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A nonexistence result of positive stable solutions to a weighted elliptic equation

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Abstract. We investigate the following weighted elliptic equation

$$-\Delta u + b(x)\cdot\nabla u = (1+|x|^2)^{\frac{\alpha}{2}}u^p \text{ in } \mathbb{R}^N,$$

where $\alpha \geq 0$, $p > 1$, $N \geq 3$ and the advection term $b(x)$ is a smooth vector field satisfying certain decay condition. We establish a Liouville-type theorem for possible stable solutions of the equation above.

1. Introduction

In this paper, we consider the following weighted elliptic equation

$$-\Delta u + b(x)\cdot\nabla u = (1+|x|^2)^{\frac{\alpha}{2}}u^p \tag{1}$$

in the whole space $\mathbb{R}^N$, where $\alpha \geq 0$, $p > 1$, $N \geq 3$ and $b(x)$ is a smooth vector field satisfying

$$\operatorname{div} b = 0 \text{ and } \kappa = \sup_{\mathbb{R}^N}|x||b(x)| < \infty. \tag{2}$$

We will establish Liouville type theorems for the class of positive stable solutions of (1).

Let us begin by recalling that in the case $b \equiv 0$, the equation (1) turns into

$$-\Delta u = (1+|x|^2)^{\frac{\alpha}{2}}u^p \text{ in } \mathbb{R}^N. \tag{3}$$

This problem is known as the weighted Lane-Emden equation which has been studied recently in a number of papers, see [10–12] and the references therein. The nonexistence of positive stable classical solutions of the problem (3) was obtained in [12].

In the case $b \not\equiv 0$ and $\alpha = 0$, the equation (1) becomes

$$-\Delta u + b(x)\cdot\nabla u = u^p \text{ in } \mathbb{R}^N, \tag{4}$$

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The equation (4) was examined in [3], in which $b(x)$ is a smooth divergence free vector field satisfying $|b(x)| \leq \frac{C}{|x|+1}$ with $0 < C$ sufficiently small. The classification of stable classical solutions was completely established.

Recently, the elliptic problems involving advection terms, i.e. $b \neq 0$, have received considerable attention, see [3, 4, 6, 7, 13–16]. In the general case where $b \neq 0$, the elliptic problems with advections have no variational structure and this requires another approach to obtain the classification of stable solutions. Recall that, see e.g. [3, 15], a classical solution u of

$$-\Delta u + b \cdot \nabla u = f(u)$$

is called stable if there exists a smooth positive function E such that

$$-\Delta E + b \cdot \nabla E \geq f'(u)E. \tag{5}$$

In the case $b \equiv 0$, based on the work of Liang-Gen Hu in [12], we provide the definition of a stable solution of the Lane-Emden weighted equation as follows:
A solution $u \in C^2(\mathbb{R}^N)$ of (3) is called *stable*, if the eigenvalue equation

$$-\Delta\phi = p(1 + |x|^2)^{\frac{\alpha}{2}} u^{\frac{p-1}{2}} \phi + \eta\phi$$

has a positive eigenvalue $\eta > 0$, with corresponding positive smooth function ϕ, which implies that

$$-\Delta\phi > p(1 + |x|^2)^{\frac{\alpha}{2}} u^{\frac{p-1}{2}} \phi. \tag{6}$$

To the best of our knowledge, the above definition is based on the original idea of Marcelo Montenegro. Note that the stability conditions (5) and (6) are not exactly the same with the equations not having a variational structure. When $b \equiv 0$, if u is a stable solution in the sense of (6), then u is stable in the sense of (5).

Recently, by generalizing Hardy inequality from [1], Cowan [3] has proved the nonexistence of stable solutions of (4) under smallness condition on b:
Theorem A.([3]) *Suppose* $3 \leq N \leq 10$ *or* $N \geq 11$ *and* $1 < p < p_c$, *where*

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & N \geq 11 \\ \infty & 3 \leq N \leq 10. \end{cases}$$

Suppose $b(x)$ *is a smooth divergence free vector field satisfying* $|b(x)| \leq \frac{C}{|x|+1}$ *with* $0 < C$ *sufficiently small. Then there is no positive stable solution of* (4).

