|_{\Omega(r_j^n,r_{j+1}^n)}\|^{\frac{1}{p_2}}\left((r_{j+1}^n)^2-(r_j^n)^2\right)^{\frac{1}{p_2}\left(1-\frac{1}{(q-2)p_2-2}\right)}.\end{aligned}$$

Then

$$\|u_n|_{\Omega(r_j^n,r_{j+1}^n)}\| \ge \widetilde{C}\left((r_{j+1}^n)^2-(r_j^n)^2\right)^{-(1-\frac{1}{(q-2)p_2-2})},$$

where $\widetilde{C} = \left(\frac{1-C\epsilon}{\overline{C}C_\epsilon C(p_1)C(M_1,\alpha)}\right)^{p_2}$. This implies that, for $\epsilon > 0$ small, $r_{j+1}^n - r_j^n$ is bounded away from 0 for each $j = 1,2,\cdots,k$.

According to Lemma 2.2, we have

$$|u_n(x)| \le C|x|^{-\frac{1}{2}}e^{-\frac{|x|^2}{8}}\|u_n\|, \text{ for all } u_n \in X_{rad}(\mathbb{R}^2).$$

Then, we see that

$$\|u_n(x)\|_{L^\infty} \le C|r_k^n|^{-\frac{1}{2}}e^{-\frac{|r_k^n|^2}{8}}\|u_n\|, \text{ for all } u_n \in X_{0,rad}(\Omega(r_k^n,\infty)). \tag{19}$$

Recalling (18), we obtain that

$$\frac{(1-C\epsilon)}{C_\epsilon C(p_1)C(M_1,\alpha)} \le \left(\int_{\Omega(r_k^n,\infty)}|u_n|^{(q-2)p_2}\right)^{1/p_2} = \left(\int_{\Omega(r_k^n,\infty)}|u_n|^{(q-2)p_2-1}|u_n|\right)^{1/p_2}.$$

Combining this inequality with (19), then

$$\frac{(1-C\epsilon)}{C_\epsilon C(p_1)C(M_1,\alpha)} \le C_1\|u_n|_{\Omega(r_k^n,\infty)}\|^{q-2}\left(C(r_k^n)^{-\frac{1}{2}}e^{-\frac{(r_k^n)^2}{8}}\right)^{\frac{1}{p_2}((q-2)p_2-1)},$$

which implies that

$$\|u_n|_{\Omega(r_k^n,\infty)}\| \ge \hat{C}\left((r_k^n)^{\frac{1}{2}}e^{\frac{(r_k^n)^2}{8}}\right)^{\frac{1}{p_2}((q-2)p_2-1)},$$

for some $\hat{C} > 0$. Therefore, we infer that r_j^n bounded away from ∞ for each $j = 1,2,\cdots,k$.

Then, there exist $0 = r_0 < r_1 < \cdots < r_k < r_{k+1} = \infty$ such that $r_j^n \to r_j$, as $n \to \infty$ for $j = 1,2,\cdots,k$.. Up to a subsequence, we may assume that $u_n \to u$ weakly in $X_{rad}(\mathbb{R}^2)$, strongly in $L_K^s(\mathbb{R}^2)$ for any $s \in [2,\infty)$, and *a.e.*

on $\mathbb{R}^2$. It follows that $u_n|_{\Omega(r_j^n,r_{j+1}^n)} \to u|_{\Omega(r_j,r_{j+1})}$ weakly in $X_{rad}(\mathbb{R}^2)$, strongly in $L_K^s(\mathbb{R}^2)$ for any $s \in [2,\infty)$, and *a.e.* on $\Omega(r_j, r_{j+1})$. Then $(-1)^j u|_{\Omega(r_j,r_{j+1})} \geq 0$. By (18), we have

$$\int_{\Omega(r_k^n,r_{k+1}^n)} |u_n|^{qp_1} \geq C > 0,$$

and so

$$\int_{\Omega(r_k,r_{k+1})} |u|^{qp_1} \geq C > 0.$$

which implies that $u|_{\Omega(r_j,r_{j+1})} \neq 0$. Thus, from Lemma 2.5, there exists $t_j > 0$ such that $t_j u|_{\Omega(r_j,r_{j+1})} \in \mathcal{N}_{r_j,r_{j+1}}$ for $j = 1, 2, \cdots, k$. Set

$$u_k^+ := \sum_{j=0}^{k} t_j u|_{\Omega(r_j,r_{j+1})}. \tag{20}$$

It is clear that $u_k^+ \in \mathcal{N}_k^+$. We claim that $I(u_k^+) = c_k^+$. Indeed, from $u_n \to u$ weakly in $X_{rad}(\mathbb{R}^2)$ and strongly in $L_K^s(\mathbb{R}^2)$ for any $s \in [2,\infty)$, we have

$$c_k^+ \leq I(u_k^+) = \sum_{j=0}^{k} I(t_j u|_{\Omega(r_j,r_{j+1})}) \leq \sum_{j=0}^{k} \liminf_{n\to\infty} I(t_j u_n|_{\Omega(r_j^n,r_{j+1}^n)}). \tag{21}$$