Motivated by the suggestions in [2], Hu [12] has established Liouville type theorem for the stable solution of (3). That is,
Theorem B.([12]) *Let* $p > \frac{4}{3}$, $\alpha > 0$ *and the space dimension satisfies*

$$N < 2 + \frac{2(2+\alpha)}{p-1}(p + \sqrt{p^2 - p}).$$

Then there does not exist a classical positive semi-stable solution of (3).

A solution $u \in C^2(\mathbb{R}^N)$ of (3) is called *semi-stable*, if the eigenvalue equation

$$-\Delta\phi = p(1 + |x|^2)^{\frac{\alpha}{2}} u^{\frac{p-1}{2}} \phi + \eta\phi$$

has a nonnegative eigenvalue $\eta \geq 0$, with corresponding positive smooth function ϕ, which indicates that

$$-\Delta\phi \geq p(1 + |x|^2)^{\frac{\alpha}{2}} u^{\frac{p-1}{2}} \phi.$$

As mentioned above, the condition for the semi-stable solution of Theorem B is not exactly the same as for the stable solution in the manuscript. This is also a new point in our study: problem (1) without variational structure and no smallness assumption in b has not been studied in the literature. The main result in this paper is concerned with the classification of stable solutions of (1) based on the condition (5) and certain sufficient conditions involving the parameters N, κ, p and α.

To the best of our knowledge, the problem (1) without variational structure, and without the smallness assumption on b have not been investigated in the literature. It's worth to mentioning that the problem $-\Delta u + b(x)\cdot\nabla u = e^u$ or $-\Delta u + b(x)\cdot\nabla u = -u^{-p}$ has been recently studied in [8]. Our purpose in this paper is to generalize some results in [3, 12] to the problems (1) in the general case. Moreover, we also relax the smallness assumption on the advection term b as in [8].

The main result in this paper is concerned with the classification of stable solutions of (1).

Theorem 1.1. *Suppose that $\alpha \geq 0, p > 1 + \frac{\kappa^2}{(N-2)^2}$, the space dimension N satisfies*

$$2 + \frac{\kappa}{\sqrt{p-1}} < N < 2 + \frac{2(\alpha+2)}{p-1}\left(p + \sqrt{p^2 - p}\right)$$

and (2) *holds with*

$$\kappa < \sqrt{(N-2)\left(\frac{4p(\alpha+2)}{p-1} + 2 - N\right) - \frac{4p(\alpha+2)^2}{(p-1)^2}}.$$

Then (1) *has no positive stable solution.*

Notice that when $b \equiv 0$, an immediate consequence of Theorem 1.1 is Theorem B.

Remark that when $\alpha = 0$ and κ small enough, our result recovers Theorem A.

The rest of this paper is devoted to the proof of our main result.

2. Proof of Theorem 1.1

For simplicity, we denote by $\int$ the integral $\int_{\mathbb{R}^N} dx$. Let us begin by establishing a key estimate.

Proposition 2.1. *Let $\alpha \geq 0, p > 1$ and assume that* (2) *holds. If u is a stable solution of* (1)*, then for $t > \frac{1}{2}$, there is a positive constant C depending on t such that*

$$\left(\frac{p}{\frac{\kappa^2}{(N-2)^2} + 1} - \frac{t^2}{2t-1}\right)\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \leq C\int u^{2t}\left(\left|\Delta(\psi^2)\right| + \left|\nabla\psi\right|^2 + |b|\left|\nabla\psi\right||\psi|\right), \tag{7}$$

for all $\psi \in C_c^\infty(\mathbb{R}^N)$.

Proof. Let u be a positive stable solution of (1). We first use the stability condition (5) with $f(u) = (1+|x|^2)^{\frac{\alpha}{2}}u^p$. By multiplying (5) by $u^{2t}\psi^2E^{-1}$ and then integrating over $\mathbb{R}^N$, one gets

$$p\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \leq \int \frac{b\cdot\nabla E}{E}u^{2t}\psi^2 - \int\frac{\Delta E}{E}u^{2t}\psi^2. \tag{8}$$

Integrating by parts the second term on the right hand side of (8), we have

$$\begin{aligned} -\int\frac{\Delta E}{E}u^{2t}\psi^2 &= -\int\frac{u^{2t}\psi^2}{E}\nabla\cdot\nabla E = \int\nabla E\cdot\nabla\left(\frac{u^{2t}\psi^2}{E}\right) \\ &= \int\nabla E\cdot\left(\frac{E\nabla(u^{2t}\psi^2) - u^{2t}\psi^2\nabla E}{E^2}\right) = \int\nabla E\cdot\left[-u^{2t}\psi^2\frac{\nabla E}{E^2} + \frac{\nabla(u^{2t}\psi^2)}{E}\right]. \end{aligned} \tag{9}$$

Then the inequality (8) becomes

$$\begin{aligned} p\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} &\le \int \frac{b\cdot\nabla E}{E}u^{2t}\psi^2 + \int \frac{\nabla E\cdot\nabla(u^{2t}\psi^2)}{E} - \int u^{2t}\psi^2\frac{|\nabla E|^2}{E^2} \\ &\le \int \frac{b\cdot\nabla E}{E}u^{2t}\psi^2 + 2\int \frac{\nabla E\cdot\nabla(u^{t}\psi)}{E}u^t\psi - \int u^{2t}\psi^2\frac{|\nabla E|^2}{E^2} \\ &\le \int u^t\psi\frac{\nabla E\cdot(bu^t\psi+2\nabla(u^t\psi))}{E} - \int u^{2t}\psi^2\frac{|\nabla E|^2}{E^2}. \end{aligned} \tag{10}$$

Using integration by parts, and the Young inequality, we also arrive at

$$p\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \le \frac{1}{4}\int \left|bu^t\psi+2\nabla(u^t\psi)\right|^2. \tag{11}$$

By combining $\operatorname{div} b = 0$ and integrating by parts we obtain

$$\begin{aligned} \int \left|bu^t\psi+2\nabla(u^t\psi)\right|^2 &= \int \left(|b|^2u^{2t}\psi^2 + 4b\cdot\nabla(u^t\psi)u^t\psi + 4\left|\nabla(u^t\psi)\right|^2\right) \\ &= \int \left(|b|^2u^{2t}\psi^2 - 2\operatorname{div} b\cdot(u^t\psi)^2 + 4\left|\nabla(u^t\psi)\right|^2\right) \\ &= \int \left(|b|^2u^{2t}\psi^2 + 4\left|\nabla(u^t\psi)\right|^2\right). \end{aligned} \tag{12}$$

Therefore, from (8) and (12), we easily get

$$p\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \le \frac{1}{4}\int \left(|b|^2u^{2t}\psi^2 + 4\left|\nabla(u^t\psi)\right|^2\right). \tag{13}$$

From (2) and Hardy's inequality [9], we get

$$\int |b|^2u^{2t}\psi^2 \le \int \frac{\kappa^2}{|x|^2}u^{2t}\psi^2 \le \frac{4\kappa^2}{(N-2)^2}\int \left|\nabla(u^t\psi)\right|^2. \tag{14}$$

Combining (13), (14), we have

$$p\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \le \frac{1}{4}\left(\frac{4\kappa^2}{(N-2)^2}\int \left|\nabla(u^t\psi)\right| + 4\int \left|\nabla(u^t\psi)\right|\right)$$

or it is rewritten as

$$p\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \le \left(\frac{\kappa^2}{(N-2)^2}+1\right)\int \left|\nabla(u^t\psi)\right|^2. \tag{15}$$