Moreover, it follows from $u_n|_{\Omega(r_j^n,r_{j+1}^n)} \in \mathcal{N}_{r_j^n,r_{j+1}^n}$ and Lemma 2.5 that

$$\sum_{j=0}^{k} \liminf_{n\to\infty} I(t_j u_n|_{\Omega(r_j^n,r_{j+1}^n)}) \leq \sum_{j=0}^{k} \liminf_{n\to\infty} I(u_n|_{\Omega(r_j^n,r_{j+1}^n)}) = \liminf_{n\to\infty} I(u_n) = c_k^+.$$

Thus, we conclude that $I(u_k) = c_k^+$, and $t_j = 1$ for all j.

Then, by the equality in (21), we obtain that $u|_{\Omega(r_j,r_{j+1})}$ is a minimizer of $\inf_{\mathcal{N}_{r_j,r_{j+1}}} I^+(u)$ with $(-1)^j u|_{\Omega(r_j,r_{j+1})} \geq 0$. By Strauss inequality, u_k is continuous except perhaps at 0. We observe that $u_k(r_j) = 0$ for $j = 1, 2, \cdots, k$. The elliptic regularity theory implies that $u_k \in C^2$ on (r_j, r_{j+1}) for any j. Then, by the strong maximum principle, we obtain that $u_k^+(0) > 0$, $(-1)^j u_k^+(x) > 0$ for $r_j < |x| < r_{j+1}$ and $j = 0, 1, 2, \cdots, k$. So u_k^+ has exactly k nodes. □

In the following, we show that the minimizer of $c_k^\pm$ are sign-changing solutions of (2), that is, if $c_k^\pm = I(u_k^\pm)$ for some $u_k^\pm \in \mathcal{N}_k^\pm$, then $I'(u_k^\pm) = 0$.

Lemma 3.2. *For each positive integer k, the minimizers of $c_k^\pm$ are critical points of I.*

Proof. We still give the proof only for the case c_k^+. We use an indirect argument. Suppose that u_k^+ is defined in (20) with $u_k^+ \in \mathcal{N}_k^+$, $c_k^+ = I(u_k^+)$ and $I'(u_k^+) \neq 0$. Then there exist $\varphi \in X_{rad}(\mathbb{R}^2)$ such that

$$I'(u_k^+)\varphi = \int_\Omega K(x)\nabla u_k^+ \nabla\varphi - \int_\Omega K(x) f(u_k^+)\varphi \leq -1.$$

Choose $\varepsilon > 0$ small such that

$$I'(\sum_{j=0}^{k} s_j u|_{\Omega(r_j,r_{j+1})} + \sigma\varphi)\varphi \leq -\frac{1}{2}, \text{for all} \sum_{j=0}^{k} |s_j - 1| + |\sigma| \leq \varepsilon,$$

and $\sum_{j=0}^{k} s_j u|_{\Omega(r_j,r_{j+1})} + \sigma\varphi$ has exactly k nodes

$$0 < r_1(\mathbf{s},\sigma) < r_2(\mathbf{s},\sigma) < \cdots < r_k(\mathbf{s},\sigma) < \infty,$$

where $r_j(\mathbf{s},\sigma)$ is continuous with respect to $\mathbf{s}$ and σ, $\mathbf{s} := (s_0, s_1, \cdots, s_k) \in \mathbb{R}^{k+1}$. Let η be a cut-off function such that

$$\eta(\mathbf{s}) = \begin{cases} 1, & \text{if } |s_j - 1| \le \frac{1}{2}\varepsilon \text{ for all } j, \\ 0, & \text{if } |s_j - 1| \ge \varepsilon \text{ for at least one } j. \end{cases}$$

We proceed to estimate $\sup_{s_j\ge 0} I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\mathbf{s})\varphi)$. If $\sum_{j=0}^k |s_j - 1| + |\sigma| \le \varepsilon$, and so $|s_j - 1| \le \varepsilon$ for all j, then

$$\begin{aligned} I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\mathbf{s})\varphi) &= I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})}) + \int_0^1 I'(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \sigma\varepsilon\eta(\mathbf{s})\varphi)\varepsilon\eta(\mathbf{s})\varphi d\sigma \\ &\le I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})}) - \frac{1}{2}\varepsilon\eta(\mathbf{s}). \end{aligned} \tag{22}$$