It is easy to see

$$\int \left|\nabla(u^t\psi)\right|^2 = \int \left(tu^{t-1}\nabla u\psi + u^t\nabla\psi\right)^2 = \int \left(t^2u^{2t-2}\left|\nabla u\right|^2\psi^2 + 2tu^{2t-1}\psi\nabla u\cdot\nabla\psi + u^{2t}\left|\nabla\psi\right|^2\right). \tag{16}$$

To estimate the first term on the right hand side of (16), we use the weak form of (1) with the test function $u^{2t-1}\psi^2$ to get

$$\int -\Delta u u^{2t-1}\psi^2 = \int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} - \int b\cdot\nabla u u^{2t-1}\psi^2. \tag{17}$$

Applying integration by parts to (17), we obtain

$$\begin{aligned}\int -\Delta u u^{2t-1}\psi^2 = \int \nabla u \cdot \nabla(u^{2t-1}\psi^2) = \int \nabla u \cdot [(2t-1)u^{2t-2}\psi^2\nabla u + 2u^{2t-1}\psi\nabla\psi] \\ =(2t-1)\int |\nabla u|^2 u^{2t-2}\psi^2 + 2\int \nabla u \cdot \nabla\psi u^{2t-1}\psi.\end{aligned} \tag{18}$$

Then the equality (17) becomes

$$(2t-1)\int |\nabla u|^2 u^{2t-2}\psi^2 = \int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} - \int b\cdot\nabla u u^{2t-1}\psi^2 - 2\int \nabla u\cdot\nabla\psi u^{2t-1}\psi. \tag{19}$$

Thus, we deduce from (16) and (19) that

$$\begin{aligned}\int \left|\nabla(u^t\psi)\right|^2 = \frac{t^2}{2t-1}\int \left(u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} - b\cdot\nabla u u^{2t-1}\psi^2\right) + \\ + \frac{2t(t-1)}{2t-1}\int \nabla u\cdot\nabla\psi u^{2t-1}\psi + \int u^{2t}\left|\nabla\psi\right|^2.\end{aligned} \tag{20}$$

Combining (15), (20) and integrating by parts the right hand side, we have

$$\begin{aligned}&\left(\frac{p}{\frac{\kappa^2}{(N-2)^2}+1} - \frac{t^2}{2t-1}\right)\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \\ &\leq -\frac{t^2}{2t-1}\int b\cdot\nabla u u^{2t-1}\psi^2 + \frac{2t(t-1)}{2t-1}\int \nabla u\cdot\nabla\psi u^{2t-1}\psi + \int u^{2t}\left|\nabla\psi\right|^2 \\ &= \left(-\frac{t^2}{2t-1}\right)\left(-\frac{1}{2t}\right)\int u^{2t} b\cdot\nabla(\psi^2) + \frac{2t(t-1)}{2t-1}\left(-\frac{1}{4t}\right)\int u^{2t}\Delta(\psi^2) + \int u^{2t}\left|\nabla\psi\right|^2 \\ &= \frac{t}{(2t-1)}\int u^{2t} b\cdot\nabla(\psi^2) - \frac{(t-1)}{2(2t-1)}\int u^{2t}\Delta(\psi^2) + \int u^{2t}\left|\nabla\psi\right|^2.\end{aligned} \tag{21}$$

Since the right hand side in the last equality of (21) is bounded from above by

$$C\left(\int u^{2t}(|b||\nabla\psi||\psi| + |\Delta(\psi^2)| + \left|\nabla\psi\right|^2)\right),$$

the Proposition follows. □

Proof. **[End of the proof of Theorem 1.1]** We will prove Theorem 1.1 by contradiction. Suppose that u is a stable solution of (1). Set

$$A := \frac{\kappa^2}{(N-2)^2}.$$

It is evident that for

$$p > 1 + A,$$

and if

$$\frac{p - \sqrt{p^2-(1+A)p}}{1+A} < t < \frac{p + \sqrt{p^2-(1+A)p}}{1+A}, \tag{22}$$

then

$$\left(\frac{p}{\frac{\kappa^2}{(N-2)^2}+1} - \frac{t^2}{2t-1}\right) > 0.$$

Indeed, (22) is the same as

$$\left(t - \frac{p - \sqrt{p^2 - (1+A)p}}{1+A}\right)\left(t - \frac{p + \sqrt{p^2 - (1+A)p}}{1+A}\right) < 0.$$

Expand the left-hand side of the inequality above is obtained

$$t^2 - t\left[\frac{\left(p - \sqrt{p^2 - (1+A)p}\right)\left(p + \sqrt{p^2 - (1+A)p}\right)}{1+A}\right] + \frac{\left(p - \sqrt{p^2 - (1+A)p}\right)\left(p + \sqrt{p^2 - (1+A)p}\right)}{(1+A)^2} < 0$$

which is equivalent to

$$t^2 - \frac{2pt}{1+A} + \frac{p}{1+A} < 0.$$

This leads to

$$t^2(1+A) - p(2t-1) < 0.$$

Divide both sides by $2t-1$ and replace A by $\frac{\kappa^2}{(N-2)^2}$, the result is given.

Thus, for any t in the range (22), we obtain from (7) that there exists a positive constant C depending on t, N and γ such that

$$\int u^{2t+p-1}\psi^2(1+|x|^2)^{\frac{\alpha}{2}} \leq C\int u^{2t}\left(\left|\Delta(\psi^2)\right| + \left|\nabla\psi\right|^2 + |b|\left|\nabla\psi\right||\psi|\right). \tag{23}$$

Let $\chi \in C_c^\infty(\mathbb{R};[0,1])$ such that $\chi = 1$ on the interval $[-1,1]$ and $\chi = 0$ outside the interval $[-2,2]$. For R large enough, we put $\varphi(x) = \chi(\frac{|x|}{R})$. Recall that $|b(x)| \leq \frac{\kappa}{|x|}$. Then it is easy to see that

$$\left(\left|\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right) \leq \frac{C}{R^2}, \tag{24}$$

here and in what follows C denotes a generic positive constant which may change from line to line and is independent of R. We now replace ψ in (23) by φ^m, where $m > 1$ is chosen later, then we get

$$\int u^{2t+p-1}\varphi^{2m}(1+|x|^2)^{\frac{\alpha}{2}} \leq C\int u^{2t}\left(\left|\Delta(\varphi^{2m})\right| + \left|\nabla(\varphi^m)\right|^2 + |b|\left|\nabla(\varphi^m)\right||\varphi^m|\right). \tag{25}$$

Note that

$$\left|\nabla(\varphi^m)\right|^2 = m^2\varphi^{2m-2}\left|\nabla\varphi\right|^2$$

and

$$|\Delta(\varphi^{2m})| \leq 2m(2m-1)\varphi^{2m-2}\left|\nabla\varphi\right|^2 + 2m\varphi^{2m-1}|\Delta\varphi|.$$

Consequently, it follows from (25) that

$$\int u^{2t+p-1}\varphi^{2m}(1+|x|^2)^{\frac{\alpha}{2}} \leq C\int u^{2t}\varphi^{2m-2}\left(\left|\varphi\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right). \tag{26}$$