If $|s_j - 1| \ge \varepsilon$ for at least one j, $\eta(t) = 0$, the above inequality is trivial. Now since $u_k^+ \in \mathcal{N}_k^+$, we have

$$I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\mathbf{s})\varphi) \le I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})}) < I(u_k^+), \text{for all } s_j \ne 1.$$

For $s_j = 1$, $j = 0, 1 \cdots, k$, from (22), we obtain that

$$I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\mathbf{1})\varphi) \le I(\sum_{j=0}^k u|_{\Omega(r_j,r_{j+1})}) - \frac{1}{2}\varepsilon\eta(\mathbf{1}) < I(u_k^+).$$

Thus, we conclude that $\sup_{s_j\ge 0} I(\sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\mathbf{s})\varphi) < I(u_k^+)$. To complete the proof, it is sufficient to find $\hat{\mathbf{s}} = (\hat{s_0}, \hat{s_1}, \cdots, \hat{s_k})$ such that $\sum_{j=0}^k \hat{s_j} u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\hat{\mathbf{s}})\varphi \in \mathcal{N}_k^+$, which contradicts the definition of c_k^+. To this end, we set $Q(\mathbf{s}) := \sum_{j=0}^k s_j u|_{\Omega(r_j,r_{j+1})} + \varepsilon\eta(\mathbf{s})\varphi$. Obviously, $Q(\mathbf{s})$ has exactly k nodes $0 < r_1(\mathbf{s}) < r_2(\mathbf{s}) < \cdots < r_k(\mathbf{s}) < \infty$ and $r_j(\mathbf{s})$ is continuous with respect to $\mathbf{s}$. Now, we consider the continuous function

$$\Upsilon_j(\mathbf{s}) := I'\left(Q(\mathbf{s})\Big|_{\Omega(r_j(\mathbf{s}),r_{j+1}(\mathbf{s}))}\right)\left(Q(\mathbf{s})\Big|_{\Omega(r_j(\mathbf{s}),r_{j+1}(\mathbf{s}))}\right),$$

where $Q(\mathbf{s})\Big|_{\Omega(r_j(\mathbf{s}),r_{j+1}(\mathbf{s}))} = \left(\sum_{i=0}^k s_i u|_{\Omega(r_i,r_{i+1})} + \varepsilon\eta(\mathbf{s})\varphi\right)\Big|_{\Omega(r_j(\mathbf{s}),r_{j+1}(\mathbf{s}))}$. For a fixed j, if $|s_j - 1| = \varepsilon$, then $\eta(\mathbf{s}) = 0$ and $r_j(\mathbf{s}) = r_j$ for all $j = 1, 2, \cdots, k$, and so $\Upsilon_j(\mathbf{s}) = I'\left(s_j u|_{\Omega(r_j,r_{j+1})}\right)\left(s_j u|_{\Omega(r_j,r_{j+1})}\right)$. A simple calculation shows that $\Upsilon_j(\mathbf{s}) > 0$ if $s_j = 1 - \varepsilon$ and $\Upsilon_j(\mathbf{s}) < 0$ if $s_j = 1 + \varepsilon$. As a consequence, using Miranda's theorem in [18], we conclude that there exists $\hat{\mathbf{s}} = (\hat{s_0}, \hat{s_1}, \cdots, \hat{s_k})$ with $\hat{s_j} \in (1 - \varepsilon, 1 + \varepsilon)$ such that $Q(\hat{\mathbf{s}}) \in \mathcal{N}_k^+$. The prove is completed. □

3.1. Proof of Theorem 1.1 and Theorem 1.2

The existence of $u_k^\pm$ with exactly k nodes follows from Lemma 3.1 and Lemma 3.2. By construction, $u_k^\pm$ is radial and $u_k^-(0) < 0 < u_k^+(0)$. Moreover, since $u_k^\pm|_{\Omega(r_j,r_{j+1})} \in \mathcal{N}_{r_j,r_{j+1}} \subseteq \mathcal{N}$, then $I(u_k^\pm) > (k+1)I(u_0^\pm)$. Finally, the conclusion $I(u_{k+1}^\pm) > I(u_k^\pm)$ follows from $I(u_k^\pm) = \sum_{j=0}^k I(u_k^\pm|_{\Omega(r_j,r_{j+1})})$ and $I(u_k^\pm|_{\Omega(r_j,r_{j+1})}) > 0$ for $j = 0, 1, \cdots, k$.

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