Applying Hölder's inequality to the right hand side of (26), one has, for $t > 0$,

$$\begin{aligned}
&\int u^{2t}\varphi^{2m-2}\left(\left|\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right) \\
&= \int u^{2t}\varphi^{2m-2}(1+|x|^2)^{\frac{\alpha}{2}\cdot\frac{2t}{2t+p-1}}\left(\left|\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right)(1+|x|^2)^{-\frac{\alpha}{2}\cdot\frac{2t}{2t+p-1}} \\
&\le \left[\int \left(u^{2t}\varphi^{2m-2}(1+|x|^2)^{\frac{\alpha}{2}\cdot\frac{2t}{2t+p-1}}\right)^{\frac{2t+p-1}{2t}}\right]^{\frac{2t}{2t+p-1}} \times \\
&\times\left[\int \left(\left(\left|\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right)(1+|x|^2)^{-\frac{\alpha}{2}\cdot\frac{2t}{2t+p-1}}\right)^{\frac{2t+p-1}{p-1}}\right]^{\frac{p-1}{2t+p-1}} \\
&= \left[\int u^{2t+p-1}\varphi^{\frac{(m-1)(2t+p-1)}{t}}(1+|x|^2)^{\frac{\alpha}{2}}\right]^{\frac{2t}{2t+p-1}} \times \\
&\times\left[\int \left(\left|\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right)^{\frac{2t+p-1}{p-1}}(1+|x|^2)^{-\frac{\alpha t}{p-1}}\right]^{\frac{p-1}{2t+p-1}}.
\end{aligned} \tag{27}$$

Let us choose m sufficiently large such that $\frac{(m-1)(2t+p-1)}{t} > 2m$. Hence, from (27), (26) and (24), we obtain

$$\int u^{2t+p-1}\varphi^{2m}(1+|x|^2)^{\frac{\alpha}{2}} \le C\int \left(\left|\Delta\varphi\right| + \left|\nabla\varphi\right|^2 + |b|\left|\nabla\varphi\right||\varphi|\right)^{\frac{2t+p-1}{p-1}}(1+|x|^2)^{\frac{\alpha}{2}} \le CR^{N-\frac{2(2t+p-1+\alpha t)}{p-1}}. \tag{28}$$

We are going to show that the exponent on the right hand side of (28) is negative provided that t is close to $\frac{p+\sqrt{p^2-(1+A)p}}{1+A}$. Then, we get

$$N - \frac{2(2t+p-1+\alpha t)}{p-1} = N - \frac{2(\alpha+2)t + 2(p-1)}{p-1} = N - 2 - \frac{2t(\alpha+2)}{p-1} < 0$$

or is it rewritten as

$$N - 2 < \frac{2t(\alpha+2)}{p-1}.$$

Therefore, we obtain

$$N - 2 < \frac{2(\alpha+2)(p+\sqrt{p^2-(1+A)p})}{(1+A)(p-1)}. \tag{29}$$

For simplicity, we set $s = N-2$. Then (29) is equivalent to

$$s < \frac{2(\alpha+2)\left(p+\sqrt{p^2-\left(1+\frac{\kappa^2}{s^2}\right)p}\right)}{(p-1)\left(1+\frac{\kappa^2}{s^2}\right)}.$$

By simple calculation, the inequality above becomes

$$(p-1)(s^2+\kappa^2) < 2(\alpha+2)\left(ps + \sqrt{p^2s^2-(s^2+\kappa^2)p}\right)$$

and then, we have

$$(p-1)(s^2+\kappa^2) - 2(\alpha+2)ps < 2(\alpha+2)\sqrt{s^2p^2-(s^2+\kappa^2)p}\,. \tag{30}$$

Case 1. If $(p-1)(s^2+\kappa^2)-2(\alpha+2)ps \leq 0$ or $\kappa^2 \leq \dfrac{2(\alpha+2)ps}{p-1} - s^2$, then (30) is true.

Case 2. Consider $(p-1)(s^2+\kappa^2)-2(\alpha+2)ps > 0$ or $\kappa^2 > \dfrac{2(\alpha+2)ps}{p-1} - s^2$. It is easy to see that (30) is equivalent to

$$(p-1)^2(s^2+\kappa^2)^2 + 4(\alpha+2)^2p^2s^2 - 4(p-1)(s^2+\kappa^2)(\alpha+2)ps < 4(\alpha+2)^2(s^2p^2-(s^2+\kappa^2)p).$$

Conspicuously, the above inequality can be rewritten as

$$(s^2+\kappa^2)(p-1)^2 - 4(\alpha+2)ps(p-1) + 4p(\alpha+2)^2 < 0.$$

This lead to

$$\kappa^2 < \frac{4ps(\alpha+2)}{p-1} - \frac{4p(\alpha+2)^2}{(p-1)^2} - s^2. \tag{31}$$

Combining (31) with the condition under consideration, we have

$$\frac{4ps(\alpha+2)}{p-1} - \frac{4p(\alpha+2)^2}{(p-1)^2} - s^2 > \frac{2(\alpha+2)ps}{p-1} - s^2.$$

We simplify the inequality above to get

$$s > \frac{2(\alpha+2)}{p-1} > 0. \tag{32}$$

Based on the initial conditions of p we infer that

$$0 < p - \sqrt{p^2-p} < 1. \tag{33}$$

Combining (31), (32) and (33) we obtain

$$s < \frac{2(\alpha+2)}{p-1}\left(p + \sqrt{p^2-p}\right). \tag{34}$$

Then inequality (30) is equivalent to

$$\kappa < \sqrt{\frac{4ps(\alpha+2)}{p-1} - \frac{4p(\alpha+2)^2}{(p-1)^2} - s^2}. \tag{35}$$

We are going to show that

$$s^2(p-1) > \frac{4ps(\alpha+2)}{p-1} - \frac{4p(\alpha+2)^2}{(p-1)^2} - s^2. \tag{36}$$

Certainly, we will prove the above by assuming the opposite.
Suppose

$$s^2(p-1) \leq \frac{4ps(\alpha+2)}{p-1} - \frac{4p(\alpha+2)^2}{(p-1)^2} - s^2.$$

It can be rewritten as

$$s^2(p-1)^2 \leq 4s(\alpha+2)(p-1) - 4(\alpha+2)^2$$

and it is equivalent to

$$[s(p-1) - 2(\alpha+2)]^2 \leq 0.$$

This leads to absurdity.
Combining (35) and (36) we have

$$\kappa < (N-2)\sqrt{p-1}$$

or it is rewritten as

$$N > 2 + \frac{\kappa}{\sqrt{p+1}}.$$

This together with (34) we get

$$2 + \frac{\kappa}{\sqrt{p-1}} < N < 2 + \frac{2(\alpha+2)}{p-1}\left(p + \sqrt{p^2-p}\right). \tag{37}$$

Two inequalities (35) and (37) are respectively equivalent to the assumption of N and κ in Theorem 1.1. Combining these two cases, we obtain that (29) holds. Finally, the exponent on the right hand side of (28) is negative when t is close to $\frac{p+\sqrt{p^2-(1+A)p}}{1+A}$. Letting $R \to \infty$ in (28), we get a contradiction. The proof is complete. □

Example 2.2. *We will provide an example of a smooth vector field b satisfying condition* (2) *for a given* $\kappa \neq 0$ *as: When N is even, we construct a smooth vector field in* $\mathbb{R}^N$ *as*

$$b(x) = \kappa\left(\frac{x_2}{1+\sum_{i=1}^{N} x_i^2}, \frac{-x_1}{1+\sum_{i=1}^{N} x_i^2}, \dots, \frac{x_N}{1+\sum_{i=1}^{N} x_i^2}, \frac{-x_{N-1}}{1+\sum_{i=1}^{N} x_i^2}\right).$$

In the case where N is odd, $b(x)$ *is chosen as*

$$b(x) = \kappa\left(\frac{x_2}{1+\sum_{i=1}^{N} x_i^2}, \frac{-x_1}{1+\sum_{i=1}^{N} x_i^2}, \dots, \frac{x_{N-1}}{1+\sum_{i=1}^{N} x_i^2}, \frac{-x_{N-2}}{1+\sum_{i=1}^{N} x_i^2}, 0\right).$$

It is not difficult to check that $b(x)$ *satisfies the condition* (2).

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